Decay to Zero in Critical Cases of Second Order Ordinary Differential Equations of Duffing Type

J. M. BALL & J. CARR

Communicated by J. SERRIN

1. Introduction

We study in this paper the decay to zero of solutions of the equation

$$\ddot{u}+\dot{u}+f(u)=0, \tag{1}$$

where f is a nonlinear C^1 function satisfying $f(0)=f'(0)=0$, $rf(r)>0$ for $r\neq 0$, and $\int_0^r f(s)\,ds\to\infty$ as $|r|\to\infty$. These conditions ensure that every solution of (1) tends to zero as $t\to\infty$. Under quite mild additional assumptions on f we give a reasonably complete description of the asymptotic behaviour of all solutions of (1). Because $f'(0)=0$ the rate of decay of solutions cannot be determined by linearization. Our assumptions are satisfied, for example, by

$$f(u)=|u|^{\alpha-1}\,|\log|u||^{\beta}\,u,$$

where $\alpha>1$ and β are constants, and by finite sums of functions of this type.

The results are typified and motivated by the case $f(u)=u^3$, which may be regarded as the special case $a=0$ of the damped Duffing equation

$$\ddot{u}+\dot{u}+au+u^3=0; \tag{2}$$

in the introduction we concentrate on this example. One application where (2) arises is in damped motion of an extensible elastic rod with hinged ends; a crude model of this has been studied by BALL [1–3] and consists of the initial-boundary value problem

$$\begin{aligned}
&\ddot{w}+\dot{w}+w_{xxxx}-\left(\beta+\frac{2}{\pi^4}\int_0^1 w_\xi^2(\xi,t)\,d\xi\right)w_{xx}=0,\\
&w=w_{xx}=0 \quad \text{at } x=0,1,\\
&w(x,0)=w_0(x), \qquad \dot{w}(x,0)=w_1(x).
\end{aligned} \tag{3}$$

In (3) $w(x,t)$ is the transverse deflection and β is a constant proportional to the tensile axial load induced when the rod is constrained to lie straight. If w_0 and w_1

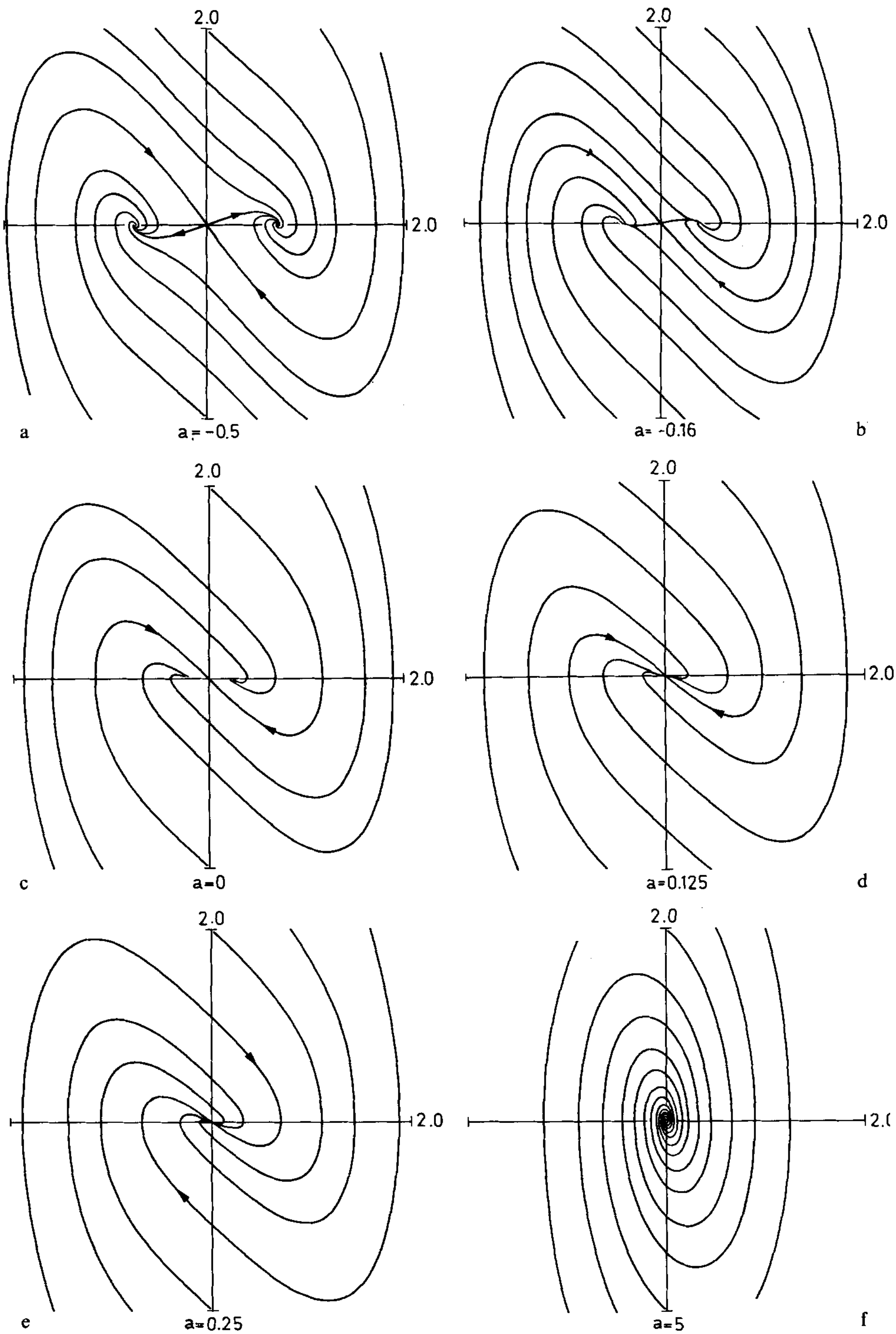

Fig. 1a – f

are scalar multiples of $\sin \pi x$, the solution of (3) has the form $w(x,t)=u(t)\sin \pi x$, where u satisfies (2) with $a=\pi^2(\pi^2+\beta)$. It follows that $a<0$, $a=0$, $a>0$ according as $\beta<-\pi^2$, $\beta=-\pi^2$, $\beta>-\pi^2$. The critical case $a=0$ in which we are interested corresponds to the situation when the axial load is compressive and exactly equal to the Euler load of the rod. In Fig. 1 are shown the $(u,\dot{u})$ phase plane diagrams for (2) corresponding to the values $a=-0.5$, $a=-0.16$, $a=0$, $a=0.125$, $a=0.25$ and $a=5$. When $a<0$ there are 3 equilibrium points, namely $u=\pm(-a)^{\frac{1}{2}}$ and $u=0$, the first two being stable and $u=0$ being unstable. The stable manifold of $u=0$ forms a separatrix, the unstable manifold consisting of two Lyapunov stable orbits connecting 0 to $\pm(-a)^{\frac{1}{2}}$. Convergence to each equilibrium point is exponential. When $a>0$ the only equilibrium point is $u=0$, convergence to it again being exponential. The local phase portrait is easily obtained by linearization. For $0<a<\frac{1}{4}$, $u=0$ is a node with two asymptotic directions, namely the lines $\dot{u}=m_{\pm}u$, where $m_{\pm}=-\dfrac{1\pm\sqrt{1-4a}}{2}$. These directions coincide when $a=\frac{1}{4}$. For $a>\frac{1}{4}$, $u=0$ is a focus. (Most of these facts are proved in [3].)

In the case $a=0$ we show that $u=0$ is a node with two asymptotic directions, exactly as for the case $0<a<\frac{1}{4}$. There are precisely two solutions approaching zero with slope -1, convergence being exponential. These two solutions correspond to the stable manifold of $u=0$ in the case $a<0$, and for $0<a<\frac{1}{4}$ they correspond to the two solutions approaching zero with slope m_-. All other solutions approach zero tangential to the u-axis, and have asymptotic form

$$u(t)=\pm\left[\frac{1}{\sqrt{2}}t^{-\frac{1}{2}}-\frac{3}{4\sqrt{2}}t^{-\frac{3}{2}}\log t+O(t^{-\frac{3}{2}})\right].$$

In particular every nonzero solution satisfies $u\dot{u}<0$ eventually (this may be proved also for $0<a\leqq\frac{1}{4}$ by the method of Theorem 2.2).

For general f the situation is qualitatively the same. Under our assumptions there are two exponential solutions, while all other solutions have asymptotic form

$$u(t)=U(t)+f(U(t))\log|f(U(t))|+O(f(U(t))),$$

where U satisfies

$$\dot{U}+f(U)=0, \qquad U(0)=\pm 1.$$

Finally we note that, among other applications, equation (1) governs the decay of travelling wave solutions $u(\xi)$, $\xi=x-ct$, $c>0$, to the nonlinear diffusion equation

$$u_t=u_{xx}+f(u).$$

2. General Behaviour of Solutions

We consider equation (1) under the following hypotheses on f:

H1. f is continuously differentiable.

H2. $rf(r)\geqq 0$ for all r, with $f(r)=0$ if and only if $r=0$.

H3. $F(r)\to\infty$ as $|r|\to\infty$, where

$$F(r)=\int_0^r f(s)\,ds.$$

For $\phi,\psi\in\mathscr{R}$ we define $V(\psi,\phi)=\frac{1}{2}\phi^2+F(\psi)$.

Theorem 2.1. *For any real u_0, u_1 there exists a unique solution $u(t)$ to* (1) *which is defined for all $t\in\mathscr{R}$, is three times continuously differentiable, and satisfies $u(0)=u_0$, $\dot{u}(0)=u_1$. Furthermore u and $\dot{u}$ tend to zero as $t\to\infty$.*

Proof. Local existence and uniqueness follows from standard theorems on ordinary differential equations. If u satisfies (1) locally in t then

$$\dot{V}(u,\dot{u})=-\dot{u}^2,$$

so that by H3 both u and $\dot{u}$ are bounded for $t\geqq 0$. Standard results now imply that u exists for all $t\geqq 0$. Existence for all $t\leqq 0$ follows similarly from the inequality $\dot{V}\geqq -2V$. For the last assertion of the theorem see HALE [5, p. 298].

Next we show that u does not oscillate.

Theorem 2.2. *Let $f'(0)=0$. Then either $u\equiv 0$ or $u\,\dot{u}<0$ for large enough t.*

Proof. Let $v=e^{t/2}u$ and suppose that u is not identically zero. Then

$$\ddot{v}-\left(\frac{1}{4}-\frac{f(u)}{u}\right)v=0,$$

so that, by Theorem 2.1, $\ddot{v}v\geqq 0$ for large enough t. Hence $v\dot{v}$ has only finitely many roots, so that u is eventually strictly positive or negative. But (1) implies that $\ddot{u}$ has the opposite sign to u when $\dot{u}=0$. Since $u\to 0$ as $t\to\infty$ it follows that $u\dot{u}<0$ for large enough t.

From now on we assume that $u>0$ for large enough t; in particular u is not the zero solution. In the rest of the paper we will make certain assumptions on the behaviour of $f(r)$ for positive r. Analogous assumptions on the behaviour of $f(r)$ for negative r lead to corresponding results for solutions u of (1) satisfying $u<0$ for large enough t. If f is odd the behaviour of these solutions can be obtained trivially from that of the eventually positive ones.

Lemma 2.3. *Let $f'(0)=0$. Then $\ddot{u}/\dot{u}$ tends either to 0 or to -1 as $t\to\infty$.*

Proof. Let $q=\ddot{u}/\dot{u}$. Differentiating (1) with respect to t we obtain the Riccati equation

$$\dot{q}+q+q^2=g(t), \tag{4}$$

where $g(t)=-f'(u(t))$ tends to zero as $t\to\infty$. By Theorem 2.2, $q(t)>-1$ for large enough t. Also $q(t)$ is bounded for $t\geqq 0$, since if not there would exist a sequence $t_n\to\infty$ with $q(t_n)\to\infty$, $\dot{q}(t_n)>0$, which contradicts (4). It is also clear from (4) that if $C\neq 0$, 1 the equation $q(t)=C$ has at most finitely many positive roots. It follows that q tends to a limit, which by (4) must be 0 or -1.

Theorem 2.4. *Let $f'(0)=0$, and let U denote the solution of the initial-value problem*

$$\dot{U}+f(U)=0, \qquad U(0)=1. \tag{5}$$

Then as $t\to\infty$ either

$$\frac{\log u(t)}{t}\to -1,$$

or

$$\frac{U^{-1}(u(t))}{t} \to 1.$$

Proof. By Lemma 2.3, $\ddot{u}/\dot{u} \to -1$ or 0. In either case, by L'Hospital's rule,

$$\lim_{t\to\infty} \frac{\ddot{u}}{\dot{u}} = \lim_{t\to\infty} \frac{\dot{u}}{u} = \lim_{t\to\infty} \frac{\dot{\overline{\log u}}}{\dot{t}} = \lim_{t\to\infty} \frac{\log u}{t}. \tag{6}$$

But if $\ddot{u}/\dot{u} \to 0$, then $\dot{u}/f(u) \to -1$, so that again by L'Hospital's rule

$$\lim_{t\to\infty} \frac{\int_1^u \frac{1}{f(r)}\, dr}{t} = -1.$$

Since $U^{-1}(u(t)) = -\int_1^{u(t)} \frac{1}{f(r)}\, dr$, the result follows at once.

3. The Exponential Solution

In this section we establish the existence of a solution u (which is unique up to parametrization) satisfying the first possibility of Theorem 2.4, namely such that $\lim_{t\to\infty} \frac{\dot{u}}{u} = \lim_{t\to\infty} \frac{\log u}{t} = -1$.

Theorem 3.1. *Let $f'(0)=0$. There exist numbers $\delta>0$, $\delta_1>0$ such that for any $y_0 \in [-\delta_1, 0)$ there is a unique solution $u(t)=u(t, y_0)$ to (1), which is defined for $t\in\mathcal{R}$, satisfies $|u(t)|+|\dot{u}(t)|<\delta$ for all $t \geqq 0$, and is such that $\dot{u}(0, y_0)=y_0$ and $\lim_{t\to\infty} \frac{\dot{u}}{u} = -1$. Furthermore $u(0, y_0) \in C^1[-\delta_1, 0)$, $\frac{du}{dy_0}(0, y_0) \to -1$ as $y_0 \to 0-$, and there exists a number $\sigma(y_0)>0$ such that, for any $\gamma \geqq 0$, u has the asymptotic form*

$$\begin{aligned} u(t) &= \sigma\, e^{-t}[1+o(t^{-\gamma})] \\ \dot{u}(t) &= -\sigma\, e^{-t}[1+o(t^{-\gamma})]. \end{aligned} \tag{7}$$

Proof. Let $x=\dot{u}$, $y=u+\dot{u}$, so that (1) reduces to the system

$$\begin{aligned} \dot{x} &= -x - f(y-x), \\ \dot{y} &= -f(y-x). \end{aligned} \tag{8}$$

The theorem is then a consequence of a result of HARTMAN [p. 296, Corollary 8.1, p. 313] and Theorems 2.1, 2.2.

If f satisfies extra conditions then more terms in the asymptotic expansion of u may be obtained. For example, if $f(r)=O(r^\alpha)$ as $r\to 0+$ for some $\alpha>1$, then

$$u(t) = \sigma[e^{-t} + O(e^{-\alpha t})] \tag{9}$$

4. Asymptotic Form of Other Solutions

We now consider solutions u satisfying the second possibility given by Theorem 2.4, namely that $U^{-1}(u(t))/t \to 1$ as $t\to\infty$. Before rigorously establishing an asymptotic representation for these remaining solutions, we indicate briefly why this representation is to be expected. We seek a solution to (1) of the form $u=U+g$. Substituting this into (1), expanding the term $f(U+g)$ in a Taylor series, and neglecting $\ddot{g}$ and powers of g greater than 1, we obtain the equation

$$\dot{g}+f'(U)\,g=f'(U)f(U), \tag{10}$$

which has solution

$$g(t)=C f(U)+f(U)\log f(U)$$

where C is an arbitrary constant. We thus expect $[u-U-f(U)\log f(U)]/f(U)$ to tend to a limit as $t\to\infty$.

We now list various extra hypotheses on $f(r)$ as $r\to 0+$ which we will need to establish this behaviour.

H 4. $\left(\dfrac{f(r)}{r}\right)^{\varepsilon}\log f(r)\to 0$ as $r\to 0+$ for all $\varepsilon>0$.

H 5. $\dfrac{f(r)}{r}\displaystyle\int_r^1 \frac{1}{f(s)}\,ds=O(1)$.

H 6. $f\in C^2(0,\delta)$ for some $\delta>0$ and

$$\frac{r^2 f''(r)}{f(r)}=O(1).$$

H 7. $\dfrac{f(k\,r)}{f(r)}\to 1$ as $r\to 0+,\ k\to 1$.

Remarks. 1. Suppose that f satisfies $f'(0)=0$, the hypothesis H 6, and the condition (weaker than H 7)

$$\limsup_{\substack{k\to 1\\ r\to 0+}}\frac{f(k\,r)}{f(r)}<\infty.$$

Then f satisfies H 7, and in addition

$$\frac{r f'(r)}{f(r)}=O(1) \tag{11}$$

as $r\to 0+$. This follows from the representation

$$\frac{f(k\,r)}{f(r)}=1+(k-1)\frac{r f'(r)}{f(r)}+\frac{(k-1)^2}{2\bar{k}^2}\,\frac{(\bar{k}\,r)^2 f''(\bar{k}\,r)}{f(\bar{k}\,r)}\,\frac{f(\bar{k}\,r)}{f(r)}$$

where $|\bar{k}-1|\leqq|k-1|$.

2. If f satisfies

$$\left|\frac{r f'(r)}{f(r)}-1\right|\geqq C>0$$

for small enough $r>0$, where C is a constant, then f satisfies H 5. To prove this it is sufficient to note that for large enough t_0 we have that

$$\frac{t-t_0}{(U/\dot{U})(t)-(U/\dot{U})(t_0)}=\frac{1}{1-(U\ddot{U}/\dot{U}^2)(t^*)},$$

where $t_0 \leqq t^* \leqq t$.

3. Examples of functions f satisfying all the hypotheses H1–H7 are given by

$$f(r)=|r|^{\alpha-1}\left|\log|r|\right|^{\beta} r,$$

where $\alpha>1$ and β are constants, and by finite sums of terms of this type: this is easy to verify using Remark 2.

Lemma 4.1. *Let f satisfy* H 5, *and suppose that* $\frac{U^{-1}(u(t))}{t} \to 1$ *as* $t\to\infty$. *Then* $\frac{u(t)}{U(t)} \to 1$ *as* $t\to\infty$.

Proof. Let $t_n\to\infty$ and let $s_n=U^{-1}(u(t_n))$. Then $\frac{s_n}{t_n}\to 1$ and we have to show that $\frac{U(s_n)}{U(t_n)}\to 1$. Let

$$\varepsilon_n=\max\left(\left|\frac{s_n}{t_n}-1\right|, \left|\frac{t_n}{s_n}-1\right|\right),$$

so that $\varepsilon_n\to 0$. For fixed n, if $s_n \geqq t_n$ then $U(s_n)=U(t_n)+\dot{U}(\zeta_n)(s_n-t_n)$ for some $\zeta_n\in[t_n, s_n]$. Therefore

$$\left|\frac{U(s_n)}{U(t_n)}-1\right| \leqq \frac{f(U(\zeta_n))}{U(\zeta_n)}\zeta_n \cdot \frac{t_n}{\zeta_n}\left(\frac{s_n}{t_n}-1\right)\frac{U(\zeta_n)}{U(t_n)} \leqq C\,\varepsilon_n,$$

where C is a constant and we have used H 5. If $t_n \geqq s_n$ we obtain by transposition

$$\left|\frac{U(s_n)}{U(t_n)}-1\right| \leqq C\,\varepsilon_n \frac{U(s_n)}{U(t_n)},$$

which implies that $\frac{U(s_n)}{U(t_n)}$ is bounded, the bound being independent of n. Thus $\frac{U(s_n)}{U(t_n)}\to 1$.

Next we prove a boundedness result for solutions of a second order linear ordinary differential equation. Although the result can be obtained from one of Bellman [4] via a transformation, this procedure is very involved. We therefore give a simpler proof.

Lemma 4.2. *Let $\varepsilon>0$ and let $g=g(t)$ satisfy the equation*

$$\ddot{g}+(1+a(t))\dot{g}+\frac{b(t)}{t}g=h(t), \tag{12}$$

where a, b, h are continuous functions satisfying $a(t)\to 0$, $b(t)\to 0$ as $t\to\infty$, and $\int_1^\infty t^{1-2\varepsilon}h^2(t)\,dt<\infty$. *Then* $\frac{g(t)}{t^\varepsilon}$ *is bounded for large* t.

Proof. Let $E(t)=t^{-2\varepsilon}[t\,\dot{g}^2+\varepsilon\, g^2]$. Then

$$t^{2\varepsilon}\dot{E}(t)=[1-2\varepsilon-2t(1+a)]\,\dot{g}^2-2\varepsilon^2\frac{g^2}{t}+[2\varepsilon-2b(t)]\,g\,\dot{g}+2t\,h(t)\,\dot{g}.$$

Using the inequalities

$$2g\,\dot{g}(\varepsilon-b)\leqq t\,\dot{g}^2(1+|b|)+\frac{g^2}{t}(\varepsilon^2+|b|)$$

and

$$h(t)\,\dot{g}\leqq h^2(t)+\frac{\dot{g}^2}{4}$$

we see that, for large enough t,

$$\dot{E}(t)\leqq 2\,t^{1-2\varepsilon}\,h^2(t)$$

from which the lemma follows.

Remark. Even if $h\equiv 0$, g need not be bounded for $t>0$, as the example $a\equiv 0$, $b(t)=(1-t)/t\log t$, $g(t)=\log t$ shows.

Theorem 4.3. *Let f satisfy the hypotheses* H1–H7. *Let* $\dfrac{U^{-1}(u(t))}{t}\to 1$. *Then as* $t\to\infty$ *we have*

$$[u-U-f(U)\log f(U)]/f(U)\to L$$

for some constant L. Conversely, given any real constant L there is a solution u which has the above asymptotic form.

Proof. Let u satisfy $\dfrac{U^{-1}(u(t))}{t}\to 1$, and write

$$u=U-\dot{U}\log|\dot{U}|+\dot{U}\,g. \tag{13}$$

Then for large enough t

$$\begin{aligned} f(u(t))=&f(U(t))+\big(\dot{U}(t)\,g(t)-\dot{U}(t)\log|\dot{U}(t)|\big)f'(U(t))\\ &+\big(\dot{U}(t)\,g(t)-\dot{U}(t)\log|\dot{U}(t)|\big)^2 f''(U^*(t))/2, \end{aligned} \tag{14}$$

where $|U(t)-U^*(t)|\leqq|u(t)-U(t)|$. (Note that $U^*(t)>0$ by Lemma 4.1.) Substituting (13) and (14) into (1) we obtain

$$\ddot{g}(t)+(1+a(t))\,\dot{g}(t)+\beta(t)\,g(t)+\gamma(t)\,g^2(t)=h(t), \tag{15}$$

where

$$\begin{aligned} &a=\frac{2\ddot{U}}{\dot{U}},\qquad \beta=\frac{\dddot{U}}{\dot{U}}-\dot{U}\log|\dot{U}|\,f''(U^*),\qquad \gamma=\dot{U}f''(U^*)/2,\\ &h=\left(\frac{\ddot{U}}{\dot{U}}\right)^2+\frac{\dddot{U}}{\dot{U}}+\frac{\ddot{U}\log|\dot{U}|}{\dot{U}}-\frac{\dot{U}(\log|\dot{U}|)^2}{2}f''(U^*). \end{aligned} \tag{16}$$

We next estimate the behaviour as $t\to\infty$ of the coefficients in (15). First, since $a(t)=-2f'(U(t))$ it follows from H4 that

$$a(t)\to 0. \tag{17}$$

Next

$$\beta = f''(U)f(U) + f'(U)^2 + f(U)\log f(U) f''(U^*)$$
$$= \frac{U^2 f''(U)}{f(U)}\left(\frac{f(U)}{U}\right)^2 + \left(\frac{Uf'(U)}{f(U)}\right)^2\left(\frac{f(U)}{U}\right)^2$$
$$+ \frac{f^2(U)\log f(U)}{U^2}\left(\frac{U^{*2} f''(U^*)}{f(U^*)}\right)\frac{f(U^*)}{f(U)}\left(\frac{U}{U^*}\right)^2$$

By Lemma 4.1, $\frac{U(t)}{U^*(t)} \to 1$ as $t \to \infty$, so that by H7, $\frac{f(U^*(t))}{f(U(t))} \to 1$ as $t \to \infty$. Also, by H5, $\frac{tf(U(t))}{U(t)}$ is bounded. It then follows from H4, H6 and (11) that for any $\varepsilon > 0$

$$t^{2-\varepsilon}\beta(t) \to 0. \tag{18}$$

Similarly it can be shown that

$$\gamma(t)[U(t)/f(U(t))]^2 \tag{19}$$

is bounded and that for any $\varepsilon > 0$

$$t^{2-\varepsilon}h(t) \to 0. \tag{20}$$

From (13), H4 and Lemma 4.1 we have

$$\frac{f(U(t))}{U(t)}g(t) \to 0. \tag{21}$$

Now let $b(t) = t[\beta(t) + \gamma(t)g(t)]$. By (18), (19), (21), H4 and H5 we see that $b(t) \to 0$. Therefore by (17), (20) and Lemma 4.2

$$g(t)/t^\varepsilon \quad \text{is bounded as } t \to \infty. \tag{22}$$

Substituting (22) back into (15) we find (using (18), (19) and (20)) that

$$\ddot{g}(t) + (1 + a(t))\dot{g}(t) = H(t), \tag{23}$$

with $H(t)$ a continuous function satisfying $t^{2-\varepsilon}H(t) \to 0$.

Solving (23) shows that $g(t)$ tends to a limit as $t \to \infty$, which completes the proof that $[u - U - f(U)\log f(U)]/f(U) \to L$ as $t \to \infty$ for some constant L. We now show that, given any L, there is a solution u which has the above asymptotic form. If g satisfies

$$\ddot{g}(t) + (1 + 2\ddot{U}/\dot{U})\dot{g}(t) = Q(g(t), t) \tag{24}$$

where $\dot{U}Q(g, t) = -\dot{U}f(U - \dot{U}\log|\dot{U}| + \dot{U}g) - \left(\frac{d}{dt} + \frac{d^2}{dt^2}\right)(U - \dot{U}\log|\dot{U}|)$, then u satisfies (1), where u is defined by (13). It will be sufficient to prove that (24) has a solution $g(t)$ with $g(t) \to L$ as $t \to \infty$. Let $C[t_0, \infty)$ be the set of bounded continuous functions on $[t_0, \infty)$ with the supremum norm. For $g \in C[t_0, \infty)$ define

$$(Tg)(t) = L - V(t)\int_{t_0}^{t} e^s \dot{U}^2(s)\,Q(g(s), s)\,ds - \int_{t}^{\infty} V(s)\,e^s \dot{U}^2(s)\,Q(g(s), s)\,ds,$$

where $V(t)=\int_t^\infty e^{-s}\dot{U}^{-2}(s)\,ds$. A straightforward but tedious argument shows that there is a constant C such that T is a contraction on $\{g\in C[t_0,\infty): \|g-L\|\leqq C\}$ for large enough t_0, that the fixed point g satisfies (24), and that $g(t)\to L$ as $t\to\infty$.

We now present an alternative proof of the first part of Theorem 4.3 which Professor P. HARTMAN communicated to us and has kindly allowed us to use here. Assume initially that H1–H3, H5 and H7 hold and that

H8. $\displaystyle\int_{0+}\frac{f'(r)^2}{f(r)}\,dr<\infty,$

H9. $\displaystyle\int_{0+}|f''(r)|\,dr<\infty,$

these two new conditions being implied by H4, H6 and H7.

In what follows C denotes a generic constant and T is chosen so that $u(T)=1$. Integrating the equation

$$\frac{-\dot{u}}{f(u)}=1+\frac{\ddot{u}}{f(u)}$$

over $[T,t]$, integrating by parts, and using the condition $\dot{u}/f(u)\to -1$ as $t\to\infty$, we deduce that

$$\begin{aligned}U^{-1}(u(t))&=t+C+\int_T^t\frac{\ddot{u}(s)\,ds}{f(u(s))}\\&=t+C+o(1)+\int_T^t f'(u)\left(\frac{\dot{u}}{f(u)}\right)^2 ds.\end{aligned}$$

Since

$$\left(\frac{\dot{u}}{f(u)}\right)^2=-\frac{\dot{u}}{f(u)}-\frac{\dot{u}\,\ddot{u}}{f^2(u)}$$

we obtain

$$U^{-1}(u(t))=t+C+o(1)-\log f(u)-\int_T^t\frac{f'(u)\,\dot{u}\,\ddot{u}}{f^2(u)}\,ds. \tag{25}$$

But

$$\int_T^t\frac{f'(u)\,\dot{u}\,\ddot{u}}{f^2(u)}\,ds=C+o(1)-\tfrac{1}{2}\int_T^t\frac{f''(u)\,\dot{u}^3}{f^2(u)}\,ds+\int_T^t\frac{f'(u)\,\dot{u}^3}{f^3(u)}\,ds. \tag{26}$$

Both integrals in (26) are absolutely convergent by H8 and H9, since, for example,

$$\int_T^t\left|\frac{f''(u)\,\dot{u}^3}{f^2(u)}\right|ds=-\int_T^t(1+o(1))\,|f''(u)|\,\dot{u}\,ds=\int_{u(t)}^{u(T)}(1+o(1))\,|f''(r)|\,dr.$$

Thus

$$u(t)=U(t+C-\log f(u(t))+o(1)),$$

and so by Lemma 4.1 and H7 we obtain the representation

$$u(t)=U(t+C-\log f(U(t))+o(1)). \tag{27}$$

It follows from (27) that

$$u(t)=U(t)+\big(C-\log f(U(t))+o(1)\big)f\big(U(t)\big)$$
$$+\tfrac{1}{2}\big(C-\log f(U(t))+o(1)\big)^2 f\big(U(t^*)\big)f'\big(U(t^*)\big),$$

where $t^*\in\big[t, t+C-\log f\big(U(t)\big)+o(1)\big]$. By using H4, H5, H7 and (11) it can now easily be shown that $[u-U-f(U)\log f(U)]/f(U)$ tends to a limit as $t\to\infty$.

In the special case $f(r)=|r|^{\alpha-1}r$, $\alpha>1$, we obtain from Theorem 4.3 the asymptotic form

$$u(t)=a_1(\alpha)\,t^{1/1-\alpha}-a_2(\alpha)\,t^{\alpha/1-\alpha}\log t+t^{\alpha/1-\alpha}\big(C+o(1)\big)$$

where $a_1(\alpha)=(\alpha-1)^{1/1-\alpha}$, $a_2(\alpha)=\alpha(\alpha-1)^{(2\alpha-1)/(1-\alpha)}$, and C is a constant.

Further terms in the asymptotic expansion may be obtained in an essentially similar way. In the case $f(r)=r^3$, for example, it can be shown that

$$u(t)=\frac{t^{-\frac{1}{2}}}{\sqrt{2}}-\frac{3t^{-\frac{3}{2}}}{4\sqrt{2}}\log t+Ct^{-\frac{3}{2}}+\frac{27t^{-\frac{5}{2}}}{32\sqrt{2}}(\log t)^2-\frac{9t^{-\frac{5}{2}}}{8\sqrt{2}}\log t$$
$$+\left(\frac{15}{8\sqrt{2}}+\frac{15C}{4}+\frac{3C^2}{\sqrt{2}}\right)t^{-\frac{5}{2}}+o(t^{-\frac{5}{2}}).$$

Only one free parameter C appears in this and similar expansions. The other free parameter anticipated in a second order equation parametrizes exponentially small terms.

Acknowledgement. We are indebted to Professor P. Hartman for his helpful comments on an earlier draft of this paper. We would also like to thank Dr. J.C. Eilbeck for help with the numerical computations associated with Figure 1.

References

1. Ball, J. M., Initial-boundary value problems for an extensible beam, J. Math. Anal. Appl. **42**, 61–90 (1973).
2. Ball, J. M., Stability theory for an extensible beam, J. Diff, Eqns. **14**, 339–418 (1973).
3. Ball, J. M., Saddle point analysis for an ordinary differential equation in a Banach space, and an application to dynamic buckling of a beam, Proc. Symp. Nonlinear Elasticity, ed., R. W. Dickey, Academic Press, 1973.
4. Bellman, R., Methods of Nonlinear Analysis, Vol. 1, Academic Press 1970.
5. Hale, J. K., "Ordinary differential equations", Wiley-Interscience, New York 1969.
6. Hartman, P., "Ordinary differential equations", Wiley, New York, 1964.

Department of Mathematics
Heriot-Watt University
Edinburgh, Scotland

(Received June 1, 1976)