Estimates of Harmonic Measure

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Communicated by J. C. C. NITSCHE

The object of this paper is a study of the relation between the harmonic measure of a set and its $(n-1)$-dimensional Hausdorff-measure, $n \geqq 2$. In this direction we have obtained the following result.

Theorem 1. *Let $D \subset R^n$ be a Lipschitz domain. Then a Borel measurable set $E \subset \partial D$ is of harmonic measure zero with respect to D if and only if E is of vanishing $(n-1)$-dimensional Hausdorff measure.*

The case $n=2$ has been settled in [10, p. 125], and the situation when D satisfies various additional conditions has been discussed in [2], [14], [15]. At the same time, even the case when D is a C^1-domain is new when $n>2$, as far as we know.

In [6] it is proved that if u is non-negative and harmonic in a Lipschitz domain D then u has a finite non-tangential limit at each point $Q \in \partial D$, except for a set of vanishing harmonic measure. Hence we have the following consequence of Theorem 1.

Theorem 2. *Suppose u is non-negative and harmonic in a Lipschitz domain D. Then u has a finite non-tangential limit at every point $Q \in \partial D$ except for a set of vanishing $(n-1)$-dimensional Hausdorff measure.*

Let σ be the surface measure of ∂D. Since Lipschitz functions are differentiable almost everywhere (see [12, p. 250]) it follows that for all points Q on ∂D outside of a set of vanishing σ-measure there is an inward unit normal, which we denote by n_Q. If $E \subset R^n$ we denote the harmonic measure of $E \cap \partial D$ with respect to D by $\omega(\cdot, E)$. For the basic properties of ω, see [5, Chapter8]. We can now formulate a more precise version of Theorem 1.

Theorem 3. *Let $D \subset R^n$, $n \geqq 3$, be a Lipschitz domain and let G denote the Green's function of D. Let $P \in D$ and put $g=G(\cdot, P)$. Then there exists a set $E \subset \partial D$ such that $\sigma(E)=0$, and for all $Q \in \partial D - E$ the limit*

$$\lim_{t \downarrow 0} (\partial/\partial n_Q) g(Q + t n_Q)$$

exists. If we denote this limit by $(\partial/\partial n) g(Q)$, then the following results hold.

(a) *If $Q \in \partial D - E$ then $0 < (\partial/\partial n) g(Q) < \infty$.*

(b) *Let σ_n be the surface measure of $\{P \in R^n : |P| = 1\}$ and define $\gamma_n = [\sigma_n(n-2)]^{-1}$. If $F \subset \partial D$ then*

$$\omega(P, F) = \gamma_n \int_F (\partial/\partial n) g(Q) \, d\sigma(Q).$$

(c) *There is a number $C>0$ such that if $P'\in\partial D$ and $0<r<1$, then*

$$\sigma(A(P',r))\int_{A(P',r)}[(\partial/\partial n)g(Q)]^2\,d\sigma(Q)\leqq C\Big[\int_{A(P',r)}(\partial/\partial n)g(Q)\,d\sigma(Q)\Big]^2,$$

where $A(P',r)=\{Q\in\partial D:|P'-Q|<r\}$.

Theorem 3 makes it possible for us to compare the harmonic measure of a set with its surface measure.

Corollary. *Let D be as above and let $P\in D$. Then there are numbers $\alpha>1/2$, $\beta>0$, and $C>0$ such that if $F\subset\partial D$ then*

$$\omega(P,F)\leqq C\big(\sigma(F)\big)^\alpha \quad \textit{and} \quad \sigma(F)\leqq C\big(\omega(P,F)\big)^\beta.$$

Remark. The results of Theorem 3 and its corollary also hold in the case $n=2$, but the proof given here for $n\geqq 3$ must be modified. For other results in the plane case, see [13].

We say that a bounded domain $D\subset R^n$ is a Lipschitz domain if to each point $Q\in\partial D$ there corresponds a coordinate system (ξ,η), $\xi\in R^{n-1}$, $\eta\in R$, and a function φ such that $|\varphi(\xi)-\varphi(\xi_1)|\leqq C|\xi-\xi_1|$ for some C and $D\cap V=\{(\xi,\eta):\varphi(\xi)<\eta\}\cap\}^{\cap V}$ for some neighborhood V of Q.

We will, from now on, assume $n\geqq 3$ unless otherwise mentioned. Let L be the class of functions in R^{n-1} such that

$$\|\varphi\|=\sup_{x\neq y}|x-y|^{-1}|\varphi(x)-\varphi(y)|<\infty, \qquad \varphi(0)=0,$$

$$\text{support } \varphi\subset\{x\in R^{n-1}:|x|<1\}.$$

We define

$$S(\varphi)=\{(x,\varphi(x)):|x|\leqq 1\}.$$

If $m>0$ we put

$$L(m)=\{\varphi\in L:\|\varphi\|<m\} \quad \text{and} \quad \Gamma(m)=\{(x,y):m|x|<y\}.$$

If $\varphi\in L(m)$ and $(x,y)\in\Gamma(m)+(\xi,\varphi(\xi))$ for some $\xi\in R^{n-1}$, then $y>\varphi(x)$. Let $\lambda=\lambda(m)=(m+2)^{-1}$. Then for all $\eta>0$

$$\{(x,y):|x|\leqq\lambda\eta(1-\lambda)\eta\leqq y\}\subset\Gamma(m). \tag{1}$$

From (1) follows the existence of numbers $A=A(m)$ and $B=B(m)$, such that $10\lambda^{-1}<B<\frac{1}{2}A$, with the following property: If $\varphi\in L(m)$ and

$$D(\varphi,m)=\{(x,y):|x|<10 \quad \text{and} \quad \varphi(x)<y<mA\},$$

then $D(\varphi,m)$ is star shaped with respect to $P_m=(0,mB)$.

If $Q\in R^n$, $r>0$, we put $B(Q,r)=\{P\in R^n:|P-Q|<r\}$.

Lemma 1. *Let $m\geqq 1$ and $\varphi\in L(m)$. Let G be the Green's function of $D=D(\varphi,m)$. Then there are numbers δ_0, C_0, and C_1, which depend only on m, with the following property. If $Q\in S(\varphi)$, $0<\rho<\delta_0$, then*

$$C_0^{-1}\rho^{n-2}G(Q+(0,C_1\rho),P_m)\leqq\omega(P_m,B(Q,\rho))\leqq C_0\rho^{n-2}G(Q+(0,C_1\rho),P_m). \tag{2}$$

Proof. Let $Q=(\xi, \varphi(\xi))\in S(\varphi)$ and put

$$C(Q, \xi)=\{(x, y): |x-\xi|<\lambda\varepsilon, \quad (1-\lambda)\varepsilon<y-\varphi(\xi)<(B+2)\varepsilon\}$$

where λ and B are as above. Then there is a $\delta=\delta(m)$ such that $0<\delta<1$ and $C(Q, \varepsilon)\subset D$ for all $Q\in S(\varphi)$ and $0<\varepsilon<\delta$. Suppose $|z|<2$ and put $P=(z, \varphi(z))$. For $0<t<1$, let

$$P_t=(x_t, y_t)=tP_m+(1-t)P.$$

If $|z-\xi|\leqq t$ then $|x_t-\xi|\leqq 2t$ and

$$(B-2)tm\leqq y_t-\varphi(\xi)\leqq(B+2)tm.$$

Choosing $t=\varepsilon m^{-1}(B-2)^{-1}=c\varepsilon$, we find that $P_t\in C_1(Q, \varepsilon)$ when $|z-\xi|\leqq t$, where

$$C_1(Q, \varepsilon)=\{(x, y): |x-\xi|\leqq\frac{\lambda}{2}\varepsilon, \quad \varepsilon\leqq y-\varphi(\xi)\leqq\tfrac{1}{2}(B+2)\varepsilon\}.$$

Let G' be the Green's function of $C(Q, \varepsilon)$. Then a change of scale shows the existence of a number $C=C(m)$ such that

$$C\varepsilon^{n-2}\inf\{G'(Q+(0, \varepsilon), P): P\in C_1(Q, \varepsilon)\}\geqq 1.$$

Now the function $u_t: P\to G(Q+(0, \varepsilon), tP_m+(1-t)P)$ is superharmonic in D. Furthermore if $P=(z, \varphi(z))\in S(\varphi)$ and $|z-\xi|\leqq c\varepsilon=t$ then

$$C\varepsilon^{n-2}u_t(P)\geqq C\varepsilon^{n-2}G'(Q+(0, \varepsilon), P_t)\geqq 1.$$

The minimum principle now gives $\omega(P, B(Q, c\varepsilon))\leqq C\varepsilon^{n-2}u_t(P)$ for all $P\in D$. Taking $P=P_m$ gives the right-hand inequality of (2).

Let $K(\rho)=\{(x, y): |x|<\rho/2, -2m\rho<y<2C_1\rho\}$. If $0<\rho<\rho_0$, there is a number $C_2=C_2(m)$ such that if $Q=(\xi, \varphi(\xi))\in S(\varphi)$ then $D(Q, \rho)\subset(K(\rho)+Q)\cap D$, where $D(Q, \rho)$ is the ball with center $Q+(0, C_1\rho)$ and radius $C_2\rho$. Since $G(P, P')\leqq|P-P'|^{2-n}$ it follows that

$$\sup\{G(P, Q+(0, C_1\rho)): P\in\partial D(Q, \rho)\}\leqq C_\rho^{2-n},$$

where C only depends on m. Let ω' be the harmonic measure of the set $\{(x, y): |x-\xi|<\rho/2, y=-2m+\varphi(\xi)\}$ with respect to $K(\rho)+Q$. Then the maximum principle implies that $\omega(P, B(A, \rho))\geqq\omega'(P)$ for all $P\in D(Q, \rho)$. Since there is a number $c>0$ depending only on m such that $\omega'(P)\geqq c$ for all $P\in D(Q, \rho)$, the maximum principle now gives

$$\rho^{n-2}G(P, a+(0, C_1\rho))\leqq C\omega(P, B(Q, \rho))$$

for all $P\in D-D(Q, \rho)$, where C only depends on m. Taking ρ_0 so small that $P_m\in D(Q, \rho)$, we obtain the left-hand inequality of (2), and the lemma is proved.

We will need the following elementary estimate.

Lemma 2. *Let $m\geqq 1$, $\varphi\in L(m)$, and $D=D(\varphi, m)$. Then there is a number $c=c(m)>0$ such that $\omega(P_m, S_0(\varphi))\geqq c$, where $S_0(\varphi)=\{(x, \varphi(x)): |x|\leqq\frac{1}{2}\}$.*

Proof. Put $\Omega=\{(x, y): |x|<\frac{1}{3}, -2m<y<(B+1)m\}$ and let v be the harmonic measure of $\{(x, y): y=-2m\}\cap\partial\Omega$ with respect to Ω. Then $\omega(\cdot, S_0(\varphi))|\Omega\geqq v$ and hence $\omega(P_m, S_0(\varphi))\geqq v(P_m)>0$; the lemma is proved.

Lemma 3. *Let $m \geqq 1$, $\varphi \in L(m)$ and $D_1(\varphi, m) = D(\varphi, m) - \overline{B(P_m, m)}$, and put $g = G(\cdot, P_m)$. Then there is a constant $C = C(m)$ such that $(\partial/\partial y)g + C \geqq 0$ in $D_1(\varphi, m)$.*

Proof. Suppose first that $\varphi \in C^\infty(R^{n-1}) \cap L(m)$. Since we have

$$0 \leqq g(P) \leqq |P - P_m|^{2-n} \qquad \text{for all } P \in D(\varphi, m)$$

it follows from the Schauder estimates that there is a constant $C = C(m)$ such that $\sup \{|(\partial/\partial y)g(P)| : P \in \partial B(P_m, m)\} \leqq C$. Since g can be extended across both $\{(x, 0) : 1 < |x| < 10\}$ and $\{(x, Am) : |x| < 10\}$ by reflexion, it follows from [1, Thm. 7.3] that

$$\sup \{|(\partial/\partial y)g(P)| : P \in \partial D(\varphi, m) - \{(x, \varphi(x)) : |x| \leqq 2\} \leqq C = C(m).$$

Since $(\partial/\partial y)g$ has non-negative boundary values on the rest of the boundary the lemma follows in this case. If $\varphi \in L(m)$ and φ is not assumed to be of class C^∞, we can find a sequence $\{\varphi_i\}$ such that

$$\varphi_i \in C_0^\infty \{x \in R^{n-1} : |x| < 1\}, \qquad \varphi_i \geqq \varphi, \qquad \|\varphi_i\| < m, \qquad \varphi_i \to \varphi \text{ uniformly.}$$

If G_i denotes the Green's function of $D(\varphi_i, m)$ and $g_i = G_i(\cdot, P_m)$, then [3, Theorem 5.15] $g_i \to g$ uniformly on compact subsets of $D(\varphi, m) - \{P_m\}$. Hence by the Poisson representation formula $(\partial/\partial y)g_i \to (\partial/\partial y)g$ uniformly on compact subsets of $D(\varphi, m) - \{P_m\}$. Therefore the lemma follows from the previous case.

Let σ denote the surface measure of $\partial D(\varphi, m)$, $\varphi \in L(m)$. Let $E \subset S(\varphi)$ and let $E' = \{x \in R^{n-1} : (x, \varphi(x)) \in E\}$. Then

$$\sigma(E) = \int_{E'} \sqrt{1 + |\text{grad } \varphi|^2}\, dx.$$

Therefore there is a number $C = C(m)$ such that

$$C^{-1} r^{n-1} \leqq \sigma(B(Q, r) \cap \partial D) \leqq C r^{n-1} \qquad \text{for } Q \in S(\varphi). \tag{3}$$

If $E \subset R^n$ we define $\sigma(E) = \sigma(E \cap \partial D(\varphi, m))$.

Lemma 4. *Let $m \geqq 1$ and $\varphi \in L(m)$. If $E \subset S(\varphi)$ and $\sigma(E) = 0$, then E has harmonic measure zero with respect to $D(\varphi, m)$.*

Proof. From (3) and Lemma 1 follows the existence of a constant $C = C(m)$ such that

$$\limsup_{r \to 0} \frac{\omega(P_m, B(Q, r))}{\sigma(B(Q, r))} \leqq C \limsup_{t \to 0} (\partial/\partial y) g(Q + (0, t)). \tag{4}$$

From Lemma 3 and the fact that $(\partial/\partial y)g$ has finite non-tangential boundary values except on a set of harmonic measure zero [6], it follows that

$$\limsup_{r \to 0} \omega(P_m, B(Q, r))/\sigma(B(Q, r)) < \infty$$

for all $Q \in S(\varphi)$ except for a set of harmonic measure zero. As in [11, Theorem 14.5], the conclusion of the lemma now follows at once.

Lemma 4 implies the existence of an $f \in L^1(S(\varphi), \sigma)$ such that

$$\omega(P_m, E) = \int_E f d\sigma$$

for all $E \subset S(\varphi)$. We notice that $f \geqq 0$ and

$$\int_{S(\varphi)} f d\sigma \leqq 1. \tag{5}$$

We will now show $f \in L^2(S(\varphi), \sigma)$.

Lemma 5. *Let $m \geqq 1$ and $\varphi \in L(m)$. Then there is a number $C = C(m)$ such that*

$$\omega(P_m, E) \leqq C\sqrt{\sigma(E)}$$

for all $E \subset S(\varphi)$.

Proof. Let $g = G(\cdot, P_m)$, where G is the Green's function of $D(\varphi, m)$. Then there is a function g_1 harmonic in $D(\varphi, m)$ such that $g(P) = |P - P_m|^{2-n} + g_1(P)$. From Lemma 3 it follows that there is a constant $C_1 = C_1(m)$ such that $(\partial/\partial y) g_1 + C_1 \geqq 0$ in $D(\varphi, m)$. Since

$$\sup\{|g_1(P)| : P \in D(\varphi, m)\} = \max\{|P - P_m|^{2-n} : P \in \partial D(\varphi, m)\}$$

there is a constant $C_2 = C_2(m)$ such that $h(P_m) \leqq C_2$ where $h = (\partial/\partial y) g_1 + C_1$. Let $0 < t < 1$. Since $D(\varphi, m)$ is star shaped with respect to P_m we have

$$h(P_m) = \int_{\partial D(\varphi, m)} h(tQ + (1-t)P_m)\,\omega(P_m, dQ) \geqq \int_{S(\varphi)} h(tQ + (1-t)P_m)\,\omega(P_m, dQ).$$

Putting

$$F(Q) = \liminf_{n\to\infty} h\left(\left(1 - \frac{1}{n}\right) Q + \frac{1}{n} P_m\right), \qquad Q \in S(\varphi),$$

we see from [6] and Lemma 3 that

$$F(Q) = \limsup_{t\to 0} h(Q + (0, t)) \quad \text{a.e.} \quad [\omega(P_m, \cdot)].$$

By (4) and the definition of h there exists a constant $C_3 = C_3(m)$ such that $C_3(F + C_3) \geqq f$ a.e. $[\omega(P_m, \cdot)]$. Fatou's lemma and (5) now gives

$$\int_{S(\varphi)} f(Q)\,\omega(P_m, dQ) = \int_{S(\varphi)} f^2 d\sigma \leqq C = C(m). \tag{6}$$

If $E \subset S(\varphi)$ we have $\omega(P_m, E) = \int_E f d\sigma \leqq C\sqrt{\sigma(E)}$ by Hölder's inequality. This proves the lemma.

Lemma 6. *Suppose D_1 and D_2 are bounded domains which are regular for the Dirichlet problem. Assume that $E \subset \partial D_1 \cap \partial D_2$ is closed and that there is an open set V with $E \subset V$ and $V \cap D_1 = V \cap D_2$. Let ω_i denote the harmonic measure of E with respect to D_i. Then $\omega_1(\cdot, E) = 0$ if and only if $\omega_2(\cdot, E) = 0$.*

Proof. Suppose $\omega_1(\cdot, E) = 0$, and notice that

$$\lim_{P\to 0} \omega_2(P, E) = 0 \qquad \text{for all } Q \in \partial D_2 - E.$$

Let $\Omega = R^n - E$; for $P \in \Omega$ define $u(P) = \omega_2(P, E)$ if $P \in \Omega \cap D_2$ and zero otherwise. Then u is continuous and subharmonic in Ω. Define $\varphi(P) = 0$ if $P \in E$ and $\varphi(P) = u(P)$ if $P \in \partial D_1 - E$. Then φ is continuous in ∂D_1. Now let v be the harmonic function in D_1 with boundary values φ. Fix a point $P_0 \in D_1$ and choose a sequence $\{U_j\}$ of open sets such that $\omega_1(P_0, U_j) \to 0$ and $U_j \supset E$. Then the maximum principle implies that $u|D_1 \leqq v + \omega_1(\cdot, U_j)$. Letting $j \to \infty$, we have $u|D_1 \leqq v$. Then since $V \cap D_1 = V \cap D_2$ and $E \subset V$, we have $\lim_{P \to Q} \omega_2(P, E) = 0$ for all $Q \in V \cap \partial D_1$ and hence $\omega_2(\cdot, E) = 0$. Since the other direction is analogous we have proved the lemma.

Let $D \subset R^n$ be a Lipschitz domain. We say that D is *simple* if there is a function $\varphi \in L(m)$ and a number $\beta > 0$ such that D is congruent to $\{\beta P : P \in D(\varphi, m)\}$. In this case, for $0 < t < 1$, we let $S(D, t)$ be the part of the boundary of D corresponding to $\{(x, \varphi(x)) : |x| \leqq t\}$. If D is a Lipschitz domain, it follows from the definition that there are finitely many simple Lipschitz domains D_i, $1 \leqq i \leqq N$, such that for each i there is an open set V_i with the property that $D_i \cap V_i = D \cap V_i$ and

$$S(D_i, 2/3) \subset V_i \cap \partial D \quad \text{and} \quad \bigcup_{i=1}^{N} S(D_i, \tfrac{1}{2}) = \partial D. \tag{7}$$

We can now prove Theorem 1.

Proof of Theorem 1. From (7) and (3) it follows that a set $E \subset \partial D$ is of vanishing $(n-1)$-dimensional Hausdorff measure if and only if $\sigma(E) = 0$, where σ is the surface measure of ∂D.

To prove Theorem 1, we see from (7) and Lemma 6 that it is sufficient to show that, if $\varphi \in L(m)$ and $E \subset S(\varphi)$, then E is of harmonic measure zero with respect to $D(\varphi, m)$ if and only if $\sigma(E) = 0$. By Lemma 4, in order to prove this equivalence it is enough to show that $\omega(\cdot, E) = 0$ implies $\sigma(E) = 0$. To prove this, we argue by contradiction.

Suppose there is a number $m \geqq 1$, an element $\varphi \in L(m)$, and a set $E \subset S(\varphi)$ such that $\sigma(E) > 0$ but $\omega(P_m, E) = 0$. Put

$$E' = \{(x, 0) : |x| < 1 \quad \text{and} \quad (x, \varphi(x)) \in E\}.$$

Let $|F|$ denote the Lebesgue measure of a set $F \subset R^{n-1}$. Then $|E'| > 0$ and we may without loss of generality assume 0 is a point of density of E', i.e.

$$\lim_{r \to 0} \frac{|E' \cap B(r)|}{|B(r)|} = 1 \quad \text{where} \quad B(r) = \{x \in R^{n-1} : |x| < r\}.$$

Put $e_r = \{x \in R^{n-1} : |x| < 1/2 \text{ and } rx \in E'\}$. Pick a Lipschitz function F in R^{n-1} such that $F(x) = 1$ for $|x| < 2/3$ and the support of F lies in $\{x \in R^{n-1} : |x| < 1\}$. Define $\varphi_r(x) = r^{-1} F(x) \varphi(rx)$. Then $\varphi_r \in L$ and $\|\varphi_r\| \leqq C \|\varphi\|$, where C is independent of r. Let k be a number such that $\sup_{0<r<1} \|\varphi\| < k$. If $E_r = \{(x, \varphi_r(x)) : x \in e_r\}$ then by Lemma 6 the harmonic measure of E_r with respect to $D(\varphi_r, k)$ is zero. From Lemma 2 we have

$$\omega(P_k, S_r) \geqq C > 0,$$

where $S_r = \{(x, \varphi_r(x)) : |x| < 1/2\}$ and C is independent of r. From Lemma 5 we have

$$\omega(P_k, S_r) \leqq C\sqrt{\sigma(S_r - E_r)} \leqq C\left|B\left(\frac{1}{2}\right) - e_r\right|^{1/2} = C\left(1 - \frac{|E' \cap B(r)|}{|B(r)|}\right)^{1/2} \to 0$$

as $r \to 0$. This yields a contradiction, and hence completes the proof of Theorem 1.

In the next lemma we shall compare the Green's functions of two Lipschitz domains with intersecting boundaries. The proof will use a result of NAIM, which was pointed out to the author by Professor PAUL GAUTHIER.

Lemma 7. *Let D_1 and D_2 be two Lipschitz domains in R^n, $n \geqq 2$, and let g_i denote the Green's function of D_i with pole at $Q_i \in D_i$, $i = 1, 2$. Suppose there is a domain $W \subset D_1 \cap D_2$ such that for some open set V we have $\bar{W} \subset V$, $V \cap D_1 = V \cap D_2$, and $Q_i \in D_i - V$, $i = 1, 2$.*
Then there is a constant $C > 0$ such that

$$g_1(P) \leqq C g_2(P) \quad \text{for all } P \in W.$$

Proof. Assume the conclusion is false. This means there is a sequence of points $P_n \in W$ such that

$$\lim_{n \to \infty} \frac{g_1(P_n)}{g_2(P_n)} = \infty.$$

We may without loss of generality assume that $\lim_{n \to \infty} P_n = Q_0$ exists. Since $Q_i \notin W$ for $i = 1, 2$, we must have $Q_0 \in \partial W \cap \partial D_1 \cap \partial D_2$. From the definition of a Lipschitz domain follows the existence of a neighbourhood U of Q_0 such that $U \subset V$ and $U \cap D_1$ is a Lipschitz domain. Let g denote the Green's function of $U \cap D_1$. Since the Martin boundary of a Lipschitz domain coincides with the Euclidean boundary [7, Theorem 4.2], it follows from the computation in [9, p. 223] that if $Q \in U \cap D_1$ then

$$\lim_{n \to \infty} \frac{g(P_n, Q)}{g_i(P_n)} = K_i(Q) - h_i(Q).$$

Here K_i is the kernel function of D_i with pole at Q_0, normalized by $K_i(Q_i) = 1$, and h_i is the harmonic function in $U \cap D_1$ with boundary values equal to $K_i(Q)$ when $Q \in \partial(U \cap D_1) \cap D_1$ and zero otherwise. Hence $h_i \leqq K_i$ in $U \cap D_1$. Suppose $h_i(Q') = K_i(Q')$ for some $Q' \in U \cap D_1$. From the maximum principle it follows then that $h_i(Q) = K_i(Q)$ for all $Q \in U \cap D_1$. Hence

$$\lim_{Q \to Q_0} K_i(Q) = \lim_{Q \to Q_0} h_i(Q) = 0.$$

Since $\lim_{Q \to P} K_i(Q) = 0$ for all $P \in \partial D_i - \{Q_0\}$ we obtain $K_i \equiv 0$, which is a contradiction. This shows that $h_i(Q) < K_i(Q)$ for all $Q \in U \cap D_1$. Hence

$$\lim_{n \to \infty} \frac{g(P_n, Q)}{g_i(P_n)} > 0$$

for all $Q \in U \cap D_1$. This gives

$$\lim_{n\to\infty} \frac{g_1(P_n)}{g_2(P_n)} < \infty,$$

which contradicts the assumption in the beginning of the proof.

We shall next compare positive harmonic functions which simultanously vanish on a part of the boundary.

Lemma 8. *Let $\varphi : R^{n-1} \to R$, $n \geqq 2$, be a Lipschitz function such that $\varphi(0)=0$. Suppose that positive numbers a, b and c have been chosen such that*

(i) $a > 2 \sup \{|\varphi(x)| : |x| \leqq 4b\}$, and

(ii) the domain $D = \{(x, y) : \varphi(x) < y < 4a,\ |x| < 4b\}$ is star shaped with respect to $P_0 = (0, c)$.

Put $D_1 = \{(x, y) : \varphi(x) < y < a,\ |x| < b\}$. Then there is a constant $C > 0$ such that if u and v are non-negative harmonic functions in D which vanish on $\{(x, \varphi(x)) : |x| \leqq 4b\}$ and which satisfy $u(P_0) \leqq v(P_0)$, then $u(P) \leqq Cu(P)$ for all P in D_1.

Proof. Let $D_j = \{(x, y) : \varphi(x) < y < ja,\ |x| < jb\}$. By a result of HUNT and WHEEDEN [7, (2.4)] there exists a constant C_1 such that

$$u(P) \leqq C_1 u(P_0) \qquad \text{for all } P \in \bar{D}_3.$$

Also from Harnack's inequality there exists a constant $C_2 > 0$ such that

$$v(P) \geqq C_2 v(P_0) \qquad \text{for all } P \in T,$$

where $T = \{(x, 3a) : |x| \leqq 3b\}$. Let g denote the harmonic measure of $\partial D_3 - \{(x, \varphi(x_1)) : |x| < 3b\}$ with respect to D_3, and let h denote the harmonic measure of T with respect to D_3. Then $u \leqq C_1 u(P_0) g$ and $v \geqq C_2 v(P_0) h$ in D_3. To prove the lemma it is now sufficient to show that there is a constant C such that $g(P) \leqq Ch(P)$ for all $P \in D_1$.

Define

$$\psi(x) = \min(\varphi(x), \alpha - \beta|x|),\ x \in R^{n-1}.$$

It is easily seen that we can choose α and β such that $\psi(x) = \varphi(x)$ for $|x| \leqq 2b$ and $\psi(x) < \varphi(x)$ for $|x| > \frac{5}{2}b$. With this choice, let

$$U_j = \{(x, y) : \psi(x) < y < ja,\ |x| < jb\}.$$

Choose a point $Q_1 \in U_4 - \bar{U}_3$ and denote by G_1 the Green's function of U_4 with pole at Q_1. We now extend g to U_3 by defining $g(P)=0$ if $P \in U_3 - D_3$. With this extension g is subharmonic in U_3. Since $\inf\{G_1(P) : P \in \partial U_3 \cap D_4\}$ is positive it follows from the maximum principle that

$$g(P) \leqq C_3 G_1(P) \qquad \text{for all } P \in U_3.$$

Let $Q_2 \in D_3 - \bar{D}_2$ and denote by G_2 the Green's function of D_3 with pole at Q_2. Let B be a ball with center at Q_2 such that $\bar{B} \subset D_3 - \bar{D}_2$. We now observe that

$$\sup\{G_2(P) : P \in \partial B\} < \infty, \qquad \inf\{h(P) : P \in \partial B\} > 0.$$

Since the boundary values of G_2 vanish on ∂D_3, it follows from the maximum principle that there is a constant C_4 such that $h(P) \geqq C_4 G_2(P)$ for all P in $D_3 - \bar{B}$.

If we now use Lemma 7 to compare G_1 and G_2 in D_1, we find that $g(P) \leqq Ch(P)$ for all $P \in D_1$, and as noted above, this proves the lemma.

The next theorem was formulated in [8, Thm. 2.2] but Professor KEMPER has pointed out to me in a conversation that the proof contains a mistake on page 253, line 1.

Theorem 4. *Let $D \subset R^n$, $n \geqq 2$, be a Lipschitz domain and let V be an open set such that $V \cap \partial D \neq \emptyset$. Suppose W is a domain such that $W \subset D$ and $\bar{W} \subset V$, and let P_0 be a point in W.*

Then there is a constant $C > 0$ such that if u and v are non-negative harmonic functions in D which vanish on $V \cap \partial D$ and satisfy $u(P_0) \leqq v(P_0)$, then $u(P) \leqq Cv(P)$ for all $P \in W$.

Proof. If Ω is congruent to a domain of the type indicated in Lemma 8, we denote by $\Gamma(\Omega)$ the part of $\partial\Omega$ corresponding to $\{(x, \varphi(x)) : |x| < b\}$. We notice that Theorem 8 follows from Harnack's inequality if $\partial W \cap \partial D = \emptyset$. Otherwise we can find finitely many domains Ω_i, each of them congruent to a domain of the form indicated in Lemma 8, such that $\bigcup_i \Gamma(\Omega_i) \supset \bar{W} \cap \partial D$. The theorem now follows by repeated application of Harnack's inequality and Lemma 8.

The proof of Theorem 3 will be based on the following lemma.

Lemma 9. *Suppose that $D \subset R^n$, $n \geqq 3$, is a Lipschitz domain and suppose further that there is an open set V and a function $\varphi \in L(m)$ such that*

$$D \cap V = D(\varphi, m) \cap V$$

and

$$S'(\varphi) = \{(x, \varphi(x)) : |x| \leqq 2/3\} \subset V \cap \partial D.$$

Let σ denote the surface measure of D, let

$$S_0(\varphi) = \{(x, \varphi(x)) : |x| \leqq \tfrac{1}{2}\},$$

and for $Q \in S(\varphi)$ let n_Q denote the unit inward normal of D, whenever it exists. For $P \in D$, define $g = G(\cdot, P)$, where G is the Green's function of D. Then the following conclusions hold.

(a) *There is a set $E \subset S_0(\varphi)$ such that $\sigma(E) = 0$,*

$$\lim_{t \downarrow 0} (\partial/\partial n_Q) g(Q + t n_Q) = (\partial/\partial n) g(Q) \text{ exists,}$$

and

$$0 < (\partial/\partial n) g(Q) < \infty \quad \text{for all } Q \in S_0(\varphi) - E.$$

(b) *If $F \subset S_0(\varphi)$ then*

$$\omega(P, F, D) = \gamma_n \int_F (\partial/\partial n) g(Q)\, d\sigma(Q).$$

(c) *There is a number $C > 0$, depending on D, φ and V, such that if $P' \in S_0(\varphi)$ and $0 < r < 1$, then*

$$\sigma(A(P', r)) \int_{A(P,r)} [(\partial/\partial n) g(Q)]^2\, d\sigma(Q) \leqq C \left[\int_{A(P,r)} (\partial/\partial n) g(Q), d\sigma(Q) \right]^2,$$

where $A(P', r) = B(P', r) \cap \partial D$.

Proof of part (a). Put $D'=D(\varphi, m)$, and let G' denote the Green's function of D'. Notice that if we take $\varepsilon\in(0,1)$ sufficiently small and put

$$D''=\{(x,y):|x|<2/3+\varepsilon, \quad \varphi(x)<y<\varphi(x)+\varepsilon\},$$

then $D''\subset D\cap D'$ and $\{P, P_m\}\subset R^n-D''$. Since g and $g'=G'(\cdot, P_m)$ are positive and harmonic in D'' and have vanishing boundary values on $\{(x,y): y=\varphi(x), |x|\leqq 2/3+\varepsilon\}$ it follows from Theorem 4 that there is a neighborhood U of $S'(\varphi)$ and a number $C>0$ such that for all $Q\in U\cap D$

(8) $$C^{-1}g'(Q)\leqq g(Q)\leqq Cg'(Q).$$

We may assume this inequality holds for all $Q\in D''$, if necessary by making ε smaller.

For $Q=(x,y)\in D$, we denote by $d(Q)$ the distance from Q to ∂D, and if $Q\in D(\varphi, m)$ we let Q^* denote the point $(x, \varphi(x))$. Pick f such that

$$\omega(P_m, E, D')=\int_E f\,d\sigma$$

for all $E\subset S(\varphi)$. If

$$f^*(Q)=\sup\left\{r^{1-n}\int_{A(Q,r)} f(P')\,d\sigma(P'): 0<r<1\right\},$$

then it follows from (6) that

(9) $$\int_{S(\varphi)} (f^*(Q))^2\,d\sigma(Q)<\infty.$$

Harnack's inequality and the Schauder estimates give

(10) $$|\text{grad}\, g(Q)|\leqq C(d(Q))^{-1}g(Q^*+(0, d(Q)))$$

for all $Q\in D''$. Hence from (7) and Lemma 1 we see that if $Q\in S'(\varphi)$ and $Q'\in D''\cap(K+Q)$, then

(11) $$|\text{grad}\, g(Q')|\leqq Cf^*(Q),$$

where $K=\{(x,y): 2m|x|<y\}$. From (11) it follows that $|\text{grad}\, g|$ is non-tangentially bounded a.e. with respect to σ at $S'(\varphi)$. This implies, using [6], that grad g has a finite non-tangential limit a.e. on $S'(\varphi)$. Hence part (a) follows from Theorem 1.

Proof of part (b). Extend g to all of R^n by putting $g\equiv 0$ outside D. Then g is continuous and subharmonic in $R^n-\{P\}$. Since

(12) $$g(Q)=|P-Q|^{2-n}-\int_{\partial D}|P'-Q|^{2-n}\omega(P, dP', D)$$

for all $Q\in R^n-(\partial D\cup\{P\})$, and $g(Q')=0$ for all $Q'\in\partial D$, Fatou's lemma implies

$$\int_{\partial D}|P'-Q'|^{2-n}\omega(P, dP', D)<\infty \qquad \text{for all } Q'\in\partial D.$$

Let $Q' \in \partial D$, and let Γ be a truncated cone with vertex at Q' such that $(i)\ \bar{\Gamma} - \{Q'\} \subset D$, and (ii) there is a number C for which $d(Q) \geqq C|Q - Q'|$ for all $Q \in \Gamma$. If $P' \in \partial D$ and $Q \in \Gamma$ then

$$|P' - Q'| \leqq |P' - Q| + |Q - Q'| \leqq |P' - Q| + C d(Q) \leqq (C+1)|P' - Q|.$$

Hence the dominated convergence theorem implies that

$$g(Q') = |P - Q'|^{2-n} - \int_{\partial D} |P' - Q'|^{2-n} \omega(P, dP', D).$$

Consequently (12) holds for all $Q \in R^n - \{P\}$. Therefore, if $h \in C_0^\infty(R^n)$ and P does not belong to the support of h,

$$\int_D g(Q) \Delta h(Q) dQ = \gamma_n^{-1} \int_{\partial D} h(Q) \omega(P, dQ, D), \tag{13}$$

where Δ denotes the Laplace operator.

Suppose now the support of h lies in the set

$$\{(x, y) : |x| \leqq 2/3,\ \varphi(x) - \varepsilon/2 < y < \varphi(x) + \varepsilon/2.\}$$

Then

$$\int_D g(Q) \Delta h(Q) dQ = \lim_{S \to 0} \int_{D_S} g(Q) \Delta h(Q) dQ, \tag{14}$$

where D_S is defined by the following procedure. Pick a function $\chi \in C_0^\infty(R^{n-1})$ and put $\varphi_S = \chi_S * \varphi$, where $\chi_S(x) = S^{1-n} \chi(S^{-1} x)$. Then

$$\|\varphi_S - \varphi\|_\infty \to 0 \text{ as } S \to 0, \qquad \sup_{S>0} \|\text{grad } \varphi_S\|_\infty < \infty,$$

$$\text{and } \lim_{S \to 0} \text{grad } \varphi_S(x) = \text{grad } \varphi(x) \text{ a.e.}$$

We now define $\psi_S = \varphi_S + 2\|\varphi_S - \varphi\|_\infty + S$ and put

$$D_S = \{(x, y) : |x| < 2/3 + \psi_S(x) < y < \psi_S(x) + 2\varepsilon/3\}.$$

Notice that if S is sufficiently small then $D_S \subset D''$. Green's formula now gives

$$\int_{D_S} g(Q) \Delta h(Q) = \int_{\Lambda_S} h(\partial/\partial n) g\, d\sigma - \int_{\Lambda_S} g(\partial/\partial n) h\, d\sigma = A(S) + B(S),$$

where we have put $\Lambda_S = \{(x, \psi_S(x)),\ |x| < 2/3 + \varepsilon\}$. We observe that $B(S) \to 0$ as $S \to 0$. Put

$$F_S(x) = (1 + |\text{grad } \psi_S|^2)^{1/2}, \qquad H_S(x) = (\text{grad } g(x, \psi_S(x), n_s(x)))$$

where n_s denotes the inward unit normal to ∂D_S at $(x, \psi_S(x))$. Then

$$A(S) = \int h(x, \psi_S(x)) H_S(x) F_S(x) dx.$$

The proof of part (a) shows that

$$H_S(x) \to (\partial/\partial n) g(x, \varphi(x)) \text{ a.e.}$$

and

$$F_S(x) \to (1 + |\text{grad } \varphi(x)|^2)^{1/2} \text{ a.e.}$$

as $S \to 0$. From (9) and (11)

$$\int_{|x|<2/3+\varepsilon} \left(\sup_{0<S<\delta} |H_S(x)| \right)^2 dx < \infty$$

if δ is sufficiently small. The dominated convergence theorem together with (14) now implies that

$$\int_D g(Q) \Delta H(Q) dQ = \gamma_n \int_{S(\varphi)} h(Q) (\partial/\partial n) g(Q) d\sigma(Q).$$

Part (b) then follows from relation (13).

Proof of part (c). Suppose $P' \in S_0(\varphi)$ and $0<r<\varepsilon$. Then from (6) the function

$$h(Q) = \int_{A(P',r)} (\partial/\partial n) g'(Q') \omega(Q, dQ', D')$$

is non-negative and harmonic in D'. Since the boundary values of h vanish outside $A(P', r)$ it follows from [7, (2.4)] and [6, p. 311] that

$$h(P_m) \leqq C\omega\big(P_m, A(P', r), D'\big) h\big(P' + (0, r)\big). \tag{15}$$

Let $D_1(\varphi, m)$ be as in Lemma 3, and let v be the harmonic measure of $\partial D_1(\varphi, m) - \{(x, \varphi(x)) : |x| < 2\}$ with respect to $D_1(\varphi, m)$. Then from Lemma 3 there is a number $C = C(m)$ such that

$$(\partial/\partial y) g'(Q) \geqq \int_{S(\varphi)} (\partial/\partial y) g'(Q') \omega(Q, dQ', D') - Cv(Q)$$

for $Q \in D_1(\varphi, m)$. From (4) and part (b) we may find a $C = C(m)$ such that

$$(\partial/\partial y) g'(Q) \geqq Ch(Q) - Cv(Q).$$

Theorem 4 implies that $v(Q) \leqq Cg'(Q)$ for all $Q \in D''$. It now follows from (10) that

$$h(Q) \leqq Cd(Q)^{-1} g'\big(Q^* + (0, d(Q))\big) \quad \text{for} \quad Q \in D''.$$

From Lemma 1 we have

$$h\big(P' + (0, r)\big) \leqq Cr^{1-n} \omega\big(P_m, D(P', r), D'\big).$$

From this estimate and (15) we find

$$\sigma\big(A(P', r)\big) \int_{A(P',r)} [(\partial/\partial n) g'(Q)]^2 d\sigma(Q) \leqq C \left(\int_{A(P',r)} (\partial/\partial n) g'(Q) d\sigma(Q) \right)^2.$$

Part (c) follows now from (8), and the lemma is proved.

Proof of Theorem 3. Covering ∂D by simple Lipschitz domains as in the proof of Theorem 1, we obtain Theorem 3 directly from Lemma 9.

Proof of the Corollary. We observe the following consequences of Theorem 3. First, from a theorem of Gehring [4] and part (c)

$$\int_{\partial D} [(\partial/\partial n) g(Q)]^p d\sigma(Q) < \infty$$

for some $p>2$. Hölder's inequality now gives the first part of the corollary. Since $\omega(P,\cdot)$ and σ are comparable in the sense of [3, p. 248], Lemma 5 of [3] then yields the second part of the conclusion.

We shall now obtain some lower bounds for the exponents appearing in the corollary. For $0<\theta<\pi$, let

$$D(\theta)=\{P=(x,y):|x|<1, \quad (x_{n-1}^2+y^2)^{1/2}\cos\theta<y<1\}$$

and define $v(P)=\operatorname{Re}(y+ix_{n-1})^{\rho(\theta)}$, where $\rho(\theta)=(2\theta)^{-1}\pi$. Then v is non-negative and harmonic in $D(\theta)$, and v has vanishing boundary values on

$$\partial' D(\theta)=\{(x,y):y=(x_{n-1}^2+y^2)^{1/2}\cos\theta\}.$$

Fix a point $P_0\in D(\theta)$ and put $g=G(\cdot,P_0)$, where G is the Green's function of $D(\theta)$. By Theorem 4 there exists a number $C>0$ and a neighborhood V of $\partial' D(\theta)\cap\{(x,y):|x|\leqq 1/2\}$ such that

$$C^{-1}v(P)\leqq g(P)\leqq Cv(P) \tag{16}$$

for all $P\in V\cap D(\theta)$. Now for $0<\varepsilon<1/2$, let

$$E(\varepsilon)=\{(x,y)\in\partial' D(\theta):|x|\leqq 1/2, \quad |x_{n-1}|\leqq\varepsilon\}$$

and notice that there is a number $C=C(\theta)$ such that

$$C^{-1}\varepsilon\leqq\sigma(E(\varepsilon))\leqq C\varepsilon.$$

From (16) follows the existence of a constant $C=C(\theta)$ such that

$$C^{-1}\varepsilon^{\rho(\theta)}\leqq\omega(P_0,E(\varepsilon))\leqq C\varepsilon^{\rho(\theta)}.$$

Let α and β be as in the corollary. Letting $\theta\to\pi$ and $\theta\to 0$ respectively, we see that, in general, $\alpha>1/2$ and $\beta>0$.

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(Received May 9, 1976; revised October 13, 1976)