Swirling Flow between Rotating Plates

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Dedicated to Professor J. L. Ericksen on his 60th Birthday

Introduction

The flow of the classical linearly viscous fluid between two infinite parallel planes rotating with constant (but different) angular velocities about a common axis has received a great deal of attention during the past 60 years (*cf.* PARTER [12]). However, until recently the assumptions which have been employed to study this problem have always led to solutions which are axisymmetric. Recently BERKER [3] in his study of the flow between parallel planes rotating with the same constant angular velocities about a common axis exhibited a one parameter family of solutions that are not axisymmetric. In this study we prove that when the planes rotate with different angular velocities about a common axis or distinct axes there is a one parameter family of solutions (for "large" viscosities).

Let Π_1 and Π_2 be two infinite planes parallel to the (x, y) plane, say Π_1 is the plane $z = -1$ while Π_2 is the plane $z = 1$. Let $a > 0$ be a fixed constant and suppose Π_1 rotates about a point $(x = 0, y = -\frac{1}{2}a, z = -1)$ with constant angular velocity Ω_{-1} while Π_2 rotates about the point $(x = 0, y = \frac{1}{2}a, z = 1)$ with constant angular velocity Ω_{+1} (*cf.* Figure 1). We suppose that a classical incompressible fluid fills the infinite space between these planes and we seek steady solutions of the Navier-Stokes equation which describe such a flow.

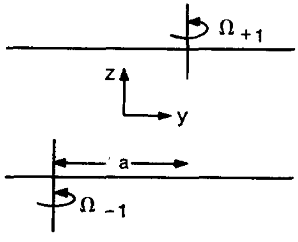

Fig. 1. Flow domain

Finally, we make the basic assumption that

$$U_z = -H(z), \tag{1.1}$$

that is, the component of velocity in the z direction is a function of z alone.

If $a = 0$ and we also assume that the flow is axisymmetric then the basic theory of VON KÁRMÁN [9] and BATCHELOR [2] leads to the following conclusions:

$$U_r = \tfrac{1}{2} r\, H'(z), \quad r = (x^2 + y^2),^{\frac{1}{2}} \tag{1.2}$$

$$U_\theta = \tfrac{1}{2} r\, G(z), \tag{1.3}$$

where the functions $\langle G(z), H(z) \rangle$ are solutions of the boundary-value problem

$$\varepsilon H^{iv} + HH''' + GG' = 0, \quad -1 < z < 1, \tag{1.4a}$$

$$\varepsilon G'' + HG' - H'G = 0, \quad -1 < z < 1 \tag{1.4b}$$

and

$$H(-1, \varepsilon) = H(1, \varepsilon) = 0 \text{ (no penetration)}, \tag{1.5a}$$

$$H'(-1, \varepsilon) = H'(1, \varepsilon) = 0 \text{ (no slip)}, \tag{1.5b}$$

$$G(-1, \varepsilon) = 2\Omega_{-1}, \quad G(1, \varepsilon) = 2\Omega_{+1}, \tag{1.5c}$$

where the positive parameter $\varepsilon > 0$ is related to the bulk viscosity.

This boundary-value problem has been studied at great length. There are many numerical studies and many formal asymptotic studies. There are also rigorous existence theorems

(i) for $\varepsilon \gg 1$ by HASTINGS [8] and ELCRAT [6],

(ii) for $\Omega_{-1} = \Omega_{+1} \neq 0$ and all $\varepsilon > 0$ by MCLEOD & PARTER [11],

(iii) for $0 < \varepsilon \ll 1$ by KREISS & PARTER [10].

The recent survey article [11] contains a reasonably up-to-date discussion of this problem.

When $\Omega_{-1} = \Omega_{+1} \neq 0$ there is one special solution (not the only solution—see [3]): the rigid body rotation given by

$$H(z, \varepsilon) \equiv 0, \quad G(z, \varepsilon) \equiv 2\Omega_{-1}. \tag{1.6}$$

It is not difficult to verify ([5], [18]) that this solution is "stable" and "isolated" relative to the von Kármán equations (1.4), (1.5). By "isolated" we mean there is a neighborhood of this solution wherein there are no other solutions and by "stable" we mean there is no bifurcation from this trivial solution; in particular, the linearized problem at this solution is non-singular. On the other hand, BERKER [3] has constructed a one-parameter family of solutions of the general steady Navier-Stokes equation which includes the rigid body motion. The rigid body motion is the *only* axisymmetric solution that belongs to BERKER's [3] family. In

Cartesian coordinates his solution takes the form (we have set $\Omega_{-1} = \Omega_{+1} = 1$)

$$U_x = -[y - g(z)], \quad U_y = [x - f(z)], \quad U_z = 0, \tag{1.7}$$

where

$$f(z) = l\left\{\frac{1-\phi(1)}{\Delta}[\phi(z) - \phi(1)] - \frac{\chi(1)}{\Delta}[\chi(z) - \chi(1)]\right\}, \tag{1.8a}$$

$$g(z) = l\left\{\frac{\chi(1)}{\Delta}[\phi(z) - \phi(1)] + \frac{1-\phi(1)}{\Delta}[\chi(z) - \chi(1)]\right\}, \tag{1.8b}$$

with

$$\phi(z) = \cosh mz \cdot \cos mz, \tag{1.9a}$$

$$\chi(z) = \sinh mz \cdot \sin mz, \tag{1.9b}$$

$$m = \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}}, \tag{1.9c}$$

$$\Delta = [1 - \phi(1)]^2 + [\chi(1)]^2 = (\cosh m - \cos m)^2, \tag{1.9d}$$

where l is an arbitrary positive constant. Observe that (1.7) shows that this solution satisfies the basic assumption (1.1).

The case $a \neq 0$ and $\Omega_{-1} = \Omega_{+1} = \Omega$ relates to the flow in the orthogonal rheometer, an instrument that is employed in determining the material moduli which characterize non-Newtonian fluids. Recently RAJAGOPAL [13] has studied the flow of general simple fluids in such a domain and RAJAGOPAL & GUPTA [15] and RAJAGOPAL & WINEMAN [17] have found exact solutions to the problem for certain non-Newtonian fluids.* RAJAGOPAL & GUPTA [16] have also established a one-parameter family of exact solution for an incompressible homogeneous fluid of second grade when $a = 0$ and $\Omega_{-1} = \Omega_{+1} = \Omega$.

When $a = 0$ and $\Omega_{-1} = \Omega_{+1}$, the solution corresponding to the usual von Kármán assumption leads to exactly one of the solutions in BERKER'S [3] one-parameter family, namely the rigid motion which is axisymmetric. In the case $a \neq 0$, and $\Omega_{-1} = \Omega_{+1}$, an exact solution has been obtained for the classical incompressible fluid by ABBOT & WALTERS [1]. The existence of such a solution is motivated in an earlier analysis by BERKER [4]. That motivates us to look for a more general class of solutions to the von Kármán problem (and to the corresponding problem when $a \neq 0$) which would reduce to the class of solutions exhibited by BERKER [3]. Thus, we seek a solution such as to have the following form in Cartesian coordinates:

$$U_x = \tfrac{1}{2}x\,H'(z) - \tfrac{1}{2}G(z)\,y + g(z), \tag{1.10a}$$

$$U_y = \tfrac{1}{2}y\,H'(z) + \tfrac{1}{2}G(z)\,x - f(z), \tag{1.10b}$$

$$U_z = -H(z). \tag{1.10c}$$

* More recently, GODDARD [7] has published results which are similar to those in [13].

In cylindrical coordinates this velocity field takes the form

$$V_r = \tfrac{1}{2}r\,H'(z) + g(z)\cos\theta - f(z)\sin\theta, \tag{1.11a}$$

$$V_\theta = \tfrac{1}{2}r\,G(z) - g(z)\sin\theta - f(z)\cos\theta, \tag{1.11b}$$

$$V_z = -H(z). \tag{1.11c}$$

Observe that, if $H = 0$ and $G = 2\Omega$ we have a velocity field of the form described by BERKER, while if $f = g = 0$ we have a velocity field of the form described by VON KÁRMÁN.

As we show in section 2, there is a solution of the Navier-Stokes equations of the form (1.10) [or (1.11)] if and only if the function $H(z)$, $G(z)$, $g(z)$, $f(z)$ are solutions of the boundary-value problem

$$\varepsilon H^{iv} + HH''' + GG' = 0, \quad -1 \leqq z \leqq 1, \tag{1.12a}$$

$$\varepsilon G'' + HG' - H'G = 0, \quad -' \leqq z \leqq 1, \tag{1.12b}$$

$$\varepsilon f''' + Hf'' + \tfrac{1}{2}H'f' - \tfrac{1}{2}H''f + \tfrac{1}{2}(Gg)' = 0, \tag{1.13a}$$

$$\varepsilon g''' + Hg'' + \tfrac{1}{2}H'g' - \tfrac{1}{2}H''g - \tfrac{1}{2}(Gf)' = 0, \tag{1.13b}$$

$$H(-1,\varepsilon) = H(1,\varepsilon) = 0, \quad \text{(no penetration)}, \tag{1.14a}$$

$$H'(-1,\varepsilon) = H'(1,\varepsilon) = 0, \quad \text{(no slip)}, \tag{1.14b}$$

$$G(-1,\varepsilon) = 2\Omega_{-1}, \quad G(1,\varepsilon) = 2\Omega_{+1}, \tag{1.14c}$$

$$f(-1,\varepsilon) = f(1,\varepsilon) = 0, \quad g(-1,\varepsilon) = -a\Omega_{-1}/2, \quad g(1,\varepsilon) = a\Omega_{+1}/2. \tag{1.15}$$

We note that equations (1.12) with boundary conditions (1.14) are exactly the nonlinear von Kármán equations for axially symmetric swirling flow for functions $\langle H(z,\varepsilon), G(z,\varepsilon)\rangle$ while the equations (1.13) with boundary conditions are *linear equations* for $\langle f(z,\varepsilon), g(z,\varepsilon)\rangle$ with coefficients depending on $\langle H(z,\varepsilon)$, $G(z,\varepsilon)\rangle$ which reflect the lack of symmetry and the possible displacement of the centers of rotation of the bounding planes. Moreover, given $\langle H(z,\varepsilon), G(z,\varepsilon)\rangle$ the equations (1.13) are a system of two third order equations with *only* four boundary conditions. Whenever there is a solution of the von Kármán equations one can ask two questions:

(i) In the case when $a = 0$, is the axisymmetric flow imbedded in a continuous one parameter family of more general solutions?

(ii) When $a \neq 0$, does this axisymmetric flow form the basis for a one parameter family of solutions of the problem for rotations about different centers?

In case (i) this is a *homogeneous* underdetermined system and the answer is *yes*! We need merely consider the additional condition

$$f'(-1) = g'(-1) = 0. \tag{1.16}$$

If this augmented homogeneous problem has a non-trivial solution $\langle f(z,\varepsilon)$, $g(z,\varepsilon)\rangle \neq \langle 0, 0\rangle$, then $\langle lf(z,\varepsilon), lg(z,\varepsilon)\rangle$ is also a solution for every real number l.

On the other hand, if the system (1.13), (1.15), (1.16) does not have a nontrivial solution; then the problem given by (1.13), (1.15) and

$$f'(-1) = l, \quad g'(-1) = 0, \tag{1.17}$$

yields a unique solution $\langle f(z, \varepsilon, l), g(z, \varepsilon, l)\rangle$ of the form

$$\langle f(z, \varepsilon, l), g(z, \varepsilon, l)\rangle = \langle lf(z, \varepsilon, 1), lg(z, \varepsilon, 1)\rangle .$$

This simple result has the following important consequence. In the classical case of two infinite parallel planes rotating about a common axis, (*i.e.*, $a = 0$) whenever there is a solution of the von Kármán equations (12), (14), this axisymmetric flow is imbedded in a one parameter family of solutions of the full Navier-Stokes equations. Thus, despite the intense interest in the von Kármán problem, within the class of *all* solutions of the Navier-Stokes equations, these special solutions are "unstable". Such a simple argument does not suffice for case (ii).

In either case one can ask a more subtle question: can we find a family $\langle f(z, \varepsilon, l), g(z, \varepsilon, l)\rangle$ which is continuous in both ε and l and (at the same time) has the geometric significance of the Berker solution for the special case $\Omega_{-1} = \Omega_{+1} = \Omega$? In other words, can we find solutions of (1.12), (1.13), (1.14), (1.15) and

$$g(0, \varepsilon, l) = 0, \quad f(0, \varepsilon, l) = l. \tag{1.18}$$

Since the system (1.13), (1.15) is linear, the answer is 'yes' for problem (i) if and only if it is also 'yes' for problems (ii).

In section 3 we answer these questions in the affirmative for *large* ε. While this result is an immediate consequence of the implicit function theorem (applied at $\varepsilon = \infty$) we will give a complete proof.

Equations of Motion

In this discussion we follow the outline of the argument given in [12, section 2]. A velocity field of the form (1.10) satisfies the constraint

$$\operatorname{div} U = \tfrac{1}{2}H'(z) + \tfrac{1}{2}H'(z) - H'(z) = 0. \tag{2.1}$$

We now turn to the equation:

$$\mu \Delta u - \varrho u \cdot \nabla u = \nabla p, \tag{2.2}$$

where μ denotes the viscosity, ϱ the density, and p is the pressure. We eliminate the pressure by taking the curl of both sides of (2.2) and obtain

$$\mu \Delta\omega - \varrho(\omega \times U) = 0, \tag{2.3a}$$

where

$$\omega = \operatorname{curl} U. \tag{2.3b}$$

A detailed calculation now yields

$$\Delta\omega = -\boldsymbol{i}\{\tfrac{1}{2}y H^{iv} + \tfrac{1}{2}x G''' - \tfrac{1}{2}f'''\} + \boldsymbol{j}\{\tfrac{1}{2}x H^{iv} - \tfrac{1}{2}y G''' + \tfrac{1}{2}g'''\} + \boldsymbol{k}\{G''\}, \tag{2.4}$$

while

$$\begin{aligned}\omega \times U = &-\boldsymbol{i}\{\tfrac{1}{2}x[(GH')' - (G'H)'] - \tfrac{1}{2}y[GG' + HH''']\\ &+ [\tfrac{1}{4}(Gg)' - \tfrac{1}{4}(fH')' + \tfrac{1}{2}(f'H)']\} + \boldsymbol{j}\{-\tfrac{1}{2}x[GG' + HH''']\\ &+ \tfrac{1}{2}y[(G'H)' - (GH')'] + [\tfrac{1}{4}(Gf)' + \tfrac{1}{4}(gH')' - \tfrac{1}{2}(g'H)']\}\\ &+ \boldsymbol{k}\{GH' - HG'\}.\end{aligned} \tag{2.5}$$

On equating the coefficients of $\boldsymbol{k}$ in (2.3a) we obtain

$$\frac{\mu}{\varrho} G'' + HG' - H'G = 0. \tag{2.6a}$$

On equating the coefficients of $\boldsymbol{i}x$ we obtain

$$\frac{\mu}{\varrho} G''' + (HG')' - (H'G)' = 0,$$

which is the same as (2.6a). On equating the coefficients of $\boldsymbol{i}y$ we obtain

$$\frac{\mu}{\varrho} H^{iv} + HH''' + GG' = 0. \tag{2.6b}$$

Finally, the zero$^{\text{th}}$-order terms in the coefficient of $\boldsymbol{i}$ yield

$$\frac{\mu}{\varrho} f''' + (H'f')' - \tfrac{1}{2}(H'f)' + \tfrac{1}{2}(Gg)' = 0, \tag{2.6c}$$

and the coefficients of $\boldsymbol{j}x$, $\boldsymbol{j}y$, and $\boldsymbol{j}$ yield the final equation

$$\frac{\mu}{\varrho} g''' + (Hg')' - \tfrac{1}{2}(H'g)' + \tfrac{1}{2}(Gf)' = 0. \tag{2.6d}$$

Thus we have established the equations (1.12) and (1.13) where

$$\varepsilon = \frac{\mu}{\varrho}.$$

We now turn our attention to the boundary conditions. Equation (1.11c) and the "no penetration" condition imply that

$$H(-1, \varepsilon) = H(1, \varepsilon) = 0. \tag{2.7a}$$

Equation (1.11b) and the conditions

$$v_\theta(r, \theta, \pm 1) = (r \pm \tfrac{1}{2}a)\, \Omega_{\pm 1}$$

yield

$$\operatorname*{Lim}_{r\to\infty} \frac{1}{r} v_\theta(r, \theta, \pm 1) = \Omega_{\pm 1} = \tfrac{1}{2}G(\pm 1).$$

Hence

$$G(-1, \varepsilon) = 2\Omega_{-1}, \quad G(1, \varepsilon) = 2\Omega_{+1}. \tag{2.7b}$$

Thus we have obtained the boundary conditions (1.14). Now, using (1.11b) and letting $r \to 0$ with a judicious choice of θ we obtain the boundary conditions (1.15).

Existence for large $\varepsilon \gg 1$

In this section we present what is essentially a standard argument for regular perturbation problems. The argument is given in some detail because we wish to emphasize the following facts.

(i) There is an $\varepsilon_0 \ll 1$ and, for all $\varepsilon \geqq \varepsilon_0$ there is a solution of the von Kármán problem (1.4), (1.5). Moreover, this solution is continuous in ε. Hence there is a curve of solutions and there is no local bifurcation of solutions of the von Kármán equation from this curve. Again, within the set of solutions of the von Kármán equations, for fixed $\varepsilon \geqq \varepsilon_0$, each of these solutions is isolated.

(ii) In this same range of ε there is a solution of the full system (1.12)–(1.15): a one parameter family of solutions $\langle H(x, \varepsilon), G(x, \varepsilon), f(x, \varepsilon, l), g(g, \varepsilon, l)\rangle$ which includes (for $l = 0$) the axisymmetric von Kármán solution. Moreover, if ε and l are both fixed, this solution is an isolated stable solution. Of course, with $a > 0$ these problems provide a one parameter family of solutions of the problem of rotation about different centers.

Our first goal is to show that in the case of the von Kármán equations a relatively simple Picard iteration scheme converges for $\varepsilon \gg 1$ and —in the nature of things— the solutions so obtained are continuous in $R = 1/\varepsilon$.

Definition. *Let $f \in C^k[-1, 1]$, $k \geqq 0$. Then*

$$\|f\|_k = \sum_{j=0}^{k} \operatorname{Max}\{|f^{(j)}(x)|: -1 \leqq x \leqq 1\}.$$

Lemma 3.1. *Consider the two boundary value problems:*

$$\theta^{(4)} = f, \quad -1 \leqq x \leqq 1, \tag{3.1a}$$

$$\theta(\pm 1) = \theta'(\pm 1) = 0, \tag{3.1b}$$

$$\psi'' = g, \quad -1 \leqq x \leqq 1, \tag{3.2a}$$

$$\psi(-1) = 2\Omega_{-1}, \quad \psi(1) = 2\Omega_{+1}. \tag{3.2b}$$

There is a constant $K_0 \geqq 1$ such that

$$\|\theta\|_4 \leqq K_0 \|f\|_0, \tag{3.3a}$$

$$\|\psi\|_2 \leqq K_0 [\|g\|_0 + 2|\Omega_{-1}| + 2|\Omega_{+1}|]. \tag{3.3b}$$

Proof. Direct integration.

Let Ω_{-1} and Ω_{+1} be given. Let

$$\sigma = 20K_0[|\Omega_{-1}| + |\Omega_{+1}|] + 1, \tag{3.4a}$$

$$R = \frac{1}{\varepsilon} \leqq R_0 = \frac{1}{16K_0\sigma}. \tag{3.4b}$$

Let

$$H_0 \equiv 0, \tag{3.5a}$$

$$G_0 = 2[\Omega_{-1} + \tfrac{1}{2}(x+1)(\Omega_{+1} - \Omega_{-1})], \tag{3.5b}$$

and consider the iterative scheme

$$H_{k+1}^{iv} = -R[H_k H_k''' + G_k G_k'], \tag{3.6a}$$

$$H_{k+1}(\pm 1) = H_{k+1}'(\pm 1) = 0, \tag{3.6b}$$

$$G_{k+1}'' = R[H_k' G_k - H_k G_k'], \tag{3.7a}$$

$$G_{k+1}(-1) = 2\Omega_{-1}, \quad G_{k+1}(1) = 2\Omega_{+1}. \tag{3.7b}$$

Lemma 3.2. *If*

$$\|H_k\|_4 \leqq \sigma, \quad \|G_k\|_2 \leqq \sigma. \tag{3.8a}$$

Then

$$\|H_{k+1}\|_4 \leqq \sigma \quad \text{and} \quad \|G_{k+1}\|_2 \leqq \sigma. \tag{3.8b}$$

Proof. From the definition of K_0 we have

$$\|H_{k+1}\|_4 \leqq 2K_0 R\sigma^2 \leqq \frac{2K_0}{16K_0\sigma} \cdot \sigma^2 = \tfrac{1}{8}\sigma \leqq \sigma \tag{3.9a}$$

and

$$\|G_{k+1}\|_2 \leqq K_0[2R\sigma^2 + 2|\Omega_{-1}| + 2|\Omega_{+1}|]$$
$$\leqq \frac{2K_0}{16K_0\sigma}\sigma^2 + 2K_0[|\Omega_{-1}| + |\Omega_{+1}|].$$

That is

$$\|G_{k+1}\|_2 \leqq \frac{\sigma}{8} + \frac{\sigma}{10} < \sigma. \tag{3.9b}$$

Lemma 3.3. *Suppose*

$$\|H_k\|_4 \leqq \sigma, \quad \|G_k\|_2 \leqq \sigma, \quad k = 0, 1, \ldots \tag{3.10}$$

Then, for $k \geqq 1$

$$(\|H_{k+1} - H_k\|_4 + \|G_{k+1} - G_k\|_2) \leqq \tfrac{1}{4}(\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2). \tag{3.11}$$

Proof.

$$(H_{k+1} - H_k)^{iv} = -R\{(H_k - H_{k-1}) H_k''' + H_{k-1}(H_{k-1}''')$$
$$+ G_k'(G_k - G_{k-1}) + G_{k-1}(G_k' - G_{k-1})\},$$
$$\|H_{k+1} - H_k\|_4 \leqq 2K_0 R_0 - \{\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2\}$$
$$= \tfrac{1}{8}\{\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2\}.$$

Thus

$$\|H_{k+1} - H_k\|_4 \leqq \tfrac{1}{4}\{\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2\}. \tag{3.12a}$$

Also

$$(G_{k+1} - G_k)'' = R[H_k' - H_{k-1}')\, G_k + (G_k - G_{k-1})\, H_{k-1}']$$
$$+ (H_k - H_{k-1})\, G_k' - (G_k' - G_{k-1}')\, H_{k-1}].$$

Hence

$$\|G_{k+1} - G_k\|_2 \leqq 2K_0 R\sigma[\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2] \tag{3.12b}$$
$$\leqq \tfrac{1}{8}\,[\|H_k - H_{k-1}\|_4 + \|G_k - G_{k-1}\|_2].$$

Adding (3.12a) and (3.12b) gives the desired result.

Theorem 3.1. *If* $R = \dfrac{1}{\varepsilon} \leqq R^0 = \dfrac{1}{16K_0\sigma}$,

$$\sigma = 20K_0[|\Omega_{-1}| + |\Omega_{+1}|] + 1,$$

then the iterative procedure (3.5)–(3.7) *converges to an isolated solution* $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ *which is continuous in* ε *for* $\varepsilon \geqq 1/R_0$.

Proof. The proof is now a standard argument based on the estimate of lemma 3.2 and lemma 3.3.

We now turn to the linear equations (1.13) with boundary conditions (1.15).

Lemma 3.4. *Consider the multi-point problem*

$$v''' = F, \qquad -1 \leqq x \leqq 1, \tag{3.13a}$$

$$v(-1) = A, \qquad v(1) = B, \qquad v(0) = c. \tag{3.13b}$$

Let (x) *be the triple integral of* F, i.e.,

$$\mathscr{F}(x) = \int_{-1}^{x} dy \int_{-1}^{y} dt \int_{-1}^{t} F(s)\, ds.$$

Then the solution of (3.13a), (3.13b) *is given by*

$$v(x) = A + \alpha(x+1) + \frac{\beta}{2}(x+1)^2 + \mathscr{F}(x), \tag{3.14a}$$

where

$$\alpha = 2c - \tfrac{3}{2}A - \tfrac{1}{2}B - 2\mathscr{F}(0) + \tfrac{1}{2}\mathscr{F}(1), \tag{3.14b}$$

$$\beta = (B + A - 2C) + 2\mathscr{F}(0) - \mathscr{F}(1). \tag{3.14c}$$

Proof. direct verification.

Corollary 3.4. *There is a constant* K_1 *such that*

$$|v| \leqq K_1[|A| + |B| + |C| + \|F\|_0].$$

Given $\langle H(x, \varepsilon), G(x, \varepsilon)\rangle$ *for* $\varepsilon \geqq \varepsilon_0$, *let us consider the iterative procedure*

$$f'''_{k+1} = 4R[-(Gg_k)' + \tfrac{1}{2}H''f_k - Hf''_k - \tfrac{1}{2}H'f'_k], \tag{3.16a}$$

$$f(-1) = 0, \quad f(0) = l, \quad f(1) = 0, \tag{3.16b}$$

$$g'''_{k+1} = R[(Gf_k)' + \tfrac{1}{2}H''g_k - Hg''_k - \tfrac{1}{2}H'g'_k], \tag{3.17a}$$

$$g(-1) = -\tfrac{1}{2}a\Omega_{-1}, \quad g(0) = 0, \quad g(1) = \tfrac{1}{2}a\Omega_{+1}. \tag{3.18a}$$

Quite clearly, the arguments above show that there is an $\varepsilon_1 \geqq \varepsilon_0$ *and, for all* $\varepsilon \geqq \varepsilon_1 \geqq \varepsilon_0$ *this multi-point problem possesses a unique solution which is continuous in* ε *and* l. *Thus we have verified all the opening remarks of this section.*

Remarks

We conclude this analysis by making a few observations on the significance of the result established in the previous section.

We have studied special solutions of the Navier-Stokes equations for a fluid contained within two infinite parallel planes each rotating with a constant angular velocity Ω_k ($k = \pm 1$). The axis of rotation may be the same or distinct. In either case we are led to a system of ordinary differential equations which contain (as a subset) the classical equations of VON KÁRMÁN [9] and BATCHELOR [2] for special axisymmetric flow about a common axis. In particular, in the classical case studied VON KÁRMÁN and BATCHELOR, if there are such special solutions, they are *never isolated solutions* when considered within the scope of the full Navier-Stokes equations. In the case of rotation about distinct axes there are many unanswered questions. However, we have shown that (contrary to most intuitive ideas) in the case of "large" viscosity, there are solutions and they are *never* isolated. While the underlying basis for these anomalies is not completely understood, we believe it is related to the fact that in this unbounded domain the velocities at large r are great.

It is also worth observing that similar results can be established in the case of the flow of a Newtonian fluid between rotating porous disks when $\varepsilon \gg 1$ (*cf.* RAJAGOPAL [14]). The only change in the problem is in the boundary condition (1.14a) and it is an easy matter to modify the arguments of Section 3 for this case.

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(Received December 15, 1983)