

# *A Theory of Constrained Mixtures with Multiple Temperatures*

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*Dedicated to Clifford Truesdell on the occasion  
of his sixtieth birthday*

## **1. Introduction**

Since the publication of TRUESDELL's fundamental article on mixture theory in 1957 and COLEMAN & NOLL's article on thermodynamics in 1963, there has been considerable interest in thermodynamic formulations of theories which describe the behavior of various types of mixtures.<sup>1</sup> A survey of certain of these formulations can be found in the article by BOWEN.<sup>2</sup>

Generally speaking, mixture theories contain the possibility of three distinct physical phenomena. These are diffusion, chemical reactions, and multiple temperatures. These phenomena are the result of momentum, mass and energy transfers, respectively, between the constituents. When one or more of these phenomena are omitted in a particular formulation, the result is a theory of mixtures with internal constraints.

This article is a thermodynamic formulation of a theory of mixtures constrained such that diffusion between the constituents is not allowed. However, mass and energy transfers are allowed. The novel feature of this formulation is the presence of energy transfer in a mixture which is not capable of diffusing. As with many continuum models with internal constraints, the results obtained here are not a special case of a mixture theory which allows diffusion. From a physical standpoint, the model presented here is felt to apply to certain composite materials where relative motion between the constituents is not possible, but the thermodynamic properties of the constituents are fundamentally different.

Section 2 contains the field equations and the entropy inequality appropriate to the constrained theory. Section 3 is concerned with the constitutive equations which define the mixture and the thermodynamic restrictions on these constitutive equations. Section 4 illustrates the results of Section 3 restricted to small displacement gradients, small coldness changes, small coldness gradients,

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<sup>1</sup> TRUESDELL [1], COLEMAN & NOLL [2].

<sup>2</sup> BOWEN [3].

and small extent of reaction changes. For simplicity, in this section the mixture is assumed to have the symmetry of an isotropic solid. The results of this section take the form of a set of coupled linear partial differential equations which govern the mixture. In Section 5, these partial differential equations are specialized further by the elimination of the possibility of chemical reactions. These simplified equations are used to investigate the propagation of one dimensional harmonic waves in a binary mixture. Among the results in this section are formulas which give phase velocities and attenuation coefficients for non-dispersive high and low frequency approximations.

## 2. Preliminaries

For a general theory of mixtures of  $\mathfrak{N}$  materials, where the combined effects of diffusion, chemical reactions and multiple temperatures are allowed, one must prescribe equations of balance and constitutive equations sufficient to determine  $\mathfrak{N}$  deformation functions,  $\mathfrak{N}$  temperature fields and  $\mathfrak{N}$  density fields. For a constrained theory of mixtures, one can avoid certain of the complications of a general theory. For example, the theory of a mixture constrained such that diffusion, chemical reactions, and multiple temperatures are not present is formally identical to the theory of a single material. As indicated in the introduction, in this article we investigate the features of a certain constrained theory of mixtures. This theory is for a mixture of  $\mathfrak{N}$  materials constrained such that there is no diffusion. Chemical reactions and the effects of multiple temperatures are allowed. The occurrence of multiple temperatures in a non-diffusing mixture is the novel feature of this formulation. In this section, the governing balance equations appropriate to this investigation shall be stated.

For a mixture of  $\mathfrak{N}$  materials constrained such that there is no diffusion, the motion of the mixture is described by a single point valued function  $\chi$  such that

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad (2.1)$$

where  $\mathbf{X}$  is the position of a particle of the mixture in its reference configuration,  $t$  is the time, and  $\mathbf{x}$  is the spatial position occupied at the time  $t$  by the particle labeled  $\mathbf{X}$ . As usual,  $\chi$  is assumed to be invertible for each time  $t$ . The function  $\chi$  and its inverse  $\chi^{-1}$  must possess certain differentiability properties. For most of our calculations, it is sufficient for  $\chi$  and  $\chi^{-1}$  to be at least of class  $C^2$ . The velocity and acceleration of  $\mathbf{X}$  at time  $t$  are defined by

$$\dot{\mathbf{x}} = \frac{\partial \chi}{\partial t}(\mathbf{X}, t) \quad (2.2)$$

and

$$\ddot{\mathbf{x}} = \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X}, t). \quad (2.3)$$

The deformation gradient is defined by

$$\mathbf{F} = \text{GRAD } \chi(\mathbf{X}, t), \quad (2.4)$$

and the velocity gradient is defined by

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \text{grad } \dot{\mathbf{x}}(\mathbf{x}, t), \tag{2.5}$$

where  $\dot{\mathbf{F}}$  denotes the material derivative of  $\mathbf{F}$ .

The mass density of the mixture is given the symbol  $\rho$ , and it is calculated from

$$\rho = \sum_{\alpha=1}^{\mathfrak{N}} \rho_{\alpha}, \tag{2.6}$$

where  $\rho_{\alpha}$  denotes the mass density of the  $\alpha^{\text{th}}$  constituent in the mixture. The mass concentration for the  $\alpha^{\text{th}}$  constituent is

$$c_{\alpha} = \rho_{\alpha} / \rho. \tag{2.7}$$

The equations of balance take the form of field equations and jump equations. For simplicity here, we shall only record the appropriate field equations. The field equations which govern the mixture are the following:<sup>1</sup>

(a) Balance of mass for the  $\alpha^{\text{th}}$  constituent

$$\dot{\rho}_{\alpha} + \rho_{\alpha} \text{tr } \mathbf{L}_{\alpha} = \hat{c}_{\alpha}. \tag{2.8}$$

(b) Balance of mass for the  $\alpha^{\text{th}}$  atomic element

$$\sum_{\alpha=1}^{\mathfrak{N}} T_{\alpha}^{\alpha} \hat{c}_{\alpha} / M_{\alpha} = 0. \tag{2.9}$$

(c) Balance of momentum for the mixture

$$\rho \ddot{\mathbf{x}} = \text{div } \mathbf{T} + \rho \mathbf{b} \tag{2.10}$$

and

$$\mathbf{T} = \mathbf{T}^T. \tag{2.11}$$

(d) Balance of energy for the  $\alpha^{\text{th}}$  constituent

$$\rho_{\alpha} \dot{\hat{\varepsilon}}_{\alpha} = \text{tr } \mathbf{T}_{\alpha}^T \mathbf{L}_{\alpha} - \text{div } \mathbf{q}_{\alpha} + \rho_{\alpha} r_{\alpha} + \hat{\varepsilon}_{\alpha} \tag{2.12}$$

and

(e) Balance of energy for the mixture

$$\sum_{\alpha=1}^{\mathfrak{N}} (\hat{\varepsilon}_{\alpha} + \hat{c}_{\alpha} \varepsilon_{\alpha}) = 0. \tag{2.13}$$

For the  $\alpha^{\text{th}}$  constituent,  $\hat{c}_{\alpha}$  is the mass supply,  $M_{\alpha}$  is the molecular weight,  $\varepsilon_{\alpha}$  is the internal energy density,  $\mathbf{T}_{\alpha}$  is the partial stress tensor,  $\mathbf{q}_{\alpha}$  is the heat flux vector,  $r_{\alpha}$  is the heat supply and  $\hat{\varepsilon}_{\alpha}$  is the energy supply. For the mixture,  $\mathbf{T}$  is the

<sup>1</sup> BOWEN [3, Chap. 1], TRUESDELL & TOUPIN [4, Sections 159, 193, 215, 243].

stress tensor and  $\mathbf{b}$  is the body force density. The stress tensor for the mixture and the partial stress tensors are related by

$$\mathbf{T} = \sum_{\alpha=1}^{\mathfrak{N}} T_{\alpha} \mathbf{T}_{\alpha}. \quad (2.14)$$

The quantities  $T_{\alpha}^a$  are integers each of which represent the number of moles of the  $\alpha^{\text{th}}$  atomic element in one mole of the  $\alpha^{\text{th}}$  constituent. If there are  $\mathfrak{A}$  atomic elements, then  $\mathfrak{S} = \text{rank} [T_{\alpha}^a] \leq \min \{\mathfrak{N}, \mathfrak{A}\}$ . It can be shown that (2.9) implies balance of mass for the mixture in the form<sup>2</sup>

$$\dot{\rho} + \rho \text{tr} \mathbf{L} = \sum_{\alpha=1}^{\mathfrak{N}} c_{\alpha} = 0. \quad (2.15)$$

Also, one can show that (2.9) and (2.8) imply that

$$\hat{c}_{\alpha} = \rho M_{\alpha} \sum_{v=1}^{\mathfrak{N}-\mathfrak{S}} P_{\alpha}^v j_v, \quad (2.16)$$

$$c_{\alpha} - c_{\alpha 0} = M_{\alpha} \sum_{v=1}^{\mathfrak{N}-\mathfrak{S}} P_{\alpha}^v \xi_v \quad (2.17)$$

and

$$\hat{\xi}_v = j_v \quad (2.18)$$

for  $v=1, \dots, \mathfrak{N}-\mathfrak{S}$ . The  $\mathfrak{N} \times (\mathfrak{N}-\mathfrak{S})$  matrix  $[P_{\alpha}^v]$  is the stoichiometric matrix and obeys

$$\sum_{\alpha=1}^{\mathfrak{N}} T_{\alpha}^a P_{\alpha}^v = 0 \quad (2.19)$$

for  $v=1, \dots, \mathfrak{N}-\mathfrak{S}$  and  $\alpha=1, \dots, \mathfrak{N}$ . The quantities  $j_v$ ,  $v=1, \dots, \mathfrak{N}-\mathfrak{S}$ , are the reaction rates; the quantities  $\xi_v$ ,  $v=1, \dots, \mathfrak{N}-\mathfrak{S}$ , are the extents of reaction and the quantities  $c_{\alpha 0}$ ,  $\alpha=1, \dots, \mathfrak{N}$ , are concentrations at some previous time. The number  $\mathfrak{N}-\mathfrak{S}$  represents the number of independent chemical reactions allowed in the mixture. For notational convenience, we shall write (2.18) in the vector form

$$\hat{\xi} = \hat{\omega}, \quad (2.20)$$

where  $\xi$  and  $\hat{\omega}$  are  $\mathfrak{N}-\mathfrak{S}$  tuples defined by

$$\xi = (\xi_1, \xi_2, \dots, \xi_{\mathfrak{N}-\mathfrak{S}}) \quad (2.21)$$

$$\hat{\omega} = (j_1, j_2, \dots, j_{\mathfrak{N}-\mathfrak{S}}). \quad (2.22)$$

In our later analysis (2.15), and (2.20) will be used in place of (2.8) and (2.9). As is well known, we can interpret our formulation as a theory of materials with internal state variables  $\xi_1, \dots, \xi_{\mathfrak{N}-\mathfrak{S}}$ . A theory of chemical reactions without diffusion and a theory of materials with internal state variables are formally equivalent.

<sup>2</sup> BOWEN [3, Section 1.3].

Certain comments should be made about the equations of motion for the constituents. Since there is no diffusion in our analysis, the equations of motion take the form

$$\rho_a \ddot{\mathbf{x}} = \text{div}_a \mathbf{T} + \rho_a \mathbf{b} + \hat{\mathbf{p}}_a, \tag{2.23}$$

where  $\mathbf{b}$  is the external body force density for the  $a^{\text{th}}$  constituent and  $\hat{\mathbf{p}}$  is the momentum supply to the  $a^{\text{th}}$  constituent. The quantity  $\mathbf{b}$  in (2.10) is related to the  $\mathbf{b}$ 's in (2.23) by

$$\mathbf{b} = \frac{1}{\rho} \sum_{a=1}^{\mathfrak{N}} \rho_a \mathbf{b}_a. \tag{2.24}$$

The momentum supplies must obey the condition

$$\sum_{a=1}^{\mathfrak{N}} \hat{\mathbf{p}}_a = \mathbf{0}. \tag{2.25}$$

Note that (2.25), (2.24), (2.6) and (2.14) can be used to derive (2.10) by simply adding the  $\mathfrak{N}$  equation (2.23). In a theory of mixtures constrained such that there is no diffusion,  $\mathfrak{N} - 1$  of the momentum supplies are indeterminate. They can be viewed as forces of constraint arising from a no diffusion constraint. The  $\mathfrak{N}^{\text{th}}$  momentum supply is determined by (2.25) which is equivalent to (2.10). Our analysis is carried out in such a fashion that we do not need to utilize (2.23).

The entropy inequality which applies to our formulation is<sup>3</sup>

$$\rho \dot{\eta} + \text{div} \sum_{a=1}^{\mathfrak{N}} \mathbf{q}_a / \theta - \sum_{a=1}^{\mathfrak{N}} \rho_a r_a / \theta \geq 0, \tag{2.26}$$

where  $\eta$  is the entropy density for the mixture and  $\theta$  is the temperature for the  $a^{\text{th}}$  constituent. If (2.12) is used to eliminate  $r$  from (2.26), the result is

$$\rho \dot{\lambda} + \sum_{a=1}^{\mathfrak{N}} \rho_a \varepsilon_a \dot{\vartheta}_a + \sum_{a=1}^{\mathfrak{N}} \vartheta_a \text{tr} \mathbf{T}^T \mathbf{L}_a + \sum_{a=1}^{\mathfrak{N}} \mathbf{q}_a \cdot \mathbf{g}_a + \sum_{a=1}^{\mathfrak{N}} \vartheta_a \hat{\varepsilon}_a \geq 0, \tag{2.27}$$

where  $\vartheta$  is the coldness of the  $a^{\text{th}}$  constituent defined by

$$\vartheta_a = 1/\theta, \tag{2.28}$$

$\mathbf{g}$  is the coldness gradient defined by

$$\mathbf{g}_a = \text{grad}_a \vartheta_a, \tag{2.29}$$

$\lambda$  is the Massieu function for the mixture defined by

$$\lambda = \eta - \frac{1}{\rho} \sum_{a=1}^{\mathfrak{N}} \rho_a \vartheta_a \varepsilon_a \tag{2.30}$$

<sup>3</sup> BOWEN [3, Section 1.6], BOWEN & GARCIA [5, Section 2].

and  $\hat{\varepsilon}$  is defined by

$$\hat{\varepsilon} = \hat{\varepsilon} + \hat{c} \varepsilon. \tag{2.31}$$

Note that from (2.13),

$$\sum_{a=1}^{\mathfrak{N}} \hat{\varepsilon} = 0. \tag{2.32}$$

### 3. Constitutive Assumptions and Thermodynamic Restrictions

The constitutive equations which define the mixture are taken to be

$$\lambda = \lambda(\vartheta, \mathbf{g}, \mathbf{F}, \xi), \tag{3.1}$$

$$\varepsilon = \varepsilon(\vartheta, \mathbf{g}, \mathbf{F}, \xi), \tag{3.2}$$

$$\mathbf{T} = \mathbf{T}(\vartheta, \mathbf{g}, \mathbf{F}, \xi), \tag{3.3}$$

$$\mathbf{q} = \mathbf{q}(\vartheta, \mathbf{g}, \mathbf{F}, \xi), \tag{3.4}$$

$$\hat{\varepsilon} = \hat{\varepsilon}(\vartheta, \mathbf{g}, \mathbf{F}, \xi), \tag{3.5}$$

and

$$\hat{\omega} = \hat{\omega}(\vartheta, \mathbf{g}, \mathbf{F}, \xi). \tag{3.6}$$

These equations describe a mixture whose mechanical response is like an elastic material but it is capable of reacting chemically, conducting heat, and transferring energy between the constituents. It follows from (3.2), (3.5), (3.6), (2.22) and (2.16) that

$$\hat{\varepsilon} = \hat{\varepsilon}(\vartheta, \mathbf{g}, \mathbf{F}, \xi). \tag{3.7}$$

From (2.11), (2.14) and (3.3), we must require that the response functions for  $\mathbf{T}$ ,  $a = 1, 2, \dots, \mathfrak{N}$ , yield a symmetric stress  $\mathbf{T}$ . However, if we require that the energy equations (2.12) be indifferent to a change of frame, then one can use (3.5) and prove that *each*  $\mathbf{T}$  must be symmetric.<sup>1</sup> In the following, we shall assume that the response functions in (3.3) have symmetric values.

In order to render (3.1) through (3.7) consistent with the entropy inequality, one must substitute these equations into (2.27). The result is

$$\sum_{a=1}^{\mathfrak{N}} \left( \rho \varepsilon + \rho \frac{\partial \lambda}{\partial \vartheta} \right) \dot{\vartheta} + \text{tr} \left\{ \sum_{a=1}^{\mathfrak{N}} \vartheta \mathbf{T} + \rho \mathbf{F} \left( \frac{\partial \lambda}{\partial \mathbf{F}} \right)^T \right\} \mathbf{L} + \rho \sum_{a=1}^{\mathfrak{N}} \frac{\partial \lambda}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} + \rho \frac{\partial \lambda}{\partial \xi} \cdot \dot{\xi} + \sum_{a=1}^{\mathfrak{N}} \mathbf{q} \cdot \mathbf{g} + \sum_{a=1}^{\mathfrak{N}} \vartheta \hat{\varepsilon} \geq 0. \tag{3.8}$$

<sup>1</sup> BOWEN [3, Section 2.5].

By the now standard thermodynamic argument, (3.8) yields<sup>2</sup>

$$\rho \varepsilon_{\alpha} = -\rho \frac{\partial \lambda}{\partial \vartheta_{\alpha}}, \tag{3.9}$$

$$\sum_{\alpha=1}^{\mathfrak{N}} \vartheta_{\alpha} T_{\alpha} = -\rho \mathbf{F} \left( \frac{\partial \lambda}{\partial \mathbf{F}} \right)^T, \tag{3.10}$$

$$\frac{\partial \lambda}{\partial \mathbf{g}_{\alpha}} = \mathbf{0} \tag{3.11}$$

and

$$\rho \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\omega}} + \sum_{\alpha=1}^{\mathfrak{N}} \mathbf{q}_{\alpha} \cdot \mathbf{g}_{\alpha} + \sum_{\alpha=1}^{\mathfrak{N}} \vartheta_{\alpha} \hat{e} \geq 0, \tag{3.12}$$

where  $\boldsymbol{\sigma}$  is defined by

$$\boldsymbol{\sigma} = \frac{\partial \lambda}{\partial \boldsymbol{\xi}}. \tag{3.13}$$

The  $\mathfrak{N} - \mathfrak{S}$  tuple  $\boldsymbol{\sigma}$  plays the role of the chemical affinity for the mixture. Actually, it is not identical to the chemical affinity of theories of reacting mixtures with a single temperature field.<sup>3</sup> Because of (3.11), it follows that

$$\lambda = \lambda(\vartheta, \mathbf{F}, \boldsymbol{\xi}), \tag{3.14}$$

$$\rho \varepsilon_{\alpha} = \rho \varepsilon_{\alpha}(\vartheta, \mathbf{F}, \boldsymbol{\xi}) = -\rho \frac{\partial \lambda}{\partial \vartheta_{\alpha}}(\vartheta, \mathbf{F}, \boldsymbol{\xi}), \tag{3.15}$$

and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\vartheta, \mathbf{F}, \boldsymbol{\xi}) = \frac{\partial \lambda}{\partial \boldsymbol{\xi}}(\vartheta, \mathbf{F}, \boldsymbol{\xi}). \tag{3.16}$$

Also, from (3.14) and (3.10), the sum  $\sum_{\alpha=1}^{\mathfrak{N}} \vartheta_{\alpha} T_{\alpha}$  is a function of  $(\vartheta, \mathbf{F}, \boldsymbol{\xi})$ .

Given (3.15), (3.10) and (2.31), it is possible to show that the energy equation (2.12) can be written

$$\sum_{\alpha=1}^{\mathfrak{N}} \rho \frac{\partial \varepsilon_{\alpha}}{\partial \vartheta_{\alpha}} \vartheta_{\alpha} + \text{tr} \left( \sum_{\alpha=1}^{\mathfrak{N}} \vartheta_{\alpha} \frac{\partial T_{\alpha}}{\partial \vartheta_{\alpha}} \right) \mathbf{L} + \rho \frac{\partial \varepsilon}{\partial \boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} = -\text{div} \mathbf{q}_{\alpha} + \hat{e}_{\alpha} + \rho r. \tag{3.17}$$

Equation (3.12) contains other results which we shall need in the next section. These results are the restrictions implied by the entropy inequality at a thermodynamic equilibrium state. For a reacting mixture there are several possible states of thermodynamic equilibrium.<sup>4</sup> In this work we shall be concerned only with a thermodynamic equilibrium state defined by  $\boldsymbol{\sigma} = \mathbf{0}$ ,  $\mathbf{g}_{\alpha} = \mathbf{g} = \dots$

<sup>2</sup> TRUESDELL [6], COLEMAN & NOLL [2], BOWEN [3, Section 2.3].

<sup>3</sup> BOWEN [7].

<sup>4</sup> BOWEN [7], TRUESDELL [6].

$= \mathbf{g} = \mathbf{0}$  and  $\vartheta = \vartheta_1 = \vartheta_2 = \dots = \vartheta_{\mathfrak{N}} \equiv \vartheta^+$  where  $\vartheta^+$  is an arbitrary positive real number. Clearly, by (2.32), the left side of (3.12) is zero in this equilibrium state. In order to obtain the simplest form of the equilibrium restrictions, we assume that (3.16) can be inverted for each  $(\vartheta, \mathbf{F}, \boldsymbol{\xi})$  to obtain

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\vartheta, \mathbf{F}, \boldsymbol{\sigma}). \tag{3.18}$$

Note, in passing, that our inversion assumption implies that

$$\det \frac{\partial^2 \lambda}{\partial \boldsymbol{\xi}^2}(\vartheta, \mathbf{F}, \boldsymbol{\xi}) \neq \mathbf{0} \tag{3.19}$$

for each  $(\vartheta, \mathbf{F}, \boldsymbol{\xi})$ . Given (3.18), we can eliminate  $\boldsymbol{\xi}$  from (3.4), (3.6) and (3.7) to obtain

$$\mathbf{q} = \mathbf{q}^*(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma}), \tag{3.20}$$

$$\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\epsilon}}^*(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma}), \tag{3.21}$$

and

$$\hat{\omega} = \hat{\omega}^*(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma}). \tag{3.22}$$

Given (3.20) through (3.22), the left side of (3.12) is a function of  $(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma})$ . If we define a function  $\Phi$  of  $(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma})$  by

$$\Phi(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma}) = \rho \boldsymbol{\sigma} \cdot \hat{\omega} + \sum_{\alpha=1}^{\mathfrak{N}} \mathbf{q} \cdot \mathbf{g} + \sum_{\alpha=1}^{\mathfrak{N}} \vartheta \hat{\boldsymbol{\epsilon}}, \tag{3.23}$$

then, (3.12) yields

$$\Phi(\vartheta, \mathbf{g}, \mathbf{F}, \boldsymbol{\sigma}) \geq 0 \tag{3.24}$$

and

$$\Phi(\vartheta^+, \mathbf{0}, \mathbf{F}, \mathbf{0}) = 0. \tag{3.25}$$

Equations (3.24) and (3.25) show that  $\Phi$  is a minimum in the thermodynamic equilibrium state and, thus,

$$\left. \frac{d\Phi}{d\beta}(\vartheta^+ + \beta \boldsymbol{\alpha}, \beta \mathbf{a}, \mathbf{F}, \beta \boldsymbol{\phi}) \right|_{\beta=0} = 0 \tag{3.26}$$

and

$$\left. \frac{d^2\Phi}{d\beta^2}(\vartheta^+ + \beta \boldsymbol{\alpha}, \beta \mathbf{a}, \mathbf{F}, \beta \boldsymbol{\phi}) \right|_{\beta=0} \geq 0 \tag{3.27}$$

for all real numbers  $\alpha_1, \dots, \alpha_{\mathfrak{N}}$ , all vectors  $\mathbf{a}_1, \dots, \mathbf{a}_{\mathfrak{N}}$ , and all  $\mathfrak{N} - \mathfrak{S}$  tuples  $\boldsymbol{\phi}$ . An elementary calculation shows that (3.26) is equivalent to the following three restrictions

$$\hat{\omega}^*(\vartheta^+, \mathbf{0}, \mathbf{F}, \mathbf{0}) = \mathbf{0}, \tag{3.28}$$

$$\mathbf{q}^*(\vartheta^+, \mathbf{0}, \mathbf{F}, \mathbf{0}) = \mathbf{0} \tag{3.29}$$



and

$$\hat{e}_a^*(\vartheta^+, \mathbf{0}, \mathbf{F}, \mathbf{0}) = 0. \tag{3.30}$$

The additional restrictions which follow from (3.27) will not be shown explicitly here. In the next section, these restrictions will be given for a linear isotropic theory.

#### 4. Linear Isotropic Theory

As an illustration of the results of Section 3, in this Section the special case of a mixture restricted to small displacement gradients, small coldness changes, small coldness gradients, and small extent of reaction changes will be investigated. For simplicity, the mixture is assumed to have the symmetry of an isotropic solid.<sup>1</sup>

The mixture is assumed to be at rest in a reference state of uniform coldness  $\vartheta_0$ , uniform deformation  $\mathbf{F}_0 = \mathbf{I}$  and uniform extent of reaction  $\xi_0$ . This state is assumed to be a thermodynamic equilibrium state and to have zero stress in each constituent. The density of the mixture is a constant denoted by  $\rho_R$ . In order that the state just described satisfies the field equations (2.10) and (2.12), we require that

$$\mathbf{b} = \mathbf{0} \tag{4.1}$$

and

$$r_a = 0. \tag{4.2}$$

In states near the reference state, the approximate constitutive equation for the Massieu function is

$$\begin{aligned} \rho_R \lambda = & \rho_R \lambda_0 - \sum_{a=1}^{\mathfrak{N}} \rho_a \varepsilon_a (\vartheta - \vartheta_0) + \frac{1}{2\vartheta_0^2} \sum_{a,b=1}^{\mathfrak{N}} c_{ab} (\vartheta_a - \vartheta_0) (\vartheta_b - \vartheta_0) \\ & - \frac{1}{2} \rho_R (\xi - \xi_0) \cdot \Phi (\xi - \xi_0) - \frac{1}{2} \vartheta_0 \hat{\lambda} (\text{tr } \mathbf{E})^2 - \vartheta_0 \hat{\mu} \text{tr } \mathbf{E}^2 \\ & - \sum_{a=1}^{\mathfrak{N}} \delta_a \cdot (\xi - \xi_0) (\vartheta_a - \vartheta_0) - \frac{1}{\vartheta_0} \sum_{a=1}^{\mathfrak{N}} \beta_a (\vartheta_a - \vartheta_0) \text{tr } \mathbf{E} \\ & - \vartheta_0 \alpha \cdot (\xi - \xi_0) \text{tr } \mathbf{E}, \end{aligned} \tag{4.3}$$

where  $\lambda_0, \varepsilon_a, c_{ab}, \Phi, \hat{\lambda}, \hat{\mu}, \delta_a, \beta_a$  and  $\alpha$  are material constants in the expansion of  $\lambda$ . These constants are related to  $\lambda$  in an obvious fashion. For example,  $c_{ab}$  and  $\Phi$  are given by

$$c_{ab} = \rho_R \vartheta_0^2 \frac{\partial^2 \lambda}{\partial \vartheta_a \partial \vartheta_b} (\vartheta_0, \mathbf{I}) = c_{ba} \tag{4.4}$$

and

$$\Phi = - \frac{\partial^2 \lambda}{\partial \xi^2} (\vartheta_0, \mathbf{I}) = \Phi^T. \tag{4.5}$$

<sup>1</sup> TRUESDELL & NOLL [8, Section 31].

Note that (3.19) tells us that the symmetric matrix  $\Phi$  is nonsingular. The symbol  $E$  in (4.3) is the classical infinitesimal strain tensor defined by

$$2E = \text{GRAD } \mathbf{w} + (\text{GRAD } \mathbf{w})^T, \tag{4.6}$$

where  $\mathbf{w}(X, t)$  is the displacement vector.

Given (4.3), it follows from (3.16) that

$$\boldsymbol{\sigma} = -\Phi(\boldsymbol{\xi} - \boldsymbol{\xi}_0) - \frac{\vartheta_0}{\rho_R} \boldsymbol{\alpha} \text{tr } E - \frac{1}{\rho_R} \sum_{\alpha=1}^n \delta_\alpha (\vartheta - \vartheta_0). \tag{4.7}$$

Note that (4.7) yields  $\boldsymbol{\sigma} = \mathbf{0}$  in the reference state. Thus, the expansion (4.3) reflects the assumption that the reference state is a thermodynamic equilibrium state. In states near the reference state, it is easily shown that (3.10) is approximated by

$$\vartheta_0 \mathbf{T} = -\rho_R \frac{\partial \lambda}{\partial E}. \tag{4.8}$$

Therefore, from (4.3), the stress  $T$  is given by

$$\mathbf{T} = \hat{\lambda}(\text{tr } E) \mathbf{I} + 2\hat{\mu}E + \frac{1}{\vartheta_0^2} \sum_{b=1}^n \beta_b (\vartheta_b - \vartheta_0) \mathbf{I} + \boldsymbol{\alpha} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_0) \mathbf{I}. \tag{4.9}$$

Equation (4.9) reflects our assumption that the reference state is stress free. It follows from (4.9) that  $\hat{\mu}$  and  $\hat{\lambda}$  are the Lamé constants and  $\beta_b$  is the coefficient of thermal expansion resulting from coldness changes in the  $b^{\text{th}}$  constituent. Finally, it follows from (3.15) and (4.3) that

$$\begin{aligned} \rho \varepsilon = \rho_0 \varepsilon_0 (1 - \text{tr } E) - \frac{1}{\vartheta_0^2} \sum_{b=1}^n c_{ab} (\vartheta_b - \vartheta_0) \\ + \frac{1}{\vartheta_0} \beta_a \text{tr } E + \delta_a \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{aligned} \tag{4.10}$$

This result shows us that the  $N \times N$  matrix  $[c_{ab}]$  is the specific heat matrix.

The approximate versions of (3.20), (3.21) and (3.22) for a linear isotropic theory are

$$\mathbf{q}_a = \frac{1}{\vartheta_0^2} \sum_{b=1}^n \kappa_{ab} \mathbf{g}_b, \tag{4.11}$$

$$\hat{\omega} = \Psi \Phi^{-1} \boldsymbol{\sigma} + \sum_{b=1}^n \zeta_b (\vartheta - \vartheta_0) \tag{4.12}$$

and

$$\hat{e} = \sum_{b=1}^n \Gamma_{ab} (\vartheta - \vartheta_0) + \rho_R \boldsymbol{\chi}_a \cdot \boldsymbol{\sigma}. \tag{4.13}$$

The coefficient  $\Psi \Phi^{-1}$  in (4.12) is given in terms of the response function  $\hat{\omega}^*$  by the formula

$$\Psi \Phi^{-1} = \frac{\partial \hat{\omega}^*}{\partial \boldsymbol{\sigma}} (\vartheta_0, \mathbf{0}, \mathbf{I}, \mathbf{0}). \tag{4.14}$$

If (4.7) is used, it easily follows that

$$\Psi = -\frac{\partial \hat{\omega}}{\partial \xi}(\vartheta_0, \mathbf{0}, \mathbf{I}, \xi_0). \tag{4.15}$$

The  $N \times N$  matrix  $[\kappa_{ab}]$  in (4.11) is the matrix of thermal conductivities. It follows from (4.11) that (3.29) is satisfied. In order that (3.28) and (3.30) be satisfied for all  $\vartheta^+$ , the coefficients  $\zeta_b$  and  $\Gamma_{ab}$  must obey the following conditions:

$$\sum_{b=1}^{\mathfrak{N}} \zeta_b = \mathbf{0} \tag{4.16}$$

and

$$\sum_{b=1}^{\mathfrak{N}} \Gamma_{ab} = 0. \tag{4.17}$$

Because of (2.32), it is also necessary that

$$\sum_{a=1}^{\mathfrak{N}} \chi_a = \mathbf{0} \tag{4.18}$$

and

$$\sum_{a=1}^{\mathfrak{N}} \Gamma_{ab} = 0. \tag{4.19}$$

If (4.11), (4.12) and (4.13) are substituted into (3.23), it is possible to show that (3.27) yields the following two inequalities:

$$\frac{1}{\vartheta_0^2} \sum_{a,b=1}^{\mathfrak{N}} \kappa_{ab} \mathbf{a}_a \cdot \mathbf{a}_b \geq 0 \tag{4.20}$$

for all vectors  $\mathbf{a}_1, \dots, \mathbf{a}_{\mathfrak{N}}$  and

$$\rho_R \phi \cdot \Psi \Phi^{-1} \phi + \rho_R \sum_{b=1}^{\mathfrak{N}} \alpha_b (\zeta_b + \chi_b) \cdot \phi + \sum_{a,b=1}^{\mathfrak{N}} \Gamma_{ab} \alpha_a \alpha_b \geq 0 \tag{4.21}$$

for all real numbers  $\alpha_1, \dots, \alpha_{\mathfrak{N}}$  and all  $\mathfrak{N} - \mathfrak{S}$  tuples  $\phi$ .

The equation of motion appropriate to the linear isotropic model can be obtained by substitution of (4.9) and (4.1) into (2.10) and formally linearizing the result. The resulting field equation is

$$\begin{aligned} \rho_R \ddot{\mathbf{w}} = & (\hat{\lambda} + \hat{\mu}) \text{grad} (\text{div } \mathbf{w}) + \hat{\mu} \text{div} (\text{grad } \mathbf{w}) \\ & + \frac{1}{\vartheta_0^2} \sum_{b=1}^{\mathfrak{N}} \beta_b \text{grad } \vartheta_b + \alpha \cdot \text{grad } \xi. \end{aligned} \tag{4.22}$$

The energy equation for each constituent can be obtained by substitution of (4.2), (4.9), (4.10), (4.11) and (4.13) into (3.17). Among the linearizations involved

in deriving the energy equation is one which approximates  $\sum_{b=1}^{\mathfrak{N}} \vartheta_b \frac{\partial \mathbf{T}_b}{\partial \vartheta_b}$  by  $\vartheta_0 \frac{\partial \mathbf{T}}{\partial \vartheta_a}$ .

Because of this approximation, a linearized expression for the partial stresses  $T_\alpha$ ,  $\alpha = 1, 2, \dots, \mathfrak{N}$ , need not be given. The linearized energy equations are

$$\begin{aligned} & \sum_{b=1}^{\mathfrak{N}} c_{ab} \vartheta_b - \vartheta_0 \beta_a \operatorname{div} \dot{\mathbf{w}} - \vartheta_0^2 \delta_a \cdot \dot{\boldsymbol{\xi}} \\ & = \sum_{b=1}^{\mathfrak{N}} \kappa_{ab} \operatorname{div} (\operatorname{grad} \vartheta_b) - \vartheta_0^2 \sum_{b=1}^{\mathfrak{N}} \Gamma_{ab} (\vartheta_b - \vartheta_0) - \rho_R \vartheta_0^2 \boldsymbol{\chi}_a \cdot \boldsymbol{\sigma}. \end{aligned} \tag{4.23}$$

Equations (4.12) and (2.20) combine to yield

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Psi} \boldsymbol{\Phi}^{-1} \boldsymbol{\sigma} + \sum_{b=1}^{\mathfrak{N}} \zeta_b (\vartheta_b - \vartheta_0). \tag{4.24}$$

Equations (4.22), (4.23), (4.24) and (4.7) are the field equations for the linearized isotropic model.

It is elementary to establish uniqueness theorems for solutions of (4.22), (4.23), (4.24) and (4.7) whenever we require certain inequalities to be valid. These inequalities are suggested by the following identity:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V (\Sigma + \frac{1}{2} \rho_R \dot{\mathbf{w}}^2) dv = \oint_{\partial V} \dot{\mathbf{w}} \cdot \mathbf{T} ds - \sum_{a=1}^{\mathfrak{N}} \oint_{\partial V} \mathbf{q} \left( 1 - \frac{\vartheta_a}{\vartheta_0} \right) \cdot ds \\ & - \int_V \left\{ \rho_R \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\omega}} + \sum_{a=1}^{\mathfrak{N}} \mathbf{q}_a \cdot \mathbf{g}_a + \sum_{a=1}^{\mathfrak{N}} \vartheta_a \hat{\boldsymbol{\epsilon}}_a \right\} dv, \end{aligned} \tag{4.25}$$

where  $V$  is a fixed region,  $\partial V$  is its boundary and  $\Sigma$  is defined by

$$\Sigma = -\frac{1}{\vartheta_0} \rho_R \left( \lambda - \lambda_0 - \sum_{a=1}^{\mathfrak{N}} \frac{\partial \lambda}{\partial \vartheta_a} (\vartheta_a - \vartheta_0) \right), \tag{4.26}$$

or, by (4.3),

$$\begin{aligned} \Sigma & = \frac{1}{2} \hat{\lambda} (\operatorname{tr} \mathbf{E})^2 + \hat{\mu} \operatorname{tr} \mathbf{E}^2 + \frac{1}{2 \vartheta_0^3} \sum_{a,b=1}^{\mathfrak{N}} c_{ab} (\vartheta_a - \vartheta_0) (\vartheta_b - \vartheta_0) \\ & + \frac{1}{2 \vartheta_0} \rho_R (\boldsymbol{\xi} - \boldsymbol{\xi}_0) \cdot \boldsymbol{\Phi} (\boldsymbol{\xi} - \boldsymbol{\xi}_0) + \boldsymbol{\alpha} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_0) \operatorname{tr} \mathbf{E}. \end{aligned} \tag{4.27}$$

It follows from (4.11), (4.12), (4.13), (4.20) and (4.21) that the last term on the right side of (4.25) cannot be greater than zero. Therefore, (4.25) yields the inequality.

$$\frac{\partial}{\partial t} \int_V (\Sigma + \frac{1}{2} \rho_R \dot{\mathbf{w}}^2) dv \leq \oint_{\partial V} \dot{\mathbf{w}} \cdot \mathbf{T} ds - \sum_{a=1}^{\mathfrak{N}} \int_{\partial V} \mathbf{q} \left( 1 - \frac{\vartheta_a}{\vartheta_0} \right) \cdot ds. \tag{4.28}$$

If  $\Sigma$  is required to be positive definite; *i.e.*

$$\Sigma > 0, \tag{4.29}$$

for all nonzero  $\mathbf{E}$ ,  $\vartheta_1 - \vartheta_0$ ,  $\vartheta_2 - \vartheta_0, \dots, \vartheta_{\mathfrak{N}} - \vartheta_0$  and  $\boldsymbol{\xi} - \boldsymbol{\xi}_0$ , then one can use (4.28) and standard arguments to establish the uniqueness of solutions to (4.22), (4.23), (4.24) and (4.7). Among the restrictions implied by (4.29) are the following:

$$\hat{\lambda} + \frac{2}{3}\hat{\mu} > 0, \quad \hat{\mu} > 0, \tag{4.30}$$

$$[c_{ab}] \text{ positive definite} \tag{4.31}$$

and

$$\Phi \text{ positive definite.} \tag{4.32}$$

### 5. One Dimensional Frozen Harmonic Waves

In this section certain features of the linear isotropic theory are illustrated by briefly investigating the propagation of one dimensional harmonic waves in a binary mixture where no chemical reactions are possible. The omission of chemical reactions makes it appropriate to refer to the harmonic waves as frozen. Thus,  $\mathfrak{N} = 2$ ,  $\mathfrak{N} - \mathfrak{S} = 0$  and the displacement vector takes the special form

$$\mathbf{w}(\mathbf{X}, t) = (w(X, t), 0, 0). \tag{5.1}$$

The assumption of no chemical reactions implies that the material coefficients  $\Phi$ ,  $\Psi\Phi^{-1}$ ,  $\alpha$ ,  $\delta_\alpha$ ,  $\zeta_\alpha$  and  $\chi_\alpha$ , for  $\alpha = 1, 2$ , are zero. The field equations for this one dimensional case can be written in the matrix forms

$$\rho_R \ddot{w} = (\hat{\lambda} + 2\hat{\mu}) \frac{\partial^2 w}{\partial X^2} + \frac{1}{\mathfrak{g}_0^2} \boldsymbol{\beta}^T \frac{\partial \mathfrak{g}}{\partial x} \tag{5.2}$$

and

$$\mathbf{C}\mathfrak{g} - \mathfrak{g}_0 \boldsymbol{\beta} \frac{\partial \dot{w}}{\partial X} = \mathbf{K} \frac{\partial^2 \mathfrak{g}}{\partial x^2} - \mathfrak{g}_0^2 \boldsymbol{\Gamma} \mathbf{E} \mathfrak{g}, \tag{5.3}$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{5.4}$$

$$\mathfrak{g} = \begin{bmatrix} \mathfrak{g} \\ 1 \\ \mathfrak{g} \\ 2 \end{bmatrix}, \tag{5.5}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}, \tag{5.6}$$

$$\mathbf{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}, \tag{5.7}$$

and

$$\mathbf{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{5.8}$$

The derivation of (5.3) from (4.23) made use of (4.17) and (4.19). These two equations show that

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \boldsymbol{\Gamma} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \boldsymbol{\Gamma} \mathbf{E}, \tag{5.9}$$

where  $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{11}$ . It follows from (4.20) that  $\mathbf{K}$ , the matrix of thermal conductivities, is positive semi-definite and from (4.21) that

$$\boldsymbol{\Gamma} \geq 0. \tag{5.10}$$

One dimensional harmonic waves are solutions of (5.2) and (5.3) of the form

$$w = a e^{i(kX - \omega t)} \tag{5.11}$$

and

$$\mathfrak{g} = \mathbf{b} e^{i(kX - \omega t)}, \tag{5.12}$$

where  $a$  is a real number,  $\mathbf{b}$  is a  $2 \times 1$  matrix,  $k$  is the wave number (possibly complex) and  $\omega$  is the frequency (a real number). Because  $\omega$  is required to be real, the waves are propagating with assigned frequency. The phase velocity  $q$  of the wave represented by (5.11) and (5.12) is defined by

$$q = \omega/R(k), \tag{5.13}$$

and the attenuation coefficient  $\phi$  is defined by

$$\phi = \mathfrak{g}(k). \tag{5.14}$$

If (5.11) and (5.12) are substituted into (5.2) and (5.3), the result is

$$((\hat{\lambda} + 2\hat{\mu})k^2 - \rho_R \omega^2) a - \frac{1}{\mathfrak{g}_0^2} ik \boldsymbol{\beta}^T \mathbf{b} = 0 \tag{5.15}$$

and

$$-\mathfrak{g}_0 k \omega \boldsymbol{\beta} a + (\mathbf{K} k^2 - i\omega \mathbf{C} + \mathfrak{g}_0^2 \Gamma \mathbf{E}) \mathbf{b} = \mathbf{0}. \tag{5.16}$$

In order that (5.15) and (5.16) yield a non-trivial solution for  $a$  and  $\mathbf{b}$ , it is necessary and sufficient for the determinant of coefficients to vanish. The solution of the resulting characteristic polynomial for  $k(\omega)$  is the dispersion relation.

The result of expanding the  $3 \times 3$  matrix of coefficients is the polynomial

$$\begin{aligned} &(\det \mathbf{K} k^4 + \mathfrak{g}_0^2 \Gamma \operatorname{tr}((\operatorname{adj} \mathbf{K}) \mathbf{E}) k^2)(u^2 k^2 - \omega^2) \\ &- i\omega k^2 \operatorname{tr}((\operatorname{adj} \mathbf{K}) \mathbf{C})(s^2 k^2 - \omega^2) \\ &- \omega^2 \det \mathbf{C}(v^2 k^2 - \omega^2) - i\omega \mathfrak{g}_0^2 \Gamma \operatorname{tr}((\operatorname{adj} \mathbf{C}) \mathbf{E})(p^2 k^2 - \omega^2) = 0, \end{aligned} \tag{5.17}$$

where  $u^2, s^2, v^2$  and  $p^2$  are positive numbers defined by

$$u^2 = \frac{\hat{\lambda} + 2\hat{\mu}}{\rho_R}, \tag{5.18}$$

$$s^2 = u^2 + \boldsymbol{\beta}^T (\operatorname{adj} \mathbf{K}) \boldsymbol{\beta} / \rho_R \mathfrak{g}_0 \operatorname{tr}(\operatorname{adj} \mathbf{K}) \mathbf{C}, \tag{5.19}$$

$$v^2 = u^2 + \boldsymbol{\beta}^T \mathbf{C}^{-1} \boldsymbol{\beta} / \rho_R \mathfrak{g}_0 \tag{5.20}$$

and

$$p^2 = u^2 + \boldsymbol{\beta}^T (\operatorname{adj} \mathbf{E}) \boldsymbol{\beta} / \rho_R \mathfrak{g}_0 \operatorname{tr}(\operatorname{adj} \mathbf{C}) \mathbf{E}. \tag{5.21}$$

In (5.17),  $\operatorname{adj} \mathbf{K}$  and  $\operatorname{adj} \mathbf{C}$  denote the adjoints of the  $2 \times 2$  matrices  $\mathbf{K}$  and  $\mathbf{C}$ . The definition (5.19) assumes  $\operatorname{tr}(\operatorname{adj} \mathbf{K}) \mathbf{C}$  is not zero. Because  $\mathbf{C}$  is symmetric and positive definite and  $\mathbf{K}$  is positive semi-definite,  $\operatorname{tr}(\operatorname{adj} \mathbf{K}) \mathbf{C}$  cannot be negative. It is possible to show that

$$u^2 \leq p^2 \leq v^2 \tag{5.22}$$

and

$$u^2 \leq s^2 \leq v^2. \tag{5.23}$$

These results have been established in a more general context by BOWEN and REINICKE.<sup>1</sup> The physical dimension of the four quantities  $u, s, v$  and  $p$  is velocity. As will be shown below, in certain cases they represent phase velocities.

It is informative to examine various limiting cases of the characteristic polynomial (5.17). The case where  $\Gamma \rightarrow \infty$  corresponds to the situation when each constituent has the same temperature. In this case, (5.17) becomes the familiar characteristic polynomial of classical thermoelasticity.<sup>2</sup> In the remainder of the section, certain of the high and low frequency limits of (5.17) will be investigated for the case when  $\Gamma$  is not infinite.

The nondispersive low frequency limits are those limits which yield a frequency independent phase velocity. It is possible to show that if  $\Gamma \neq 0$ , then (5.17) yields the following nondispersive low frequency approximation:

$$\frac{k^2}{\omega^2} = \frac{1}{p^2} \left\{ 1 + i \left\{ \left( 1 - \frac{u^2}{p^2} \right) \frac{\text{tr}(\text{adj } \mathbf{K}) \mathbf{E}}{p^2 \text{tr}(\text{adj } \mathbf{C}) \mathbf{E}} + \left( \frac{v^2}{p^2} - 1 \right) \frac{1}{\Gamma \text{tr}(\mathbf{C}^{-1} \mathbf{E})} \right\} \omega \right\} + O(\omega^2). \quad (5.24)$$

For this mode of propagation, the phase velocity is given by

$$q = \pm p + O(\omega) \quad (5.25)$$

and, for  $q$  positive, the attenuation coefficient is the positive number

$$\phi = O(\omega^2) = \frac{\omega^2}{2p} \left\{ \left( 1 - \frac{u^2}{p^2} \right) \frac{\text{tr}(\text{adj } \mathbf{K}) \mathbf{E}}{p^2 \text{tr}(\text{adj } \mathbf{C}) \mathbf{E}} + \left( \frac{v^2}{p^2} - 1 \right) \frac{1}{\Gamma \text{tr} \mathbf{C}^{-1} \mathbf{E}} \right\} + O(\omega^3). \quad (5.26)$$

It is possible to show that  $p$  is the isentropic wave speed familiar from classical thermoelasticity. In a rough way, low frequency corresponds to large time. With this correspondence in mind, one can view (5.25) as showing that after a large time there is one mode of propagation whose velocity is unaffected by the nonequilibrium effects of heat conduction and energy transfer. Equation (5.26) shows that the attenuation coefficient is second order in frequency and, while small, depends upon the heat conduction and energy transfer. In the limit as  $\Gamma \rightarrow \infty$ , (5.26) reduces to the classical thermoelasticity result for waves propagating in a material with thermal conductivity  $\text{tr}(\text{adj } \mathbf{K}) \mathbf{E} = \kappa_{11} + \kappa_{12} + \kappa_{21} + \kappa_{22}$  and specific heat  $\text{tr}(\text{adj } \mathbf{C}) \mathbf{E} = c_{11} + 2c_{12} + c_{22}$ .

If  $\Gamma = 0$ , one does not obtain (5.24). In this case, the result is

$$\frac{k^2}{\omega^2} = \frac{1}{v^2} \left\{ 1 + i \left\{ \left( 1 - \frac{s^2}{v^2} \right) \frac{\text{tr}(\mathbf{C}^{-1} \mathbf{K})}{v^2} \right\} \omega \right\} + O(\omega^2) \quad (5.27)$$

for the mode which is nondispersive in its low frequency limit. In this case, the squared phase velocity is approximately  $v^2$  and the attenuation coefficient is, for positive phase velocity, a positive number.

The nondispersive high frequency approximations to (5.17) are the following:

(i) if  $\det \mathbf{K} \neq 0$

$$\frac{k^2}{\omega^2} = \frac{1}{u^2} \left\{ 1 + i(s^2 - u^2) \text{tr } \mathbf{K}^{-1} \mathbf{C} \frac{1}{\omega} \right\} + O(1/\omega^2) \quad (5.28)$$

<sup>1</sup> BOWEN & REINICKE [9, Section 3].

<sup>2</sup> CHADWICK [10].

(ii) if  $\det \mathbf{K} = 0$  and  $\text{tr}(\text{adj } \mathbf{K}) \mathbf{C} \neq 0$

$$\frac{k^2}{\omega^2} = \frac{1}{s^2} \left\{ 1 + i \left\{ \left( 1 - \frac{u^2}{s^2} \right) \frac{\vartheta_0^2 \Gamma \text{tr}(\text{adj } \mathbf{K}) \mathbf{E}}{\text{tr}(\text{adj } \mathbf{K}) \mathbf{C}} + (v^2 - s^2) \frac{1}{\text{tr } \mathbf{K} \mathbf{C}^{-1}} \right\} \frac{1}{\omega} \right\} + O(1/\omega^2) \quad (5.29)$$

and

(iii) if  $\mathbf{K} = \mathbf{0}$

$$\frac{k^2}{\omega^2} = \frac{1}{v^2} \left\{ 1 + i \left\{ \left( 1 - \frac{p^2}{v^2} \right) \vartheta_0^2 \Gamma \text{tr } \mathbf{C}^{-1} \mathbf{E} \right\} \frac{1}{\omega} \right\} + O(1/\omega^2). \quad (5.30)$$

The phase velocity and attenuation coefficients for each case can be read off from (5.28) through (5.30). In each case, the attenuation coefficients are positive for the wave with positive phase velocity.

Because high frequency nondispersive limits for progressive waves yield the same phase velocities and attenuation coefficients as occur with acceleration waves, (5.28) through (5.30) imply formulas for the decay of acceleration waves propagating in a mixture defined by (5.2) and (5.3).

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### References

1. TRUESDELL, C., Sulle Basi Della Thermomeccanica, Rend. Lincei (8) **22**, 33-38, 158-166 (1957).
2. COLEMAN, B., & W. NOLL, The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity, Arch. Rational Mech Anal., **13**, 176-178 (1963).
3. BOWEN, R. M., Theory of Mixtures. Continuum Physics, Vol. III (ed. A. C. Eringen). New York-London: Academic Press 1976.
4. TRUESDELL, C., & R. TOUPIN, The Classical Field Theories. Handbuch der Physik, Band III/1 (ed. S. Flügge). Berlin-Göttingen-Heidelberg: Springer 1960.
5. BOWEN, R. M., & D. J. GARCIA, On the Thermodynamics of Mixtures with Several Temperatures, Int. J. Eng. Sci., **8**, 63-83 (1970).
6. TRUESDELL, C., Rational Thermodynamics. New York: McGraw-Hill 1969.
7. BOWEN, R. M., Thermochemistry of Reacting Materials. J. Chem. Phys., **49**, 625-1637 (1968); **50**, 4601-4602 (1969).
8. TRUESDELL, C., & W. NOLL, The Nonlinear Field Theories of Mechanics, Handbuch der Physik, Band III/3 (ed. S. Flügge). Berlin-Heidelberg-New York: Springer 1965.
9. BOWEN, R. M., & K. M. REINICKE, Plane Progressive Waves in a Binary Mixture of Linear Elastic Materials, J. Appl. Mech. **45**, 493-499 (1978).
10. CHADWICK, P., Thermoelasticity, The Dynamic Theory, Progress in Solid Mechanics, Vol. 1 (ed. I. Sneddon and R. Hill). Amsterdam: North Holland 1960.

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