Can One Measure the Temperature of a Curve?

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Abstract

The entropy of a plane curve is defined in terms of the number of intersection points with a random line. The Gibbs distribution which maximizes the entropy enables one to define the temperature of the curve. At 0 temperature, the curve reduces to a straight segment. At high temperature, the curve is somewhat chaotic and "behaves like a perfect gas".

We attempt to show that thermodynamic formalism can be used for the study of plane curves. The curves we discuss have finite length, unlike MANDELBROT's fractal curves [1], yet we feel our approach to the mathematics is not far from his.

1. Random lines

Consider a plane curve Γ of finite length and let $\Omega(\Gamma)$ be the set of straight lines D which intersect Γ. Denote by $M(\Gamma)$ the family of probability measures on $\Omega(\Gamma)$. Suppose one tosses, in the sense of tossing a coin, a straight line D on Γ; then we let $|\Gamma \cap D|$ represent the number of intersecting points. It is well known that, for large N and randomly chosen lines $D_1, \ldots, D_N$, the average

$$\frac{1}{N} \sum_{k=1}^{N} |\Gamma \cap D_k|$$

is approximately equal to $2|\Gamma|/|\delta K|$, where $|\Gamma|$ is the length of Γ and $|\delta\Gamma|$ the length of the boundary of the convex hull K of Γ, see [4]. (Notice that for all curves, $2|\Gamma|/|\delta\Gamma| \geqq 1$, with equality occurring if and only if Γ is a straight segment.)

The usual way to justify this is to endow $\Omega(\Gamma)$ with the "natural" probability measure defined as follows. Let

$$x \cos\theta + y \sin\theta - \varrho = 0$$

be the Cartesian equation of the straight line D. If we identify couples (ϱ, θ) and $(-\varrho, \theta + \pi)$, the set of all straight lines appears as a Möbius variety on which we can define the Lebesgue measure $d\varrho\, d\theta$. The natural probability is then

$$dp = \frac{d\varrho\, d\theta}{\text{meas } \Omega(\Gamma)}.$$

A theorem of Steinhaus [4] (see also Santalo [3]) asserts now that the expectation of the number of intersecting points of D with Γ is

$$\int_{D \in \Omega(\Gamma)} |\Gamma \cap D|\, dp(D) = \frac{2\,|\Gamma|}{|\delta K|}.$$

The underlying assumption in this "proof" is that the natural measure p is the only one which mimics reality. We propose to enlarge the situation and to consider all probability measures $m \in M(\Gamma)$ such that

$$\int_{D \in \Omega(\Gamma)} |\Gamma \cap D|\, dm(D) = \frac{2\,|\Gamma|}{|\delta K|},$$

or more conveniently

$$\sum_{k=1}^{\infty} k m_k = \frac{2\,|\Gamma|}{|\delta K|},$$

where

$$m_k = m\{D \in \Omega(\Gamma) / |D \cap \Gamma| = k\}.$$

We denote by $M^*(\Gamma)$ the set of all such probabilities.

2. Entropy

Let $m \in M^*(\Gamma)$. The m-entropy of Γ can be defined as

$$S_m(\Gamma) = -\sum_{k=1}^{\infty} m_k \log m_k.$$

Gibb's equilibrium measure $g \in M^*(\Gamma)$ maximizes the entropy; the determination of g is classic. Use of Lagrange multipliers leads to

$$g_k = C \cdot e^{-\beta k}$$

where C is determined by the condition $\sum_{k=1}^{\infty} g_k = 1$. Hence

$$C^{-1} = \sum_{k=1}^{\infty} e^{-\beta k} = \frac{1}{e^{\beta} - 1},$$

this being the partition function. As for β, we have

$$\sum_{k=1}^{\infty} k g_k = \frac{2\,|\Gamma|}{|\delta K|}$$

whence

$$\beta = \log \frac{2\,|\Gamma|}{2\,|\Gamma| - |\delta K|}. \tag{1}$$

The g-entropy S_g, which we now denote by S, is then

$$S = -\sum_{k=1}^{\infty} g_k \log g_k = \log \frac{2\,|\Gamma|}{|\delta K|} + \frac{\beta}{e^\beta - 1}. \tag{2}$$

Notice that $g_k \neq 0$ for all $k \geqq 1$ so that, strictly speaking, our computation is only valid for the class of curves which can be intersected in a countable number of points by a suitable straight line D. Yet the above entropy makes sense for all rectifiable curves. We shall agree to extend equality (2) to all rectifiable curves.

The number S will be called the (intrinsic) entropy of Γ; it plays the same role as the topological entropy in dynamical systems.

It should be underlined at this point that $\beta > 0$, whence

$$0 \leqq \frac{\beta}{e^\beta - 1} < 1.$$

We see easily that $S \geqq 0$ for all Γ, and $S = 0$ if and only if Γ is a straight line. As an exercise, one can verify that if Γ is a portion of an algebraic curve of degree ν, then $S \leqq 1 + \log \nu$. The higher the entropy, the higher the degree, so in some sense entropy measures complexity.

3. Temperature

Physicists usually identify the exponent β in Gibb's measure with the inverse of the absolute temperature T. (Actually $\beta = (k_0 T)^{-1}$, where k_0 is the Boltzmann constant which fixes the scale of temperature; here we choose $k_0 = 1$). Equation (1) gives the temperature of the curve Γ, namely

$$T = \left(\log \frac{2\,|\Gamma|}{2\,|\Gamma| - |\delta K|}\right)^{-1}. \tag{3}$$

The temperature T is obviously non-negative for all curves, and is 0 if and only if $2\,|\Gamma| = |\delta K|$. In other words, only straight segments exist at $T = 0$, and then $S = 0$. (This is in accord with Nernst's thermodynamic assumption.)

4. Volume and pressure

To pursue the parallel with thermodynamics, one is led to identify the length of Γ with its "volume" V. The "pressure" P is then defined by $|\delta K|^{-1}$. The higher the pressure, the more Γ is confined to a small area. Equation (3) becomes

$$T = \left(\log \frac{2PV}{2PV - 1}\right)^{-1},$$

or alternatively

$$2PV = (1 - e^{-1/T})^{-1}.$$

This equation of state contains Boyle's law for high temperatures. Indeed, as T tends to infinity $PV \sim \frac{1}{2} T$, so at high temperatures curves behave as perfect gases.

We are aware of the limitations of our presentation. The definition of P and V (or of the product PV) is obviously artificial and boils down to the fact that the partition function depends on one variable only.

Another way to underline the difference with traditional thermodynamics is the following. Define the heat Q by

$$dQ = T\,dS.$$

Now

$$S = \log \frac{2|\Gamma|}{|\delta K|} + \frac{\beta}{e^\beta - 1} = \log \frac{e^\beta}{e^\beta - 1} + \frac{\beta}{e^\beta - 1},$$

and hence

$$dQ = \frac{1}{\beta}\, dS = d\left(\frac{1}{e^\beta - 1}\right).$$

On the other hand, the total energy U of the curve is by definition

$$U = \sum_{k=1}^{\infty} k g_k = \frac{2|\Gamma|}{|\delta K|} = \frac{e^\beta}{e^\beta - 1}.$$

Hence $dU = dQ$, whereas in traditional thermodynamics $dU = dQ - P\,dV$.

5. Local temperature and spirals

Let Γ be a curve and suppose $A \in \Gamma$. Let $\widehat{MN}$ be a subarc containing A. We define the upper temperature $T^*(A)$ and the lower temperature $T_*(A)$ at the point A by

$$T^*(A) = \limsup_{M \to A, N \to A} T(\widehat{MN}), \qquad T_*(A) = \liminf_{M \to A, N \to A} T(\widehat{MN}).$$

In case of equality, we say that the temperature is $T(A)$. If A is an endpoint of Γ, the definitions are modified to the form

$$T^*(A) = \limsup_{M \to A} T(\widehat{MA}), \qquad T_*(A) = \liminf_{M \to A} T(\widehat{MA}).$$

These last definitions hold whether or not the endpoint A belongs to Γ.

We apply these definitions to the particular case of finite length spirals Γ which converge to the center 0 without ever reaching it. Typically, let $\varrho = f(\theta)$ be the polar equation of such a spiral. *We assume that f decreases to* 0 *as* θ *in-*

creases to infinity, that f is continuous and differentiable, and that

$$\int_0^\infty (f^2(\theta) + f'^2(\theta))^{\frac{1}{2}} \, d\theta < \infty .$$

Let $M \in \Gamma$ have the polar coordinates $(\omega, f(\omega))$.

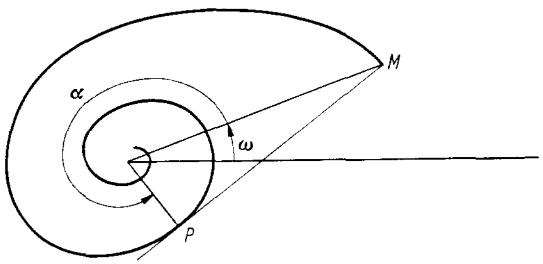

Fig. 1

The convex hull of the portion $\widehat{M0}$ of the spiral consists of an arc $\widehat{MP}$ and a chord $\overline{MP}$. The polar angle of P is $\omega + \alpha$, where $\alpha = \alpha(\omega) \in (\pi, 2\pi)$. A rather tedious computation then leads to

$$T(\widehat{M0}) = \left(\log \frac{\gamma(\omega)}{\gamma(\omega) - 1}\right)^{-1},$$

where

$$\gamma(\omega) = 2 \frac{\int_\omega^\infty [f^2(\theta) + f'^2(\theta)]^{\frac{1}{2}} \, d\theta}{[f^2(\omega) - 2f(\omega) f(\alpha + \omega) \cos \alpha + f^2(\alpha + \omega)]^{\frac{1}{2}} + \int_\omega^{\alpha+\omega} [f^2(\theta) + f'^2(\theta)]^{\frac{1}{2}} \, d\theta} .$$

The temperature at the center is obtained by letting ω tend to infinity. We give three examples.

Example 1. The exponential spiral $\varrho = a^{-\theta}$, $a > 1$. Then

$$T(0) = \left(\log \frac{2}{1 + \cos \alpha}\right)^{-1},$$

where α is the unique solution of

$$a^{-\alpha} = \cos \alpha - \sin \alpha \log a, \qquad \pi < \alpha < 2\pi .$$

In particular, for the spiral $\varrho = e^{-\theta}$ we obtain $T(0) = .5296 \ldots$

Example 2. Let $\varrho = \theta^{-\lambda}$, $\lambda > 1$. The temperature of the spiral at its center is infinite. This seems reminiscent of galaxies.

Example 3. The spiral $\varrho = \exp(-\theta^2)$. The temperature at the center is 0.

6. Nonzero temperature points

In the previous paragraph we gave examples of curves whose local temperature vanishes at all points except one. We shall now show that, *for all rectifiable curves Γ, almost all points $M \in \Gamma$ have zero temperature yet the set of strictly positive upper temperature points can be dense and uncountable.* This will be illustrated by an example. Both of these results are mere translations of the following property concerning rectifiable curves.

Let Γ be a planar rectifiable curve. For all points $M \in \Gamma$ define

$$d(M) = \limsup_{\substack{P \to M, Q \to M \\ P \in \Gamma, Q \in \Gamma}} \left| \frac{\widehat{PQ}}{\overline{PQ}} \right|$$

where $\widehat{PQ}$ and $\overline{PQ}$ represent respectively the arc and the chord, and where $|\cdot|$ denotes length.

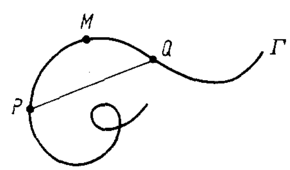

Fig. 2

Observe that

$$2\,|\overline{PQ}| \leqq |\delta K(PQ)| \leqq |\overline{PQ}| + |\widehat{PQ}|,$$

where $\delta K(PQ)$ is the boundary of the convex hull of the arc $\widehat{PQ}$. Then

$$\frac{2\,|\widehat{PQ}|}{|\widehat{PQ}| + |\overline{PQ}|} \leqq \frac{2\,|\widehat{PQ}|}{|\delta K(PQ)|} \leqq \frac{|\widehat{PQ}|}{|\overline{PQ}|}$$

and hence

$$\left(\log \frac{2d(M)}{d(M)-1}\right)^{-1} \leqq T^*(M) \leqq \left(\log \frac{d(M)}{d(M)-1}\right)^{-1}.$$

Consequently $T^*(M) = 0$ if and only if $d(M) = 1$, and $d(M) = \infty$ implies $T^*(M) \geqq (\log 2)^{-1}$.

Theorem. (i) *For almost all $M \in \Gamma$ we have $d(M) = 1$.*
(ii) *There exists a rectifiable curve Γ for which the set*

$$\{M \in \Gamma / d(M) = \infty\}$$

is dense and uncountable.

Remark. The measure involved in this theorem is Lebesgue measure on the curve Γ. We shall assume that all curves under consideration have finite length. The first part of the theorem is well-known; see, for example, [2] page 27.

Proof of (ii). We construct a rectifiable curve Γ for which the set

$$\{M \in \Gamma / d(M) = \infty\}$$

is dense and uncountable.

For $n \geqq 1$ and $1 \leqq j \leqq 2^{n-1}$, we consider the interval

$$I_{n,j} = [(2j-1)\,2^{-n} - 4^{-n^2}, (2j-1)\,2^{-n} + 4^{-n^2})$$

and the real continuous function $g_{n,j}$ defined on $[0, 1]$ by

$$g_{n,j}(t) = \begin{cases} |t - (2j-1)\,2^{-n}|^{1+(1/n)} \sin \dfrac{2\pi}{t-(2j-1)\,2^{-n}} & \text{for } t \in I_{n,j} \\ 0 & \text{for } t \notin I_{n,j}. \end{cases}$$

The function

$$g_n = \sum_{j=1}^{2^{n-1}} g_{n,j}$$

is obviously continuous on the unit interval and

$$\sup_{0 \leqq t \leqq 1} |g_n(t)| \leqq 4^{-n^2-n}.$$

The series

$$f = \sum_{n=1}^{\infty} g_n$$

converges uniformly on $[0, 1]$ and defines a continuous curve

$$\Gamma = \{(t, f(t))/t \in [0, 1]\}.$$

We shall show that Γ has the desired properties.

To this end, we introduce the variation V_h of a real function h on an interval I, that is

$$V_h(I) = \int_I |dh| \leqq |I| \sup_{x \in I} |h'(x)| \qquad \text{(if } h \text{ is derivable).}$$

Notice that the length of the curve $\{(x, h(x))/x \in I\}$ is bounded from above by $|I| + V_h(I)$, a fact we shall use later.

We first compute $V_{g_n}(I_{n,j})$, namely

$$\begin{aligned} V_{g_n}(I_{n,j}) = V_{g_{n,j}}(I_{n,j}) &= \sum_{\substack{i \in \mathbf{Z} \\ |i+\frac{1}{4}| \geq 4^{n^2}}} |i + \tfrac{1}{4}|^{-1-1/n} + \sum_{\substack{i \in \mathbf{Z} \\ |i+\frac{3}{4}| \geq 4^{n^2}}} |i + \tfrac{3}{4}|^{-1-1/n} \\ &= 4(1 + o(1)) \sum_{i \geq 4^{n^2}} i^{-1-1/n} = 4(1 + o(1)) \int_{4^{n^2}}^{\infty} x^{-1-1/n}\,dx \\ &= 4^{-n+1} n(1 + o(1)). \end{aligned}$$

For a fixed n, the intervals $I_{n,j}$ are disjoint. Therefore, letting I denote the unit interval, we obtain

$$V_{g_n}(I) = 2^{n-1} V_{g_n}(I_{n,j}) = 2^{-n+1} n(1 + o(1)).$$

Hence

$$V_f(I) \leqq \sum_{n=1}^{\infty} V_{g_n}(I) = \sum_{n=1}^{\infty} n2^{-n+1}(1 + o(1)) < \infty,$$

which establishes that Γ is rectifiable.

In order to describe the set of M's for which $d(M) = \infty$, we shall need a lower bound for V_f. Put

$$f_n = \sum_{m=1}^{n-1} g_m, \qquad r_n = \sum_{m=n+1}^{\infty} g_m .$$

The trivial observation that $f = g_n + f_n + r_n$ now implies

$$V_f(I_{n,j}) \geqq V_{g_n}(I_{n,j}) - V_{f_n}(I_{n,j}) - V_{r_n}(I_{n,j}).$$

We are therefore led to estimate $V_{g_m}(I_{n,j})$ in order to find an upper bound for $V_{f_n}(I_{n,j})$.

Suppose $m < n$. For all j, k we have

$$|(2j - 1)\, 2^{-n} - (2k - 1)\, 2^{-m}| \geqq 2^{-n}.$$

Hence $I_{n,j}$ intersects at most one $I_{m,k}$ (exactly one if $2m^2 \leqq n$). Therefore

$$V_{g_m}(I_{n,j}) = \begin{cases} V_{g_{m,k}}(I_{n,j}) & \text{if there exists a (unique) } k \text{ such that } I_{n,j} \cap I_{m,k} \neq \emptyset \\ 0 & \text{if } I_{n,j} \cap I_{m,k} = \emptyset, \quad \text{for all } k. \end{cases}$$

Furthermore

$$V_{g_{m,k}}(I_{n,j}) \leqq 2 \cdot 4^{-n^2} \sup_{t \in J} |g'_{m,k}(t)|,$$

where

$$J \subset \{t / |t - (2k - 1)\, 2^{-m}| \geqq 2^{-n} - 4^{-n^2}\}.$$

But

$$\begin{aligned} |g'_{m,k}(t)| &\leqq \left(1 + \frac{1}{n}\right) |t - (2k - 1)\, 2^{-n}|^{1/n} \\ &\quad + |t - (2k - 1)\, 2^{-n}|^{1+1/n} \cdot 2\pi\, |t - (2k - 1)\, 2^{-m}|^{-2} \\ &\leqq 2\pi \cdot 2^n (1 + o(1)) \qquad \text{for } t \in J, \end{aligned}$$

whence

$$V_{g_{m,k}}(I_{n,j}) \leqq \pi 2^{2+n-2n^2}(1 + o(1))$$

where the term $o(1)$ is independent of m.

We now look for an upper bound for $V_{r_n}(I_{n,j})$, that is

$$\begin{aligned} V_{r_n}(I_{n,j}) &= V_{r_{2n^2-1}}(I_{n,j}) \leqq V_{r_{2n^2-1}}(I) \\ &\leqq \sum_{m=2n^2}^{\infty} m2^{-m+1}(1 + o(1)) \leqq 2^{-2n^2+3} n^2 (1 + o(1)). \end{aligned}$$

Finally, collecting the various results,

$$\begin{aligned} V_f(I_{n,j}) &\geqq V_{g_n}(I_{n,j}) - V_{f_n}(I_{n,j}) - V_{r_n}(I_{n,j}) \\ &\geqq n4^{-n+1}(1 + o(1)) - 4\pi n 2^{n-2n^2}(1 + o(1)) - 8n^2 2^{-2n^2}(1 + o(1)) \qquad (4) \\ &\geqq n4^{-n+1}(1 + o(1)). \end{aligned}$$

Let $P_{n,j} \in \Gamma$ be the point with coordinates $x = (2j - 1)\, 2^{-n} - 4^{-n^2}$, $y = 0$, and let $Q_{n,j} \in \Gamma$ be the point $x = (2j - 1)\, 2^{-n} + 4^{-n^2}$, $y = 0$, so that $P_{n,j}$ and $Q_{n,j}$ are the endpoints of the interval $I_{n,j}$. We claim that

$$\lim_{n\to\infty} \frac{|\widehat{P_{n,j}Q_{n,j}}|}{|\overline{P_{n,j}Q_{n,j}}|} = \infty .$$

Indeed,

$$|\widehat{P_{n,j}Q_{n,j}}| \geqq V_f(I_{n,j}) \geqq n4^{-n+1}(1 + o(1))$$

from (4).

On the other hand, since

$$|\overline{P_{n,j}Q_{n,j}}| = 2 \cdot 4^{-n^2}$$

we have

$$\lim_{n\to\infty} \frac{|\widehat{P_{n,j}Q_{n,j}}|}{|\overline{P_{n,j}Q_{n,j}}|} \geqq \lim_{n\to\infty} \frac{n4^{-n+1}}{2 \cdot 4^{-n^2}} = \infty .$$

The set Ω of points of Γ whose abscissae belong to infinitely many $I_{n,j}$ is dense and uncountable. We have therefore shown that $d(M) = \infty$ for all $M \in \Omega$.

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