Loeb Solutions of the Boltzmann Equation

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Abstract

Existence problems for the Boltzmann equation constitute a main area of research within the kinetic theory of gases and transport theory. The present paper considers the spatially periodic case with L^1 initial data. The main result is that the Loeb subsolutions obtained in a preceding paper are shown to be true solutions. The proof relies on the observation that monotone entropy and finite energy imply Loeb integrability of non-standard approximate solutions, and uses estimates from the proof of the H-theorem. Two aspects of the continuity of the solutions are also considered.

1. The Equation

The non-linear Boltzmann equation describes among other things the macroscopic behaviour of rarefied gases, when the molecules interact by elastic collisions. The molecules move in a region of R^3 with some suitable conditions on the boundary. For simplicity we take the region as R^3/Z^3 and we adopt periodic boundary conditions. Let F be the density of the molecules in phase-space $M \equiv R^3/Z^3 \times R^3$. The expected number of molecules at time t in a region A of M is then

$$\int_A F(x, v, t)\, dx\, dv.$$

Given two molecules of initial velocities v_1, v_2, and initially separated in space, let $v_1', v_2' \equiv J(v_1, v_2, u)$ be the velocities of the molecules after collision, and with J a C^1-mapping, to be further specified below. To describe the details of the collision process we introduce a plane P orthogonal to $v_2 - v_1$, and at rest with respect to the first molecule. In this plane the impact parameter u is the vector from the first molecule to the point of intersection with P of the straight line from the second molecule at time $-\infty$ in the direction of $v_2 - v_1$. Let $B \subsetneq P$ denote the set of values of u for which one has collision. Any radial cut-off will do, but we shall for convenience use

$$B = \{u \in R^2; |u| \leqq \pi^{-\frac{1}{2}}\}.$$

Also hard potentials with an angular cut-off can be treated by the same methods. Set

$$p: R^3 \times R^3 \to R^3, \quad (v_1, v_2) \to v_1 + v_2,$$

$$T: R^3 \times R^3 \to R, \quad (v_1, v_2) \to |v_1|^2 + |v_2|^2,$$

$$\Sigma: R^3 \times R^3 \to R^3 \times R^3, \quad (v_1, v_2) \to (v_2, v_1).$$

On physical grounds J is restricted by

(1) $\quad p \circ J_u = p, \quad T \circ J_u = T, \quad \Sigma \circ J_u = J_u \circ \Sigma, \quad J \circ J = \text{identity}.$

Using a statistical hypothesis and retaining only binary collisions, BOLTZMANN expressed $\partial F/\partial t$ through a balance between the density of the number of molecules entering a region of collision and leaving it, as

(2) $\quad \partial F/\partial t(x, v_1, t) + v_1 \cdot \nabla_x F(x, v_1, t,) = QF(x, v_1, t) \quad (t > 0).$

Here ∇_x is the gradient with respect to the position $x \in R^3/Z^3$, and Q is called the collision operator:

(3) $\quad QF(x, v_1) \equiv \int_{R^3 \times B} (F(x, v_1') F(x, v_2') - F(x, v_1) F(x, v_2)) k(v_1, v_2, u) \, dv_2 \, du,$

which we shall sometimes abbreviate to

$$QF(x, v_1) \equiv \int (F \otimes F' - F \otimes F) k \, dv_2 \, du.$$

The non-negative, measurable function k is required to satisfy

(4) $\quad k \circ \Sigma = k \circ J_u = k,$

together with

(5) $\quad k(v_1, v_2) \leqq C_k(1 + |v_1|^\lambda + |v_2|^\lambda)$

for some constant λ with $0 \leqq \lambda < 2$. For a more complete discussion of (1)–(5) see [C] or [TM].

In the physical case with radial cut-off we take

$$k(v_1, v_2, u) = |v_2 - v_1| \quad \text{for} \quad u \in B, \quad k(v_1, v_2, u) = 0 \quad \text{otherwise}.$$

In the discussion below we also consider a truncated version k^n,

$$k^n(v_1, v_2, u) = k(v_1, v_2, u) \quad \text{for} \quad u \in B \quad \text{and} \quad v_1^2 + v_2^2 \leqq n^2,$$

$$k^n(v_1, v_2, u) = 0 \quad \text{otherwise}.$$

The Boltzmann equation (2) is to be solved under an initial condition

$$F(x, v, 0) = F_0(x, v).$$

Computing formally in (2), we conclude that $F \geqq 0$ if $F_0 \geqq 0$, and that

$$\int F_0 \, dx \, dv = \int F \, dx \, dv, \qquad \int v^2 F_0 \, dx \, dv = \int v^2 F \, dx \, dv$$

for all t, so that mass and energy are conserved quantities. Also by formal computation, the H-quantity (an entropy function) is decreasing as t increases. Here

$$HF(t) \equiv \int F \log F \, dx \, dv.$$

Thus a natural setting for the Boltzmann equation is in a space of functions F such that

$$F \geqq 0, \qquad F, v^2 F, F \log F \in L^1(M). \tag{6}$$

2. A Truncated Version

Before considering the full equation (2) with initial values satisfying (6), we shall first look at a simpler case of a truncated, integrated Boltzmann equation,

$$F(x + tv_1, v_1, t) = F_0(x, v_1) + \int_0^t Q_n F(x + sv_1, v_1, s) \, ds \tag{7}$$

with

$$Q_n(F(x, v_1) \equiv \int_{R^3 \times B} (\langle F(x, v_1') F(x, v_2') \rangle - \langle F(x, v_1) F(x, v_2) \rangle) \, k^n(v_1, v_2, u) \, dv_2 \, du$$

and

$$\langle F \otimes F \rangle \equiv \begin{cases} F \otimes F \text{ if } |F \otimes F| \leqq n, \\ n \text{ sign } F \otimes F \text{ otherwise.} \end{cases}$$

In this case the collision operator is Lipschitz-continuous and the solution is easy to obtain.

Theorem 1. *For any initial value F_0 in L_+^∞, there exists a unique nonnegative solution of* (7) *in L^∞. The solution conserves mass and energy, and the H-function is nonincreasing.*

Proof. It is enough to consider F_0 with support in

$$R^3/Z^3 \times \{v; |v| \leqq n\}.$$

For any functions F, G

$$|\langle F(x, v_1) F(x, v_2) \rangle - \langle G(x, v_1) G(x, v_2) \rangle|$$
$$\leqq |F(x, v_1)| \cdot |F(x, v_2) - G(x, v_2)| + |G(x, v_2)| \cdot |F(x, v_1) - G(x, v_1)|.$$

Since the integration in Q_n is over bounded sets, it follows that

$$\|Q_n F - Q_n G\|_\infty \leqq K(\|F\|_\infty + \|G\|_\infty) \cdot \|F - G\|_\infty$$

for some constant K. Thus Q_n is locally Lipschitz-continuous in L_∞, and so there exists a unique local solution F of (7). But

$$\|Q_n F\|_\infty \leqq 8\pi n^6 C_k \equiv K',$$

and so

$$\|F(t)\|_\infty \leqq \|F_0\|_\infty + tK'.$$

Thus F exists for all $t > 0$. To complete the proof it remains to verify that $F \geqq 0$.

By using L^1-differentiation it is easy to show that F is the only solution of the equation

$$G(x + v_1 t, v_1, t) = \exp(-H(x, v_1, t))\, F_0(x, v_1) + \int_0^t \exp(-H(x, v_1, t) + H(x, v_1, s))\, \tilde{Q}_n G(x + v_1 s, v_1, s)\, ds \equiv T_t \tilde{Q}_n G \tag{8}$$

where

$$H(x, v_1, t) \equiv \int_0^t \int_B \int_{R^3} F(x + v_1 s, v_2, s)\, k^n(v_1, v_2, u)\, dv_2\, du\, ds,$$

and

$$\tilde{Q}_n F(x, v_1) \equiv \int \{\langle F \otimes F' \rangle (x, v_1, v_2, u) + F \otimes F(x, v_1, v_2) - \langle F \otimes F \rangle (x, v_1, v_2)\}\, k^n(v_1, v_2, u)\, dv_2\, du.$$

We note that $\tilde{Q}_n$ has a structure similar to Q_n, but it is also order-preserving in the sense that

$$\tilde{Q}_n G_1 \geqq \tilde{Q}_n G_2 \geqq 0 \quad \text{if} \quad G_1 \geqq G_2 \geqq 0.$$

For small t, the solution G of (8) is the limit of the increasing sequence

$$G_1 = 0, \qquad G_j = T_t \tilde{Q}_n G_{j-1}, \qquad j \geqq 2.$$

It follows that the solution of (8), hence of (7), satisfies $F \geqq 0$ for small t. By a continuation argument we get $F \geqq 0$ for all $t > 0$.

The conservation of mass and energy follow from integration of (7) and (7) multiplied by v^2. It can be shown from (7) by a straightforward calculation that the mapping

$$F \log F : R_+ \to L^1(M)$$

is differentiable, if for some $\varepsilon > 0$, $F_0 > \varepsilon \exp(-v^2)$ when $|v| \leqq n$. From these statements the formal proof of the H-theorem can be made rigorous (*cf.* [Ar1]). The solution F_ε of (7) with initial value $F_0 + \varepsilon \exp(-v^2)$ satisfies $\lim_{\varepsilon \to 0} F_\varepsilon(t) = F(t)$ in $L^\infty(M)$, uniformly on any interval $0 \leqq t \leqq T$. We conclude that the mapping

$$HF : t \to HF(t) = \int_M F(x, v, t) \log F(x, v, t)\, dx\, dv$$

is non-increasing as a function of t. $\square$

3. Non-Standard Aspects

One way to study the Boltzmann equation using non-standard methods is to start from the truncated case of Theorem 1, but within an enlarged universe $V(^*R)$. That approach has suggested a new concept of solution, based on the Loeb integral, which in fact until now has yielded the only results on existence far from equilibrium. Before describing them we shall in this section give as a background a quick sketch of some relevant facts about non-standard analysis.

In non-standard analysis, proper ordered extensions $^*R$ of the field R of real numbers are used. A simple example can be constructed by fixing a free ultrafilter Δ on the natural numbers N. Two sequences of real numbers $\{r_i\}$ and $\{s_i\}$ are equivalent if $r_i = s_i$ for all i in an element U of Δ. The equivalence classes form the non-standard reals. An element $a \in {}^*R$ is infinitesimal if $|a| < c$ for every $0 < c \in R$. $^*R$ contains non-zero infinitesimal numbers, and we sometimes write $a \approx 0$ to denote that a is infinitesimal. The standard part of $a \in {}^*R$ is

$$\operatorname{st} a \equiv {}^\circ a \equiv \begin{cases} c & \text{if} \quad a \in {}^*R, \quad c \in R, \quad a - c \approx 0, \\ \infty & \text{if} \quad a > n \quad (n \in N), \\ -\infty & \text{if} \quad a < -n \quad (n \in N). \end{cases}$$

The near-standard part of $^*R$ is

$$\operatorname{ns} {}^*R = \{x \in {}^*R;\ {}^\circ x \in R\}.$$

The elements of ns $^*R$ are said to be finite, and the other elements of $^*R$ are called infinite. The superstructure on a set $S = S_0$ is by definition

$$V(S) = \bigcup_{n<\infty} S_n,$$

where S_{n+1} is the set of all subsets of $\bigcup_{j=0}^{n} S_j$ $(n \in N)$. Along with $^*R$ a map

$$^* : V(R) \to V(^*R)$$

is given with

$$^*r = r \quad (r \in R),$$

and satisfying the transfer principle. This states that if $S_1, \ldots, S_n \in V(R)$, and if E is an elementary statement true of $S_1, \ldots, S_n$ in $V(R)$, then it is true of $^*S_1, \ldots, {}^*S_n$ in $V(^*R)$. (A statement is elementary if it is built up from ε and $=$, using the propositional connectives and bounded quantifiers $\forall x \in y$, $\exists x \in y$.) Standard sets in $V(^*R)$ are *-images of sets in $V(R)$. Internal sets in $V(^*R)$ are elements of standard sets in $V(^*R)$.

We next describe the relevant measure spaces, starting from a standard set X in a denumerably comprehensive superstructure $V(^*R)$. (In what follows, X will be taken as $^*M$.) Denumerably comprehensive means that if $S \in V(R)$, $A_n \in {}^*S$

($n \in N$), then the sequence $\{A_n; n \in N\}$ is the restriction to N of an internal function from $^*N$ into $^*S$.

Let Ψ be an internal algebra of internal subsets of X (i.e. $A, B \in \Psi \Rightarrow A \cup B$, $X - A \in \Psi$), and let ν be an internal, finitely additive mapping from Ψ into $^*R_+$ (*i.e.* $\nu(A \cup B) = \nu(A) + \nu(B)$ if $A \cap B = \emptyset$). Define $L\nu(A) = {}^0\nu(A)$ $(A \in \Psi)$, and take $\sigma\Psi$ as the smallest σ-algebra (in the standard sense) of subsets of X with $\sigma\Psi \supset \Psi$. With

$$L\nu(B) = \inf_{A \in \Psi,\ B \subset A} L\nu(A),$$

we obtain a standard measure space $(X, \sigma\Psi, L\nu)$. The Loeb space $(X, L\Psi, L\nu)$ is by definition the completion of $(X, \sigma\Psi, L\nu)$.

We say that

$$f: X \to {}^*R$$

is S-integrable, if

i) f is Ψ-measurable, ii) $\int_X (|f| - |f| \wedge w)\, d\nu \approx 0 \quad (w \in {}^*N - N)$,

iii) $\int_X (|f| \wedge w^{-1})\, d\nu \approx 0 \quad (w \in {}^*N - N)$.

Below we use the following properties of this concept of integration.

Proposition. *Suppose $f: X \to {}^*R$ is S-integrable, $g: X \to {}^*R$ is Ψ-measurable and $|g(x)| \leqq |f(x)|$ for all $x \in X$. The g is S-integrable.*

Proposition. *Suppose $f: X \to {}^*R$ is S-integrable. Then ${}^\circ f$ is $L\nu$-integrable and*

$${}^\circ\int_A f\, d\nu = \int_A {}^\circ f\, dL\nu$$

for any $A \in \Psi$.

The reader is urged to consult [L] and [An] for a more complete discussion.

Let us also point out that existence results often can be obtained in the denser Loeb context, even when there are no solutions in the standard setting. The following control problem is a simple example.

Example. Minimize the cost function

$$J(u) = \int_0^T (|y(t)|^2 - |u(t)|^2)\, dt$$

under the constraint

$$-1 \leqq u(t) \leqq 1, \qquad t \in [0, T],$$

and with

$$y(t) = \int_0^t u(s)\, ds.$$

Evidently

$$J(u) \geqq \int_0^T -|u|^2 dt \geqq \int_0^T -1\, dt = -T.$$

Choosing

$$u_n(t) = \begin{cases} 1 & \text{if} \quad 0 \leqq t < T/2n \\ -1 & \text{if} \quad T/2n \leqq t < T/n, \end{cases}$$

and extending it periodically, we obtain

$$0 \leqq y_n(t) \leqq 1/2n,$$

$$-T \leqq J(u_n) \leqq \int_0^T ((T/2n)^2 - 1)\, dt \to -T \quad (n \to \infty).$$

On the other hand

$$J(u) = -T$$

only if $|u| = 1$, together with $y = 0$, which implies that $u = 0$. Hence the infimum is never attained.

But the corresponding Loeb problem on $*[0, T]$ has infinitely many solutions. Let $L\,dt$ denote the Loeb measure obtained from $*dt$. For $n \in *N - N$ let $\tilde{u}(t) = u_n(t)$ be defined as above. Then

$$|\tilde{y}(t)| = \left|\int_0^t \tilde{u}(t)\, L\, dt\right| = \left|{}^\circ\!\int_0^t \tilde{u}(t)\, *dt\right| \leqq {}^\circ|T/2n| = 0,$$

and so

$$J(\tilde{u}) = \int_0^T |\tilde{y}|^2 - |\tilde{u}|^2\, L\, dt = \int_0^T 0 - 1\, L\, dt = -T. \qquad \square$$

This example has a standard interpretation ([Cu1], [Cu2]] within the theory of relaxed controls (as measure-valued functions instead of point-valued ones). As for the Loeb solutions of (2) on ns $*M$ discussed below, such an interpretation does not hold due to the product structure within the collision operator. It seems as if any condensed interpretation with respect to M would have to invoke further properties from the Boltzmann equation set-up.

4. Spatially Dependent Loeb Solutions of the Boltzmann Equation

We shall start from theorem 1 extended by transfer for $n \in *N - N$. Our aim is to substitute $\langle f \otimes f \rangle$ in (7) by $f \otimes f$, $*R^3$ by ns $*R^3$, and change the *Lebesgue integrals into Loeb integrals. Let f be the solution of the non-standard cutoff Boltzmann equation (7) with initial condition

$$f_0(x, v_1) = *F_0(x, v_1) \wedge n + n^{-1} \exp(-v^2).$$

Here F_0 lies in the positive cone of $L^1(M)$ and has finite energy and entropy. A wider class of non-negative, *Lebesgue-measurable initial data with finite mass, energy, and entropy, together with a condition at infinity, can also be handled by the methods below.

Theorem 2. *Suppose* $0 \leqq \lambda < 2$ *in* (5). *Then* $°f$ *is a solution of the Boltzmann equation in the following sense; for Loeb a.e.* $(x, v_1) \in \text{ns}\,{}^*M$

$$(9) \qquad °f(x + tv_1, v_1, t) = F_0 \circ \text{st}\,(x, v_1) +$$

$$+ \int_0^t \int_{\text{ns}^*R^3 \times B} °f(x + sv_1, v_1', s)\, °f(x + sv_1, v_2', s)\, k \circ \text{st}\,(v_1, v_2, u)\, L\, dv_2\, du\, ds$$

$$- \int_0^t {}^{\circ}\!\!\int_{\text{ns}^*R^3 \times B} °f(x + sv_1, v_1, s)\, °f(x + sv_1, v_2, s)\, k \circ \text{st}\,(v_1, v_2, u)\, L\, dv_2\, du\, ds.$$

This theorem is an immediate consequence of the following three lemmas.

Lemma 3. *For Loeb a.e.* $(x, v_1) \in \text{ns}\,{}^*M$, *the function*

$$(v_2, u, s) \to f(x + sv_1, v_2, s)\, k^n(v_1, v_2, u)$$

is S-integrable, and

$$°\int_0^t \int_{{}^*R^3 \times B} f(x + sv_1, v_2, s)\, k^n(v_1, v_2, u)\, {}^*dv_2\, du\, ds =$$

$$\int_0^t \int_{\text{ns}^*R^3 \times B} °f(x + sv_1, v_2, s)\, k \circ \text{st}\,(v_1, v_2, u)\, L\, dv_2\, du\, ds.$$

Sketch of proof. Essentially we only have to check the definition of S-integrability. But i) is immediate and ii) is a consequence of mass conservation and the finiteness of the H-function. Finally iii) is implied by the conservation of energy. □

For a complete proof see [Ar2].

Lemma 4. *For Loeb a.e.* $(x, v_1) \in \text{ns}\,{}^*M$

$$(10) \quad °\int_0^t \int_{{}^*R^3 \times B} \langle f \otimes f \rangle\, (x + sv_1, v_1, v_2, s)\, k^n(v_1, v_2, u)\, {}^*dv_2\, du\, ds =$$

$$= \int_0^t \int_{\text{ns}^*R^3 \times B} °f(x + sv_1, v_1, s)\, °f(x + sv_1, v_2, s)\, k \circ \text{st}\,(v_1, v_2, u)\, L\, dv_2\, du\, ds.$$

Sketch of proof. By the first non-standard proposition of the previous section, the product of a bounded *Lebesgue-measurable function and an S-integrable one is S-integrable. From (8) it is easy to see that the mapping $s \to f(x + sv_1, v_1, s)$

is bounded on $[0, t]$ for Loeb a.e. $(x, v_1) \in \text{ns}\ {}^*M$. Using this fact and lemma 3 we conclude that

$$f(x + sv_1, v_1, s) f(x + sv_1, v_2, s)\, k^n(v_1, v_2, u)$$

is S-integrable in (v_2, u, s) for Loeb a.e. $(x, v_1) \in \text{ns}\ {}^*M$. This also holds for $\langle f \otimes f \rangle$, since

$$0 \leqq \langle f \otimes f \rangle \leqq f \otimes f.$$

Hence essentially the lemma is a consequence of the second non-standard proposition from the preceding section. □

For a complete proof see [Ar2].

Lemma 5. *For Loeb a.e.* $(x, v_1) \in \text{ns}\ {}^*M$

$$(11)\quad {}^\circ\int_0^t \int_{{}^*R^3 \times B} \langle f \otimes f \rangle (x + sv_1, v_1', v_2', s)\, k^n(v_1, v_2, u)\, {}^*dv_2\, du\, ds =$$

$$= \int_0^t \int_{\text{ns}\,{}^*R^3 \times B} {}^\circ f(x + sv_1, v_1', s)\, {}^\circ f(x + sv_1, v_2', s)\, k \circ \text{st}\,(v_1, v_2, u)\, L\, dv_2\, du\, ds.$$

Proof. The following discussion holds for Loeb a.e. $(x, v_1) \in \text{ns}\ {}^*M$. Let χ be the characteristic function of any set

$$\{(v_1, v_2);\, v_1^2 + v_2^2 \leqq m^2\} \quad (m \in N).$$

Since the integrals

$$\int \chi f(x + sv_1, v_j', s)\, {}^*dx\, dv_1\, dv_2\, du\, ds \quad (j = 1, 2)$$

both are finite, it follows that

$$\chi f(x + sv_1, v_1', s) \quad \text{and} \quad \chi f(x + sv_1, v_2', s)$$

are finite for Loeb a.e. $(v_2, u, s) \in \text{ns}\ {}^*R^3 \times B \times [0, t]$. Thus

$${}^\circ\langle f \otimes f \rangle (x + sv_1, v_1', v_2', s) = {}^\circ f(x + sv_1, v_1', s)\, {}^\circ f(x + sv_1, v_2', s)$$

for Loeb a.e. $(v_2, u, s) \in \text{ns}\ {}^*R^3 \times B \times [0, t]$.

We next cousider the S-integrability in (v_2, u, s) of $\langle f \otimes f' \rangle k^n$. Since $\langle f \otimes f \rangle k^n$ is S-integrable, it is enough to prove for some finite $j > 1$ that the function $\langle f \otimes f' \rangle k^n$ is S-integrable on the set

$$\Omega_j = \{(v_2, u, s) \in {}^*R^3 \times B \times [0, t];\, \langle f \otimes f' \rangle \geqq j \langle f \otimes f \rangle\}.$$

But on Ω_2

$$0 \leqq \langle f \otimes f' \rangle k^n \leqq 2(\langle f \otimes f' \rangle - \langle f \otimes f \rangle)\, k^n,$$

and it suffices to consider the S-integrability of the right member on Ω_2. We only have to check ii), iii). Now

$$f \otimes f' \geqq j f \otimes f$$

on Ω_j for $j>1$, and

$$(12)\quad 0 \leqq \int_{\Omega_j} (\langle f\otimes f'\rangle - \langle f\otimes f\rangle)\, k^n\, {}^*dv_2\, du\, ds$$

$$\leqq (\log j)^{-1} \int_{\Omega_j} (\langle f\otimes f'\rangle - \langle f\otimes f\rangle) \log (f\otimes f'/f\otimes f)\, k^n\, {}^*dv_2\, du\, ds$$

$$\leqq C/\log j,$$

with C finite.

We can majorize a.e. by a finite C in the last member of (12) by following the usual proof of the H-theorem. Indeed, we use the fact the integrand is positive, and the inequality

$$\int_A (\langle f\otimes f'\rangle - \langle f\otimes f\rangle)\, k^n \log (f\otimes f'/f\otimes f)\, {}^*dx\, dv_1\, dv_2\, du\, ds$$

$$\leqq \left\{\int_{{}^*M} (f_0 \log f_0 + \exp(-v^2) + v^2 f_0)\, {}^*dx\, dv\right\} \in \text{ns}\, {}^*R_+,$$

when

$$A = {}^*(M\times R^3\times B\times R_+).$$

To prove ii) we let $w\in {}^*N - N$ and set

$$\Omega = \{(v_2, u, s)\in \Omega_2; \langle f\otimes f'\rangle\, k^n > w\}.$$

It follows from (12) that

$$\int_\Omega {}^*dv_2\, du\, ds \approx 0,$$

and from $f_0>0$ together with (8), that

$$\int_\Omega \langle f\otimes f\rangle\, k^n\, {}^*dv_2\, du\, ds > 0 \quad \text{if} \quad \int_\Omega {}^*dv_2\, du\, ds > 0.$$

Then

$$0 < j_0^{-2} \equiv \int_\Omega \langle f\otimes f\rangle\, k^n\, {}^*dv_2\, du\, ds \approx 0,$$

since $\langle f\otimes f\rangle$ is S-integrable. Hence

$$\int_{\Omega-\Omega_{j_0}} \langle f\otimes f'\rangle\, k^n\, {}^*dv_2\, du\, ds \approx 0.$$

As for the remaining part of Ω,

$$\int_{\Omega_{j_0}} (\langle f\circ f'\rangle - \langle f\otimes f\rangle)\, k^n\, {}^*dv_2\, du\, ds \approx 0$$

by (12). This proves ii).

To prove iii) we let $w\in {}^*N - N$. It follows from $f_0>0$, and the S-integrability of $\langle f\otimes f\rangle\, k^n$, that

$$0 < j_0^{-2} \equiv \int_{\Omega_2} \langle f\otimes f\rangle\, k^n \wedge w^{-1}\, {}^*dv_2\, du\, ds \approx 0.$$

Hence

$$\int_{\Omega_2-\Omega_{j_0}} \langle f\otimes f'\rangle\, k^n \wedge w^{-1}\, {}^*dv_2\, du\, ds \approx 0.$$

As for the remaining part of Ω_2, by (12)

$$0 \leqq \int_{\Omega_{j_0}} \langle f\otimes f'\rangle\, k^n \wedge w^{-1}\, {}^*dv_2\, du\, ds \leqq \int_{\Omega_{j_0}} \langle f\otimes f'\rangle\, k^n\, {}^*dv_2\, du\, ds < 2C/\log j_0 \approx 0.$$

This proves iii), and so the S-integrability.

Finally to obtain (11) we shall also check that

$$\lim_{\nu\to\infty} {}^\circ\!\int_{A_\nu} \langle f\otimes f'\rangle\, k^n\, {}^*dv_2\, du\, ds = 0,$$

where

$$A_\nu = \{(v_2, u, s)\in \Omega_2;\, |v_2| \geqq \nu\}.$$

Suppose the limit equals $\varepsilon > 0$. By (12) we can choose $j\in N$, such that

$${}^\circ\!\int_{\Omega_j} (\langle f\otimes f'\rangle - \langle f\otimes f\rangle)\, k^n\, {}^*dv_2\, du\, ds < \varepsilon/2.$$

This however, leads to a contradiction, since by lemma 4.

$$\lim_{\nu\to\infty} {}^\circ\!\int_{A_\nu} \langle f\otimes f'\rangle\, k^n\, {}^*dv_2\, du\, ds = 0. \qquad \square$$

Proof of Theorem 2. As stated above, f is a non-standard solution of (7). To obtain (9), we now only have to apply the standard part mapping to (7) and use (10 and (11). □

5. Comments

It follows from the corresponding properties of f that ${}^\circ f$ conserves mass and first moments. Also the energy and the H-function are bounded from above by their initial values.

If $F: R_+ \to L^1_+(M)$ is a solution of the Boltzmann equation., then $F\circ \mathrm{st}$ is a Loeb solution with respect to ns$^*M$. Thus if uniqueness holds in the corresponding Loeb problem (9), then the Loeb solution is an extension of the solution on M to the denser space $^*M$. So far such Loeb uniqueness has only been obtained using ideas from classical proofs, as for the spatially homogeneous case with finite fourth moments and for some spatially inhomogeneous cases with data close to equilibrium.

As noted in [Ar2], the solution ${}^\circ f$ of (9) is t-continuous in the sense of [TM] p. 343, i.e. for Loeb a.e. $(x, v_1)\in \mathrm{ns}\,{}^*M$, given $t > 0$ and $\eta\in R_+$, there is a $\delta\in R_+$, such that

$$|{}^\circ f(x+tv_1, v_1, t) - {}^\circ f(x+t'v_1, v_1, t')| < \eta \quad \text{if } |t-t'| < \delta.$$

Moreover

Theorem 3. *The solution* $^\circ f$ *of* (9) *is S-continuous as a mapping*

$$^\circ f\colon \mathrm{ns}\ {}^*R_+ \to \mathrm{Loeb}\ L^1(\mathrm{ns}\ {}^*M),$$

i.e. *given* $t \geqq 0$ *and finite, and* $\eta \in R_+$, *there is a* $\delta \in R_+$, *such that*

$$\int_{\mathrm{ns}\,{}^*M} |f(x+tv_1, v_1, t) - f(x+t'v_1, v_1, t')|\, L\, dx\, dv_1 < \eta \tag{13}$$

provided $|t-t'| < \delta$.

Proof. It follows from the conservation of energy that we have only to prove (13) when the integration is over an arbitrary bounded *Lebesgue-measurable subset of ns $^*M$. Given such a subset A, outside of a *Lebesgue-measurable subset $E \subset A$ of arbitrarily small measure, the two integrands in (9) are each Loeb integrable with respect to all variables. It is a consequence of the H-theorem that the part of the integral in (13) coming from E can be made arbitrarily small uniformly with respect to t and t' by taking the measure of E small enough.

To integrate in (13) over $A - E$ with say $t < t'$, we set

$$G_\nu = \{(x, v_1, v_2, u, s) \in (A - E) \times \mathrm{ns}\ {}^*R^3 \times B \times [t, t']; |v_2| \leqq \nu\},$$

$$G = \cup\ G_\nu, \quad \nu \in N.$$

Now

$$\int_G {}^\circ f \otimes {}^\circ f k \circ \mathrm{st}\, L\, dx\, dv_1\, dv_2\, du\, ds = \lim_{\nu\to\infty} \int_{G_\nu} {}^\circ f \otimes {}^\circ f k \circ \mathrm{st}\, L\, dx\, dv_1\, dv_2\, du\, ds.$$

Since the measure of G_ν tends to zero when t' tends to t,

$$\int_{G_\nu} {}^\circ f \otimes {}^\circ f k \circ \mathrm{st}\, L\, dx\, dv_1\, dv_2\, du\, ds$$

also tends to zero, when t' tends to t.

An analogous discussion of convergence can be carried through for

$$\int_G {}^\circ f \otimes {}^\circ f'\, k \circ \mathrm{st}\, L\, dx\, dv_1\, dv_2\, du\, ds.$$

This completes the proof of the theorem. □

Note added in proof. I thank P. LOEB for bringing to my attention a simplification used in the above proofs.

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