On the Quasi-static Approximation in Dynamic Linear Viscoelasticity

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1. Introduction and statement of the results

It is a difficult task to solve the equations of dynamic linear viscoelasticity and for that reason investigators often prefer to work with the simpler quasi-static equations obtained by ignoring inertia[1]. We may expect the solution of the quasi-static equations to approximate closely to the solution of the full equations if the traction changes slowly, but it is not clear what meaning we should assign to the qualifications 'closely' and 'slowly' and it is desirable to prove theorems which make them precise[2]. My purpose is to show that certain estimates on the kinetic energy, related to those I obtained in a study of the decay of energy [3], can be made to yield theorems of the required kind, at least when the body is one-dimensional and one of its ends is fixed while the other is subjected to a prescribed traction.

In order to state and prove the results we use the following conventions and notation. All functions of the time t are understood to be defined on the line $-\infty < t < \infty$, with the expectation of the creep function $k(t)$ which is defined only for $t \geqq 0$, and all functions of x and t are understood to be defined on the strip $0 \leqq x \leqq a$, $-\infty < t < \infty$. All relations which involve an order symbol O or o hold as $t \to \infty$. We denote derivatives with respect to x by writing x as a subscript and we denote derivatives with respect to t by a prime: thus $f_x(x, t) = \partial f(x, t)/\partial x$, $f'(x, t) = \partial f(x, t)/\partial t$, and $k'(t) = dk(t)/dt$. If $f(t)$ is continuous and bounded we write

$$\underline{M}(f) = \liminf_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds, \quad \overline{M}(f) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds$$

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[1] LEITMAN & FISHER's article [1] gives a thorough account of both the dynamic theory and the quasi-static approximation.

[2] DUVAUT & LIONS [2] (section 6.7) prove such a theorem for Kelvin-Voigt materials but there is no overlap between their theorem and my own results which, in their terminology, are concerned with materials with long memory.

for its lower and upper mean values, respectively, while if $\underline{M}(f) = \overline{M}(f)$ we denote the common value, called the mean value, by $M(f)$. If $f(x, t)$ is continuous we write

$$\underline{M}(f, x) = \liminf_{t\to\infty} \frac{1}{t} \int_0^t f(x, s)\, ds, \quad \overline{M}(f, x) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t f(x, s)\, ds$$

whenever these lower and upper limits are finite and if $\underline{M}(f, x) = \overline{M}(f, x)$ we denote the common value by $M(f, x)$.

We identify the body with the interval $0 \leqq x \leqq a$, we suppose that its density ϱ is a positive constant, and we suppose that its viscoelastic response is described by a creep function $k(t)$ which is C^2 and meets the requirements (i) there is a finite number, denoted by $k(\infty)$, such that $k(t) \to k(\infty)$ as $t \to \infty$, (ii) $k'(t) \to 0$ as $t \to \infty$, (iii) $k(t) > 0$, $k'(t) > 0$, $k''(t) < 0$, (iv) the integral $\varkappa = \int_0^\infty (k(\infty) - k(t)\, dt$ converges, (v) $\log (k(\infty) - k(t))$ is a convex function of t.

We note that

$$\int_0^\infty k'(t)\, dt = k(\infty) - k(0), \quad \int_0^\infty |k''(t)|\, dt = k'(0),$$

that

$$0 \leqq (k(\infty) - k(t))^2 \frac{d^2}{dt^2} \log (k(\infty) - k(t)) = (k(\infty) - k(t))\, |k''(t)| - k'(t)^2$$

and, hence, that

$$\int_0^\infty \frac{k'(t)^2}{|k''(t)|}\, dt \leqq \varkappa.$$

The requirements (i), (ii), (iii), (iv), and (v) say, roughly speaking, that $k(t)$ is a positive and concave function which increases steadily to the finite equilibrium value $k(\infty)$. They allow $k(t)$ to be of the form

$$\nu - \sum_{n=1}^{N} \lambda_n \exp(-\mu_n t),$$

where the λ's and μ's are positive constants and $\nu > \sum_{n=1}^{N} \lambda_n$, but they prohibit purely elastic reponse ($k(t)$ identically constant) and, as we shall see, there is good reason why they should do so.

We denote the displacement by $u(x, t)$ and the stress by $\sigma(x, t)$ and we suppose that the strain $\varepsilon(x, t) = u_x(x, t)$, the velocity $u'(x, t)$, and the stress $\sigma(x, t)$ are C^3 functions, that the stress and the stress-rate are uniformly bounded on the interval $t \leqq 0$ *i.e.* there is a constant δ such that

$$|\sigma(x, t)| + |\sigma'(x, t)| < \delta \quad (0 \leqq x \leqq a, \quad t \leqq 0),$$

and that the equation of motion

$$\sigma_x = \varrho u'',$$

the creep law

$$\varepsilon(x, t) = k(0)\,\sigma(x, t) + \int_{-\infty}^{t} k'(t-s)\,\sigma(x, s)\,ds,$$

and the boundary conditions

$$u(0, t) = 0, \quad \sigma(a, t) = \tau(t),$$

are satisfied.

The hypotheses on the creep function and on the stress history ensure the convergence of the improper integral which appears in the creep law; they also ensure the validity of a number of differentiations of improper integrals which we shall need to perform.

We note that the traction must satisfy $|\tau(t)| + |\tau'(t)| < \delta$ $(t \leqq 0)$.

In the quasi-static approximation, the displacement $\hat{u}(x, t)$, the strain $\hat{\varepsilon}(x, t) = \hat{u}_x(x, t)$, and the stress $\hat{\sigma}(x, t)$ are determined by replacing the equation of motion by the equation

$$\hat{\sigma}_x = 0,$$

and retaining the creep law

$$\hat{\varepsilon}(x, t) = k(0)\,\hat{\sigma}(x, t) + \int_{-\infty}^{t} k'(t-s)\,\hat{\sigma}(x, s)\,ds,$$

and the boundary conditions

$$\hat{u}(0, t) = 0, \quad \hat{\sigma}(a, t) = \tau(t).$$

We can integrate these equations immediately and we find that

$$\hat{\sigma}(x, t) = \tau(t),$$

$$\hat{\varepsilon}(x, t) = k(0)\,\tau(t) + \int_{-\infty}^{t} k'(t-s)\,\tau(s)\,ds,$$

$$\hat{u}(x, t) = x\left(k(0)\,\tau(t) + \int_{-\infty}^{t} k'(t-s)\,\tau(s)\,ds\right).$$

Our task is to examine the extent to which $\hat{\sigma}(x, t)$, $\hat{\varepsilon}(x, t)$, and $\hat{u}(x, t)$ approximate $\sigma(x, t)$, $\varepsilon(x, t)$, and $u(x, t)$, respectively, on the interval $t \geqq 0$. Theorems 1 and 2 say that under mild restrictions on $\tau(t)$ the quasi-static approximation always delivers the correct mean values of $\sigma(x, t)$, $\varepsilon(x, t)$, and $u(x, t)$, whatever the stress history prior to $t = 0$.

Theorem 1. *If $\tau(t)$, $\tau'(t)$, and $\tau''(t)$ are bounded on the interval $t \geqq 0$, then, for each x in $0 \leqq x \leqq a$,*

$$\underline{M}(\sigma, x) = \underline{M}(\hat{\sigma}, x) = \underline{M}(\tau),$$

$$\underline{M}(\varepsilon, x) = \underline{M}(\hat{\varepsilon}, x) = k(\infty)\,\underline{M}(\tau),$$

$$\underline{M}(u, x) = \underline{M}(\hat{u}, x) = xk(\infty)\,\underline{M}(\tau),$$

and the equations obtained from these by replacing $\underline{M}$ with $\overline{M}$ also hold.

Theorem 2 is an obvious corollary of theorem 1; since it asserts the existence of the mean values of σ, ε, and u it can be regarded as an ergodic theorem.

Theorem 2. *If, in addition, the mean value $M(\tau)$ exists, then the mean values $M(\sigma, x)$, $M(\varepsilon, x)$, $M(u, x)$, $M(\hat{\sigma}, x)$, $M(\hat{\varepsilon}, x)$, and $M(\hat{u}, x)$ exist for each x in $0 \leqq x \leqq a$, and*

$$M(\sigma, x) = M(\hat{\sigma}, x) = M(\tau),$$

$$M(\varepsilon, x) = M(\hat{\varepsilon}, x) = k(\infty)\, M(\tau),$$

$$M(u, x) = M(\hat{u}, x) = xk(\infty)\, M(\tau).$$

In order for theorems 1 and 2 to hold we must restrict the creep function so as to exclude elastic bodies. To see this we observe that if $k(t)$ is identically equal to a positive constant λ the stress $\sigma = \varepsilon/\lambda = u_x/\lambda$, and the displacement satisfies the wave equation $c^2 u_{xx} = u''\left(c = 1/\sqrt{\varrho\lambda}\right)$ and the boundary conditions $u(0, t) = 0$, $u_x(a, t) = \lambda\tau(t)$. To be consistent with our hypotheses $u(x, t)$ must be C^4, there must be a constant δ such that

$$|u_x(x, t)| + |u_x'(x, t)| < \lambda\delta \quad (0 \leqq x \leqq a, t \leqq 0),$$

and $\tau(t)$, $\tau'(t)$, and $\tau''(t)$ must be bounded for $t \geqq 0$. However, it is not difficult to construct such a $u(x, t)$ and, at the same time, arrange that $M(\tau)$ exists while $M(u, x)$ exists only at $x = 0$, in conflict with theorem 2. An example is provided by taking

$$u(x, t) = f(ct + a + x) - f(ct + a - x)$$

where $f(x) = 0$ $(x \leqq 0)$ and

$$f(x) = 27\left(\sin\frac{\pi x}{2a} - \frac{\pi x}{2a}\cos\frac{\pi x}{2a}\right) - \left(\sin\frac{3\pi x}{2a} - \frac{3\pi x}{2a}\cos\frac{3\pi x}{2a}\right) \quad (x \geqq 0)$$

The function $f(x)$ has continuous derivatives of order 4 even at $x = 0$ and it is easily checked that $u(x, t)$ satisfies the requirements set out earlier in this paragraph. Moreover,

$$\lambda\tau(t) = f'(ct + 2a) + f'(ct) = \frac{9\pi^2}{2a}\left(\sin\frac{3\pi ct}{2a} - 3\sin\frac{\pi ct}{2a}\right) \quad (t \geqq 0)$$

and so $M(\tau) = 0$. On the other hand, a straightforward calculation shows that

$$\frac{1}{t}\int_0^t u(x, s)\, ds = 27\sin\frac{\pi ct}{2a}\sin\frac{\pi x}{2a} + \sin\frac{3\pi ct}{2a}\sin\frac{3\pi x}{2a} + o(1),$$

and, thus, that $M(u, x)$ exists only when $x = 0$. Of course, the failure of theorem 2 in this instance is to be ascribed to resonance; the traction has period $4a/c$, which is a characteristic period of the elastic free vibration problem: $c^2 u_{xx} = u''$, $u(0, t) = 0$, $u_x(a, t) = 0$.

Our third and fourth theorems require somewhat strengthened hypotheses

Theorem 3. *Suppose that $\varepsilon(x, t)$, $u'(x, t)$, and $\sigma(x, t)$ are C^4, suppose that there is a constant δ such that*

$$|\sigma(x, t)| + |\sigma'(x, t)| + |\sigma''(x, t)| < \delta \quad (0 \leqq x \leqq a, t \leqq 0)$$

and suppose that $\tau'(t)$, $\tau''(t)$, and $\tau'''(t)$ are bounded on the interval $t \geqq 0$. Then, for every x in $0 \leqq x \leqq a$,

$$\overline{M}((\hat{\sigma} - \sigma)^2, x) \leqq \gamma \varrho^2 a^2 (a - x)\,(k'(0)^2\, \overline{M}(\tau'^2) + \varkappa^2 \overline{M}(\tau'''^2)),$$

$$\overline{M}((\hat{\varepsilon} - \varepsilon)^2, x) \leqq k(\infty)^2\, \gamma \varrho^2 a^3 (a - x)\,(k'(0)^2\, \overline{M}(\tau'^2) + \varkappa^2 \overline{M}(\tau'''^2)),$$

$$\overline{M}((\hat{u} - u)^2, x) \leqq k(\infty)^2\, \gamma \varrho^2 a^3 x (a x - \tfrac{1}{2} x^2)\,(k'(0)^2\, \overline{M}(\tau'^2) + \varkappa^2 \overline{M}(\tau'''^2)),$$

where γ is the dimensionless constant $32(k(\infty)^2 + k(0)^2)^2/\pi^2(k(\infty) - k(0))^4$.

In the light of theorems 1 and 2, it is natural to interpret the upper mean value $\overline{M}((\hat{\sigma} - \sigma)^2, x)$ as a variance and this interpretation suggests a corollary to theorem 3 which resembles Tchebychev's inequality in probability theory.

Given any large number A, let $S(A, x)$ be the set $\{t : t \geqq 0 \text{ and } |\hat{\sigma}(x, t) - \sigma(x, t)| > A\}$. If m is Lebesgue measure we can regard

$$\frac{1}{t}\, m(S(A, x) \cap [0, t]) = \frac{m(S(A, x) \cap [0, t])}{m([0, t])}$$

as the fraction of time in the interval $[0, t]$ at which $\hat{\sigma}$ and σ differ at x by more than A, and we can regard

$$F(A, x) = \limsup_{t \to \infty} \frac{1}{t}\, m(S(A, x) \cap [0, t])$$

as the fraction of the total time at which $\hat{\sigma}$ and σ differ at x by more than A. Since

$$\int_0^t (\hat{\sigma}(x, s) - \sigma(x, s))^2\, ds \geqq \int_{S(A,x) \cap [0,t]} (\hat{\sigma}(x, s) - \sigma(x, s))^2\, ds \geqq A^2 m(S(A, x) \cap [0, t])$$

we deduce, on dividing by t and letting $t \to \infty$, the inequality

$$\overline{M}((\hat{\sigma} - \sigma)^2, x) \geqq A^2 F(A, x)$$

which provides an upper bound for the fraction $F(A, x)$.

Theorem 4. *The fraction $F(A, x)$ does not exceed*

$$\gamma \varrho^2 a^3 (a - x)\,(k'(0)^2\, \overline{M}(\tau'^2) + \varkappa^2\, \overline{M}(\tau'''^2))/A^2.$$

Clearly, there are two further results of this type, one for the strain and one for the displacement, which can be proved in just the same way.

2. Proof of theorem 1

The chief difficulty in proving theorem 1 is the technical one of showing that the kinetic energy

$$K(t) = \frac{1}{2}\varrho \int_0^a u'(x, t)^2\, dx$$

is $O(t)$. If, for the moment, we take it that this estimate[3] is correct we can complete the proof in the following way.

First, we integrate the equation of motion $\sigma_x = \varrho u''$ and we appeal to the boundary condition $\sigma(a, t) = \tau(t)$ and find the equation

$$\sigma(x, t) = \tau(t) - \varrho \int_x^a u''(y, t)\, dy \tag{1}$$

for the stress.

A further integration tells us that

$$\int_0^t \sigma(x, s)\, ds = \int_0^t \tau(s)\, ds + \varrho \int_x^a (u'(y, 0) - u'(y, t))\, dy$$

$$= \int_0^t \tau(s)\, ds - \varrho \int_x^a u'(y, t)\, dy + O(1),$$

where, according to the Schwarz inequality,

$$\left(\int_x^a u'(y, t)\, dy\right)^2 \leqq (a - x) \int_x^a u'(y, t)^2\, dy \leqq a \int_0^a u'(y, t)^2\, dy = \frac{2a}{\varrho} K(t) = O(t).$$

In this way we arrive at the order relation[3]

$$\frac{1}{t} \int_0^t \sigma(x, s)\, ds = \frac{1}{t} \int_0^t \tau(s)\, ds + o(1),$$

which, since $\hat{\sigma}(x, t) = \tau(t)$, is enough to prove the equations $\underline{M}(\sigma, x) = \underline{M}(\hat{\sigma}, x) = \underline{M}(\tau)$ and the equations obtained by replacing $\underline{M}$ with $\overline{M}$.

Next, we write the creep law as

$$\varepsilon(x, t) = k(0)\, \sigma(x, t) + \int_0^t k'(t - s)\, \sigma(x, s)\, ds + \eta(x, t),$$

where

$$\eta(x, t) = \int_{-\infty}^0 k'(t - s)\, \sigma(x, s)\, ds,$$

[3] From our point of view, this estimate represents the crucial difference between the viscoelastic body and the elastic body. As the example considered in section 1 shows, the kinetic energy of an elastic body subjected to resonant traction behaves in such a way that $\limsup_{t\to\infty} K(t)/t^2$ is positive.

and, in view of the hypothesis on the stress history,

$$|\eta(x,t)| \leqq \delta \int_{-\infty}^{0} k'(t-s)\,ds = \delta(k(\infty)-k(t)).$$

An integration tells us that

$$\int_0^t \varepsilon(x,s)\,ds = k(0)\int_0^t \sigma(x,s)\,ds + \int_0^t\int_0^s k'(s-\xi)\,\sigma(x,\xi)\,d\xi\,ds + \int_0^t \eta(x,s)\,ds$$

and when we interchange the orders of integration in the double integral we find that

$$\int_0^t \varepsilon(x,s)\,ds = \int_0^t k(t-s)\,\sigma(x,s)\,ds + \int_0^t \eta(x,s)\,ds.$$

We now substitute for the stress from (1) and integrate by parts and we get

$$\begin{aligned}\int_0^t \varepsilon(x,s)\,ds &= \int_0^t k(t-s)\,\tau(s)\,ds - \varrho\int_0^t\int_x^a k(t-s)\,u''(y,s)\,dy\,ds + \int_0^t \eta(x,s)\,ds\\ &= \int_0^t k(t-s)\,\tau(s)\,ds - \varrho k(0)\int_x^a u'(y,t)\,dy + \varrho k(t)\int_x^a u'(y,0)\,dy\\ &\quad - \varrho\int_0^t\int_x^a k'(t-s)\,u'(y,s)\,dy\,ds + \int_0^t \eta(x,s)\,ds\end{aligned}$$

and, hence,

$$\int_0^t \varepsilon(x,s)\,ds = k(\infty)\int_0^t \tau(s)\,ds + \theta(x,t),$$

where

$$\begin{aligned}\theta(x,t) = \int_0^t (k(t-s)-k(\infty)\,\tau(s))\,ds - \varrho k(0)\int_x^a u'(y,t)\,dy\\ + \varrho k(t)\int_x^a u'(y,0)\,dy - \varrho\int_0^t\int_x^a k'(t-s)\,u'(y,s)\,dy\,ds + \int_0^t \eta(x,s)\,ds.\end{aligned}$$

If we choose a constant B such that $|\tau(t)| \leqq B$ $(t \geqq 0)$ and estimate each of the terms in the sum which defines $\theta(x,t)$ we find that

$$\begin{aligned}|\theta(x,t)| &\leqq B\int_0^t (k(\infty)-k(s))\,ds + k(0)\,(2a\varrho K(t))^{\frac{1}{2}} + k(t)\,(2a\varrho K(0))^{\frac{1}{2}}\\ &\quad + (2a\varrho)^{\frac{1}{2}}\int_0^t k'(t-s)\,K(s)^{\frac{1}{2}}\,ds + \delta\int_0^t (k(\infty)-k(s))\,ds\\ &\leqq \varkappa(B+\delta) + k(0)\,(2a\varrho K(t))^{\frac{1}{2}} + k(\infty)\,(2a\varrho K(0))^{\frac{1}{2}}\\ &\quad + (2a\varrho)^{\frac{1}{2}}\int_0^t k'(t-s)\,K(s)^{\frac{1}{2}}\,ds = O(t^{\frac{1}{2}}) + (2a\varrho)^{\frac{1}{2}}\int_0^t k'(t-s)\,K(s)^{\frac{1}{2}}\,ds.\end{aligned}$$

Since

$$\int_0^t k'(t-s)\, s^{\frac{1}{2}}\, ds \leqq t^{\frac{1}{2}} \int_0^t k'(t-s)\, ds = (k(t) - k(0))\, t^{\frac{1}{2}} \leqq (k(\infty) - k(0))\, t^{\frac{1}{2}}$$

we have

$$\int_0^t k'(t-s)\, K(s)^{\frac{1}{2}} ds = O\left(\int_0^t k'(t-s)\, s^{\frac{1}{2}}\, ds\right) = O(t^{\frac{1}{2}})$$

and we have shown that $\theta(x, t) = O(t^{\frac{1}{2}})$ uniformly in x, and, hence, that

$$\frac{1}{t} \int_0^t \varepsilon(x, s)\, ds = k(\infty) \frac{1}{t} \int_0^t \tau(s)\, ds + o(1)$$

uniformly in x. This order relation implies the equations $\underline{M}(\varepsilon, x) = k(\infty)\, \underline{M}(\tau)$, $\overline{M}(\varepsilon, x) = k(\infty)\, \overline{M}(\tau)$ and, moreover, if we integrate the order relation and appeal to its uniformity and to the boundary condition $u(0, t) = 0$ we find that

$$\frac{1}{t} \int_0^t u(x, s)\, ds = x k(\infty) \frac{1}{t} \int_0^t \tau(s)\, ds + o(1)$$

and, hence, that $\underline{M}(u, x) = x k(\infty)\, \underline{M}(\tau)$, $\overline{M}(u, x) = x k(\infty)\, \overline{M}(\tau)$.

On the other hand, the quasi-static strain can be written as

$$\hat{\varepsilon}(x, t) = k(0)\, \tau(t) + \int_0^t k'(t-s)\, \tau(s)\, ds + \hat{\eta}(t),$$

where

$$\hat{\eta}(t) = \int_{-\infty}^0 k'(t-s)\, \tau(s)\, ds$$

and

$$|\hat{\eta}(t)| \leqq \delta \int_{-\infty}^0 k'(t-s)\, ds = \delta(k(\infty) - k(t)).$$

Thus

$$\begin{aligned}\int_0^t \hat{\varepsilon}(x, s)\, ds &= k(0) \int_0^t \tau(s)\, ds + \int_0^t \int_0^s k'(s-\xi)\, \tau(\xi)\, d\xi + \int_0^t \hat{\eta}(s)\, ds \\ &= \int_0^t k(t-s)\, \tau(s)\, ds + \int_0^t \hat{\eta}(s)\, ds,\end{aligned}$$

and

$$\int_0^t \hat{\varepsilon}(x, s)\, ds - k(\infty) \int_0^t \tau(s)\, ds = \int_0^t (k(t-s) - k(\infty))\, \tau(s)\, ds + \int_0^t \hat{\eta}(s)\, ds.$$

Accordingly,

$$\left|\int_0^t \hat{\varepsilon}(x, s)\, ds - k(\infty) \int_0^t \tau(s)\, ds\right| \leqq (B + \delta) \int_0^t (k(\infty) - k(s))\, ds \leqq \varkappa(B + \delta)$$

and, since $\hat{u}(x,t) = x\hat{\varepsilon}(x,t)$,

$$\left| \int_0^t \hat{u}(x,s)\,ds - xk(\infty) \int_0^t \tau(s)\,ds \right| \leqq x\varkappa(B+\delta).$$

It follows that

$$\frac{1}{t}\int_0^t \hat{\varepsilon}(x,s)\,ds = k(\infty)\frac{1}{t}\int_0^t \tau(s)\,ds + o(1),$$

$$\frac{1}{t}\int_0^t \hat{u}(x,s)\,ds = xk(\infty)\frac{1}{t}\int_0^t \tau(s)\,ds + o(1),$$

and we deduce the equations $\underline{M}(\hat{\varepsilon}, x) = k(\infty)\,\underline{M}(\tau)$, $\underline{M}(\hat{u}, x) = xk(\infty)\,\underline{M}(\tau)$ and the corresponding equations obtained by replacing $\underline{M}$ with $\overline{M}$. In short, the theorem is proved once we have established that the kinetic energy is $O(t)$ and we turn to establishing that fact.

We introduce the functions

$$K_1(t) = \frac{1}{2}\varrho \int_0^a u''(x,t)^2\,dx,$$

$$\psi(x,t) = \frac{1}{2}k(0)\,\sigma(x,t)^2 + \frac{1}{2}\int_{-\infty}^t k'(t-s)\,\sigma(x,s)^2\,ds,$$

$$\psi_1(x,t) = \frac{1}{2}k(0)\,\sigma'(x,t)^2 + \frac{1}{2}\int_{-\infty}^t k'(t-s)\,\sigma'(x,s)^2\,ds,$$

$$\Psi(t) = \int_0^a \psi(x,t)\,dx, \quad \Psi_1(t) = \int_0^a \psi_1(x,t)\,dx,$$

$$E(t) = K(t) + \Psi(t), \quad E_1(t) = K_1(t) + \Psi_1(t),$$

of which $\Psi(t)$ is a free energy, $E(t)$ is the total energy, and $E_1(t)$ is a higher-order energy. These functions and the kinetic energy have the properties:

$$K \leqq \frac{2a^2\varrho}{\pi^2}\int_0^a \varepsilon'^2\,dx, \tag{2}$$

$$u'(a,t)^2 \leqq a\int_0^a \varepsilon'(x,t)^2\,dx \leqq 2ak(\infty)\,E_1(t), \tag{3}$$

$$\tau(t)\,u'(a,t) = K'(t) + \int_0^a \sigma(x,t)\,\varepsilon'(x,t)\,dx, \tag{4}$$

$$\tau'(t)\,u''(a,t) = K_1'(t) + \int_0^a \sigma'(x,t)\,\varepsilon''(x,t)\,dx, \tag{5}$$

$$\int_0^a \sigma\varepsilon'\,dx \geqq \Psi' + \frac{1}{2k'(0)}\int_0^a (\varepsilon' - k(0)\,\sigma')^2\,dx, \tag{6}$$

$$\int_0^a \sigma'\varepsilon''\,dx \geqq \Psi_1' + \frac{1}{2\varkappa}\int_0^a (\varepsilon' - k(\infty)\,\sigma')^2\,dx. \tag{7}$$

In order to prove (2) we need only note that the boundary condition $u(0, t) = 0$ implies that $u'(0, t) = 0$. A known theorem[4] then implies the inequality

$$\int_0^a u'^2\,dx \leqq \frac{4a^2}{\pi^2}\int_0^a u_x'^2\,dx = \frac{4a^2}{\pi^2}\int_0^a \varepsilon'^2\,dx,$$

which is equivalent to (2).

Next we observe that our hypotheses permit us to differentiate the creep law and when we do so we find that

$$\varepsilon'(x, t) = k(0)\,\sigma'(x, t) + k'(0)\,\sigma(x, t) + \int_{-\infty}^{t} k''(t-s)\,\sigma(x, s)\,ds, \tag{8}$$

and, on integrating by parts, that

$$\varepsilon'(x, t) = k(0)\,\sigma'(x, t) + \int_{-\infty}^{t} k'(t-s)\,\sigma'(x, s)\,ds, \tag{9}$$

The Schwarz inequality implies that

$$\begin{aligned}(\varepsilon'(x, t) - k(0)\,\sigma'(x, t))^2 &\leqq \int_{-\infty}^{t} k'(t-s)\,ds \int_{-\infty}^{t} k'(t-s)\,\sigma'(x, s)^2\,ds\\ &= (k(\infty) - k(0))\int_{-\infty}^{t} k'(t-s)\,\sigma'(x, s)^2\,ds\\ &= 2(k(\infty) - k(0))\left(\psi_1(x, t) - \frac{1}{2}k(0)\,\sigma'(x, t)^2\right)\end{aligned}$$

and, therefore, we have

$$\begin{aligned}\psi_1 &\geqq \frac{1}{2}k(0)\,\sigma'^2 + \frac{(\varepsilon' - k(0)\,\sigma')^2}{2(k(\infty) - k(0))}\\ &= \frac{1}{2k(\infty)}\varepsilon'^2 + \frac{k(0)\,(\varepsilon' - k(\infty)\,\sigma')^2}{2k(\infty)\,(k(\infty) - k(0))}\\ &\geqq \frac{1}{2k(\infty)}\varepsilon'^2,\end{aligned}$$

and when we integrate with respect to x we arrive at the inequalities

$$\int_0^a \varepsilon'^2\,dx \leqq 2k(\infty)\,\Psi_1 \leqq 2k(\infty)\,E_1.$$

On the other hand, the fact that $u'(0, t) = 0$ tells us that

$$u'(a, t) = \int_0^a \varepsilon'(x, t)\,dx$$

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[4] See, for example, HARDY, LITTLEWOOD &, PÓLYA [4], theorem 256.

and a further appeal to the Schwarz inequality yields

$$u'(a,t)^2 \leqq a \int_0^a \varepsilon'(x,t)^2\,dx,$$

and we have proved (3).

Equation (4) is a standard result: we prove it by multiplying the equation of motion $\sigma_x = \varrho u''$ through by u' to get the equation

$$(\sigma u')_x = \varrho u' u'' + \sigma \varepsilon'$$

and we then integrate with respect to x and use the boundary conditions.

We prove equation (5) in almost the same way: we differentiate the equation of motion with respect to t and multiply through by u'' to get the equation

$$(\sigma' u'')_x = \varrho u'' u''' + \sigma' \varepsilon''$$

and then we integrate with respect to x and use the boundary conditions again.

To prove (6) we differentiate the equation which defines $\psi(x,t)$ and find that

$$\psi'(x,t) = k(0)\,\sigma(x,t)\,\sigma'(x,t) + \frac{1}{2}k'(0)\,\sigma(x,t)^2 + \frac{1}{2}\int_{-\infty}^{t} k''(t-s)\,\sigma(x,s)^2\,ds$$

and it is a straightforward matter to deduce, with the help of the formula (8) for $\varepsilon'(x,t)$, that

$$\sigma(x,t)\,\varepsilon'(x,t) = \psi'(x,t) + \frac{1}{2}\int_{-\infty}^{t} |k''(t-s)|\,(\sigma(x,s)-\sigma(x,t))^2\,ds.$$

It also follows from (8) that

$$\varepsilon'(x,t) - k(0)\,\sigma'(x,t) = \int_{-\infty}^{t} k''(t-s)\,(\sigma(x,s)-\sigma(x,t))\,ds$$

and, thus, the Schwarz inequality implies that

$$\begin{aligned}(\varepsilon'(s,t) - k(0)\,\sigma'(x,t))^2 &\leqq \int_{-\infty}^{t} |k''(t-s)|\,ds \int_{-\infty}^{t} |k''(t-s)|\,(\sigma(x,s)-\sigma(x,t))^2\,ds \\ &= k'(0)\int_{-\infty}^{t} |k''(t-s)|\,(\sigma(x,s)-\sigma(x,t))^2\,ds \\ &= 2k'(0)\,(\sigma(x,t)\,\varepsilon'(x,t) - \psi'(x,t)).\end{aligned}$$

In other words,

$$\sigma\varepsilon' \geqq \psi' + \frac{1}{2k'(0)}(\varepsilon' - k(0)\,\sigma')^2$$

and we arrive at (6) when we integrate with respect to x.

To prove (7) we differentiate the formula (9) for $\varepsilon'(x,t)$ and the equation which defines $\psi_1(x,t)$ and obtain the equations

$$\varepsilon''(x,t) = k(0)\,\sigma''(x,t) + k'(0)\,\sigma'(x,t) + \int_{-\infty}^{t} k''(t-s)\,\sigma'(x,s)\,ds,$$

$$\psi_1'(x,t) = k(0)\,\sigma'(x,t)\,\sigma''(x,t) + \frac{1}{2}k'(0)\,\sigma'(x,t)^2 + \frac{1}{2}\int_{-\infty}^{t} k''(t-s)\,\sigma'(x,s)^2\,ds,$$

from which the identity

$$\sigma'(x,t)\,\varepsilon''(x,t) = \psi_1'(x,t) + \frac{1}{2}\int_{-\infty}^{t} |k''(t-s)|\,(\sigma'(x,s)-\sigma'(x,t))^2\,ds$$

follows. On the other hand, the formula (9) also tells us that

$$\varepsilon'(x,t) - k(\infty)\,\sigma'(x,t) = \int_{-\infty}^{t} k'(t-s)\,(\sigma'(x,s)-\sigma'(x,t))\,ds$$

and if we use the Schwarz inequality we see that

$$\begin{aligned}(\varepsilon'(x,t)-k(\infty)\,\sigma'(x,t))^2 &\leqq \int_{-\infty}^{t}\frac{k'(t-s)^2}{|k''(t-s)|}\,ds\int_{-\infty}^{t}|k''(t-s)|\,(\sigma'(x,s)-\sigma'(x,t))^2\,ds\\ &= \int_{0}^{\infty}\frac{k'(s)^2}{|k''(s)|}\,ds\int_{-\infty}^{t}|k''(t-s)|\,(\sigma'(x,s)-\sigma'(x,t))^2\,ds\\ &\leqq \varkappa\int_{-\infty}^{t}|k''(t-s)|\,(\sigma'(x,s)-\sigma'(x,t))^2\,ds\\ &= 2\varkappa(\sigma'(x,t)\,\varepsilon''(x,t)-\psi_1'(x,t))\end{aligned}$$

and, therefore, that

$$\sigma'\varepsilon'' \geqq \psi_1' + \frac{1}{2\varkappa}(\varepsilon' - k(\infty)\,\sigma')^2.$$

An integration with respect to x now leads us to conclude (7).

In order to prove the required estimate for the kinetic energy we combine (4) with (6) and obtain the inequality

$$\tau(t)\,u'(a,t) \geqq E'(t) + \frac{1}{2k'(0)}\int_0^a (\varepsilon'(x,t)-k(0)\,\sigma'(x,t))^2\,dx.$$

Thus

$$E(t) + \frac{1}{2k'(0)}\int_0^t\int_0^a (\varepsilon'(x,s)-k(0)\,\sigma'(x,s))^2\,dx\,ds \leqq E(0) + \int_0^t \tau(s)\,u'(a,s)\,ds,$$

where the Schwarz inequality and the inequality (3) imply

$$\begin{aligned}\int_0^t \tau(s)\,u'(a,s)\,ds &\leqq \left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t u'(a,s)^2\,ds\right)^{\frac{1}{2}}\\ &\leqq a^{\frac{1}{2}}\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}},\end{aligned}$$

and so

$$\begin{aligned}E(t) &+ \frac{1}{2k'(0)}\int_0^t\int_0^a (\varepsilon'(x,s)-k(0)\,\sigma'(x,s))^2\,dx\,ds\\ &\leqq E(0) + a^{\frac{1}{2}}\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}. \qquad (10)\end{aligned}$$

Next, we combine (5) with (7) and obtain the inequality

$$\tau'(t)\,u''(a,t) \geqq E_1'(t) + \frac{1}{2\varkappa}\int_0^a (\varepsilon'(x,t) - k(\infty)\,\sigma'(x,t))^2\,dx$$

and, hence, the inequality

$$E_1(t) + \frac{1}{2\varkappa}\int_0^t\int_0^a (\varepsilon'(x,s) - k(\infty)\,\sigma'(x,s))^2\,dx\,ds \leqq E_1(0) + \int_0^t \tau'(s)\,u''(a,s)\,ds. \tag{11}$$

Now

$$\int_0^t \tau'(s)\,u''(a,s)\,ds = \tau'(t)\,u'(a,t) - \tau'(0)\,u'(a,0) - \int_0^t \tau''(s)\,u'(a,s)\,ds,$$

where

$$-\int_0^t \tau''(s)\,u'(a,s)\,ds \leqq a^{\frac{1}{2}}\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}$$

and, according to (3) and the arithmetic-geometric mean inequality,

$$\tau'(t)\,u'(a,t) \leqq \frac{1}{2}\,ak(\infty)\,\tau'(t)^2 + \frac{1}{2ak(\infty)}\,u'(a,t)^2 \leqq \frac{1}{2}\,ak(\infty)\,\tau'(t)^2 + E_1(t).$$

This last inequality permits us to cancel a term $E_1(t)$ from each side of (11) to get

$$\frac{1}{2\varkappa}\int_0^t\int_0^a (\varepsilon'(x,s) - k(\infty)\,\sigma'(x,s))^2\,dx\,ds \leqq E_1(0) - \tau'(0)\,u'(a,0) + \frac{1}{2}\,ak(\infty)\tau'(t)^2$$
$$+ a^{\frac{1}{2}}\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}. \tag{12}$$

We use the inequalities (10) and (12) in the following way. Since

$$(k(\infty) - k(0))\,\varepsilon' = k(\infty)\,(\varepsilon' - k(0)\,\sigma') - k(0)\,(\varepsilon' - k(\infty)\,\sigma'),$$

the Cauchy-Schwarz inequality tells us that

$$(k(\infty) - k(0))^2\,\varepsilon'^2 \leqq (k(\infty)^2 + k(0)^2)\,((\varepsilon' - k(0)\,\sigma')^2 + (\varepsilon' - k(\infty)\,\sigma')^2)$$

and, hence, that

$$\frac{(k(\infty) - k(0)^2)}{k(\infty)^2 + k(0)^2}\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds$$
$$\leqq 2k'(0)\left(E(0) + a^{\frac{1}{2}}\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}\right)$$
$$+ 2\varkappa\left(E_1(0) - \tau'(0)\,u'(a,0) + \frac{1}{2}\,ak(\infty)\,\tau'(t)^2\right.$$
$$\left.+ a^{\frac{1}{2}}\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}\right).$$

In other words, we have proved that

$$\int_0^t \int_0^a \varepsilon'(x,s)^2\,dx\,ds$$
$$\leqq \left(c_1\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}} + c_2\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\right)\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}} + c_3 + c_4\tau'(t)^2,$$

where c_1, c_2, c_3, and c_4 are known constants. However, the arithmetic-geometric mean inequality and the inequality $(p+q)^2 \leqq 2(p^2+q^2)$ tell us that

$$\left(c_1\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}} + c_2\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\right)\left(\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds\right)^{\frac{1}{2}}$$
$$\leqq \frac{1}{2}\left(c_1\left(\int_0^t \tau(s)^2\,ds\right)^{\frac{1}{2}} + c_2\left(\int_0^t \tau''(s)^2\,ds\right)^{\frac{1}{2}}\right)^2 + \frac{1}{2}\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds$$
$$\leqq c_1^2\int_0^t \tau(s)^2\,ds + c_2^2\int_0^t \tau''(s)^2\,ds + \frac{1}{2}\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds$$

and, thus, we have

$$\frac{1}{2}\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds \leqq c_1^2\int_0^t \tau(s)^2\,ds + c_2^2\int_0^t \tau''(s)^2\,ds + c_3 + c_4\tau'(t)^2. \tag{13}$$

Since $\tau(t)$, $\tau'(t)$, and $\tau''(t)$ are bounded on the interval $t \geqq 0$ we see that

$$\int_0^t\int_0^a \varepsilon'(x,s)^2\,dx\,ds = O(t)$$

and, therefore, that the right-hand side of (10) is $O(t)$. Thus $E(t) = O(t)$ and, since $K(t) \leqq E(t)$, we have $K(t) = O(t)$ and the proof of theorem 1 is complete.

3. Proof of theorem 3

In the course of proving theorem 1 we derived two subsidiary results which will enable us to prove theorem 3. Indeed, if we combine the inequalities (2) and (13) we find that the kinetic energy satisfies the estimate

$$\int_0^t K(s)\,ds \leqq \frac{4a^2\varrho}{\pi^2}\left(c_1^2\int_0^t \tau(s)^2\,ds + c_2^2\int_0^t \tau''(s)^2\,ds + c_3 + c_4\tau'(t)^2\right) \tag{14}$$

in which

$$k(\infty) - k(0))^2\,c_1 = 2a^{\frac{1}{2}}(k(\infty)^2 + k(0)^2)\,k'(0),$$
$$(k(\infty) - k(0))^2\,c_2 = 2a^{\frac{1}{2}}(k(\infty)^2 + k(0)^2)\,\varkappa,$$
$$(k(\infty) - k(0))^2\,c_3 = 2(k(\infty)^2 + k(0)^2)\,(k'(0)\,E(0) + \varkappa(E_1(0) - \tau'(0)\,u'(a,0)))$$
$$(k(\infty) - k(0))^2\,c_4 = a(k(\infty)^2 + k(0)^2)\,k(\infty)\,\varkappa.$$

Now let us suppose that the strengthened hypotheses of theorem 3 are in force. As we have seen, the equation of motion implies that $\sigma'_x = \varrho u'''$ and the strain-rate $\varepsilon'(x, t)$ is determined by the stress-rate $\sigma'(x, t)$ through the creep law (9). Thus, we can repeat our arguments but with u, ε, and σ replaced by u', ε', and σ', respectively, and in this way we shall arrive at an estimate which is a counterpart to (14) and will have the form

$$\int_0^t K_1(s)\,ds \leqq \frac{4a^2\varrho}{\pi^2}\left(c_1^2\int_0^t \tau'(s)^2\,ds + c_2^2\int_0^t \tau'''(s)^2\,ds + c_5 + c_4\tau''(t)^2\right)$$

in which the constant c_5 has replaced c_3. If we return to the formula (1) for the stress and recall that $\hat{\sigma}(x, t) = \tau(t)$, we find that

$$(\hat{\sigma}(x, t) - \sigma(x, t))^2 = \varrho^2\left(\int_x^a u''(y, t)^2\,dy\right) \leqq \varrho^2(a - x)\int_x^a u''(y, t)^2\,dy$$
$$\leqq 2\varrho(a - x)\,K_1(t)$$

and, hence, that

$$\int_0^t (\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds$$
$$\leqq \frac{8}{\pi^2}\varrho^2 a^2(a - x)\left(c_1^2\int_0^t \tau'(s)^2\,ds + c_2^2\int_0^t \tau'''(s)^2\,ds + c_5 + c_4\tau''(t)^2\right).$$

When we divide through by t and let $t \to \infty$ and use the fact that $\tau'(t)$, $\tau''(t)$, and $\tau'''(t)$ are bounded on the interval $t \geqq 0$ we see that $\overline{M}((\hat{\sigma} - \sigma)^2, x)$ cannot exceed

$$\frac{8}{\pi^2}\varrho^2 a^2(a - x)\,(c_1^2\overline{M}(\tau'^2) + c_2^2\overline{M}(\tau'''^2))$$

and this is just the advertised estimate for $\overline{M}(\hat{\sigma} - \sigma)^2, x)$.

To prove the estimate for $\overline{M}((\hat{\varepsilon} - \varepsilon)^2, x)$ we observe that the creep laws

$$\varepsilon(x, t) = k(0)\,\sigma(x, t) + \int_{-\infty}^t k'(t - s)\,\sigma(x, s)\,ds,$$

$$\hat{\varepsilon}(x, t) = k(0)\,\hat{\sigma}(x, t) + \int_{-\infty}^t k'(t - s)\,\hat{\sigma}(x, s)\,ds$$

imply that

$$\hat{\varepsilon}(x, t) - \varepsilon(x, t)$$
$$= k(0)\,(\hat{\sigma}(x, t) - \sigma(x, t)) + \int_0^t k'(t - s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds + \eta(x, t)$$

and that

$$(\hat{\varepsilon}(x, t) - \varepsilon(x, t) - \eta(x, t))^2 = k(0)^2\,(\hat{\sigma}(x, t) - \sigma(x, t))^2 + 2k(0)\,(\hat{\sigma}(x, t) - \sigma(x, t))$$
$$\times \int_0^t k'(t - s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds + \left(\int_0^t k'(t - s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds\right)^2,$$

where

$$\eta(x, t) = \int_{-\infty}^{0} k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds$$

and, since

$$|\hat{\sigma}(x, s) - \sigma(x, s)| = |\tau(s) - \sigma(x, s)| \leqq |\tau(s)| + |\sigma(x, s)| < 2\delta (s \leqq 0),$$

$$|\eta(x, t)| \leqq 2\delta \int_{-\infty}^{0} k'(t-s)\,ds = 2\delta(k(\infty) - k(t)),$$

and

$$\int_0^t \eta(x, s)^2\,ds \leqq 4\delta^2 \int_0^t (k(\infty) - k(s))^2\,ds \leqq 4\delta^2 k(\infty) \int_0^t (k(\infty) - k(s))\,ds$$
$$\leqq 4\delta^2 k(\infty)\,\varkappa.$$

The arithmetic-geometric mean inequality tells us that

$$2k(0)\,(\hat{\sigma}(x, t) - \sigma(x, t)) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds$$

$$= 2k(0) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,(\hat{\sigma}(x, t) - \sigma(x, t))\,ds$$

$$\leqq k(0) \int_0^t k'(t-s)\,((\hat{\sigma}(x, s) - \sigma(x, s))^2 + (\hat{\sigma}(x, t) - \sigma(x, t))^2)\,ds$$

$$= k(0) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds + k(0)\,(k(t) - k(0))\,(\hat{\sigma}(x, t) - \sigma(x, t))^2$$

$$\leqq k(0) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds + k(0)\,(k(\infty) - k(0))\,(\hat{\sigma}(x, t) - \sigma(x, t))^2$$

and the Schwarz inequality that

$$\left(\int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))\,ds\right)^2$$

$$\leqq \int_0^t k'(t-s)\,ds \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds$$

$$= (k(t) - k(0)) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds$$

$$\leqq (k(\infty) - k(0)) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds$$

and, thus, we have

$$(\hat{\varepsilon}(x, t) - \varepsilon(x, t) - \eta(x, t))^2 \leqq k(0)\,k(\infty)\,(\hat{\sigma}(x, t) - \sigma(x, t))^2$$
$$+ k(\infty) \int_0^t k'(t-s)\,(\hat{\sigma}(x, s) - \sigma(x, s))^2\,ds$$

and if we integrate and interchange the orders of integration in the resulting double integral we find that

$$\begin{aligned}
\int_0^t (\hat{\varepsilon}(x,s) - \varepsilon(x,s) - \eta(x,s))^2\,ds \\
&\leqq k(0)\,k(\infty) \int_0^t (\hat{\sigma}(x,s) - \sigma(x,s))^2\,ds \\
&\quad + k(\infty) \int_0^t \int_0^s k'(s-\xi)\,(\hat{\sigma}(x,\xi) - \sigma(x,\xi))^2\,d\xi\,ds \\
&= k(\infty) \int_0^t k(t-s)\,(\hat{\sigma}(x,s) - \sigma(x,s))^2\,ds \\
&\leqq k(\infty)^2 \int_0^t (\hat{\sigma}(x,s) - \sigma(x,s))^2\,ds.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\int_0^t (\hat{\varepsilon}(x,s) - \varepsilon(x,s))^2\,ds \\
&\quad = \int_0^t (\hat{\varepsilon}(x,s) - \varepsilon(x,s) - \eta(x,s))^2 + 2\eta(x,s)\,(\hat{\varepsilon}(x,s) - \varepsilon(x,s) - \eta(x,s)) \\
&\qquad + \eta(x,s)^2)\,ds \\
&\quad \leqq \int_0^t (\hat{\varepsilon}(x,s) - \varepsilon(x,s) - \eta(x,s))^2\,ds \\
&\qquad + 2\left(\int_0^t \eta(x,s)^2\,ds\right)^{\frac{1}{2}} \left(\int_0^t (\hat{\varepsilon}(x,s) - \varepsilon(x,s) - \eta(x,s))^2\,ds\right)^{\frac{1}{2}} + \int_0^t \eta(x,s)^2\,ds \\
&\quad \leqq k(\infty)^2 \int_0^t (\hat{\sigma}(x,s) - \sigma(x,s))^2\,ds + 4\delta k(\infty)^{\frac{3}{2}} \varkappa^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^t (\hat{\sigma}(x,s) - \sigma(x,s))^2\,ds\right)^{\frac{1}{2}} + 4\delta^2 k(\infty)\,\varkappa \\
&\quad \leqq k(\infty)^2 \frac{8}{\pi^2} \varrho^2 a^2 (a-x) \left(c_1^2 \int_0^t \tau'(s)^2\,ds + c_2^2 \int_0^t \tau'''(s)^2\,ds\right) + O(t^{\frac{1}{2}}) \qquad (15)
\end{aligned}$$

and we obtain the required estimate for $\overline{M}((\hat{\varepsilon} - \varepsilon)^2, x)$ on dividing through by t and letting $t \to \infty$.

Lastly, we note that, since $u(0,t) = \hat{u}(0,t) = 0$,

$$\hat{u}(x,t) - u(x,t) = \int_0^x (\hat{\varepsilon}(y,t) - \varepsilon(y,t))\,dy,$$

and, hence, the Schwarz inequality implies that

$$(\hat{u}(x,t) - u(x,t))^2 \leqq x \int_0^x (\hat{\varepsilon}(y,t) - \varepsilon(y,t))^2\,dy.$$

It is easily checked, however, that the order relation (15) holds uniformly in x and so

$$\int_0^t (u(x,s)-u(x,s))^2\,ds \leqq x\int_0^t\int_0^x (\hat{\varepsilon}(y,s)-\varepsilon(y,s))^2\,dy\,ds$$

$$= x\int_0^x\int_0^t (\hat{\varepsilon}(y,s)-\varepsilon(y,s))^2\,ds\,dy$$

$$\leqq \frac{8}{\pi^2}k(\infty)^2\varrho^2a^2x\left(ax-\frac{1}{2}x^2\right)\left(c_1^2\int_0^t \tau'(s)^2\,ds + c_2^2\int_0^t \tau'''(s)^2\,ds\right)+O(t^{\frac{1}{2}})$$

and this last estimate is enough to give us the stated bound on $\overline{M}((\hat{u}-u)^2, x)$.

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