The Linearization Principle for the Stability of Solutions of Quasilinear Parabolic Equations, I.

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§ 1. Introduction

The aim of this paper is to prove existence, uniqueness and stability theorems for an abstract quasilinear equation, in such a way that the results can be applied to the equations governing the motion of viscoelastic three-dimensional bodies. For such bodies one can justify the Energy Criterion, which has been widely used in elastic stability theory. This program will be carried out in a companion paper [19]. To my knowledge, results as accurate and general as these were known only for semilinear equations [11], [12], [13]. The simplest example of a quasilinear parabolic equation is the nonlinear heat equation, which can be written in the form

$$(1)\qquad \frac{\partial u}{\partial t} - a_{ij}(x, u, \vec{\nabla} u)\frac{\partial^2 u}{\partial x_i\, \partial x_j} = f(x, u, \vec{\nabla} u)$$

$$\forall x \in \Omega, \quad t \geqq 0,$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. Here and throughout this paper the summation convention is used. We derive (1) below. There we identify u and a_{ij}. If the temperature is fixed on $\partial\Omega$, then u satisfies the Dirichlet condition

$$(2)\qquad u(x, t) = 0 \quad \forall x \in \partial\Omega, \quad t \geqq 0.$$

If $\partial\Omega$ is insulated, then u satisfies

$$(3)\qquad Q_i(x, u, \vec{\nabla} u)\, n_i(x) = 0 \quad \forall x \in \partial\Omega, \quad t \geqq 0,$$

where we denote by n_i the components of the normal to $\partial\Omega$ and by Q_i those of the heat flux vector, which is a nonlinear function of the temperature and of its gradient. Unfortunately our techniques have not yet proved capable of handling (3).

Our main tool is P. E. SOBOLEVSKII's theory of evolution operators [21], which generalizes the theory of holomorphic semigroups. First, the system (1), (2) is considered as an abstract equation in a Banach space X:

$$\frac{du(t)}{dt} + T(u(t))\, u(t) = f(u(t)) \tag{4}$$

where $T(u)$ is a family of closed linear operators that depend nonlinearly on u and that have a domain D that is dense in X and is independent of u. To take account of the quasilinear character of the equation (4), we assume that the operator $T(u)$ and the second member $f(u)$ are well defined not only when u is in the domain D, but also when u is in an intermediate space between D and X. The idea underlying our approach is to seek the solution as a fixed point in a suitable functional space of the map $u \to v$, where v is the solution of the nonhomogeneous linear equation (5)

$$\frac{dv(t)}{dt} + T(u(t))\, v(t) = f(u(t)). \tag{5}$$

Since the operator $A(t) = T(u(t))$ is not independent of time, equation (5) cannot be solved by the theory of semigroups. We therefore use the theory of P. E. SOBOLEVSKII [21]. The evolution operator $U(t, \tau)$ is a linear operator defined for $t \geqq \tau$ by the initial value problem

$$\frac{\partial}{\partial t} U(t,\tau) + A(t)\, U(t,\tau) = 0 \quad \text{for} \quad t > \tau, \quad U(\tau,\tau) = Id.$$

SOBOLEVSKII has proved the existence of the family $\{U(t,\tau)\}$ and has obtained a variation of constants formula that permits us to transform the differential equation (5) into an integral equation. But in order to obtain a solution of (4) by the contracting map theorem, we must know how the evolution operator varies with $u(t)$.

In the same way, to prove the stability result, the exponential decay property of the operator $U(t,\tau)$ is needed. Both of these essential points, which are not considered in SOBOLEVSKII'S paper, are studied in Section 3. In Section 4 the theorems on abstract quasilinear equations are proved by the methods just described. Some properties of intermediate spaces and fractional powers of an unbounded operator are recalled in the Appendix. In the second section the main results are stated and are applied to the nonlinear heat equation.

In the application to the concrete equations (1), (2), the spaces D and X are chosen as follows:

$$X = L^p(\Omega), \qquad D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

where $W^{2,p}(\Omega)$ is the classical Sobolev space of functions having their derivatives up to order 2 in $L^p(\Omega)$. $W_0^{1,p}(\Omega)$ is similarly defined, but the functions of this space must vanish at the boundary. For (1) to have a meaning, it is necessary that the functions of D have a bounded gradient. A Sobolev embedding theorem [20] implies that this is true if and only if p is greater than the dimension N of the open set Ω. Other possible spaces would be the small Nikolskii spaces [5]. For these DA PRATO & GRISVARD [4] have even proved a variation of constants formula that could permit an analysis not requiring the evolution operator theory. Our abstract result could not be applied with the Hölder spaces because the domain D would not be dense and also because the initial data would necessarily satisfy a nonlinear compatibility condition at the boundary, whereas SOBOLEVSKII's theory requires a linear space as a domain D. The boundary condition (3) is not considered for the same reason. Of course there exist other evolution operator theories [8], [10] which could be used for the study of nonlinear boundary conditions, but they are intricate and their results are not as accurate as SOBOLEVSKII's.

One may ask whether the theories of SOBOLEVSKII [21] and of LIONS-PEETRE [15], which use interpolation spaces, could be profitably replaced in our analysis by more direct methods relying on energy estimates. The use of energy estimates to obtain uniform bounds on $\vec{\nabla} u$, which are critical in the analysis and which rely on embedding theorems, becomes very difficult when the nonlinearities are "strong" and when the dimension N exceeds 1. On the other hand, the most refined properties of solutions of parabolic equations, such as the estimate (20), follow from semigroup theory and not from energy estimates. By using semigroup theory I have obtained a result [19, Theorem 1] for three-dimensional viscoelasticity that is sharper than that obtained by EBIHARA [22] for a corresponding one-dimensional problem with a linear elastic response. I believe that it might be possible to establish results similar to those stated here by using an interpolation theory simpler than that of LIONS-PETREE in a Hilbert space setting. Such procedures might be able to handle boundary conditions of the form (3). Nevertheless, I feel that the semigroup approach will remain more powerful than energy methods for this class of problems.

The results of this paper were announced in [18].

§ 2. The main results and an application

a) The main results

We now state an existence and uniqueness theorem and a stability theorem for the initial value problem; these are the two main results of this paper. The interpolation spaces in the sense of LIONS-PEETRE [15] (described in the Appendix) are denoted by $[D, X]_{\theta,p}$ and their norms by $\| \, \|_{\theta,p}$. We introduce the following set:

$$\Sigma(\omega, \beta) = \{\lambda \in \mathbb{C} \mid |\operatorname{Arg} \lambda| \leqq \pi/2 + \omega \quad \text{or} \quad \operatorname{Re} \lambda \geqq -\beta\}.$$

If the numbers ω and β are positive, this set is the union of a sector and a triangle.

Theorem 1 (Existence and Uniqueness).

Let X and D be Banach spaces with D dense in X. Let the norm of X be denoted by $\|\cdot\|$. Let $0 < \theta \leqq 1$, $0 \leqq \theta' < 1$, $1 \leqq p \leqq \infty$. For each u in a neighborhood of 0 in $[D, X]_{\theta,p}$ let $T(u): D \to X$ be a closed linear operator. Let f be a nonlinear map from a neighborhood of 0 in D into $[D, X]_{\theta',p}$. Suppose that

(i) *There exist positive numbers* ω, β *and* C *such that* $\Sigma(\omega, \beta)$ *is contained in the resolvent set of* $T(0)$ *and*

$$\|[T(0) + \lambda]^{-1}\| \leqq C/[1 + |\lambda|] \quad \forall \lambda \in \Sigma(\omega, \beta).$$

(ii) *For any given x in D, the map* $u \to T(u)\, x$ *from a neighborhood of* 0 *in* $[D, X]_{\theta,p}$ *into X is differentiable and there are numbers* $\eta > 0$, $C > 0$ *such that its derivative* $[T'(u) -]\, x$ *satisfies the Hölder condition*

$$\|[T'(u_2)\, v - T'(u_1)\, v]\, x\| \leqq C(\|u_2 - u_1\|_{\theta,p})^{\eta}\, \|v\|_{\theta,p}\, \|x\|_D.$$

(iii) *There is a number* $C > 0$ *such that f satisfies the Lipschitz condition*

$$\|f(u_2) - f(u_1)\|_{\theta',p} \leqq C\, \|u_2 - u_1\|_D.$$

Then there exist positive numbers ε *and* t_0 *such that the initial value problem*

$$\frac{du}{dt} + T(u)\, u = f(u), \qquad u(0) = u_0 \tag{6}$$

has a unique solution

$$u \in C([0, t_0], D) \cap C^1([0, t_0], X) \tag{7}$$

provided that $\|u_0\|_D \leqq \varepsilon$.

Theorem 2 (Linearization principle). *Suppose that the hypotheses of Theorem 1 hold and furthermore that there is a number* $C > 0$ *such that*

$$\|f(u)\|_{\theta',p} \leqq C\, \|u\|_D^2. \tag{iv}$$

Then the solution u of (6) *can be extended to any positive time t and the equilibrium solution* $u = 0$ *is exponentially stable with respect to the norm of D. More precisely, if* $\|u_0\|_D$ *is sufficiently small, then there are positive numbers C and* β *such that*

$$\|u(t)\|_D \leqq C\, \|u_0\|_D \exp(-\beta t). \tag{8}$$

Remarks. 1) Theorem 2 is a justification of the linearization principle, because, by Assumption (iv), the linearized equation in a neighborhood of $u = 0$ is

$$\frac{du}{dt} + T(0)\, u = 0. \tag{9}$$

Assumption (i), ensuring that the spectrum of $T(0)$ lies to the left of the line $\operatorname{Re} \lambda = -\beta$, implies the exponential decay of the solutions of (9).

2) The other Assumptions (ii) and (iii) are technical; they characterize the quasi-linear character of the equation (6).

3) The initial value problem can be solved in the same way when the operator T and the second member f are Hölder continuous functions of time.

b) Application to the nonlinear heat equation

Let us consider a rigid heat conductor that occupies a domain Ω of the Euclidean space $\mathbb{R}^N$. The energy balance gives the equation

$$\varrho \frac{de}{dt} = r - \operatorname{div} \vec{q} \tag{10}$$

where ϱ is the mass density, e the internal energy per unit mass, r the heat supply per unit volume and $\vec{q}$ the heat flux vector. It is assumed that the internal energy e depends only on the temperature T while the heat flux vector and the heat supply are functions of the temperature and its gradient. Hence the nonlinear heat equation is obtained as

$$\varrho e'(T) \frac{\partial T}{\partial t} = r(T, \vec{\nabla} T) - \operatorname{div} \vec{q}(T, \vec{\nabla} T). \tag{11}$$

Let $T_0(x)$ be a temperature field that represents either the temperature field at time 0 or a stationary solution whose stability is to be studied. We let

$$u(x, t) = T(x, t) - T_0(x),$$

$$a_{ij}(x, u, \vec{\nabla} u) = -\frac{\partial}{\partial(\nabla T)_j} Q_i(T_0 + u, \vec{\nabla} T_0 + \vec{\nabla} u)/\varrho e'(T_0 + u).$$

Then equation (11) can be put into the form

$$\frac{\partial u}{\partial t} = a_{ij}(x, u, \vec{\nabla} u) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + g(x, u, \vec{\nabla} u). \tag{12}$$

The requirement that the temperature be prescribed on the boundary of Ω yields

$$u(x, t) = 0 \quad \forall x \in \partial\Omega, \quad t \geqq 0. \tag{13}$$

Theorem 3.

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $\partial\Omega$ be in C^2. Let the functions $(x, u, G) \to a_{ij}(x, u, G), g(x, u, G)$ be defined for x in $\bar{\Omega}$ and for (u, G) in a neighborhood of 0 in $\mathbb{R} \times \mathbb{R}^N$. Let there be a number $\eta > 0$ such that the functions $(u, G) \to a_{ij}(x, u, G)$ are in $C^{1,\eta}$ uniformly for x in $\bar{\Omega}$ and let there be a number $\varepsilon > 0$ such that the functions $(u, G) \to g(x, u, G)$, $[g(x, u, G) - g(x', u, G)]/|x - x'|^\varepsilon$ are in C^2 uniformly for x, x' in $\bar{\Omega}$. Let $a_{ij}(x, 0, 0)$ be uniformly elliptic: there exists a $C_0 > 0$ such that

$$a_{ij}(x, 0, 0)\, \xi_i \xi_j \geqq C_0 \xi_i \xi_j \quad \forall x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^N.$$

If $p > N$, if the initial data are sufficiently small in $W^{2,p}(\Omega)$, and if the initial data vanish on $\partial\Omega$, then equations (12) and (13) have a unique solution in the class

$$C([0, t_0], W^{2,p}(\Omega)) \cap C^1([0, t_0], L^p(\Omega)).$$

Furthermore, if $u = 0$ satisfies (12) and (13) and if all solutions of the linearized equation decay exponentially, then the solution of (12) and (13) exists for all positive *times t and the equilibrium state $u = 0$ is exponentially stable with respect to the $W^{2,p}$-norm.*

Remark 4. The main consequence of this theorem can be formulated as follows. Let $T_0(x)$ be a stationary solution of the heat equation. If this stationary solution, the boundary of the body and the constitutive laws are sufficiently smooth and if the coefficients of the linearized equation are uniformly elliptic, then the linearization principle can be applied to the study of the stability of the solution $T_0(x)$.

Proof of Theorem 3. We rewrite the heat equation in a slightly different form:

$$\text{(14)} \qquad \frac{\partial u}{\partial t} - a_{ij}(x, u, \vec{\nabla} u)\frac{\partial^2 u}{\partial x_i \, \partial x_j} - b_i(x)\frac{\partial u}{\partial x_i} - c(x)\, u + ku$$
$$= g(x, 0, 0) + ku + R(x, u, \vec{\nabla} u)$$

with

$$k \geqq 0,$$

$$b_i(x) = \frac{\partial g}{\partial(\nabla u)_i}(x, 0, 0), \qquad c(x) = \frac{\partial g}{\partial u}(x, 0, 0).$$

Then $R(x, u, \vec{\nabla} u)$ is the second order remainder in the Taylor's expansion of $g(x, \ldots)$. For the application of Theorems 1 and 2, we set

$$X = L^p(\Omega), \qquad D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$T(u)\, v = -a_{ij}(x, u, \vec{\nabla} u)\frac{\partial^2 v}{\partial x_i \, \partial x_j} - b_i(x)\frac{\partial v}{\partial x_i} - c(x)\, v + kv.$$

$f(u)$ is the expression given in the second member of (14). The uniform ellipticity of $(a_{ij}(x, 0, 0))$ implies that Gårding's inequality is valid: There exist numbers $C_1 > 0$, $\lambda \geqq 0$ such that

$$\int_\Omega a_{ij}(x, 0, 0)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \geqq C_1 \|u\|_{H^1}^2 - \lambda \|u\|_{L^2}^2 \qquad \forall u \in H^1(\Omega).$$

Then using this inequality and a classical criterion [9, p,. 490] we can easily prove that the operator $T(0)$ generates a holomorphic semigroup that decays exponentially in the space $L^2(\Omega)$, if k is chosen sufficiently large. The estimation (i) of Theorem 1 is a classical result, which can be obtained either by an energy method [2] , [21], or by a more general process [1] (which will be also used in a forthcoming paper [19]). Furthermore $T(0)$ has a compact resolvent because D is compactly embedded in X. Thus the spectrum of $T(0)$ consists of isolated eigenvalues accumulating only at infinity [9]. If the solutions of the linearized equation decay exponentially and if we set $k = 0$, the spectrum of $T(0)$ is located to the left of a line $\operatorname{Re} \lambda = -\beta$ $(\beta > 0)$ so the estimation (i) holds even for $k = 0$.

In order that Assumption (ii) of Theorem 1 be true, it is sufficient that the map $u \to a_{ij}(x, u, \vec{\nabla} u)$ be in $C^{1,\eta}$ from $[D, X]_{\theta,p}$ into $C(\bar{\Omega})$. By means of an embedding theorem [16] we find that

$$W^{s,p}(\Omega) \subset C^1(\bar{\Omega}) \quad \text{for} \quad 2 > s > 1 + N/p,$$

and by means of an interpolation theorem [7] we find that

$$[D, X]_{\theta,p} = W^{2(1-\theta),p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{for} \quad 0 < \theta < \frac{1}{2}.$$

We conclude that Assumption (ii) is valid if the mapping $(u, G) \to a_{ij}(x, u, G)$ is in $C^{1,\eta}$, uniformly for $x \in \bar{\Omega}$; this is a hypothesis of Theorem 3.

To prove Assumptions (iii) and (iv), we may use the following embedding theorem [20] and interpolation theorem [7]:

$$W^{1,p}(\Omega) \subset C^{\varepsilon}(\Omega) \subset W^{\varepsilon,p}(\Omega) \quad \text{for} \quad \varepsilon = 1 - N/p,$$

$$[D, X]_{1-\varepsilon'/2,p} = W^{\varepsilon',p}(\Omega) \quad \text{for} \quad \varepsilon' < 1 - N/p.$$

Furthermore, the hypotheses on $g(.,.,.,.)$ allow us to show that

$$\| R(x, u_2, \vec{\nabla} u_2) - R(x_1, u_1, \vec{\nabla} u_1)\|_{C^{\varepsilon}(\Omega)} \leqq C[\| u_2 - u_1\|_{C^{\varepsilon}(\Omega)} + \|\vec{\nabla} u_2 - \vec{\nabla} u_1\|_{C^{\varepsilon}(\Omega)}],$$

$$\| R(x, u, \vec{\nabla} u)\|_{C^{\varepsilon}(\Omega)} \leqq C[\| u\|^2_{C^{\varepsilon}(\Omega)} + \|\vec{\nabla} u\|^2_{C^{\varepsilon}(\Omega)}].$$

All these properties imply Assumptions (iii) and (iv). Then Theorem 3 is a consequence of Theorems 1 and 2.

§ 3. The nonhomogeneous equations

We first recall Sobolevskii's results about the nonhomogeneous equation

$$\frac{du}{dt} + A(t)\, u = f(t), \qquad u(0) = u_0. \tag{15}$$

Proposition 4. *Let X and D be two Banach spaces such that D is dense in X and is continuously embedded in X. Let* $\{A(t)\}$ *be a family of closed linear operators on X that are defined for* $t \in [0, t_0]$ *and have the common domain D. Let there be numbers* $C > 0$, $\alpha \in [0, 1]$ *such that*

$$\| [A(t) - A(\tau)]\, x\| \leqq C\, |t - \tau|^{\alpha}\, \|x\|_D \qquad \forall t, \tau \in [0, t_0],\ \forall x \in D. \tag{16}$$

Let there exist positive numbers ω *and* β *such that the set* $\Sigma(\omega, \beta)$ *is contained in the resolvent set of all the operators* $-A(t)$ *and such that there exists a* $C > 0$ *for which the resolvent satisfies the estimate*

$$\| [A(t) + \lambda]^{-1}\| \leqq C/[1 + |\lambda|] \qquad \forall \lambda \in \Sigma(\omega, \beta). \tag{17}$$

Then there exists an evolution operator $U(t, \tau)$ *that is defined and is strongly continuous when* $0 \leqq \tau \leqq t \leqq t_0$, *that is uniformly differentiable when* $\tau + t_1 \leqq$

$t \leqq t_0$ $(t_1 > 0)$ and that satisfies

$$\frac{\partial U(t,\tau)}{\partial t} + A(t)\, U(t,\tau) = 0, \tag{18}$$

$$U(t,s)\, U(s,\tau) = U(t,\tau), \qquad U(t,t) = I, \tag{19}$$

$$\| A^{\gamma}(t)\, U(t,\tau)\, A^{-\delta}(\tau) \| \leqq C(t-\tau)^{\delta-\gamma}, \qquad 0 \leqq \delta \leqq \gamma < 1 + \alpha.^{\star} \tag{20}$$

Furthermore, if the initial data u_0 are in D and if f is in the class $C([0, t_0], [D, X]_{\theta',p})$ *for* $0 \leqq \theta' < 1$, *then the variation of constants formula*

$$u(t) = U(t,0)\, u_0 + \int_0^t U(t,s) f(s)\, ds \tag{21}$$

gives the unique solution of Equation (15) *and*

$$u(t) \in C([0,t_0], D) \cap C^1([0,t_0], X). \tag{22}$$

The inequality (17) implies that the operator $A(t)$ generates a holomorphic semigroup $\exp[-sA(t)]$. The fundamental estimate (20) is a generalization of classical estimates for holomorphic semigroups. It will be noted that the range of possible values for γ and δ depends on the differentiability of $A(t)$.

But SOBOLEVSKII established the variation of constants formula (21) under the further assumption that

$$f \in C^{\eta}([0,t_0], X), \qquad \eta > 0.$$

We now give a short proof of (21). First (18), (19) and the strong continuity property allow us to show that $U(t,0)\, u_0$ is a solution of the homogeneous equation in the class (22). It remains to check that

$$v(t) = \int_0^t U(t,s) f(s)\, ds$$

satisfies (22) and (15) with a zero initial data. Note that $v(t)$ is in D by the estimate (20) and Corollary A2 of the Appendix. The proof of the D-continuity of $v(t)$ is left to the reader. Let h be a small positive number. Then

$$[v(t+h) - v(t)]/h = \int_0^1 U(t+h, t+\lambda h) f(t+\lambda h)\, d\lambda$$
$$+ \int_0^t [U(t+h,s) - U(t,s)] f(s)/h\, ds.$$

Clearly, the first integral converges to $f(t)$ in X when h goes to zero. By using (18), (20) and Corollary A2 of the Appendix, we find that

$$\| [U(t+h,s) - U(t,s)] f(s)/h \| \leqq C(t-s)^{-1+\theta'-\varepsilon} \|f(s)\|_{\theta',p}$$

$\star$ The definition and the main properties of the fractional powers of an operator are recalled in the Appendix.

for $\varepsilon > 0$. If $t > s$, then $[U(t+h, s) - U(t, s)]f(s)/h$, converges to $-A(t)\,U(t, s)$ $f(s)$ when h goes to zero. Then $v(t)$ possesses a right derivative in X, which is given by

$$D^+v(t) = f(t) - \int_0^t A(t)\,U(t,s) f(s)\,ds = f(t) - A(t)\,v(t).$$

But a classical argument [9] allows us to prove that $v(t)$ is differentiable and is the solution of (15).

We now study the decay of the evolution operator $U(t, \tau)$ when $t - \tau$ goes to infinity.

Proposition 5. *Let $\{A(t)\}$ be a family of unbounded operators satisfying the conditions of Proposition* 4 *for any positive t_0. Suppose that there exists an operator A_∞, which also satisfies* (17), *and a positive number ε such that*

$$\|[A(t) - A_\infty]\,x\| \leqq \varepsilon\,\|x\|_D \quad \forall x \in D, \tag{23}$$

$$\|[A(t) - A(s)]\,x\| \leqq \varepsilon\,|t-s|^\alpha\,\|x\|_D \quad \forall x \in D \tag{24}$$

for any $t, s \in [0, t_1]$. If ε is sufficiently small, then

$$\|A(t)\,U(t,\tau)\,A^{-\gamma}(\tau)\| \leqq C\,|t-\tau|^{-1+\gamma}\exp[-\beta(t-\tau)] \tag{25}$$

$$\forall t, \tau, \gamma \text{ satisfying } 0 \leqq \tau \leqq t \leqq t_1, \quad 0 \leqq \gamma \leqq 1,$$

where the constant C depends on ε and γ and on the constants related to the operator A_∞, but not on t_1. If Assumptions (23) *and* (24) *are valid for any positive t and s, then* (25) *is also valid.*

Remark. By a classical perturbation result, [9] p. 497, it is sufficient that A_∞ satisfy (17) for λ in $\Sigma(\omega, \beta)$ in order that $A(t)$ satisfy also (17) for λ in $\Sigma(\omega_\varepsilon, \beta_\varepsilon)$ with

$$0 < \omega_\varepsilon < \omega \quad \text{and} \quad 0 < \beta_\varepsilon < \beta.$$

Proof of Proposition 5. Since resolvent set of $-A_\infty$ is open, it contains a set $\Sigma(\omega, \beta_1)$ with $\beta_1 > \beta$. Moreover,

$$\|(\lambda + A_\infty)^{-1}\| \leqq C/(1+|\lambda|) \quad \forall \lambda \in \Sigma(\omega, \beta_1).$$

It follows from the previously quoted perturbation theorem that

$$\|[\lambda + A(t)]^{-1}\| \leqq C/(1+|\lambda|) \quad \forall \lambda \in \Sigma(\omega', \beta_2)$$

with $\beta_1 > \beta_2 > \beta$, $\omega > \omega' > 0$, and C independent of t. The semigroup can be represented by Dunford's integral:

$$\exp[-sA(t)] = (2i\pi)^{-1} \int_\Gamma e^{\lambda s}[A(t) + \lambda]^{-1}\,d\lambda.$$

Following [9] we then obtain

$$\|\exp[-sA(t)]\| \leqq C\exp(-\beta_3 s),$$

$$\|A(t)\exp[-sA(t)]\| \leqq C\exp(-\beta_3 s)/s \tag{26}$$

with $\beta_2 > \beta_3 > \beta$ and C independent of t and s. Because $A(t)$ commutes with the semigroup $\exp[-sA(t)]$ and the operators $A(t)\,A^{-1}(\tau)$ are uniformly bounded, we obtain

$$\|A(t)\exp[-(t-\tau)\,A(t)]\,A^{-1}(\tau)\| \leqq C\exp[-\beta_3(t-\tau)]. \tag{27}$$

We now derive the estimate (25) when $\gamma = 1$. In this case the norm of the operator $W(t,\tau) \equiv A(t)\,U(t,\tau)\,A^{-1}(\tau)$ must be bounded by $C\exp[-\beta(t-\tau)]$. SOBOLEVSKII [21] remarked that the operator

$$\phi(s) = \exp[-(t-s)\,A(t)]\,U(s,\tau)\,A^{-1}(\tau)$$

is strongly differentiable. The integration of $\phi'(s)$ between τ and t shows that $W(t,\tau)$ is the solution of the Volterra integral equation

$$\begin{aligned} W(t,\tau) &= A(t)\exp[-(t-\tau)\,A(t)]\,A^{-1}(\tau) \\ &\quad + \int_\tau^t A(t)\exp[-(t-s)\,A(t)]\,[A(t)-A(s)]\,A^{-1}(s)\,W(s,\tau)\,ds. \end{aligned} \tag{28}$$

The estimates (24) and (26) show that the kernel of this equation is bounded above by

$$C\,\varepsilon(t-s)^{\alpha-1}\exp[-\beta_3(t-s)].$$

A norm is introduced for the bounded operators in X that depend on two parameters t and τ:

$$|||W||| = \sup_{t_2 \leqq \tau \leqq t \leqq t_3} \|W(t,\tau)\exp[\beta(t-\tau)]\|.$$

Equation (28) and the estimates (24), (26), (27) imply that there exist two positive constants C_0 and C_1 such that

$$|||W||| \leqq C_0/2 + C_1\,\varepsilon\int_{t_2}^{t_3}(t_3-s)^{\alpha-1}\exp[(\beta-\beta_3)(t_3-s)]\,ds\,|||W|||.$$

Because the integral is bounded by a constant C_2 independent of t_2 and t_3, it follows that

$$|||W||| \leqq C_0/2 + C_1C_2\,\varepsilon\,|||W|||.$$

If ε is smaller than $\frac{1}{2}\,C_1C_2$, then $|||W|||$ is smaller than C_0. Therefore

$$\|W(t,\tau)\| \leqq C_0\exp[-\beta(t-\tau)] \tag{29}$$

and the latter is the same as (25) with $\gamma = 1$. For a complete proof of (25), we note that

$$\begin{aligned} &\|A(t)\,U(t,\tau)\,A^{-\gamma}(\tau)\| \\ &\qquad \leqq C(t-\tau)^{-1+\gamma}\exp[-\beta(t-\tau)] \text{ when } 0 \leqq t-\tau \leqq 1. \end{aligned} \tag{30}$$

If $t-\tau$ is greater than 1, the semigroup identity (19) implies that

$$A(t)\,U(t,\tau)\,A^{-\gamma}(\tau) = W(t,\tau+1)\,A(\tau+1)\,U(\tau+1,\tau)\,A^{-\gamma}(\tau), \tag{31}$$

and (29), (30), (31) give the desired result

$$\|A(t)\, U(t,\tau)\, A^{-\gamma}(\tau)\| \leqq C \exp[-\beta(t-\tau)]) \text{ when } t-\tau \geqq 1.$$

The following proposition shows how the evolution operator is changed if the generator $A(t)$ is suitably perturbed.

Proposition 6. *Let $A(t)$ and $\mathbf{A}(t)$ be two unbounded operators that satisfy the conditions of Proposition 4 with the same domain D. Let the positive number Δ, which characterizes the closeness of $\mathbf{A}(t)$ to $A(t)$, be defined by*

$$\|[A(t)-\mathbf{A}(t)]\, x\| \leqq C\Delta \,\|x\|_D \quad \forall t \in [0, t_0],\ x \in D, \tag{32}$$

$$\|[A(t)-\mathbf{A}(t)-A(s)+\mathbf{A}(s)]\, x\| \leqq C\Delta\, |t-s|^{\alpha}\, \|x\|_D \quad \forall t, s \in [0, t_0],\ x \in D \tag{33}$$

If t_0 is smaller than a number t_1 independent of Δ, t and τ, then the corresponding evolution operators $U(t,\tau)$ and $\mathbf{U}(t,\tau)$ are close in the following sense:

$$\|A(t)\, U(t,\tau)\, A^{-1}(\tau) - \mathbf{A}(t)\, \mathbf{U}(t,\tau)\, \mathbf{A}^{-1}(\tau)\| \leqq C\Delta, \tag{34}$$

$$\|[U(t,\tau)-\mathbf{U}(t,\tau)]\, x\|_D \leqq C\Delta\, \|x\|_D \quad \forall x \in D, \tag{35}$$

$$\|A(t)\, U(t,\tau)\, A^{-\gamma}(\tau) - \mathbf{A}(t)\, \mathbf{U}(t,\tau)\, \mathbf{A}^{-\gamma}(\tau)\| \leqq C\Delta (t-\tau)^{\gamma-1} \quad \textit{for } 0<\gamma \leqq \alpha, \tag{36}$$

$$\|[U(t,\tau)-\mathbf{U}(t,\tau)]\, x\|_D \leqq C\Delta (t-\tau)^{\gamma-1}\, \|x\|_{1-\gamma',p} \quad \forall \gamma, \gamma', p: 0<\gamma<\gamma' \leqq \alpha,\ \ 1 \leqq p \leqq \infty,\ \forall x \in [D, X]_{1-\gamma',p} \tag{37}$$

where C denotes various constants independent of t, τ and Δ.

Proof of Proposition 6.

a) *Proof of* (34) *and* (35). The following abbreviated notation will be used:

$$\delta W(t,\tau) = W(t,\tau) - \mathbf{W}(t,\tau),\ \delta A(t) = A(t) - \mathbf{A}(t) \ldots$$

Formula (28) implies that

$$\begin{aligned}
\delta W(t,\tau) &= \delta\{A(t) \exp[-(t-\tau)\, A(t)]\, A^{-1}(\tau)\} \\
&\quad + \int_\tau^t \delta\{A(t) \exp[-(t-s)\, A(t)]\}\, [A(t)-A(s)]\, A^{-1}(s)\, W(s,\tau)\, ds \\
&\quad + \int_\tau^t A(t) \exp[-(t-s)\, A(t)]\, \delta[A(t)-A(s)]\, A^{-1}(s)\, W(s,\tau)\, ds \\
&\quad + \int_\tau^t A(t) \exp[-(t-s)\, A(t)]\, [A(t)-A(s)]\, \delta A^{-1}(s)\, W(s,\tau)\, ds \\
&\quad + \int_\tau^t A(t) \exp[-(t-s)\, A(t)]\, [A(t)-A(s)]\, A^{-1}(s)\, \delta W(t,\tau)\, ds.
\end{aligned} \tag{38}$$

Let us assume for a moment that we have proved the three estimates

$$\| [\delta A^{-1}(s)]\, x \| \leqq C\Delta\, \|x\|_D, \tag{39}$$

$$\| \delta\{A(t) \exp[-(t-\tau)\, A(t)]\} \| \leqq C\Delta/(t-\tau), \tag{40}$$

$$\| \delta\{A(t) \exp[-(t-\tau)\, A(t)]\, A^{-1}(\tau)\} \| \leqq C\Delta. \tag{41}$$

Then the following inequality can be deduced from (38), (16), (17), (20), (33), (39), (40), (41):

$$\| \delta W(t,\tau) \| \leqq C_0 \Delta + C_1 \Delta \int_\tau^t (t-s)^{\alpha-1}\, ds + C_2 \int_\tau^t (t-s)^{\alpha-1}\, ds \sup_{\tau \leqq s \leqq t} \| \delta W(s,\tau) \|$$

Hence $\| \delta W(t,\tau) \| \leqq C\Delta$ if $0 \leqq t-\tau \leqq t_1 \equiv [\alpha/2C_2]^{1/\alpha}$, which is exactly (34). Inequality (35) follows easily from (34), (32) and (39).

Now the three estimates (39), (40), (41) are to be proved. Inequality (39) is a special case of the following estimates for the resolvent

$$\| \delta\{[\lambda + A(t)]^{-1}\} \| \leqq C\Delta/(1+|\lambda|) \quad \forall \lambda \in \Sigma(\omega,\beta), \tag{42}$$

$$\| \delta\{[\lambda + A(t)]^{-1}\}\, x \|_D \leqq C\Delta\, \|x\| \quad \forall \lambda \in \Sigma(\omega,\beta), \tag{43}$$

which come from (32), (17) and the formula

$$\delta\{[\lambda + A(t)]^{-1}\} = [\lambda + A(t)]^{-1} [A(t) - A(t)]\, [\lambda + A(t)]^{-1}. \tag{44}$$

By using (42) and the integral form of the semigroup

$$\exp[-sA(t)] = (2i\pi s)^{-1} \int_{\Gamma'} e^{\lambda'} \left[\frac{\lambda'}{s} + A(t)\right]^{-1} d\lambda'$$

where Γ' joins $\infty \times (-ie^{-i\omega})$ to $\infty \times ie^{i\omega}$, we obtain (40) and

$$\| \delta\{\exp[-sA(t)]\} \| \leqq C\Delta. \tag{45}$$

Finally, (41) is a consequence of (39), (32), (45) and of

$$\delta\{A(t) \exp[-(t-\tau)\, A(t)]\, A^{-1}(\tau)\}$$
$$= \delta\{\exp[-(t-\tau)\, A(t)]\}\, A(t)\, A^{-1}(\tau) + \exp[-(t-\tau)\, A(t)]\, \delta[A(t)\, A^{-1}(\tau)].$$

b) *Proof of* (36). We let $W_\gamma(t,\tau) = A(t)\, U(t,\tau)\, A^{-\gamma}(\tau)$. Then equation (28) implies that $W_\gamma(t,\tau)$ is the solution of the Volterra integral equation

$$W_\gamma(t,\tau) = A(t) \exp[-(t-\tau)\, A(\tau)]\, A^{-\gamma}(\tau) + \int_\tau^t A(t) \exp[-(t-s)\, A(t)]\, [A(t) - A(s)]\, A^{-1}(s)\, W_\gamma(s,\tau)\, ds, \tag{46}$$

because the two members of (46) coincide on the dense subset $D[A^{1-\gamma}(\tau)]$. The proof of (36) is similar to that of (34) since the following estimate will now be established:

(47) $$\| \delta\{A(t) \exp[-(t-\tau) A(t)] A^{-\gamma}(\tau)\}\| \leqq C\Delta(t-\tau)^{\gamma-1}.$$

First, the definition of the fractional power given in the Appendix yields

$$A^{-\gamma}(t) - A^{-\gamma}(\tau) = \frac{\sin(\pi\gamma)}{\pi} \int_0^\infty \lambda^{-\gamma}[\lambda + A(t)]^{-1}[A(\tau) - A(t)]\,[\lambda + A(\tau)]^{-1}\, d\lambda.$$

Using (16) and (17), we obtain

(48) $$\|A^{-\gamma}(t) - A^{-\gamma}(\tau)\| \leqq C\,|t-\tau|^{\alpha}, \quad 0<\gamma<1.$$

In the same way (A3) yields

(49) $$0<\varrho<\gamma<1 \Rightarrow \|A^{\varrho}(t)\,\delta[A^{-\gamma}(t)]\| \leqq C\Delta,$$

(50) $$0<\gamma<1 \Rightarrow \|\delta[A^{-\gamma}(t) - A^{-\gamma}(\tau)]\| \leqq C\Delta\,|t-\tau|^{\alpha}.$$

We now invoke the following lemma, which will be proved below.

Lemma 7.

$$\|\delta\{A^{\varrho}(t)\exp[-sA(t)]\}\| \leqq C\Delta s^{-\varrho} \text{ for } 0<\varrho<1.$$

Then (47) is a consequence of Lemma 7, of (48), (50) and (40) and of the identity

$$A(t)\exp[-(t-\tau)A(t)]\,A^{-\gamma}(\tau) = A^{1-\gamma}(t)\exp[-(t-\tau)A(t)] + A(t)\exp[-(t-\tau)A(t)]\,[A^{-\gamma}(\tau) - A^{-\gamma}(t)].$$

c) *Proof of* (37). We have

(51) $$\|[U(t,\tau) - U(t,\tau)]\,x\|_D \leqq C\{\|[A(t)\,U(t,\tau) - A(t)\,U(t,\tau)]\,x\| + \|[A(t) - A(t)]\,U(t,\tau)\,x\|\}.$$

Let γ'' be such that $0<\gamma<\gamma''<\gamma'$. Then

$$\begin{aligned}\|\delta[A(t)\,U(t,\tau)]\,x\| &\leqq \|\delta[A(t)\,U(t,\tau)\,A^{-\gamma''}(\tau)]\,A^{\gamma''}(\tau)\,x\| \\ &\quad + \|\{A(t)\,U(t,\tau)\,A^{-\gamma}(\tau)\}\{A^{\gamma}(\tau)\,\delta[A^{-\gamma''}(\tau)]A^{\gamma''}(\tau)\,x\| \\ &\leqq C\Delta(t-\tau)^{\gamma-1}\,\|A^{\gamma''}(\tau)\,x\| \quad \text{(by (36) and (49))},\end{aligned}$$

(52) $$\|\delta[A(t)\,U(t,\tau)]\,x\| \leqq C\Delta(t-\tau)^{\gamma-1}\,\|x\|_{1-\gamma',p} \text{ (by Corollary A2)}.$$

Inequality (37) follows from (51), (52), (32) and (20).

Proof of Lemma 7.

We first prove that

(53) $$\|\delta[A^{\varrho}(\lambda + A)^{-1}]\| \leqq C\Delta\,|\lambda|^{\varrho-1} \quad \forall\varrho\in\,]0,1[,\ \lambda\in\Sigma(\omega,\beta).$$

If we let $\sigma = 1 - \varrho$, then (53) is equivalent to

$$\|\delta[A^{-\sigma}A(\lambda + A)^{-1}]\| \leqq C\Delta\,|\lambda|^{-\sigma}, \quad \text{or}$$

$$\left\|\int_0^\infty \mu^{-\sigma}\delta[(A+\mu)^{-1}A(A+\lambda)^{-1}]\,d\mu\right\| \leqq C\Delta\,|\lambda|^{-\sigma}. \tag{54}$$

Using (42) and (43) we can show that

$$\left\|\int_{|\lambda|/2}^\infty \mu^{-\sigma}\,\delta[(A+\mu)^{-1}A(A+\lambda)^{-1}]\,d\mu\right\|$$

$$\leqq \int_{|\lambda|/2}^\infty \mu^{-\sigma}\{\|\delta[(A+\mu)^{-1}]\,A(A+\lambda)^{-1}\| + \|(A+\mu)^{-1}\,\delta[A(A+\lambda)^{-1}]\|\}\,d\mu$$

$$\leqq C\Delta \int_{|\lambda|2}^\infty \mu^{-(1+\sigma)}\,d\mu \leqq C\Delta\,|\lambda|^{-\sigma}.$$

Next, the resolvent identity

$$(\lambda-\mu)\,(A+\lambda)^{-1}\,(A+\mu)^{-1} = (A+\mu)^{-1} - (A+\lambda)^{-1}$$

implies that

$$\left\|\int_0^{|\lambda|/2} \mu^{-\sigma}\,\delta[A(A+\mu)^{-1}\,(A+\lambda)^{-1}]\,d\mu\right\| \leqq C\Delta \int_0^{|\lambda|/2} \mu^{-\sigma}\,|\lambda-\mu|^{-1}\,d\mu \leqq C\Delta\,|\lambda|^{-\sigma}.$$

These inequalities give (54) and (53). Finally Dunford's integral and (53) yield the required inequality:

$$\|\delta\{A^\varrho \exp[-As]\}\| \leqq (2\pi s)^{-1} \int_{\Gamma'} e^{\lambda'} \left\|\delta\left\{A^\varrho\left(\frac{\lambda'}{s} + A\right)^{-1}\right\}\right\| d\lambda' \leqq C\Delta s^{-\varrho}.$$

§ 4. Proof of the main results

a) Proof of Theorem 1

The interpolation inequality (A6) implies that if a solution $u(t)$ of (6) is in the class (7), then it also satisfies

$$\|u(t) - u(s)\|_{\theta,p} \leqq C\,\|u(t)-u(s)\|_D^{1-\theta}\,\|u(t)-u(s)\|^\theta,$$

$$\|u(t) - u(s)\|_{\theta,p} \leqq C\,|t-s|^\theta \operatorname*{Max}_{0\leqq t\leqq t_0} \|u(t)\|_D^{1-\theta} \operatorname*{Max}_{0\leqq t\leqq t_0} \left\|\frac{du}{dt}(t)\right\|^\theta. \tag{55}$$

Thus it is sufficient, especially for uniqueness, to seek $u(t)$ in the Banach space

$$E = C^\theta([0, t_0], [D, X]_{\theta,p}) \cap C([0, t_0], D)$$

with norm defined by

$$|||u||| = \operatorname*{Max}_{0\leqq t\leqq t_0} \|u(t)\|_D + \operatorname*{Sup}_{0\leqq s,t\leqq t_0} \{\|u(t)-u(s)\|_{\theta,p}\,|t-s|^{-\theta}\}.$$

Now it is obvious that the solutions of (6) satisfying (7) are fixed points in E of the map $u \to v$ where v is the solution of the nonhomogeneous equation

$$\frac{dv(t)}{dt} + T(u(t))\, v(t) = f(u(t)), \quad v(0) = u_0. \tag{56}$$

If u is in E and x is in D, then the map $t \to T(u(t))\, x$ from $[0, t_0]$ into X is Hölder continuous with exponent θ by Assumption (ii). Let

$$A(t) = T(u(t)).$$

Then by a previously quoted perturbation theorem [9] p. 497, the operator $A(t)$ satisfies the estimate (16). Proposition 4 permits us to define an evolution operator $U_u(t, s)$ such that (18) and (19) hold. By Assumption (iii) the map $t \to f(u(t))$ is continuous from $[0, t_0]$ into $[D, X]_{\theta',p}$. Then the variation of constants formula (21) gives $v(t)$:

$$v(t) = U_u(t, 0)\, u_0 + \int_0^t U_u(t, s) f(u(s))\, ds \equiv \mathscr{F}(u_0, u). \tag{57}$$

Furthermore, $\mathscr{F}(u_0, u)$, which has just been defined, is a map from a neighborhood of 0 in $D \times E$ into E. Now, by the Contracting Map Theorem [6], it is sufficient that the map $\mathscr{F}(u_0, \cdot)$ is contracting in E in a neighborhood of 0 (uniformly in u_0) and that $|||\mathscr{F}(0, 0)|||$ is sufficiently small, in order that Theorem 1 hold. We define

$$y(t) \equiv \mathscr{F}(0, 0) = \int_0^t U_0(t, s) f(0)\, ds = \int_0^t \exp\left[-T(0)\,(t - s)\right] f(0)\, ds.$$

Let $\theta'' \in \,]\theta', 1[$. Using (15) and Corollary A2, we find that

$$\|y(t)\|_D \leqq \|f(0)\|_{\theta',p}\, C t_0^{1-\theta''}$$

and by the equation (56) that

$$\left\|\frac{dy}{dt}(t)\right\| \leqq C\, \|f(0)\|_{\theta',p}.$$

Therefore (56) yields

$$|||\mathscr{F}(0, 0)||| \leqq C\, \|f(0)\|_{\theta',p}\, [t_0^{1-\theta''} + t_0^{(1-\theta'')(1-\theta')}]$$

and $|||\mathscr{F}(0, 0)|||$ is sufficiently small if t_0 is.

Now let u_1 and u_2 be two functions of E that are close to zero. Let us denote by $v_1 = \mathscr{F}(u_0, u_1)$, $v_2 = \mathscr{F}(u_0, u_2)$ the two functions of E that are obtained by (57).

First, $U_{u_1}(t, s) - U_{u_2}(t, s)$ is estimated by means of Proposition 6. Assumption (ii) leads to following inequality

$$\|\{T(u_2(t)) - T(u_1(t))\}\, x\| \leqq C\, \|u_2(t) - u_1(t)\|_{\theta,p}\, \|x\|_D \leqq C\, |||u_2 - u_1|||\, \|x\|_D.$$

Hence $A(t) \equiv T(u_2(t))$ and $A(t) \equiv T(u_1(t))$ are such that the condition (32) of Proposition 6 holds with $\Delta = |||u_2 - u_1|||$. On the other hand, if we define

$$\delta T \equiv T(u_2(t)) - T(u_1(t)) - T(u_2(s)) + T(u_1(s)),$$

then

$$\delta T = \int_0^1 \{[T'(\lambda u_2(t) + (1-\lambda)\, u_1(t)) - T'(\lambda u_2(s) + (1-\lambda)\, u_1(s))] \cdot (u_2(t) - u_1(t))$$
$$+ T'(\lambda u_2(s) + (1-\lambda)\, u_1(s)) \cdot (u_2(t) - u_1(t) - u_2(s) + u_1(s))\}\, d\lambda.$$

Assumption (ii) is used again to obtain the estimate

$$\forall t, s \in [0, t_0],\ x \in D \qquad \|(\delta T)\, x\| \leqq C\, |||u_2 - u_1|||\, |t - s|^{\theta\eta}\, \|x\|_D.$$

Therefore, $A(t)$ and $A(t)$ satisfy the second condition (33) of Proposition 6 with $\alpha = \theta\eta$. The estimates (35) and (37) yield

(58) $$\|[U_{u_2}(t, 0) - U_{u_1}(t, 0)]\, u\|_D \leqq C\, |||u_2 - u_1|||\, \|u_0\|_D$$

and

(59) $$\|[U_{u_2}(t, s) - U_{u_1}(t, s)] f\|_D \leqq C\, |||u_2 - u_1|||\, (t-s)^{-\theta_2}\, \|f\|_{\theta_1, p} \quad \text{for}$$
$$1 > \theta_2 > \theta_1 = \operatorname{Max}\{\theta', 1 - \theta\eta\}.$$

The formulae (57), (58), (59) lead to

(60) $$\|v_2(t) - v_1(t)\|_D \leqq C\, |||u_2 - u_1|||\, [\|u_0\|_D + t_0^{1-\theta_2}].$$

From (56), (60) and Assumptions (ii) and (iii) we find that

(61) $$\left\| \frac{dv_2(t)}{dt} - \frac{dv_1(t)}{dt} \right\| \leqq C\, |||u_2 - u_1|||.$$

By combining (60) and (61) with the interpolation inequality (A6), we obtain the desired estimate

$$|||v_2 - v_1||| \leqq C\, |||u_2 - u_1|||\, [\|u_0\|_D + t_0^{1-\theta_2}]^{1-\theta},$$

which shows that the map $\mathscr{F}(u_0, \cdot)$ is contracting if $\|u_0\|_D$ and t_0 are sufficiently small.

b) Proof of Theorem 2

The existence of $u(t)$ has been proved for $t \in [0, t_0]$. If $\|u(t_0)\|_D$ is smaller than ε, Theorem 1 can be used again to prove the existence of $u(t)$ for $t \in [t_0, 2t_0]$ and so on. To obtain the existence theorem for any positive t_0, it is sufficient to show that

$$\|u(t)\|_D \leqq \varepsilon \qquad \forall n \geqq 1, \qquad \forall t \in [0, nt_0]$$

and in order to ensure that this inequality hold, it is sufficient to prove the estimate (8). Let

$$|u|_{t_1} = \operatorname*{Max}_{0 \leqq t \leqq t_1} \|u(t) \exp(\beta t)\|_D.$$

(8) is equivalent to

(62) $$\exists \eta > 0 \text{ such that } \forall t_1 > 0, \quad \|u_0\|_D \leqq \eta \Rightarrow |u|_{t_1} \leqq C \|u_0\|_D.$$

If $\|u(t)\|_D$ is not greater than a sufficiently small number ε, then $\left\|\frac{du}{dt}(t)\right\|$ remains small by equation (6) and by Assumptions (ii) and (iv), and $|||u|||$ also remains small by the interpolation inequality (A6). Thus it can be shown that

$$\|[T(u(t)) - T(0)]\,x\| \leqq \varepsilon' \|x\|_D,$$

$$\|[T(u(t)) - T(u(s))]\,x\| \leqq \varepsilon' |t-s|^\theta \|x\|_D,$$

where ε' can be made as small as necessary by making ε small. Then Proposition 5 can be applied so that the operator $U(t, \tau)$ must satisfy the estimate (25). We set $v = u$ in equation (57) and use (25) and Assumption (iv) to obtain

$$|u|_{t_1} \leqq C_0[\|u_0\|_D + |u|_{t_1}^2 \int_0^{t_1} \exp(-\beta s)\,(t_1 - s)^{-\theta}\,ds].$$

But the integral is bounded independently of t_1. Thus

$$|u|_{t_1} \leqq C_0 \|u_0\|_D + C_1 |u|_{t_1}^2,$$

and it is obvious that (62) holds with $\eta \leqq 1/4C_0C_1$, because $|u|_{t_1}$ depends continuously on t_1.

Appendix

a) Fractional power of an operator (cf. [14] § 14)

Let A be a closed operator in the Banach space X, and let the domain D of A be dense in X. It is assumed that A is of positive type, which means that A has a resolvent $(\lambda + A)^{-1}$ for any real non-negative λ that satisfies

(A1) $$(\lambda + A)^{-1} \leqq C/(1+\lambda) \quad \forall \lambda \geqq 0.$$

Thus the fractional power A^α ($\alpha \in \mathbb{R}$) can be defined in the following manner. For $\alpha \in]0, 1[$, the operator $A^{-\alpha}$, which is bounded, is given by

(A2) $$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}\,d\lambda.$$

For each such α we define

$$A^\alpha = (A^{-\alpha})^{-1}$$

and for all real α we define A^α inductively by

$$A^{1+\alpha} = AA^\alpha.$$

Note that SOBOLEVSKII [21] used an alternative definition, but both definitions are known to be equivalent [14]. It can be proved that

$$A^{\alpha+\beta} = A^{\alpha} A^{\beta} \quad \forall \alpha, \beta \in \mathbb{R}.$$

The main property of fractional powers of an operator is the interpolation inequality:

$$\|A^{\alpha} x\| \leqq C(\alpha) \|x\|^{1-\alpha} \|Ax\|^{\alpha} \quad \forall x \in D, \alpha \in [0, 1].$$

The latter permits us to prove a property of the resolvent. Indeed, (A1) implies that the operators $A(\lambda + A)^{-1}$ are uniformly bounded. Therefore

(A3) $\|A^{\alpha}(\lambda + A^{-1})\| \leqq C/(1 + \lambda)^{\alpha}$ by the interpolation inequality.

If the operator generates a strongly continuous and bounded semigroup, then

(A4) $$\|[\exp(-At) - I]\, u\| \leqq C(\alpha)\, t^{\alpha} \|A^{\alpha} u\|.$$

Furthermore, if the semigroup is holomorphic, then

(A5) $$\|A \exp(-At)\, u\| \leqq C(\alpha)\, t^{\alpha-1} \|A^{\alpha} u\|, \quad 0 \leqq \alpha \leqq 1.$$

In spite of those good properties, there is a great disadvantage in the theory: the space $D(A^{\alpha})$ is an intermediate space between D and X, but it is not known if it is typical of the pair of spaces D and X or if it depends on the operator A. That is why we have also used the interpolation theory in the sense of LIONS-PEETRE [15].

b) Interpolation theory

Let D and X be two Banach spaces such that D is dense in X, with a continuous embedding. The interpolation theory following LIONS-PEETRE defines a two-parameter family of interpolation spaces, which are denoted by $[D, X]_{\theta,p}$ $0 < \theta < 1$, $1 \leqq p \leqq \infty$, in agreement with the notation of BUTZER-BERENS [3].

We recall only a few properties of these spaces:

$$[D, X]_{\theta',p'} \subset [D, X]_{\theta,p} \quad \text{if} \quad \theta' < \theta \quad \text{or if} \quad \theta' = \theta \quad p' \leqq p,$$

(A6) $$\|x\|_{\theta,p} \leqq C(\theta, p) \|x\|_D^{1-\theta} \|x\|_X^{\theta} \quad \forall \theta \in]0, 1[, x \in D.$$

To compare these spaces with $D(A^{\alpha})$, the following result is needed.

Proposition A 1 ([15] *Propositions* 4-1-1 *and* 4-1-2).

Let Y be an intermediate Banach space $(D \subset Y \subset X)$. In order that

$$[D, X]_{\theta,1} \subset Y \subset [D, X]_{\theta,\infty},$$

it is necessary and sufficient that

(A7) $$\exists C \text{ such that } \|x\|_Y \leqq C \|x\|_X^{1-\theta} \|x\|_D^{\theta} \quad \forall x \in D,$$

(A8) $\exists C > 0$ *such that for each* $x \in Y$ *and for each* $t > 0$ *there is an* $x_0 \in X$ *and an* x_1 *in* D *with*

$$x = x_0 + x_1,$$

$$\|x_0\|_X \leqq C t^{-\theta} \|x\|_Y, \quad \|x_1\| \leqq C t^{1-\theta} \|x\|_Y.$$

Corollary A2.

Let an unbounded operator A in a Banach space X generate a holomorphic semigroup. Then

$$[D(A), X]_{\theta,1} \subset D(A^{1-\theta}) \subset [D(A), X]_{\theta,\infty} \qquad \forall \theta \in]0, 1[.$$

Proof. It is sufficient to prove that the space $D(A^{1-\theta})$ satisfies the properties (A7) and (A8). (A7) comes from the interpolation inequality for fractional powers of an operator. For any x in $D(A^{1-\theta})$ and t positive we set

$$x_1 = \exp(-At)\,x, \qquad x_\theta = [I - \exp(-At)]\,x.$$

Then (A8) is a consequence of (A4) and (A5).

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