Stress-Free Joints and Polycrystals

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Dedicated to Jerry Ericksen on the Occasion of his 60^{th} Birthday

Introduction

Introduction

Suppose two or more stress-free homogeneous solid bodies are joined firmly together along various surfaces at a temperature θ_0. As the temperature is changed, the joined body will deform in some way with null traction at its boundary. Generally, we expect the joined body to build up stress unless some special conditions are satisfied by the orientation and constitution of the bodies, and by the shapes of the dividing surfaces.

ERICKSEN [1] has given a description of stress-free joints for two homogeneous bodies of the same material,* but of different orientation, under a mild condition on the material response. It is the purpose of this paper to analyze stress-free joints in bodies composed of different materials and in joints composed of more than two bodies. ERICKSEN'S condition is essentially that a single, homogeneous, unstressed body, made of the same material as the pair, deforms homogeneously with a known Cauchy-Green strain as the temperature θ changes. He points out that a similar theory would arise if θ were interpreted as the moisture content. In this way, some of the solutions can be illustrated by joints used commonly by cabinetmakers.

* Precisely, the two Cauchy-Green strain tensors describing the deformation of the bodies with changes in temperature are equal, when referred to appropriate reference configurations.

The problem involves two parts: the description of how the unstressed bodies deform, and the formulation of the notion that the bodies continue to stick together as θ changes. The latter is strictly analagous to certain rules which arise in the study of coherent phase transformations. JAMES [2] finds "rules of coherence" for phase transformations in which a parent phase and several variants of a daughter phase meet at a point. Although these rules are purely kinematic, they place quite severe restrictions on the arrangements of the phases. In this paper I use these rules and other ideas from joinery to describe the simplest stress-free joints consisting of more than two bodies.

The possibility that a polycrystal can build up stress with changes of temperature seems ignored in the current understanding of solidification and annealing, except in ceramics [3] where the stress built up during cooling can break up the material. Theories of annealing, for example, rely heavily on the energetics of grain boundaries. It seems likely in some cases that bulk energies associated with the stresses built up in the grains themselves could be as important as the surface energies. It is intriguing to note that many minerals, especially those of lower symmetry, do not occur naturally in polycrystalline forms. However, even the familiar hexagonal or tetragonal polycrystalline metals will build up stress unless some rather special conditions are met (*cf.* § 4 and 6).

To extend the results to alloys, it is desirable to have a treatment of stress-free joints in which the bodies are composed of different materials. In § 5 I analyze joints of this kind for two bodies, and some results for several bodies of different material are given in § 6. The question of whether two bodies of different material can be joined to remain stress-free partly reduces to a question of whether two triangular domains in a certain plane overlap.

For two bodies of the same material, there is a prescription for making a stress-free joint with a given dividing plane which works for all materials. There is only one such universal prescription. In § 6 I find all stress-free polycrystals which are universal in this sense.

Some of the results of § 5 are obtained from a linearized version of the problem. Although thermal stress is usually associated with infinitesimal deformation, it is desirable *not* to linearize the equations, and I do so only where it seems unavoidable. The reason for this is that the nonlinear equations bear a formal similarity to the equations used to find twinned configurations. Thus, the analysis delivers information about twinning.

1. Formulation

Let two bodies* occupy homogeneous, unstressed, disjoint reference configurations $\mathscr{R}$ und $\hat{\mathscr{R}}$ at the temperature θ_0. If we allow the temperature to vary over some set $\mathscr{I}$ containing θ_0, but we continue to apply null traction to the

* A hat shall always denote quantities associated with the second body; if more than two bodies are considered, subscripts 1, 2, ... will be associated with them. The subscript 0 will always indicate that the quantity is associated with the reference configuration.

boundary of each body, each body will deform in some manner. This deformation will not be unique, since a rigid rotation and translation of either body, at any value of θ in $\mathcal{I}$, will not affect the given conditions. However, it is plausible to assume that the deformation of each body is homogeneous and is governed by unique functions $U(\theta)$, $\hat{U}(\theta)$ representing the right stretch tensors as functions of $\theta \in \mathcal{I}$. The solutions of the appropriate boundary value problems do in fact deliver unique right stretch tensors for a large class of stable constitutive relations, as discussed in § 2. In any case, I shall assume the deformations of the bodies are of the forms

$$\begin{aligned} y &= F(\theta)\, x + c(\theta) = R(\theta)\, U(\theta)\, x + c(\theta), \qquad x \in \mathcal{R}, \\ y &= \hat{F}(\theta)\, x + c(\theta) = \hat{R}(\theta)\, \hat{U}(\theta)\, x + \hat{c}(\theta), \qquad x \in \hat{\mathcal{R}}, \end{aligned} \tag{1.1}$$

for some rotations $R(\theta)$, $\hat{R}(\theta)$ and translations $c(\theta)$, $\hat{c}(\theta)$ which can be any functions of θ in $\mathcal{I}$ whose values are proper orthogonal tensors and vectors.

I wish to join the bodies at the temperature θ_0. I assume that on $\partial\mathcal{R}$ and on $\partial\hat{\mathcal{R}}$ congruent surfaces have been cut, so that by a suitable rotation R_0^T and translation c_0 of $\hat{\mathcal{R}}$, the two surfaces can be brought into coincidence. This may involve restricting attention to subsets of $\mathcal{R}$ and $\hat{\mathcal{R}}$, but this will not affect any of the other assumptions we have made. Also, no greater generality would be obtained by allowing *both* bodies to translate and rotate before joining them, since a subsequent rotation and translation of the joined bodies could always be done so as to bring the first body back into the configuration $\mathcal{R}$, again, without affecting any of the other assumptions.

Let the dividing surface be $\mathcal{S}$ and let $\hat{\mathcal{R}}' = R_0^T \hat{\mathcal{R}} + c_0$. A natural expression of the idea that the bodies are joined firmly along $\mathcal{S}$ is the statement that the deformation of $\hat{\mathcal{R}} \cup \hat{\mathcal{R}}'$ is continuous for all θ in $\mathcal{I}$. The condition of continuity is simply that for all $x \in \mathcal{S}$ and all $\theta \in \mathcal{I}$,

$$(\hat{R}(\theta)\, \hat{U}(\theta)\, R_0 - R(\theta)\, U(\theta))\, x + (\hat{c}(\theta) - c(\theta)) = 0. \tag{1.2}$$

Equation (1.2) implies that $\hat{c}(\theta) = c(\theta)$ for all $\theta \in \mathcal{I}$; without loss of generality, we put $\hat{c}(\theta) = c(\theta) = 0$. Similarly, if (1.2) is satisfied with rotations $\hat{R}(\theta)$, $R(\theta)$, we can premultiply it by $R^T(\theta)$, define the *relative rotation* $\bar{R}(\theta)$ by $\bar{R}(\theta) = R^T(\theta)\, \hat{R}(\theta)$, and write (1.2) in the equivalent form

$$(\bar{R}(\theta)\, \hat{U}(\theta)\, R_0 - U(\theta))\, x = 0, \qquad \forall\, x \in \mathcal{S}. \tag{1.3}$$

One problem to be studied is the following: *given* $\hat{U}(\theta)$ *and* $U(\theta)$, $\theta \in \mathcal{I}$, *find a constant rotation* R_0 *and a temperature-dependent rotation* $\bar{R}(\theta)$, $\theta \in \mathcal{I}$, *such that* (1.3) *is satisfied for all* x *on the surface* $\mathcal{S}$.

By differentiating (1.3) with respect to x in various directions tangent to $\mathcal{S}$, we see that $\bar{R}\hat{U}R_0 - U$ has at least a two-dimensional null-space. Thus, there are two alternatives implied by (1.3):

(i) The null-space of $\bar{R}(\theta)\, \hat{U}(\theta)\, R_0 - U(\theta)$ is three-dimensional *for all* $\theta \in \mathcal{I}$, so $\bar{R}(\theta)\, \hat{U}(\theta)\, R_0 = U(\theta)$ and $\mathcal{S}$ is *any* surface; or

(ii) The null-space of $\bar{R}(\theta)\,\hat{U}(\theta)\,R_0 - U(\theta)$ is two dimensional for at least one value of θ in $\mathscr{I}$, so $\mathscr{S}$ is a plane*.

Therefore, if $\mathscr{R}$ and $\hat{\mathscr{R}}'$ are joined along a curved interface, either the deformation gradients referred to $\mathscr{R}$ and $\hat{\mathscr{R}}'$ are equal for all θ in $\mathscr{I}$, or the joint builds up stress with changes in temperature. Since every stress-free joint which can be made with a curved interface can also be made with a plane one, I shall formulate the problem in terms of plane interfaces. To render the analysis complete, I shall simply point out the cases in which (i) is fulfilled.

With this understanding in mind, I shall assume that $\mathscr{S}$ is a plane with a normal n_0 and that c_0 has been so chosen that $x = \mathbf{0}$ lies on $\mathscr{S}$. Given $U(\theta)$ and $\hat{U}(\theta)$, a *solution* of (1.3) then can be regarded as a triple $(\bar{R}(\theta), R_0, n_0)$. However, equation (1.3) shows that for any such solution, $\bar{R}(\theta)$ is uniquely determined by R_0 and n_0. Thus, if two bodies are joined, a **stress-free joint** shall be a pair (R_0, n_0) consisting of a constant rotation R_0 and a constant unit vector n_0 which solve (1.3). Without yet introducing specific constitutive relations, we shall say that the two bodies are of the **same material** if $U(\theta) = \hat{U}(\theta)$ for all $\theta \in \mathscr{I}$. This definition entails the idea that the two bodies have identical thermal response when referred to the reference configurations $\mathscr{R}$ and $\hat{\mathscr{R}}$.

A different form of equation (1.3) is sometimes useful: if $\mathscr{S}$ is a plane with normal n_0, equation (1.3) is satisfied if and only if for each $\theta \in \mathscr{I}$ there is a vector $a(\theta)$, the *amplitude*, such that

$$\bar{R}(\theta)\,\hat{U}(\theta)\,R_0 = U(\theta) + a(\theta) \otimes n_0. \tag{1.4}$$

The case (i) above is equivalent to the condition $a(\theta) = \mathbf{0}$ for all $\theta \in \mathscr{I}$.

For stress-free joints consisting of $f > 2$ bodies, the definitions are strictly analagous. Here, we shall treat the topology informally (see § 6 for precise statements). Let f bodies be joined at e plane interfaces with normals $n_{01}, \ldots, n_{0e}$ in polyhedral reference configurations $\mathscr{R}'_1, \ldots, \mathscr{R}'_f$. Let positive, symmetric tensor functions $U_1(\theta), \ldots, U_f(\theta)$ be given. Suppose there are constant rotations $R_{01}, \ldots, R_{0f}$ such that $U_i(\theta)$ is the unique right stretch tensor corresponding to zero stress, measured relative to the reference configuration $R_{0i}\mathscr{R}'_i$ (no sum). The collection $(R_{01}, \ldots, R_{0f}; \mathscr{R}'_1, \ldots, \mathscr{R}'_f)$ will be termed a *stress-free joint* if there are relative rotations $\bar{R}_1(\theta) = \mathbf{1}$, $\bar{R}_2(\theta), \ldots, \bar{R}_f(\theta)$ and translations $c_1(\theta) = \mathbf{0}$, $c_2(\theta), \ldots, c_f(\theta)$ such that the function

$$y = \bar{R}_i(\theta)\,U_i(\theta)\,R_{0i}x + c_i(\theta) \qquad x \in \mathscr{R}'_i, \qquad i = 1, \ldots, f \text{ (no sum)} \tag{1.5}$$

is a continuous function of x on $\mathscr{R}'_1 \cup \ldots \cup \mathscr{R}'_f$ for each θ in $\mathscr{I}$. Notice that (1.5) implies jump conditions like (1.3) or (1.4) at each interface; however, these jump conditions are not sufficient for the existence of a continuous deformation $y(\cdot, \theta)$ even when $f = 3$. See § 6 for details.

While stress-free polycrystals are not ordinarily formed by rotating and joining bodies at a certain temperature, they are often pictured as being composed of randomly oriented crystallites whose thermal response would be the same after

* Recall that $\mathscr{S}$ is independent of θ.

suitable rotations. Stress-free polycrystals *of one material* will satisfy $U_1(\theta) = \dots = U_f(\theta)$ for all θ in $\mathscr{I}$. If $U_1(\theta_0) = \dots = U_f(\theta_0) = \mathbf{1}$ we may interpret $R_{01}^T, \dots, R_{0f}^T$ as the orientations of crystallites or grains at the temperature $\theta_0 \in \mathscr{I}$.

2. Connections with Twinning, Symmetry and Stability

It is useful to introduce constitutive relations in order to assess the assumptions involved. Non-linear thermoelasticity is a good prototype.

Let $W = \tilde{W}(F, \theta)$ be the stored energy function of a homogeneous thermoelastic body referred to a reference configuration $\mathscr{R}$. By Galilean invariance, W is given by a function of $U = (F^T F)^{\frac{1}{2}}$ and θ only:

$$W = \hat{W}(U, \theta). \tag{2.1}$$

The equations of equilibrium for a body subject to zero stress everywhere are

$$\frac{\partial \hat{W}}{\partial U}(U, \theta) = \mathbf{0}. \tag{2.2}$$

We would like to know if there are reasonable assumptions on $\hat{W}$ which would imply uniqueness of the stable solutions $U(\theta)$ of (2.2). 'Stable' will be interpreted in the sense of Gibbs. These assumptions will clearly have to rule out some phase transformations, since energy functions appropriate for a body which can change phase* will have the property that (2.2) is not uniquely invertible at the transformation temperature.

We begin by considering one body. A homogeneous deformation $y(x, \theta)$, with $U(\theta) = (\nabla y^T \nabla y)^{\frac{1}{2}}$ satisfying (2.2), is stable according to the Gibbs criterion if for each fixed θ in $\mathscr{I}$,

$$\int_{\mathscr{R}} \hat{W}(U(\theta), \theta)\, dx \leqq \int_{\mathscr{R}} \hat{W}((\nabla z^T \nabla z)^{\frac{1}{2}}, \theta)\, dx, \tag{2.3}$$

for all continuous, piecewise differentiable functions $z(x)$ whose gradients lie in the domain of $\hat{W}(\cdot, \theta)$. We can take $\nabla z = S =$ const. in (2.3) and conclude that

$$\hat{W}(U(\theta), \theta) \leqq \hat{W}(S, \theta) \tag{2.4}$$

holds for all positive, symmetric S in the appropriate domain. The condition (2.4) is necessary and sufficient for (2.3).

The condition (2.4) is not sufficient that $U(\theta)$ be the unique solution of (2.2); however, if $\hat{W}$ satisfies the stronger condition,

$$\hat{W}(U(\theta), \theta) < \hat{W}(S, \theta) \quad \forall\, S = S^T > 0, \quad S \neq U(\theta), \tag{2.5}$$

* To include these materials, it may only be necessary to restrict $\mathscr{I}$ so as not to include the transformation temperature. In a more general treatment, we would allow $U(\theta)$ to become multi-valued at the transformation temperature.

then $U(\theta)$ is clearly the only solution of (2.2) calculated from a stable deformation. Thus, thermoelasticity, taken with the inequality (2.5), justifies the assumptions of § 1 if we restrict attention to strictly stable deformations.

Let the *point group* $\mathscr{P}$ be defined as the set of all constant orthogonal tensors Q such that for all θ in $\mathscr{I}$ and all U in the domain of $\hat{W}$

$$\hat{W}(QUQ^T, \theta) = \hat{W}(U, \theta). \tag{2.6}$$

If $\hat{W}$ satisfies (2.6) in addition to (2.5), we shall have

$$QU(\theta)\,Q^T = U(\theta) \quad \forall\, Q \in \mathscr{P}, \quad \forall\, \theta \in \mathscr{I}. \tag{2.7}$$

Thus the inequality (2.5) appears to be the origin of the fact that cubic crystals often suffer dilations with changes of θ and that crystals of lower symmetry have functions $U(\theta)$ with two or more distinct eigenvalues. The most general forms of $U(\theta)$ consistent with (2.7) for all of the crystallographic point groups can be found in the literature (*cf.* COLEMAN & NOLL [4]).

If $U(\theta)$ is continuous, if $U(\theta_0) = 1$, and if the linear elasticity tensor is positive-definite at θ_0, a local version of the preceding argument can be given. In particular, (2.7) follows from these local assumptions with $\mathscr{I}$ restricted to an interval about θ_0.

While the assumption of stability (2.5) does yield (2.6) and (2.7), and therefore permits a relation between $U(\theta)$ and the symmetry of a body to be established, it is ill motivated. We really should base the argument upon a concept of stability for the *joined* body. To this end, consider f bodies joined in reference configurations $\mathscr{R}_1', \ldots, \mathscr{R}_f'$. For simplicity, assume that the bodies are of the same material (*cf.* § 1). Assume that the response of this material relative to the reference configuration $\mathscr{R}$ is governed by the stored energy function $\hat{W}(U, \theta)$ introduced at the beginning of this section. Let $(R_{01}, \ldots, R_{0f}; \mathscr{R}_f', \ldots, \mathscr{R}_f')$ be a stress-free joint arising from a deformation $y(x, \theta)$ whose stability we judge according to the Gibbs criterion. Omitting the details, we can show that $y(x, \theta)$ is stable according to the Gibbs criterion if for each $\theta \in \mathscr{I}$,

$$\sum_{i=1} \int_{\mathscr{R}_i'} \{\hat{W}(R_{0i}\, \nabla z^T\, \nabla z R_{0i}^T, \theta) - \hat{W}(U(\theta), \theta)\}\, dx \geqq 0, \tag{2.8}$$

holds for all continuous, piecewise differentiable $z(x)$. The assumption that $z(x)$ is continuous expresses the notion that the test fields do not allow the joined body to break apart. My point is that the first term in (2.8) has a special structure, and it seems possible that (2.8) could be fulfilled for all appropriate $z(x)$ while (2.3) fails for some $z(x)$. However, (2.3) is equivalent to (2.4), and (2.4) implies (2.8). *In this sense, the absolute stability of a stress-free crystal implies the absolute stability of any stress-free polycrystal made of the same material.*

To clarify the connection between stress-free joints and twinning, we return to the definition of the point group (2.6). Molecular calculations show that (2.6) is actually satisfied by some nonorthogonal Q. Such Q, referred to as 'lattice invariant deformations' by materials scientists, can be found by considering the transformations which map a periodic set of lattice points into itself. Assume that $\hat{W}$ is consistent with such a molecular theory and let (2.6) be satisfied by all Q in a group $\mathcal{G}$. In the simplest case, a pairwise homogeneous deformation, specified by deformation gradients F and $F + a \otimes n_0$, is a twin [5] if there is a rotation $\bar{R}$ with $\bar{R}^2 = 1$ and an $H \in \mathcal{G}$ such that

$$\bar{R}FH = F + a \otimes n_0. \tag{2.9}$$

In general H need not be orthogonal, in which case (2.9) and (1.4) are not equivalent. However, some twins are described by (2.9) with an orthogonal H (or, equivalently, with H similar to an orthogonal tensor), and the analysis which follows bears upon these. It is interesting to note that when (1.4) is solved for $\bar{R}$, R_0, a and n_0 (*cf.* § 4), many of the solutions turn out to have the property $\bar{R}^2 = 1$.

3. Basic Analysis

Assuming without loss of generality (*cf.* § 2, alternative (ii), and the discussion which follows it) that $\mathcal{S}$ is a plane with normal n_0, we seek stress-free joints (R_0, n_0). Given positive symmetric tensors $\hat{U}(\theta)$ and $U(\theta)$, $\theta \in \mathcal{I}$, R_0 and n_0 must satisfy

$$\bar{R}(\theta)\, \hat{U}(\theta)\, R_0 x = U(\theta)\, x \qquad \forall\, x \perp n_0, \tag{3.1}$$

for each $\theta \in \mathcal{I}$ and for some relative rotation $\bar{R}(\theta)$.

Since the left (or right) hand side of (3.1) is linear in x, we can assume without loss of generality that $|x| = 1$. Then, the set of vectors given by either side of (3.1) describes an ellipse in a plane with normal

$$n(\theta) = \frac{U^{-1}(\theta)\, n_0}{|U^{-1}(\theta)\, n_0|}. \tag{3.2}$$

We shall begin by calculating all ellipses obtained from $y = Ux$, $|x| = 1$ $x \cdot n_0 = 0$, for various values of the unit vector n_0. Let $C = U^2$. Since an ellipse in $\mathbb{E}^3$ is determined up to a rotation by its major and minor axes, it will be helpful to calculate these. Thus, we shall be interested in extrema of

$$y \cdot y = x \cdot Cx \tag{3.3}$$

subject to the constraints

$$x \cdot x = 1 \quad \text{and} \quad x \cdot n_0 = 0. \tag{3.4}$$

We shall use repeatedly the fact that either $y \cdot y$ is the same for all values of x satisfying the constraints, or $y \cdot y$ has exactly two extrema on the set (3.4), one maximum and one minimum. Let Lagrange multipliers λ, 2μ be associated with the constraints $(3.4)_1$, $(3.4)_2$. The extrema are attained at values of x which satisfy

$$\begin{aligned} 0 &= \tfrac{1}{2} \nabla_x (y \cdot y - \lambda x \cdot x - 2\mu x \cdot n_0), \\ &= Cx - \lambda x - \mu n_0. \end{aligned} \tag{3.5}$$

Equivalently,

$$(C - \lambda 1)\, x = \mu n_0. \tag{3.6}$$

If we premultiply (3.6) by $\operatorname{adj}(C - \lambda 1)$, we get

$$(\det(C - \lambda 1))\, x = \mu\, (\operatorname{adj}(C - \lambda 1))\, n_0, \tag{3.7}$$

and if we take the scalar product of (3.7) with n_0 and use $(3.4)_2$, we get

$$\mu n_0 \cdot (\operatorname{adj}(C - \lambda 1)\, n_0) = 0. \tag{3.8}$$

Equation (3.8) implies that either $\mu = 0$ or

$$n_0 \cdot (\operatorname{adj}(C - \lambda 1)\, n_0 = 0. \tag{3.9}$$

The condition $\mu = 0$ is subsumed by (3.9). To see this, notice that if $\mu = 0$ λ is an eigenvalue and x is an eigenvector of C (*cf.* (3.6)). Thus the null space of $\operatorname{adj}(C - \lambda 1)$ is the set of all vectors perpendicular to x, and by $(3.4)_2$ n_0 belongs to this set. Thus, if $\mu = 0$, (3.9) is satisfied.

I shall denote by a and b the major and minor axes of the ellipse corresponding to definite values of C and n_0; $a_0 = U^{-1}a$, $b_0 = U^{-1}b$ shall be the unit vectors in the reference configuration which are deformed into a and b. If $|a| = |b|$, the ellipse is a circle; in this case a and b will denote any pair of perpendicular vectors whose termini lie on the circle. If we take the scalar product of (3.6) with x and use (3.4), we get

$$\lambda = x \cdot Cx. \tag{3.10}$$

Thus the Lagrange multiplier λ can be interpreted as the squared length of $y = Ux$. The symbols λ_a and λ_b will denote the values of λ given by (3.10) when $x = a_0$ and $x = b_0$, respectively. That is, $\lambda_a = |a|^2$ and $\lambda_b = |b|^2$. The notation is summarized by Figure 1.

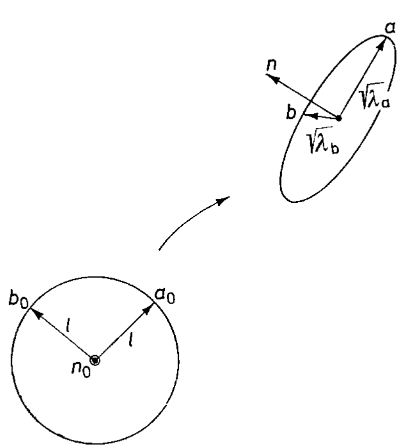

Fig. 1. Notation.

The vectors a_0 and b_0 are always orthonormal. To see this, evaluate (3.6) at $x = a_0$, take the scalar product of the result with b_0, use the constraint $(3.4)_2$ to eliminate $n_0 \cdot b_0$, and then use the fact that $a \cdot b = a_0 \cdot Cb_0 = 0$.

We return now to (3.9). Let C have the spectral representation

$$C = c_1 e_1 \otimes e_1 + c_2 e_2 \otimes e_2 + c_3 e_3 \otimes e_3 \tag{3.11}$$

with $0 < c_1 \leqq c_2 \leqq c_3$. Expressed relative to the basis $\{e_i\}$, the condition (3.9) becomes

$$n_{01}^2(c_2 - \lambda)(c_3 - \lambda) + n_{02}^2(c_1 - \lambda)(c_3 - \lambda) + n_{03}^2(c_1 - \lambda)(c_2 - \lambda) = 0 \tag{3.12}$$

which yields the following quadratic equation for λ:

$$\begin{aligned} \lambda^2 - \lambda\{(c_2 + c_3)\, n_{01}^2 + (c_1 + c_3)\, n_{02}^2 + (c_1 + c_2)\, n_{03}^2\} \\ + \{c_2 c_3 n_{01}^2 + c_1 c_3 n_{02}^2 + c_1 c_2 n_{03}^2\} = 0. \end{aligned} \tag{3.13}$$

According to the fact stated just after equation (3.4), equation (3.13) has at least one real root. Suppose it is a repeated root. Then, since the ellipse is a circle, there is a one-parameter family of unit vectors x which solve (3.6). This is possible only if $C - \lambda 1$ is not invertible. Thus, we have

Lemma 1. *If the ellipse defined by* $y = Ux$, $|x| = 1$, $x \cdot n_0 = 0$, *is a circle, then its radius is an eigenvalue of* U.

To proceed further, it is helpful to look at special cases.

Case 1. $c_1 < c_2 < c_3$

First, suppose the roots of (3.13) coincide. Then, by Lemma 1 $\lambda_a = \lambda_b = c_k$ for some $k \in \{1, 2, 3\}$. Let l and m be the indices of the remaining eigenvalues. If we substitute $\lambda = c_k$ into (3.12), we get

$$n_{0k}^2(c_l - c_k)(c_m - c_k) = 0, \tag{3.14}$$

which implies that $n_{0k} = 0$. Now if we solve (3.6) for x, we get (in the spectral basis)

$$\begin{aligned} x_l &= \mu(c_l - c_k)^{-1}\, n_{0l}, \\ x_m &= \mu(c_m - c_k)^{-1}\, n_{0m}, \end{aligned} \tag{3.15}$$

while x_k is unrestricted. Without loss of generality* we can assume $\mu \neq 0$. If we combine (3.15) with the constraint $(3.4)_2$ and use the conditions $\mu \neq 0$ and $n_{0k} \neq 0$, we get

$$(c_l - c_k)^{-1}\, n_{0l}^2 = (c_m - c_k)^{-1}\, n_{0m}^2. \tag{3.16}$$

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* If $\mu = 0$ then (3.15) uniquely determines x, $|x| = 1$. Since we have assumed the roots of (3.13) coincide, there is a nontrivial one-parameter family of x given by (3.15), which therefore must be obtained under the condition $\mu \neq 0$.

Equation (3.16) implies that $k = 2$. Thus $\lambda = c_2$. Without loss of generality, say that $l = 1$ and $m = 3$. From (3.16) and the condition $|\boldsymbol{n}_0| = 1$, it follows that

$$n_{01}^2 = \frac{c_2 - c_1}{c_3 - c_1}, \quad n_{03}^2 = \frac{c_3 - c_2}{c_3 - c_1} \tag{3.17}$$

and (3.14) gives,

$$n_{02} = 0. \tag{3.18}$$

The equations (3.17) and (3.18) give exactly two planes upon which the ellipses are circles, and these circles have radii equal to $\sqrt{c_2}$.

Now we turn to the case in which (3.13) has distinct roots $\lambda_a > \lambda_b$. We can write (3.13) in form

$$\lambda^2 - \lambda A + B = 0, \tag{3.19}$$

where

$$\begin{aligned} A &= \tilde{A}(C, \boldsymbol{n}_0) = (c_2 + c_3)\, n_{01}^2 + (c_1 + c_3)\, n_{02}^2 + (c_1 + c_2)\, n_{03}^2, \\ B &= \tilde{B}(C, \boldsymbol{n}_0) = c_2 c_3 n_{01}^2 + c_1 c_3 n_{02}^2 + c_1 c_2 n_{03}^2. \end{aligned} \tag{3.20}$$

For applications to joinery, it will be useful to understand the relation between (λ_a, λ_b) and $\boldsymbol{n}_0$. We have $\lambda_a + \lambda_b = A$, $\lambda_a \lambda_b = B$, which show that λ_a and λ_b determine A and B uniquely. The equations (3.20) and the condition $\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 = 1$ constitute linear equations for determination of n_{01}^2, n_{02}^2, n_{03}^2. The determinant of this system is $(c_3 - c_1)(c_1 - c_2)(c_3 - c_2)$, which does not vanish. Thus, we have

Lemma 2. *If $c_1 < c_2 < c_3$ are given and $\lambda_a > \lambda_b$, then the squared components of the unit normal vector $\boldsymbol{n}_0$ relative to the basis $\{\boldsymbol{e}_i\}$ are determined uniquely by λ_a and λ_b. Symbolically,*

$$n_{0i}^2 = f_i(\lambda_a, \lambda_b; c_1, c_2, c_3). \tag{3.21}$$

In fact, the functions f_i can be extended continuously to include $\lambda_a = \lambda_b = c_2$, the values of these functions being given by (3.17) and (3.18). At this point, (3.6) could be used to calculate $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$. Also, some further facts concerning the domain of the f_i are useful when bodies of different material are joined, but we defer these until § 5.

Summary $c_1 < c_2 < c_3$.

1. Ellipses, $\lambda_a > \lambda_b$

$$\boldsymbol{n}_0 = n_{0i} \boldsymbol{e}_i, \quad n_{0i}^2 = f_i(\lambda_a, \lambda_b; c_1, c_2, c_3). \tag{3.22}$$

2. Circles, $\lambda_a = \lambda_b$

$$\begin{aligned} \boldsymbol{n}_0 &= n_{0i}\boldsymbol{e}_i, \quad n_{01}^2 = \frac{c_2 - c_1}{c_3 - c_1}, \\ n_{02}^2 &= 0, \\ n_{03}^2 &= \frac{c_3 - c_2}{c_3 - c_1}. \end{aligned} \tag{3.23}$$

Case 2. $c_1 = c_2 \neq c_3$

In this case, (3.12) becomes

$$(n_{01}^2 + n_{02}^2)(c_1 - \lambda)(c_3 - \lambda) + n_{03}^2(c_1 - \lambda)^2 = 0. \tag{3.24}$$

Equation (3.24) is satisfied with $\lambda = c_1$. Then, (3.6) yields either $\mu \neq 0$ and $\boldsymbol{n}_0 = \pm\boldsymbol{e}_3$, or $\mu = 0$ and $\boldsymbol{n}_0$ unrestricted. If $\mu \neq 0$, the ellipse is a circle and $\boldsymbol{a}_0$, $\boldsymbol{b}_0$ are any orthonormal vectors in the $\boldsymbol{e}_1 - \boldsymbol{e}_2$ plane. If $\mu = 0$, we have $\boldsymbol{x} \cdot \boldsymbol{e}_3 = 0$, from (3.6).

Now suppose $\lambda \neq c_1$. Then (3.24) becomes

$$(n_{01}^2 + n_{02}^2)(c_3 - \lambda) + n_{03}^2(c_1 - \lambda) = 0, \tag{3.25}$$

which implies that

$$\lambda = (c_1 - c_3)\, n_{03}^2 + c_3. \tag{3.26}$$

Equation (3.26) delivers one value of λ; the other value must therefore follow from $\mu = 0$, which is covered above.

Summary $c_1 = c_2 < c_3$.

1. Ellipses, $\lambda_a > \lambda_b$

$$\begin{aligned} \boldsymbol{n}_0 \neq \pm\boldsymbol{e}_3, \quad \lambda_b &= c_1, \\ \lambda_a &= (c_1 - c_3)\, n_{03}^2 + c_3. \end{aligned} \tag{3.27}$$

2. Circles, $\lambda_a = \lambda_b$

$$\begin{aligned} \boldsymbol{n}_0 = \pm\boldsymbol{e}_3, \quad \lambda_b &= c_1, \\ \lambda_a &= c_1. \end{aligned} \tag{3.28}$$

(If $c_1 = c_2 > c_3$, exchange a and b above).

Case 3. $c_1 = c_2 = c_3$. All circles are ellipses of radius c_1.

To make stress-free joints using these results, we begin with (3.1) and impose the restriction $|\boldsymbol{x}| = 1$. Fix an $\boldsymbol{n}_0$ independent of θ and calculate an orthonormal set $\{\boldsymbol{a}_0(\theta), \boldsymbol{b}_0(\theta), \boldsymbol{n}_0\}$ such that $\boldsymbol{a}(\theta) = \boldsymbol{U}(\theta)\, \boldsymbol{a}_0(\theta)$ and $\boldsymbol{b}(\theta) = \boldsymbol{U}(\theta)\, \boldsymbol{b}_0(\theta)$ are principal axes of an ellipse in the plane with normal* $\boldsymbol{n}(\theta)$. Let $\lambda_a(\theta)$ and $\lambda_b(\theta)$ be the corresponding lengths. Equation (3.1) imposes the restriction that

$$\bar{\boldsymbol{R}}(\theta)\,\hat{U}(\theta)\,\boldsymbol{R}_0\boldsymbol{a}_0(\theta) \quad \text{and} \quad \bar{\boldsymbol{R}}(\theta)\,\hat{U}(\theta)\,\boldsymbol{R}_0\boldsymbol{b}_0(\theta) \tag{3.29}$$

are principal axes of the *same* ellipse with the same lengths $\lambda_a(\theta)$ and $\lambda_b(\theta)$. Let

$$\begin{aligned} \hat{\boldsymbol{a}}_0(\theta) &= \boldsymbol{R}_0\boldsymbol{a}_0(\theta), \\ \hat{\boldsymbol{b}}_0(\theta) &= \boldsymbol{R}_0\boldsymbol{b}_0(\theta), \\ \hat{\boldsymbol{a}}(\theta) &= \bar{\boldsymbol{R}}^T(\theta)\,\boldsymbol{a}(\theta), \\ \hat{\boldsymbol{b}}(\theta) &= \bar{\boldsymbol{R}}^T(\theta)\,\boldsymbol{b}(\theta). \end{aligned} \tag{3.30}$$

It follows from the remarks made above that $\hat{\boldsymbol{n}}_0 = \hat{\boldsymbol{a}}_0 \wedge \hat{\boldsymbol{b}}_0$ is independent of θ and that $\hat{\boldsymbol{a}} = \hat{U}(\theta)\,\hat{\boldsymbol{a}}_0$ and $\hat{\boldsymbol{b}} = \hat{U}(\theta)\,\boldsymbol{b}_0$ are principal axes, with lengths $\lambda_a(\theta)$ and $\lambda_b(\theta)$, of an ellipse in the plane with normal

$$\hat{\boldsymbol{n}} = \frac{\hat{U}^{-1}(\theta)\,\hat{\boldsymbol{n}}_0}{|\hat{U}^{-1}(\theta)\,\hat{\boldsymbol{n}}_0|} \tag{3.31}$$

In summary, necessary conditions that (3.1) *be satisfied are that there be orthonormal triads* $\{\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{n}_0\}$ *and* $\{\hat{\boldsymbol{a}}_0, \hat{\boldsymbol{b}}_0, \hat{\boldsymbol{n}}_0\}$, *related by a constant rotation* $\boldsymbol{R}_0$, *having the property that* $\boldsymbol{n}_0$ *is constant, and such that* $U\boldsymbol{a}_0$, $U\boldsymbol{b}_0$ *and* $\hat{U}\hat{\boldsymbol{a}}_0$, $\hat{U}\hat{\boldsymbol{b}}_0$ *represent principal axes of congruent ellipses in the planes with normals* $\boldsymbol{n}$ *and* $\hat{\boldsymbol{n}}$, *respectively.*

These conditions are also sufficient, since they guarantee the existence of $\bar{\boldsymbol{R}}(\theta)$ such that (3.1) is satisfied for $\boldsymbol{x} = \boldsymbol{a}_0$ and $\boldsymbol{x} = \boldsymbol{b}_0$, which in turn implies that (3.1) is satisfied for all $\boldsymbol{x}$ perpendicular to $\boldsymbol{n}_0$. Notice that we shall not expect stress-free joints to be unique, since if $\{\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{n}_0\}$ serves as an orthonormal triad satisfying the appropriate conditions, then so will $\{\pm\boldsymbol{a}_0, (\pm)\boldsymbol{b}_0, ((\pm))\,\boldsymbol{n}_0\}$. Similar ambiguity arises with $\bar{\boldsymbol{R}}(\theta)$.

The procedure for finding stress-free joints will be the following. Fix $\boldsymbol{n}_0 =$ *const.* and write down all orthonormal sets $\{\boldsymbol{a}_0(\theta), \boldsymbol{b}_0(\theta), \boldsymbol{n}_0\}$ such that $U(\theta)\,\boldsymbol{a}_0(\theta)$ and $U(\theta)\,\boldsymbol{b}_0(\theta)$ are principal axes of an ellipse in the plane with normal $\boldsymbol{n}(\theta)$ for all θ in $\mathscr{I}$. Calculate $\lambda_a(\theta)$ and $\lambda_b(\theta)$. Using the results of this section, find all ellipses based on $\hat{U}(\theta)$ whose principal axes have lengths $\lambda_a(\theta)$ and $\lambda_b(\theta)$. Calculate $\{\hat{\boldsymbol{a}}_0(\theta), \hat{\boldsymbol{b}}_0(\theta), \hat{\boldsymbol{n}}_0\}$ for each of these ellipses, accounting for ambiguity in principal axes. If the two orthonormal triples are related by a *constant* relation $\boldsymbol{R}_0$, then $(\boldsymbol{R}_0, \boldsymbol{n}_0)$ is a stress-free joint.

4. Joining Bodies of the Same Material

If we restrict attention to bodies of the same material, then $U(\theta) = \hat{U}(\theta)$ for all θ in $\mathscr{I}$. This case was treated by ERICKSEN [1] using different methods. Let $C(\theta) = U(\theta)^2 = \sum c_i\boldsymbol{e}_i \otimes \boldsymbol{e}_i$, $c_1 \leqq c_2 \leqq c_3$.

* *Cf.* (3.2.) In some examples it happens that $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ depend nontrivially upon θ; *cf.* § 4.

To organize the results, it will be convenient to list some properties that $C(\theta)$ can have which are relevant to problems in joinery.

(i) C has distinct eigenvalues $c_1 < c_2 < c_3$ for some value of θ in $\mathscr{I}$.

$(\text{ii})_{uv}$ There are constant orthonormal vectors $\{\boldsymbol{u}, \boldsymbol{v}\}$ such that $\boldsymbol{u} \cdot \boldsymbol{C}\boldsymbol{v} = 0$ for all values of θ in $\mathscr{I}$.

$(\text{iii})_k$ The axis containing the eigenvector $\boldsymbol{e}_k$ is independent of θ.

$(\text{iv})_\pm$ The condition $c_3 \neq c_1$ is fulfilled and for appropriate choices of the eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_3$,

$$\left[\frac{c_2 - c_1}{c_3 - c_1}\right]^{\frac{1}{2}} \boldsymbol{e}_1 \pm \left[\frac{c_3 - c_2}{c_3 - c_1}\right]^{\frac{1}{2}} \boldsymbol{e}_3$$

is independent of θ.

(v) The eigenvalues of C satisfy $c_1 = c_2$ for all θ in $\mathscr{I}$, and $c_3 \neq c_1$ for some θ in $\mathscr{I}$.

(vi) The eigenvector $\boldsymbol{e}_3$ lies in a constant plane for all values of θ in $\mathscr{I}$.

(vii) The eigenvalues of C are all equal for all values of θ in $\mathscr{I}$.

I shall give only conditions sufficient that a particular stress-free joint exists. While it might seem possible at first to give an exhaustive treatment, there are certain unusual stress-free joints, arising from rather special functions $U(\theta)$, which I have not been able to organize in a simple way. To understand some of these unusual cases, the reader may try to find a constant $\boldsymbol{R}_0$ given by various combinations of the right hand sides of (4.3) below, while permitting $\boldsymbol{e}_i(\theta)$ to be an adjustable function of θ.

First suppose C satisfies (i), but none of the other conditions $(\text{ii})_{uv}$ through (vii). Let $\boldsymbol{n}_0 = \text{const.}$ be given ($\boldsymbol{n}_0$ does not satisfy (3.23) for all θ in $\mathscr{I}$ because $(\text{v})_\pm$ is not fulfilled). Equation (3.22) shows that the squared components n_{0i}^2 relative to the basis $\{\boldsymbol{e}_i\}$ are determined by the lengths $\lambda_a \neq \lambda_b$ of the major and minor axes. Equation (3.6) shows that a_{0i}^2 and b_{0i}^2 are similarly determined. These facts can be summarized concisely if we define $\boldsymbol{Q}_0 = \boldsymbol{1}$ and $\boldsymbol{Q}_i$ as a 180° rotation about an axis through $\boldsymbol{e}_i$:

$$\begin{aligned} \boldsymbol{Q}_0 &= \boldsymbol{1}, \\ \boldsymbol{Q}_i &= -\boldsymbol{1} + 2\boldsymbol{e}_i \otimes \boldsymbol{e}_i, \quad i = 1, 2, 3 \text{ (no sum)}. \end{aligned} \tag{4.1}$$

Then $\boldsymbol{R}_0$ must satisfy

$$\boldsymbol{R}_0 \begin{Bmatrix} \boldsymbol{n}_0 \\ \boldsymbol{a}_0 \\ \boldsymbol{b}_0 \end{Bmatrix} = \pm \boldsymbol{Q}_i \begin{Bmatrix} \boldsymbol{n}_0 \\ (\pm)\, \boldsymbol{a}_0 \\ ((\pm))\, \boldsymbol{b}_0 \end{Bmatrix}, \quad i = 0, 1, 2, 3. \tag{4.2}$$

The notations $(\pm)$ and $((\pm))$ in (4.2) indicate that the sets of $\pm$ signs are not associated. By accounting for the fact that $\boldsymbol{R}_0$ is a rotation, we can reduce (4.2) to

$$\boldsymbol{R}_0 = \begin{cases} \boldsymbol{Q}_i(-\boldsymbol{1} + 2\boldsymbol{n}_0 \otimes \boldsymbol{n}_0), & \text{or} \\ \boldsymbol{Q}_i(-\boldsymbol{1} + 2\boldsymbol{a}_0 \otimes \boldsymbol{a}_0), & \text{or} \\ \boldsymbol{Q}_i(-\boldsymbol{1} + 2\boldsymbol{b}_0 \otimes \boldsymbol{b}_0), & \end{cases} \quad i = 0, 1, 2, 3. \tag{4.3}$$

In particular, (4.3) yields

$$R_0 n_0 = \begin{cases} \pm n_0 \\ \pm(-n_0 + 2e_i(e_i \cdot n_0)), \quad i = 1, 2, 3, \text{ (no sum)}. \end{cases} \tag{4.4}$$

A sufficient condition that $R_0 n_0 = \text{const.}$ is $i = 0$. Returning now to (4.3) with $i = 0$, we are left with $R_0 = \mathbf{1}$ or

$$R_0 = -\mathbf{1} + 2n_0 \otimes n_0 \tag{4.5}$$

or

$$R_0 = -\mathbf{1} + 2a_0 \otimes a_0 \tag{4.6}$$

or

$$R_0 = -\mathbf{1} + 2b_0 \otimes b_0. \tag{4.7}$$

If R_0 is given by (4.5), it is constant, but this is not true of (4.6) and (4.7). To see this, note that Ua_0 and Ub_0 are principal axes of an ellipse; thus, $Ua_0 \cdot Ub_0 = a_0 \cdot Cb_0 = 0$, which is not fulfilled because $(\text{ii})_{uv}$ is not fulfilled. Thus, given the constant vector n_0 the stress-free joints

$$\begin{gathered} (\mathbf{1}, n_0), \\ (-\mathbf{1} + 2n_0 \otimes n_0, n_0) \end{gathered} \tag{4.8}$$

exist in materials for which only (i) is met.

More generally, (4.8) *gives the only stress-free joints possible in every homogeneous material,* that is, possible for all choices of $U(\theta)$ and $\mathscr{S}$. The relative rotation corresponding to the stress-free joint $(-\mathbf{1} + 2n_0 \otimes n_0, n_0)$ is

$$\bar{R}(\theta) = -\mathbf{1} + 2n(\theta) \otimes n(\theta), \tag{4.9}$$

$n(\theta)$ being given by (3.2).

Now suppose $C(\theta)$ satisfies $(\text{ii})_{uv}$. We still can take $i = 0$ in (4.3) and get (4.5), (4.6) and (4.7), but now there are constant R_0 satisfying (4.6) and (4.7). In fact, if we put $a_0 = u$, $b_0 = v$ and $n_0 = a_0 \wedge b_0$, then we have the additional stress-free joints

$$\begin{gathered} (-\mathbf{1} + 2u \otimes u, u \wedge v), \\ (-\mathbf{1} + 2v \otimes v, u \wedge v). \end{gathered} \tag{4.10}$$

The relative rotations corresponding to $(4.10)_{1,2}$ are $-\mathbf{1} + 2U(\theta)\, u \otimes U(\theta)\, u$ and $-\mathbf{1} + 2U(\theta)\, v \otimes U(\theta)\, v$, respectively. In summary, for each pair of vectors u, v satisfying $(\text{ii})_{u,v}$ we obtain the stress-free joints (4.10).

It might seem from (4.4) that new stress-free joints would arise with $i \neq 0$ if e_i were merely contained in a θ-independent plane (with $(\text{iii})_i$ not satisfied). These turn out to lead back to special cases of (4.8).

Now assume $(\text{iii})_k$ is met. Equation (4.3) delivers constant R_0 with $i = k$. More generally, if

$$(R_0, n_0) \tag{4.11}$$

is any stress-free joint with a corresponding relative rotation $\bar{R}(\theta)$, then

$$(Q_k R_0, n_0) \tag{4.12}$$

is also a stress-free joint with a corresponding relative rotation $\bar{R}(\theta)\, Q_k$.

Now suppose $(\mathrm{iv})_+$ is fulfilled. Let

$$n_0 = \left[\frac{c_2 - c_1}{c_3 - c_1}\right]^{\frac{1}{2}} e_1 + \left[\frac{c_3 - c_2}{c_3 - c_1}\right]^{\frac{1}{2}} e_3. \tag{4.13}$$

Equations (3.23) or (3.28) show that $\lambda_a = \lambda_b$. Thus we can choose R_0 to be any constant rotation which maps a pair of orthonormal vectors in a plane normal to n_0 to any other orthonormal pair in the same plane. A general expression for a rotation of this kind is

$$R_0 = \hat{Q} Q_{n_0}, \tag{4.14}$$

where $\hat{Q}$ is either $\mathbf{1}$ or a 180° rotation about a fixed vector normal to n_0 and where Q_{n_0} is any constant rotation with axis n_0. Every rotation of this kind gives rise to a stress-free joint (R_0, n_0). Similar statements hold for $(\mathrm{iv})_-$, and if both $(\mathrm{iv})_+$ and $(\mathrm{iv})_-$ hold, (4.12) can be invoked with $i = 1$.

Now suppose $C(\theta)$ satisfies (v) but neither $(\mathrm{iii})_3$ nor (vi). Equation (3.28) is not fulfilled because the axis through e_3 is not constant. Thus, $\lambda_a \neq \lambda_b$ and $n_0 \wedge e_3 \neq 0$ hold for at least one value of θ in $\mathscr{I}$.

Let n_0 be a fixed vector. It follows from (3.27) and (3.6) that at values of θ where $n_0 \wedge e_3 \neq 0$ we have

$$b_0 = \frac{\pm(n_0 \wedge e_3)}{|n_0 \wedge e_3|} \tag{4.15}$$

and

$$a_0 = (\pm)\,(n_0 \wedge b_0). \tag{4.16}$$

The constant rotation R_0 satisfies $(3.20)_2$ in particular, which yields

$$\hat{n}_0 \wedge e_3 = \pm R_0(n_0 \wedge e_3). \tag{4.17$_\pm$}$$

To get (4.17), we have used the fact that $|\hat{n}_0 \cdot e_3| = |n_0 \cdot e_3|$, which follows from (3.27).

If $e_3(\theta)$ is parallel to n_0 for *some* θ in $\mathscr{I}$, or if $(4.17)_+$ holds in part of $\mathscr{I}$ and $(4.17)_-$ holds in the balance of $\mathscr{I}$, some stress-free joints arise which seem not easily described. I shall seek stress-free joints following from (4.17) with one sign on all of $\mathscr{I}$ and with $n_0 \wedge e_3 \neq 0$ on all of $\mathscr{I}$.

Equation $(4.17)_+$ is equivalent to the equation

$$(R_0^T - \mathbf{1})\, e_3 = \xi n_0 \tag{4.18}$$

for some scalar ξ which may depend upon θ. Let $e = \text{const.}$ be a unit vector on the axis of R_0. If we take the scalar product of (4.18) with e, we get

$$\xi = 0 \quad \text{or} \quad n_0 \cdot e = 0. \tag{4.19}$$

The former implies that e_3 is parallel to e, a condition we have forbidden by assuming the negative of (iii)$_3$. The latter is a Fredholm condition for (4.18); it shows that there is a constant vector f and a scalar α such that

$$e_3 = \alpha e + \xi f, \tag{4.20}$$

which implies the other forbidden condition (vi). Notice that the same conclusions are reached if we put $\xi = 0$ on part of $\mathscr{S}$ and $n_0 \cdot e = 0$ on the balance of $\mathscr{S}$.

Equation (4.17)$_-$ is equivalent to the equation

$$(R_0^T + 1)\, e_3 = \xi n_0. \tag{4.21}$$

If $R_0^T + 1$ is invertible, equation (4.21) implies that the axis through e_3 is constant. Thus, to avoid satisfying (iii)$_3$, $R_0^T + 1$ must fail to be invertible: there is a (not necessarily constant) vector m such that

$$R_0 m = -m. \tag{4.22}$$

Equation (4.22) implies that $e \cdot m = 0$, while equation (4.21) implies that $\xi n_0 \cdot m = 0$. If $n_0 \cdot m = 0$ (the Fredholm condition), we find from (4.21) that

$$e_3 = \alpha m + \xi f, \tag{4.23}$$

f being a constant vector and α being a scalar. According to (4.23), e_3 lies in a fixed plane unless the axis through m is not constant. But if the axis through m is not constant, then n_0 and e must be parallel. The latter leads immediately back to (4.8). On the other hand, if $\xi = 0$ on a subset of $\mathscr{S}$, then R_0 sends both e_3 and m to their negatives on this subset. This implies that either e_3 is parallel to m, which yields the forbidden condition (vi) when combined with (4.23), or R_0 is of the form

$$-1 + 2e \otimes e, \qquad e \cdot e_3 = 0. \tag{4.24}$$

But by the opposite of (vi), (4.24)$_2$ cannot hold on all of $\mathscr{S}$. Keeping this in mind, we put (4.24)$_1$ into (4.21); we find that $e = n_0$, which again leads back to (4.8).

Therefore, if $n_0 \wedge e_3 \neq 0$ on $\mathscr{S}$, if (4.17)$_+$ or (4.17)$_-$ holds on $\mathscr{S}$, and if $C(\theta)$ satisfies (v) but neither (iii)$_3$ nor (vi), the only stress-free joints which exist are those (given by (4.8)) which exist in every homogeneous material.

Now suppose $C(\theta)$ satisfies (v) and (vi). Equations (4.19)$_2$ and (4.24)$_2$ can now be fulfilled, leading to stress-free joints. To describe these, let e_3 lie in the constant plane $\mathscr{P}$ with the constant normal r. Let n_0 be any fixed unit vector in $\mathscr{P}$ and let $e = r \wedge n_0$. Then (4.18) is satisfied with $R_0 = -1 + 2e \otimes e$ and we obtain the stress-free joints

$$(-1 + 2e \otimes e, n_0), \qquad e = r \wedge n_0. \tag{4.25}$$

Also, equation (4.24) admits solutions of the form $R_0 = -1 + 2r \otimes r$, which leads to the stress free-joints

$$(-1 + 2r \otimes r, n_0). \tag{4.26}$$

The joints given in (4.25) and (4.26) also could have been obtained by other procedures. For example, the conditions (v) and (vi) imply that $(\text{ii})_{uv}$ is satisfied with $\boldsymbol{u} = \boldsymbol{e}$, $\boldsymbol{v} = \boldsymbol{r}$.

Now suppose (v) and $(\text{iii})_3$ are fulfilled. First suppose $\boldsymbol{n}_0 \wedge \boldsymbol{e}_3 \neq 0$. All joints can now be read off directly as solutions of (4.18) or (4.22). To describe these joints, let $\hat{\boldsymbol{e}}$ be a constant unit vector satisfying $\hat{\boldsymbol{e}} \cdot \boldsymbol{n}_0 = 0$, $\hat{\boldsymbol{e}} \cdot (\boldsymbol{n}_0 \wedge \boldsymbol{e}_3) = 0$, let $\boldsymbol{e}_1$ be a fixed vector normal to $\boldsymbol{e}_3$, and let $\boldsymbol{Q}_3^\zeta$ be a rotation of any angle ζ having its axis parallel to $\boldsymbol{e}_3$. The following represent all stress-free joints under the restrictions (v) and $(\text{iii})_3$ and the assumption $\boldsymbol{n}_0 \wedge \boldsymbol{e}_3 \neq 0$:

$$\begin{aligned} &(\boldsymbol{Q}_3^\zeta, \boldsymbol{n}_0), \\ &(\boldsymbol{Q}_3^\zeta(-\boldsymbol{1} + 2\boldsymbol{e}_1 \otimes \boldsymbol{e}_1), \boldsymbol{n}_0), \\ &(\boldsymbol{Q}_3^\zeta(-\boldsymbol{1} + 2\boldsymbol{n}_0 \otimes \boldsymbol{n}_0), \boldsymbol{n}_0), \\ &(\boldsymbol{Q}_3^\zeta(-\boldsymbol{1} + 2\hat{\boldsymbol{e}} \otimes \hat{\boldsymbol{e}}), \boldsymbol{n}_0). \end{aligned} \tag{4.27}$$

If $\boldsymbol{n}_0 = \boldsymbol{e}_3$, we obtain only the stress free-joints $(4.27)_1$ and $(4.27)_2$. For either of the joints $(4.27)_1$ or $(4.27)_2$, the dividing surface need not be a plane (*cf.* § 1).

Finally, if (vii) is satisfied, every pair $(\boldsymbol{R}_0, \boldsymbol{n}_0)$ is a stress-free joint, and the dividing surfaces need not be planar.

Some of the assumptions (i) through (vii) are illustrated by materials of definite symmetry, if we accept the connection between symmetry and stability which leads to (2.7). The condition (i) is associated with crystals of lower symmetry: triclinic, monoclinic and orthorhombic crystals. It is interesting to note that $(\text{iv})_\pm$ is fulfilled for a material of orthorhombic symmetry with constant coefficients of thermal expansion: $c_i(\theta) = k_i\theta$. The assumption (v) is associated with tetragonal, hexagonal and transversely isotropic materials, while (vii) is associated with cubic and isotropic materials.

5. Joining Bodies of Different Materials

We now seek congruent ellipses in planes with normals $\hat{\boldsymbol{n}}$ and $\boldsymbol{n}$ obtained from $\hat{C}(\theta) = \hat{U}(\theta)^2$ and from $C(\theta) = U(\theta)^2$. The squared lengths of the major and minor axes are roots of (3.19) in both cases. Thus, the conditions that two ellipses obtained by the procedure described in § 3 are congruent are simply

$$\begin{aligned} \tilde{A}(C(\theta), \boldsymbol{n}_0) &= \tilde{A}(\hat{C}(\theta), \hat{\boldsymbol{n}}_0), \\ \tilde{B}(C(\theta), \boldsymbol{n}_0) &= \tilde{B}(\hat{C}(\theta), \hat{\boldsymbol{n}}_0), \end{aligned} \tag{5.1}$$

the functions $\tilde{A}$ and $\tilde{B}$ being defined by (3.20). Given $C(\theta)$ and $\hat{C}(\theta)$ we seek constant unit vectors $\boldsymbol{n}_0$ and $\hat{\boldsymbol{n}}_0$ which satisfy (5.1), this being a necessary condition that there be a stress-free joint.

It will be convenient for the purpose of solving (5.1) to view $\{\tilde{A}(C, \boldsymbol{n}_0), \tilde{B}(C, \boldsymbol{n}_0)\}$ as a point in the A–B plane. As $\boldsymbol{n}_0$ runs over all unit vectors, $\{\tilde{A}(C, \boldsymbol{n}_0), \tilde{B}(C, \boldsymbol{n}_0)\}$

will determine a locus of points in the A–B plane. The two regions found in this way by using $\hat{C}(\theta)$ and then $C(\theta)$ will overlap, if (5.1) has a solution $(\boldsymbol{n}_0, \hat{\boldsymbol{n}}_0)$.

To find the regions of the A–B plane just described, we write out (3.20) and the condition $\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 = 1$:

$$\begin{aligned} A &= (c_2 + c_3)\, n_{01}^2 + (c_1 + c_3)\, n_{02}^2 + (c_1 + c_2)\, n_{03}^2, \\ B &= c_2 c_3 n_{01}^2 + c_1 c_3 n_{02}^2 + c_1 c_2 n_{03}^2, \\ 1 &= n_{01}^2 + n_{02}^2 + n_{03}^2. \end{aligned} \tag{5.2}$$

Assume $0 < c_1 \leqq c_2 \leqq c_3$. Recall that $A = \lambda_a + \lambda_b$ and $B = \lambda_a \lambda_b$. The equations (5.2) are linear equations for $(n_{01}^2, n_{02}^2, n_{03}^2)$. Equation (5.2) defines a triangle in $\mathbb{R}^3$, the triple $(n_{01}^2, n_{02}^2, n_{03}^3)$ being a typical point in this space. The equations $(5.2)_{1,2}$ then can be viewed as a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined on a triangle, whose range must therefore be a triangle, a line segment, or a point. To find the appropriate figure it is sufficient to find its vertices by putting two of n_{0i} equal to 0, and the remaining one equal to 1. If C has distinct eigenvalues, we obtain the hatched region shown in Figure 2. The triangle is always nondegenerate if the eigenvalues are distinct. It becomes a line segment if two eigenvalues are equal, and a point if all three are equal (the co-ordinates of the vertices shown in Figure 2 are always maintained).

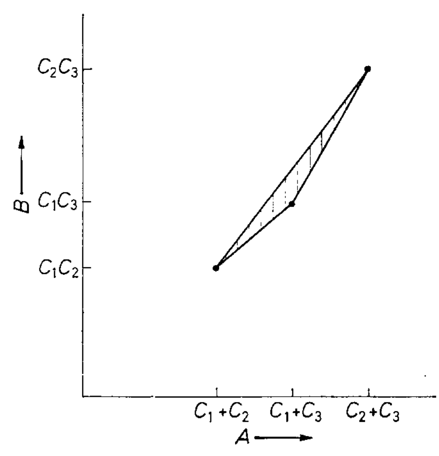

Fig. 2. The range of equation (5.2).

Let triangles be drawn in this way for both $C(\theta)$ and $\hat{C}(\theta)$, for a fixed value of θ. As explained above, a necessary condition that there be a stress-free joint is that the two triangles overlap. If this is true for all $\theta \in \mathscr{I}$, further analysis in the spirit of § 4 will be necessary to determine whether the appropriate *constant* rotation $\boldsymbol{R}_0$ and unit vector $\boldsymbol{n}_0$ can be found.

However, in certain special cases the "test of overlapping triangles" will be sufficient. We may wish to join the bodies at one temperature and have them stress-free at another single temperature θ_1. In this case we let $\mathscr{I}$ consist of θ_1 only and notice that any appropriate $(\boldsymbol{R}_0, \boldsymbol{n}_0)$ will be constant on $\mathscr{I}$. In this case it would be easy to write down all stress-free joints, but I shall not do so. The same kind of simplification is achieved by linearizing the problem, so that $C(\theta)$ and $\hat{C}(\theta)$ are replaced by constant tensors.

6. Stress-Free Polycrystals

In this section we consider stress-free joints consisting of more than two bodies. Recall the formal definition of a stress-free joint $(\boldsymbol{R}_{01}, \ldots, \boldsymbol{R}_{0f}; \mathscr{R}'_1, \ldots, \mathscr{R}'_f)$ given in Section 1. Let the positive symmetric functions $\boldsymbol{U}_1(\theta), \ldots, \boldsymbol{U}_f(\theta)$, $\theta \in \mathscr{I}$ be given.

Part of the problem of finding stress-free polycrystals concerns the conditions under which a set of deformation gradients $\{\boldsymbol{F}_1(\theta), \ldots, \boldsymbol{F}_f(\theta)\}$ defined respectively on the regions $\{\mathscr{R}'_1, \ldots, \mathscr{R}'_f\}$ are gradients with respect to $\boldsymbol{x}$ of a continuous deformation $\boldsymbol{y}(\boldsymbol{x}, \theta)$. These deformation gradients must be of the special form

$$\boldsymbol{F}_i(\theta) = \bar{\boldsymbol{R}}_i(\theta)\, \boldsymbol{U}_i(\theta)\, \boldsymbol{R}_{0i}; \tag{6.1}$$

however, the first results I shall describe do not depend on the particular form of the deformation gradients.

A treatment of general configurations of the regions $\mathscr{R}'_1, \ldots, \mathscr{R}'_f$ is made difficult by two facts. The first is that even if jump conditions like (1.3) are satisfied across every plane of discontinuity, there may not exist a continuous deformation. Figure 3 shows a particularly simple two-dimensional example of this kind (which could be made three-dimensional) consisting of three regions surrounding a hole. There are no distinct constant deformation gradients $\{\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3\}$ defined on $\{\mathscr{R}'_1, \mathscr{R}'_2, \mathscr{R}'_3\}$ that are gradients of a continuous deformation, even though there is a set of three deformation gradients satisfying the jump conditions at each interface. I omit the details. The second fact is that even if a continuous function $\boldsymbol{y}(\boldsymbol{x}, \theta)$ is found, it may not be invertible. A theorem of BALL [6] shows that if the function $\boldsymbol{y}(\boldsymbol{x}, \theta)$ is not invertible, it is also not invertible when restricted to $\partial(\mathscr{R}'_1 \cup \ldots \cup \mathscr{R}'_f)$. Here I have no readily available information on how $\partial(\mathscr{R}'_1 \cup \ldots \cup \mathscr{R}'_f)$ is deformed.

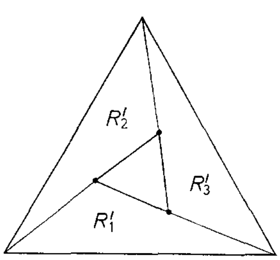

Fig. 3. Each pair of bodies forms a stress-free joint but the collection does not form a stress-free polycrystal unless all three deformation gradients are equal.

For these reasons I shall consider a special class of regions $\mathscr{R}'_1, \ldots, \mathscr{R}'_f$ which, nevertheless, describes a sufficiently small neighborhood of any point in a polycrystal quite generally. Consider a sphere $\mathscr{S}$ with center P. Place a finite number of points (vertices), v of them in all, on $\partial\mathscr{S}$ and join them pairwise by e arcs of great circles (edges), so as to divide up $\partial\mathscr{S}$ into f regions (faces). Assume that the two endpoints of each edge are distinct vertices, that edges only intersect at their endpoints, and that the set of points consisting of edges and vertices is connected. Then Euler's relation $f - e + v = 2$ is satisfied. Now construct line segments joining each vertex to P. Each edge and the line segments which connect its terminal vertices to P bound a part of a plane in $\mathscr{S}$. These parts of planes (interfaces) divide $\mathscr{S}$ into f connected regions; I shall assume the regions constructed in this way are $\mathscr{R}'_1, \ldots, \mathscr{R}'_f$. A subdivision of the sphere carried out according to these rules will be termed a **partition** of $\mathscr{S}$.

One measure of the "simplicity" of a partition is the number e of interfaces present, since the relation $f - e + v = 2$ implies that it is an upper bound for both f and v. I shall consider the following problem: *Given e, are there any partitions with e interfaces, having constant deformation gradients* $\{\mathbf{F}_1, \ldots, \mathbf{F}_f\}$ *defined on* $\{\mathscr{R}'_1, \ldots, \mathscr{R}'_f\}$, *such that the* $\mathbf{F}_1, \ldots, \mathbf{F}_f$ *are gradients of a continuous deformation* $\mathbf{y}(\mathbf{x})$ *and such that* $\mathbf{F}_i \neq \mathbf{F}_j$ *when* $\mathscr{R}'_i$ *and* $\mathscr{R}'_j$ *share an interface*? A partition admitting a continuous deformation of this kind will be termed *coherent*. Clearly, coherent partitions are the only partitions of which stress-free polycrystals can be composed. It might seem at first that ideas of continuity alone provide only rather weak restrictions on stress-free polycrystals, but the restrictions turn out to be quite significant.

Partitions of $\mathscr{S}$ partly avoid the difficulties with general arrangements discussed above. First, the jump conditions (1.3), imposed at each of the e interfaces, are necessary and sufficient for the existence of a continuous deformation. Second, if there is one such continuous deformation, there are always invertible continuous deformations, although these may not have the special property (6.1). See JAMES [2] for details.

The solution of the problem discussed above is more easily described if we note that given a coherent partition with certain values of e, f and v, we can always make another coherent partition in a trivial way by adding a vertex to the interior of an existing edge, thereby increasing e and v each by one. Partitions formed in this way clutter the description. To avoid them, it is convenient to confine attention to partitions without *removable vertices*, a removable vertex* being one at which exactly two edges meet which themselves do not intersect at another vertex.

Excluding partitions with removable vertices, a list of all coherent partitions with $e = 2, 3, 4, 5, 6, 7$ is given in Figure 4. The partitions are defined explicitly from Figure 4 by naming the vertices, edges and faces and by listing their topological relationships (*i.e.*, "a is on the boundary of b") and by accounting for certain restrictions mentioned below. These are: the two interfaces are parallel in the coherent partitions with $e = 2$, no two of the three interfaces are parallel

* This definition is explained further by JAMES [2]. Also, an algorithm by which any *given* partition can be judged coherent is given in [2].

in coherent partitions with $e = 3$, no two of the three interfaces are parallel at any line segment where exactly three interfaces meet in coherent partitions with $e = 6$ or $e = 7$. There are no further restrictions. Thus, the precise angles between the interfaces are not determined by coherence in most cases.

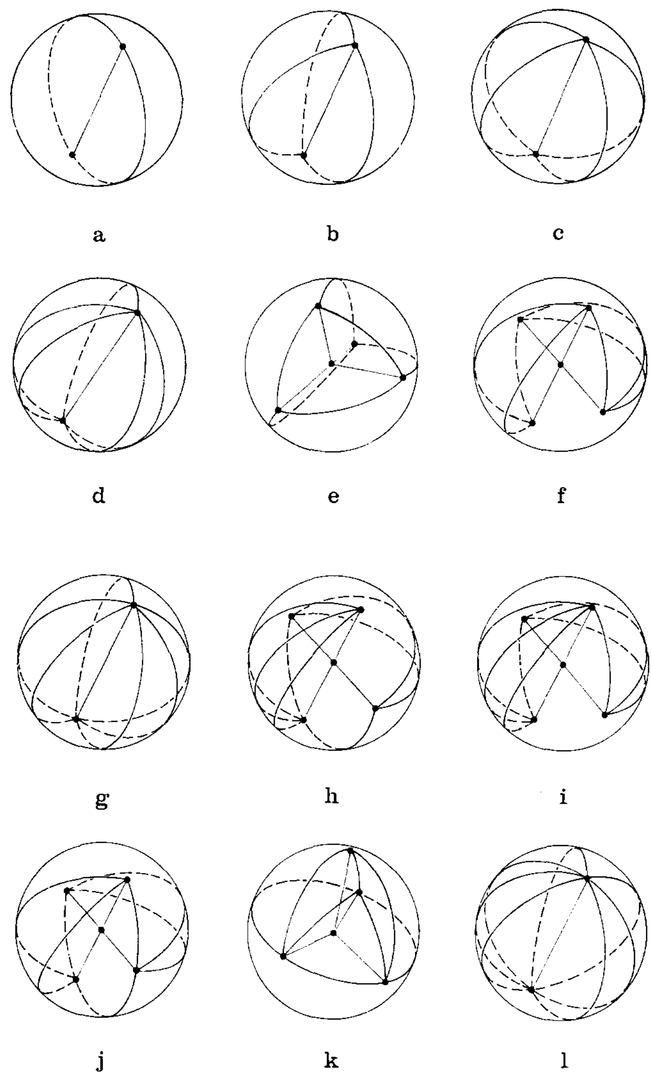

Fig. 4. Coherent arrangements with up to seven surfaces of discontinuity.

Stronger results are obtained if we make use of the special forms of the deformation gradients given by (6.1). For example, consider stress-free partitions of one material. Then, since $\boldsymbol{U}_1(\theta) = \ldots = \boldsymbol{U}_f(\theta)$, the equations (6.1) imply that $\det \boldsymbol{F}_i(\theta) = v(\theta)$, $i = 1, \ldots, f$, for some function $v(\theta)$. Suppose $\partial\mathcal{R}'_i$ and

$\partial\mathscr{R}'_j$ share the k^{th} interface which has amplitude $\boldsymbol{a}_k$ (*cf.* (1.4)) and normal $\boldsymbol{n}_{0k}$. Then

$$\boldsymbol{F}_i(\theta) - \boldsymbol{F}_j(\theta) = \boldsymbol{a}_k \otimes \boldsymbol{n}_{0k}, \tag{6.2}$$

implying that

$$\boldsymbol{F}_i(\theta)\,\boldsymbol{F}_j^{-1}(\theta) = \mathbf{1} + \boldsymbol{a}_k \otimes \boldsymbol{F}_j^{-T}(\theta)\,\boldsymbol{n}_{0k}. \tag{6.3}$$

If we take the determinant of (6.3) and use the equivalent of (3.2), we get

$$\boldsymbol{a}_k \cdot \boldsymbol{n}_k = 0, \tag{6.4}$$

$\boldsymbol{n}_k$ being the normal to the image of the k^{th} interface under $\boldsymbol{y}(\boldsymbol{x}, \theta)$ (the deformed k^{th} interface).

This result has some immediate consequences for special coherent partitions. For example, it is known (*cf.* [2]) that the coherent partitions with $e = 6$, except for the ones represented by Figure 4g, all have the property that all six amplitudes are parallel. Notice that for any of the coherent partitions of Figures 4e and 4f, the deformed interfaces give rise to a basis of normal vectors. But if (6.4) is satisfied for a basis $\{\boldsymbol{n}_k\}$ and parallel amplitudes, then all six amplitudes must vanish, a condition forbidden by the definition of a coherent partition. *Thus, the partitions of Figures* 4e *and* 4f *cannot support stress-free polycrystals of one material if all interfaces are surfaces of discontinuity.*

Partitions in the class 4e occur commonly in polycrystals with randomly oriented grains, as discussed by SMITH [7].

Another result of this kind concerns coherent partitions with $e = 3$, which also must have parallel amplitudes. For a coherent partition with $e = 3$, let $\boldsymbol{a}$ be a unit vector parallel to the amplitudes. If this partition also belongs to a stress-free polycrystal, $\boldsymbol{a}$ will be parallel to the "axis" (*cf.* Figure 4b) of the partition. *Thus an interface in a stress-free coherent partition cannot be bounded by two nonparallel line segments each of which is on the boundary of exactly three regions.*

Similar but more complicated rules can be derived for other partitions. These rules suggest that it might be quite difficult to make stress-free polycrystals in materials of lower symmetry. I have written out a number of examples using partitions with $e = 3$ and $e = 4$, and then I have used these to build up some space-filling stress-free polycrystals. These do not seem to lend themselves to an orderly classification, so I shall not describe them. In every case, I found it necessary to use stress-free joints other than the ones given by (4.8). This suggests that materials for which the only stress-free joints are given by (4.8) have stress-free polycrystals of a trivial kind: layers separated by parallel planes.

The following calculations worked out in collaboration with S. SPECTOR show that such layered bodies are indeed the only stress-free polycrystals of one material, possible in all materials.

Let us begin with the stress-free joints $(-\mathbf{1} + 2\boldsymbol{n}_0 \otimes \boldsymbol{n}_0, \boldsymbol{n}_0)$ and try to arrange them in a coherent partition. An unlayered polycrystal will have at least one line segment at which $n > 2$ regions meet, so it is natural to study partitions for which $v = 2$, $f = n$, $e = n$; these consist of n leaves meeting along an axis (*e.g.* Figures 4b, c, d, g, l). Let the regions $\mathscr{R}'_i$ and $\mathscr{R}'_{i+1}$ and the interfaces be numbered consecutively, so that the i^{th} interface separates the regions $\mathscr{R}'_i$ and $\mathscr{R}'_{i+1}$ and the n^{th} interface separates the regions $\mathscr{R}'_n$ and $\mathscr{R}'_1$.

Assuming that the regions $\mathscr{R}'_1, \ldots, \mathscr{R}'_n$ are given, we seek necessary conditions that $(\boldsymbol{R}_{01}, \ldots, \boldsymbol{R}_{0n}; \mathscr{R}'_1, \ldots, \mathscr{R}'_n)$ be a stress-free joint for some choice of the rotations $\boldsymbol{R}_{01}, \ldots, \boldsymbol{R}_{0n}$. Let $\boldsymbol{F}_1(\theta)$ be the deformation gradient in $\mathscr{R}'_1$. According to (6.1) and § 4, the remaining deformation gradients can be obtained by induction:

$$\boldsymbol{F}_{i+1}(\theta) = \boldsymbol{R}_i(\theta)\,\boldsymbol{F}_i(\theta)\,\boldsymbol{R}_{0i}, \quad i = 1, \ldots, n-1 \text{ (no sum)}. \tag{6.5}$$

In (6.5) we have from (4.8) and (4.9),

$$\begin{aligned} \boldsymbol{R}_i(\theta) &= -\mathbf{1} + 2\boldsymbol{n}_i(\theta) \otimes \boldsymbol{n}_i(\theta), \\ \boldsymbol{R}_{0i} &= -\mathbf{1} + 2\boldsymbol{n}_{0i} \otimes \boldsymbol{n}_{0i} \text{ (no sum)}, \end{aligned} \tag{6.6}$$

$\boldsymbol{n}_{0i}$ and $\boldsymbol{n}_i(\theta)$ being unit normals to the i^{th} interface and the i^{th} deformed interface, respectively. Notice that all $\boldsymbol{F}_i(\theta)$ are determined by $\mathscr{R}'_1, \ldots, \mathscr{R}'_n$ and $\boldsymbol{F}_1(\theta)$. By construction, the joints $(\boldsymbol{R}_{0i}, \boldsymbol{n}_{0i})$, $i = 1, \ldots, n-1$, each associated with the pair of bodies $\mathscr{R}'_{i-1}\mathscr{R}'_i$, are stress-free. Thus, the condition that the polycrystal be stress-free is simply

$$\boldsymbol{R}_n(\theta)\,\boldsymbol{F}_n(\theta)\,\boldsymbol{R}_{0n} = \boldsymbol{F}_1(\theta), \tag{6.7}$$

which, by the use of (6.5), becomes

$$\boldsymbol{R}_n(\theta) \ldots \boldsymbol{R}_1(\theta)\,\boldsymbol{F}_1(\theta)\,\boldsymbol{R}_{01} \ldots \boldsymbol{R}_{0n} = \boldsymbol{F}_1(\theta). \tag{6.8}$$

Retracing the steps of the argument, we see that equation (6.8) is necessary and sufficient that a partition of the form $e = n$, $f = n$, $v = 2$ be a stress-free polycrystal (of one material).

If we premultiply (6.8) by its transpose and let $\boldsymbol{C}_1(\theta) = \boldsymbol{F}_1(\theta)^T \boldsymbol{F}_1(\theta)$, we get

$$\boldsymbol{R}_{0n}^T \ldots \boldsymbol{R}_{01}^T \boldsymbol{C}_1(\theta)\,\boldsymbol{R}_{01} \ldots \boldsymbol{R}_{0n} = \boldsymbol{C}_1(\theta). \tag{6.9}$$

We can choose a material for which $\boldsymbol{C}_1(\theta)$ has distinct eigenvalues and θ-dependent eigenvectors. Then (6.9) is satisfied if and only if

$$\boldsymbol{R}_{01} \ldots \boldsymbol{R}_{0n} = \mathbf{1}. \tag{6.10}$$

Returning to (6.8), we conclude that

$$\boldsymbol{R}_1(\theta) \ldots \boldsymbol{R}_n(\theta) = \mathbf{1}. \tag{6.11}$$

Let $\boldsymbol{m}_0$ be a unit vector on the axis of the partition: $\boldsymbol{m}_0 \cdot \boldsymbol{n}_{0i} = 0$, $i = 1, \ldots, n$. Equation (6.10) implies that

$$\boldsymbol{R}_{01} \ldots \boldsymbol{R}_{0n}\boldsymbol{m}_0 = \boldsymbol{m}_0, \tag{6.12}$$

while the definition $(6.6)_2$ implies that

$$\boldsymbol{R}_{01} \ldots \boldsymbol{R}_{0n}\boldsymbol{m}_0 = \pm\boldsymbol{m}_0, \tag{6.13}$$

the minus sign occuring if and only if n is odd. It follows that there must be an even number of regions.

Assuming n is even, we group the terms on the left hand side of (6.10) in consecutive pairs. Notice that each of these $n/2$ pairs has the property that $\boldsymbol{m}_0$ lies on its axis of rotation, *i.e.*

$$\boldsymbol{R}_{0i}\boldsymbol{R}_{0(i+1)}\,\boldsymbol{m}_0 = \boldsymbol{m}_0. \tag{6.14}$$

Let $\phi_i \in (0, 2\pi)$ denote the angle between the $(i-1)^{st}$ interface and the i^{th} interface measured from the i^{th} interface by an arc which does not cross any other interface. A short calculation gives the angle of rotation for the pair $\boldsymbol{R}_{0i}\boldsymbol{R}_{0(i+1)}$ as $2\phi_{i+1}$, the sense of rotation being from $i+1$ to i. Since each of these pairs has the same axis of rotation, equation (6.10) is *equivalent* to the simple equation

$$\phi_2 + \phi_4 + \ldots + \phi_n = \pi. \tag{6.15}$$

In words, the angles determined by the planes bounding the even (or odd) numbered regions are complementary.

The condition (6.15) can always be satisfied by choosing the regions $R_1', \ldots, R_n'$ appropriately. In doing so, no restrictions are imposed on $\boldsymbol{F}_1(\theta)$.

Let $\mu_i^\theta \in (0, 2\pi)$ be the angle between the deformed $(i-1)^{st}$ and the i^{th} interfaces determined in the manner described above. By the same argument as that which leads from (6.10) to (6.15), we can show that (6.11) is equivalent to the statement

$$\mu_2^\theta + \mu_4^\theta + \ldots \mu_n^\theta = \pi. \tag{6.16}$$

Thus, (6.15) and (6.16) comprise necessary and sufficient conditions that the polycrystals being studied are stress-free.

It is not difficult to choose a function $\boldsymbol{C}_1(\theta)$ which makes (6.16) fail. This can be done by taking an appropriate pair of consecutive even numbered deformed regions and by assuring that the sum of their included angles exceeds π. I omit the details.

To understand the kind of restriction imposed by (6.16), it is helpful to consider partitions for which

$$\phi_1 = \phi_2 = \ldots = \phi_n. \tag{6.17}$$

A short calculation of the deformed angles based upon (6.5) and (6.6) shows that these must satisfy

$$\mu_1^\theta = \mu_2^\theta = \ldots = \mu_n^\theta \tag{6.18}$$

for all θ in $\mathscr{I}$. But (6.18) implies that

$$\mu_1^\theta = \frac{2\pi}{n} = \text{const.}, \tag{6.19}$$

which, in turn, shows that $\boldsymbol{C}_1(\theta)^{-1}$ must leave a pair of constant vectors unsheared:

$$\frac{\boldsymbol{n}_{01}\cdot S(\theta)\,\boldsymbol{n}_{02}}{(\boldsymbol{n}_{01}\cdot S(\theta)\,\boldsymbol{n}_{01})^{\frac{1}{2}}\,(\boldsymbol{n}_{02}\cdot S(\theta)\,\boldsymbol{n}_{02})^{\frac{1}{2}}} = \boldsymbol{n}_{01}\cdot\boldsymbol{n}_{02} = \cos\left[\frac{2\pi}{n}\right] \quad \forall\, \theta \text{ in } \mathscr{I} \tag{6.20}$$

In this expression, $S(\theta) = \boldsymbol{C}_1(\theta)^{-1}$.

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