Monotonic Decreasing and Asymptotic Behavior of the Kinetic Energy for Weak Solutions of the Navier-Stokes Equations in Exterior Domains

GIOVANNI P. GALDI & PAOLO MAREMONTI

To James Serrin on his 60th birthday

Introduction

Existence of weak solutions of the three-dimensional Navier-Stokes problem was first proved by J. LERAY in the case of the Cauchy problem, *cf.* LERAY (1934). As is well known, these solutions are important in that they are the only solutions which, so far, are known to exist for all times, without restriction on the data. Unfortunately, however, the question of whether they are classical, in an ordinary sense, is still open, even though partial conclusions regarding regularity are avilable: LERAY (1934), SCHEFFER (1976, 1980), CAFFARELLI, KOHN, & NIRENBERG (1982). Subsequently, E. HOPF (1951), using a different technique, constructed weak solutions for a general initial-boundary value problem. However, HOPF's solution (even for the Cauchy problem) has weaker properties than LERAY's solution. Among others, we refer to the "energy inequality" for the velocity field $\boldsymbol{u}(x, t)$, *i.e.*

$$\text{(I)} \qquad \int_\Omega u^2(x, t)\, dx - \int_\Omega u^2(x, s)\, dx \leqq -2 \int_s^t \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}\, dx\, d\tau$$

for $s = 0$, *for almost all* $s > 0$ *and for all* $t \geqq s$.
Actually, unlike those of LERAY, the solutions constructed by HOPF satisfy (I) if Ω is a bounded domain, while if Ω is unbounded, they obey the following weaker relation[1]

$$\text{(II)} \qquad \int_\Omega u^2(x, t)\, dx - \int_\Omega u^2(x, 0)\, dx \leqq -2 \int_0^t \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{u}\, dx\, d\tau$$

for all $t \geqq 0$.

---

[1] Notice that if $\boldsymbol{u}(x, t)$ is a regular solution in the ordinary sense, it certainly verfies (I) with the *equality* sign. So far, regular global solutions are known to exist only for "small" data: LADYZHENSKAYA (1969).

On the other hand the validity of the inequality (I) has several remarkable implications. In particular, it ensures that the kinetic energy $(1/2) \int_\Omega u^2(x, t)\, dx$ is a monotonically decreasing function of time, which is what has to be expected on physical grounds. This should be contrasted with (II), which does not rule out the possibility of an (unacceptable) increasing energy. Moreover, the existence of weak solutions with decreasing kinetic energy allows one to formulate asymptotic stability theorems in a very large class of perturbations: GALDI (1975), MASUDA (1975), HEYWOOD (1980), MAREMONTI (1984). In this respect, it is worth noticing that the authors of the above papers suppose that HOPF's solutions satisfy (I) instead of (II), in the case of an exterior domain[2]. Because of this oversight, the results of GALDI (1975), MASUDA (1975), HEYWOOD (1980), MAREMONTI (1984a, b) remain formal for such domains. Finally, we observe that the validity of (I) is intimately related to the regularity of weak solutions through the fundamental uniqueness teorem of LERAY-SATHER-SERRIN (see SERRIN (1962)).

This paper concerns some properties of solutions to Navier-Stokes equations in exterior (three-dimensional) domains. Our motivation is twofold. On the one hand, the aim is to prove the existence of global (weak) solutions verifying the energy relation (I). On the other, we wish to study the asymptotic behavior in time of their kinetic energy. Precisely, employing a method introduced by LERAY (1934) and with the aid of some estimates for solutions to the *linear* Navier-Stokes equations given in SOLONNIKOV (1977) (see Section 1), in Section 2 we prove the existence of global solutions corresponding to arbitrary initial data from a Sobolev space of suitable fractional order. These solutions, which are more regular than HOPF's solutions (having time derivatives and second-order spatial derivatives) satisfy the Navier-Stokes equations almost everywhere and possess a kinetic energy which decreases monotonically in time. We note that for the Cauchy problem their existence was established by LADYZENSKAYA (1969). Thereafter we study the asymptotic behavior in time of the kinetic energy of a weak solution satisfying (I). As is known, this problem was set by LERAY (1934, p. 248) a long time ago, and only recently has it begun to receive satisfactory answers: KATO (1984), MAREMONTI (1984b), MASUDA (1984), GALDI & RIONERO (1985), SCHONBEK (1985). In Section 3 we prove that the kinetic energy tends to zero and, what is more, we conclude that the order of decay is related to the summability of the initial data. These results, which rely on those of MAREMONTI (1986) and on a new estimate for weak solutions in exterior domains (*cf.* Lemma 3.1), show that if the initial data belong to the Lebesgue space $L^2(\Omega) \cap L^q(\Omega)$, $1 \leqq q \leqq 3/2$, the kinetic energy decays like $t^{-2\beta}$ where $\beta = (2 - q)/4q$. This behavior is further improved if Ω is the whole space. We notice that our results either improve or contain as particular cases those of KATO (1984), MAREMONTI (1984), MASUDA (1984), SCHONBEK (1985), MAREMONTI (1986).

---

[2] The validity of (I) in unbounded domains has been recently explicitly questioned in MASUDA (1984).

1. Preliminaries and notations

Let Ω be a domain of the three-dimensional Euclidean space $\mathbb{R}^3$, exterior to ν ($\geqq 0$) C^2-smooth, compact subregions, We designate by R_0 the diameter of $\bar{\Omega}^c$. For $\Omega' \subseteq \Omega$ and $s > 0$ we set $\Omega'_s \equiv \Omega' \times (0, s)$. By $L^p(\Omega')$ $p \in [1, \infty]$ we denote the Lebesgue space of functions on Ω'. The symbol $L^p(\Omega'_s)$ has an analogous meaning. The norm in $L^p(\Omega)$ [respectively, in $L^p(\Omega_s)$] will be indicated by $|\cdot|_p$ [respectively, by $|\cdot|_{p,\Omega_s}$]. $W^{m,p} = W^{m,p}(\Omega)$ $m \geqq 0$ denotes the Sobolev space of order (m, p) of functions on Ω, and $|\cdot|_{m,p}$ is its associated norm. Furthermore, $\mathscr{W}_p(\Omega_s)$ is the space of functions u on Ω_s having p^{th} power summable (generalized) derivatives of the first and second order with respect to $x \in \Omega$ and of the first order with respect to $t \in (0, s)$. An equivalent norm [SOLONNIKOV (1977)], in $\mathscr{W}_p(\Omega_s)$ is

$$|u|_{\mathscr{W}_p(\Omega_s)} \equiv |u|_{p,\Omega_s} + \sum_{i,j} \left|\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right|_{p,\Omega_s} + \left|\frac{\partial u}{\partial t}\right|_{p,\Omega_s}.$$

For an open set A in $\mathbb{R}^n$ we let $C_0^\infty(A)$ be the set of indefinitely differentiable functions of compact support in A. Moreover, denoting by $\mathscr{C}_0^\infty(\Omega)$ the class of solenoidal functions from $C_0^\infty(\Omega)$, we indicate by $\mathring{J}(\Omega)$ the completion of $\mathscr{C}_0^\infty(\Omega)$ in $L^2(\Omega)$ and by $\mathring{J}^{2-2/p,p}, (\Omega)$ the completion of $\mathscr{C}_0^\infty(\Omega)$ in the norm

$$\|u\|_{2-2/p,p} \equiv \begin{cases} |u|_p + \langle u\rangle^{(2-2/p)} & \text{for } p \neq 3/2 \\ |u|_p + \langle u\rangle^{(2-2/p)} + |u\varrho^{\frac{2}{p}-2}|_p & \text{for } p = 3/2 \end{cases}$$

where $\varrho = \varrho(x)$ is the distance from x to $\partial\Omega$ and

$$\langle u\rangle^{2-2/p} \equiv \begin{cases} \left(\int\limits_\Omega \int\limits_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{1+2p}} \, dx \, dy\right)^{\frac{1}{p}} & \text{for } p < 2 \\ \left(\int\limits_\Omega \int\limits_\Omega \frac{|\nabla u(x) - \nabla u(y)|^p}{|x-y|^{1+p}} \, dx \, dy\right)^{\frac{1}{p}} & \text{for } p > 2 \\ |\nabla u|_2 & \text{for } p = 2. \end{cases}$$

For the elementary properties of the space introduced above, we refer the reader to MIRANDA (1978) and SOLONNIKOV (1977). Finally, if X is a Banach space, by $L^p(0, s; X)$ we denote the class of functions u from $(0, s)$ in X such that $\int\limits_0^s |u|_X^p \, d\xi < \infty$, where $|\cdot|_x$ is the X-norm. We remark explicitly that, depending on the context, we shall use the same symbol for a space of scalar, vector or tensor-valued functions. Also, by the symbol C we denote a generic constant whose numerical value is inessential to our aims; it may have several values in a single computation. Typically, we may have $2C < C$ in the same line.

Now, for $T > 0$, consider in Ω and during the time interval $[0, T]$ the motion of a viscous, incompressible fluid, governed by the Navier-Stokes equations.

If $u(x, t), \pi(x, t)$ represent velocity and pressure fields of that motion, they must satisfy the system

$$(1.1)\qquad \begin{aligned} &\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla \pi + \Delta u \\ &\nabla \cdot u = 0 \quad \text{in } \Omega_T, \end{aligned}$$

where the kinematic viscosity has been set equal to one, without loss of generality. Along with (1.1) we shall consider the following initial conditions and boundary conditions:

$$(1.2)\qquad \begin{aligned} &u(x, 0) = u_0(x) \quad x \in \Omega, \\ &u(y, t) = 0 \quad (y, t) \in \partial\Omega \times (0, T). \end{aligned}$$

In this section, however, we will be interested in giving some results concerning the *linearized* version of problem (1.1)–(1.2), which we shall write in the following form:

$$(1.3)\qquad \begin{aligned} &\frac{\partial w}{\partial t} = \Delta w - \nabla p + f, \\ &\nabla \cdot w = 0 \quad \text{in } \Omega_T, \\ &w(x, 0) = w_0(x) \quad x \in \Omega, \\ &w(y, t) = 0 \quad (y, t) \in \partial\Omega \times (0, T), \end{aligned}$$

where $f = f(x, t)$ is a given "body force". We have the following results.

Lemma 1.1. *For any $f \in L^q(\Omega_T)$, $w_0 \in \mathring{J}^{2-2/q,q}(\Omega)$, $q > 1$, problem (1.3) admits for all $T > 0$ a unique solution in $\mathscr{W}_q(\Omega_T)$ satisfying the following estimate:*

$$(1.4)\qquad |w|_{\mathscr{W}_q(\Omega_T)} + |\nabla p|_{q,\Omega_T} \leqq C(T)\{|f|_{q,\Omega_T} + \|w_0\|_{2-2/q,q}\},$$

where C depends only on T.

Proof. It is a consequence of Theorem 4.2 and Remark 2 on p. 495 of the paper of SOLONNIKOV (1977).

Lemma 1.2. *Assume $f \in C_0^\infty(\Omega_T)$ and $w_0 \in \mathscr{C}_0^\infty(\Omega)$. Then there exists a unique (smooth) solution $w(x, t)$ of (1.3) which is in $\mathscr{W}_q(\Omega_T)$ and satisfies the estimate: (1.4) for all $q > 1$.*

Proof. In the light of Lemma 1.1 the proof is reduced to show the following uniqueness result. Let w_1, p_1 and w_2, p_2 be two solutions of (1.3) corresponding to the same data f and w_0 and satisfying (1.4) with exponents q_1 and q_2 respectively; then, $w_1 \equiv w_2$ and $\nabla p_1 = \nabla p_2$. Assume $q_1 < q_2$ and set $w_1 - w_2 = w$, $p_1 - p_2 = p$. We thus conclude that w, p is a solution of the problem (1.3) with $f = w_0 = 0$. Denote by ω a smooth neighborhood of $\partial\Omega$; obviously ∇p,

$\frac{\partial^2 w}{\partial x_i \, \partial x_j} \in L^{q_1}(0, T; L^{q_1}(\Omega \cap \omega))$ for all $i, j = 1, 2, 3$. Moreover, since p satisfies

$$\Delta p = 0 \quad \text{in} \quad (\Omega - \omega) \times \{t\},$$

$$\frac{dp}{dn} = \Delta w \cdot n \quad \text{on} \quad \partial(\Omega - \omega) \times \{t\},$$

from SOLONNIKOV (1977) [lemma 2.1], we conclude that $|\nabla p|_{q_1, \Omega_T} < C$. Uniquenes, therefore, follows as a paricular case of the theorems proved by GALDI & MAREMONTI (1984), GALDI & MAREMONTI (1986).

We end this section by proving an inequality of Sobolev type.

Lemma 1.3. *Let $f \colon \Omega \to \mathbb{R}$ with $\nabla f \in L^p(\Omega)$, $p < 3$. Then there is a constant f_0 such that $\psi = f - f_0$ satisfies the following inequalities:*

$$R^{3-p} \int_{S(1)} |\psi(R, \gamma)|^p \, d\gamma \leqq C \int_{|x| \geqq R} |\nabla f|^p \, dx, \tag{1.5}$$

$$|\psi|_q \leqq C \, |\nabla f|_p, \tag{1.6}$$

where $1 < p < 3$, $q = 3p/(3 - p)$ and $S(1)$ is the unit sphere.

Proof. The first part of the lemma and the estimate (1.5) are a generalization of an analogous result proved by PAYNE & WEINBERGER (1957) for $p = 2$. However, the proof given by these authors remains unchanged in our case $1 < p < 3$ and therefore, it will be omitted. As far as (1.6) is concerned, we proceed as follows. Denote by φ a smooth "cut-off" function such that $\varphi(x) = 0$ for $|x| \leqq R$ and $\varphi(x) = 1$ for $|x| \geqq 2R$ $(R > R_0)$, and set $\tilde{\psi} = \varphi \psi$. For $\tilde{\psi}$ we have the representation

$$\tilde{\psi}(x) = \frac{1}{4\pi} \sum_{j=1}^{3} \int_{|x-y| \leqq \varrho} \frac{\partial \tilde{\psi}}{\partial y_j} \frac{\partial}{\partial y_j} \left(\frac{1}{|x - y|} \right) dy - \frac{1}{4\pi} \int_{|x-y| = \varrho} \tilde{\psi} \frac{\partial}{\partial n} \left(\frac{1}{|x - y|} \right) d\sigma, \tag{1.7}$$

for $\varrho > R$. In virtue of (1.5), (1.7) furnishes in the limit as $\varrho \to \infty$

$$\tilde{\psi}(x) = \frac{1}{4\pi} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial \tilde{\psi}}{\partial y_j} \frac{\partial}{\partial y_j} \left(\frac{1}{|x - y|} \right) dy.$$

By well known estimates on Riesz potentials, *cf.* MIRANDA (1978), we thus obtain the Sobolev inequality for $\tilde{\psi}$,

$$|\tilde{\psi}|_q \leqq C \, |\nabla \tilde{\psi}|_p, \quad q = \frac{3p}{3 - p}. \tag{1.8}$$

From (1.8) we easily have

$$\left(\int_{|x| \geqq 2R} |\psi|^q \, dx \right)^{\frac{1}{q}} \leqq C \left\{ |\nabla f|_p + \left(\int_{R \leqq |x| \leqq 2R} |\psi|^p \, dx \right)^{\frac{1}{p}} \right\}. \tag{1.9}$$

Applying Poincaré's inequality, *cf.* MIRANDA (1978), to the second integral on the right-hand side of (1.9), we deduce

$$(1.10)\qquad \left(\int_{|x|\geqq 2R} |\psi|^q\,dx\right)^{\frac{1}{q}} \leqq C\left\{|\nabla f|_p + \left(\int_{\{|x|=R\}\cup\{|x|=2R\}} |\psi|^p\,d\sigma\right)^{\frac{1}{p}}\right\}.$$

Therefore, from (1.10) and (1.5) it follows that

$$(1.11)\qquad \left(\int_{|x|\geqq 2R} |\psi|^q\,dx\right)^{\frac{1}{q}} \leqq C\,|\nabla f|_p.$$

To complete the proof of the theorem we now choose another smooth "cut-off" function $\xi(x)$ such that $\xi(x)=1$ for $|x|\leqq 2R$ and $\xi(x)=0$ for $|x|\geqq 3R$. Since $\xi\psi$ is of compact support in $\bar{\Omega}$ we can apply the results of SOLONNIKOV & SCADILOV (1973) [Lemma 1 and the remark following it], to obtain

$$|\xi\psi|_q \leqq C\,|\nabla(\xi\psi)|_p$$

from which it follows that

$$(1.12)\qquad \left(\int_{|x|\leqq 2R} |\psi|^q\,dx\right)^{\frac{1}{q}} \leqq C\left\{|\nabla f|_p + \left(\int_{2R\leqq|x|\leqq 3R} |\psi|^p\,dx\right)^{\frac{1}{p}}\right\} \leqq C\,|\nabla f|_p,$$

where in the last step we have used the Hölder inequality (recall that $q>p$) and (1.11). Therefore, estimate (1.6) is a consequence of (1.11) and (1.12).

Remark. We notice that lemma 1.2 remains valid whatever be the number n of space dimensions, provided $p<n$ and $q=np/(n-p)$.

2. Existence theorem

The aim of this section is to prove a theorem of existence of global (weak) solutions to the problem (1.1). Precisely, denoting by $H^{9/10,5/4}(\Omega)$ the completion of $\mathscr{C}_0^\infty(\Omega)$ in the norm of $W^{9/10,5/4}(\Omega)$, we have

Theorem 2.1. *Assume* $\boldsymbol{u}_0\in H^{9/10,5/4}(\Omega)$. *Then there are functions* $\boldsymbol{u}(x,t)$, $p(x,t)$ *enjoying the following properties for all* $T>0$

(i) $\boldsymbol{u}\in L^\infty(0,T;\mathring{J}(\Omega))\cap L^2(0,T;\mathring{J}^{1,2}(\Omega))$;
$\boldsymbol{u}\in \mathscr{W}_{5/4}(\Omega_T)$; $\nabla p\in L^{5/4}(\Omega_T)$;

(ii) $\boldsymbol{u},p$ *satisfy* (1.1) *a.e. in* Ω_T;

(iii) $\boldsymbol{u},p$ *satisfy the "energy inequality" in the following form*:

$$|\boldsymbol{u}(t)|_2^2 \leqq |\boldsymbol{u}(s)|_2^2 - 2\int_s^t |\nabla\boldsymbol{u}(\tau)|_2^2\,d\tau$$

for $s=0$, *for almost all* $s>0$ *and for all* $t\geqq s$;

(iv) *$u(x, t)$ can be redefined on a set of zero t-measure in such a way that* $\lim_{t\to s^+} |u(t) - u(s)|_2 = 0$ *for all* $s \in [0, T]$.

Before proving the theorem, we wish to make the following remarks.

(a) The solutions provided by theorem 2.1 are more regular than HOPF's solutions [HOPF (1951)], and, in particular, their kinetic energy decreases monotonically with time. To our knowledge, this is the first example of global solutions which enjoy this property in an exterior domain $\Omega \neq \mathbb{R}^3$ *without restrictions on the "size" of the data.* For the Cauchy problem, solutions satisfying (iii) were constructed by LERAY (1934).

(b) In the case $\Omega = \mathbb{R}^3$ and Ω bounded, theorem 2.1 has already been proved by O. A. LADYZHENSKAYA (1969).

(c) The regularity of the initial data can be weakened to $u_0 \in Y(\Omega) = \mathring{J}^{2/5,5/4}(\Omega) \cap \mathring{J}(\Omega)$, provided one can prove that $\mathscr{C}_0^\infty(\Omega)$ is dense in $Y(\Omega)$ (endowed with its natural norm).

(d) As far as the smoothness of our solutions is concerned, we can give only results of partial regularity. In fact, on the one hand, because of the validity of (iii) and of the results of HEYWOOD (1980) concerning existence of classical solutions (local in time or global for small data), it is possible to prove a "théorème de structure" in the sense of LERAY (*cf.* HEYWOOD (1980)). On the other hand, by suitably modifying the construction, one can show that our solutions verify a "generalized energy inequality" in the sense specified in CAFFARELLI, KOHN & NIRENBERG (1982). Therefore they would have further (partial) regularities along the lines of CAFFARELLI, KOHN & NIRENBERG (1982) (Theorem B).

The method we shall employ to prove theorem 2.1 is that introduced by J. LERAY (1934, chapter V), to construct his "solutions turbolentes". We thus begin to consider for all $n = 1, 2, \ldots,$ the following initial-boundary value problem

$$\begin{aligned} &\frac{\partial U}{\partial t} + U_{(n)} \nabla U = \Delta U - \nabla \tau, \\ &\nabla \cdot U = 0 \quad \text{in } \Omega_T, \\ &U(x, 0) = U_{0n}(x) \quad x \in \Omega, \\ &U(y, t) = 0 \quad (y, t) \in \partial\Omega \times (0, T), \end{aligned} \tag{2.1}$$

where

$$U_{(n)}(x, t) \equiv \int_{\mathbb{R}^3} J_{1/n}(x - y)\, U(y, t)\, dy$$

is a (spatial) "mollification" of U, and $\{U_{0n}(x)\}$ is a sequence of functions from $\mathscr{C}_0^\infty(\Omega)$ converging to $u_0(x)$ in the space $H^{9/10,5/4}$. For the system (2.1) it is not hard to prove the existence of a global regular solution. In fact, recalling that

$$\sup_{\Omega} |U_{(n)}(x, t)| \leq c(n)\, |U(t)|_2 \tag{2.2}$$

where $c(n)$ depends only on ($c(n) \to \infty$ as $n \to \infty$), we may show that $U(x, t)$ obeys the following (formal) *a priori* estimates

$$\tfrac{1}{2}\frac{d}{dt}|U(t)|_2^2 = -|\nabla U(t)|_2^2,$$

$$\frac{d}{dt}|\nabla U(t)|_2^2 \leqq -\left|\frac{\partial U}{\partial t}\right|_2^2 + c^2(n)\,|U(t)|_2^2\,|\nabla U(t)|_2^2, \tag{2.3}$$

$$\frac{d}{dt}|\nabla U(t)|_2^2 \leqq -|P\Delta U|_2^2 + c^2(n)\,|U(t)|_2^2\,|\nabla U(t)|_2^2$$

P being the orthogonal projection of $L^2(\Omega)$ onto $\mathring{J}(\Omega)$ (LADYZHENSKAYA (1969)). Relations (2.3) are easily obtained after multiplication of (2.1) by U (estimate (2.3)), by $\frac{\partial U}{\partial t}$ (estimate (2.3)) and by $-P\Delta U$ (estimate (2.3)) and employing inequality (2.2). Using (2.3) and the method of HEYWOOD (1980), one thus proves the existence of classical solutions $U(x; t, n)$, $\tau(n; x, t)$ of (2.1) satisfying (in particular) the following properties for all n:

$$U \in L^\infty(0, T; \mathring{J}) \cap L^\infty([0, T]; \mathring{J}^{1,2}),$$

$$\frac{\partial U}{\partial t}, \frac{\partial^2 U}{\partial x_i\,\partial x_j}, \nabla\tau \in L^2(\Omega_T)$$

and the estimate (2.3).

We now wish to obtain estimates of the solution U, τ which hold *uniformly in* n. To this end, let us consider the linearized problem (1.3) with $f(n; x, t) = U_{(n)} \cdot \nabla U$ and $w_0 = U_{0n}$. It is well known that (*cf.* LADYZHENSKAYA (1969)),

$$|f|_{5/4,\Omega_T} \leqq \sqrt{4}\,|U_{0n}|_2^{\frac{5}{2}}; \tag{2.4}$$

therefore, the system (1.3) will admit a (unique) solution $w = w(n; x, t)$ satisfying the estimate (1.4) with $q = 5/4$. Let us prove that $w = U$. To this end, we shall follow an argument of LADYZHENSKAYA (1969). Setting $v = w - U$, one readily proves that v satisfies the following identity:

$$\int_{\Omega_T} v \cdot \left(\frac{\partial \phi}{\partial t} + \Delta\phi\right) dx\,dt = 0 \tag{2.5}$$

for all solenoidal ϕ such that ϕ, $\frac{\partial \phi}{\partial t}$, $\frac{\partial^2 \phi}{\partial x_i\,\partial x_j} \in L^q(\Omega_T)$ for all $q > 1$, ϕ vanishes on $\partial\Omega$ and $\phi(x, T) = 0$. In virtue of lemma 1.2 we may choose $\phi(x, t)$ as a solution of the adjoint problem of (1.3) corresponding to a "body force" $F(x, t) \in C_0^\infty(\Omega \times (0, T))$ and zero data at time T. Evidently, from (2.5) follows

$$\int_{\Omega_T} v(x, t) \cdot F(x, t)\,dx\,dt = 0$$

which, by the arbitrariness of F, implies $v \equiv 0$. From what we have just proved, we deduce that for all n, $U(n; x, t)$ satisfies the estimate (1.4), which by (2.4) can be written as

$$\text{(2.6)}\quad \begin{aligned} |U|_{\mathscr{W}_{5/4}(\Omega_T)} + |\nabla\tau|_{5/4,\Omega_T} &\leqq C(T)\{|U_{0n}|_2^2 + \|U_{0n}\|_{2/5,5/4}\} \\ &\leqq C(T)\{|U_{0n}|^2_{9/10,5/4} + |U_{0n}|_{9/10,5/4}\}, \end{aligned}$$

where C depends only on T. In the last step of the inequality (2.6) we have used the embedding $H^{9/10,5/4} \hookrightarrow \mathring{J}$ (*cf.* MIRANDA (1978)).

We wish now to derive an estimate which, along with (2.3), will allow us to prove the energy inequality (iii) of the theorem. Following LERAY (1934), we introduce the "cut-off" function $g \in C^\infty(\mathbb{R}^3)$, $g(\xi) = 0$ for $|\xi| < 1$ and $g(\xi) = 1$ for $|\xi| \geqq 2$, and set $g_R(x) = g(x/R)\,(R > R_0)$. Multiplying (2.1) by $g_R U$ and integrating by parts, we obtain

$$\text{(2.7)}\quad \begin{aligned} \tfrac{1}{2}\frac{d}{dt}|g_R^{1/2}U(t)|_2^2 \leqq& -|g_R^{1/2}\nabla U(t)|_2^2 + \tfrac{1}{2}\||\Delta g_R|^{1/2}U(t)|_2^2 \\ &+ \int_\Omega \nabla g_R \cdot U_{(n)}|U|^2\,dx + \int_\Omega \nabla g_R \cdot U\tau\,dx. \end{aligned}$$

The following inequalities hold:

$$\text{(a)}\quad \begin{aligned} \int_{\Omega_T} \nabla g_R \cdot U_{(n)}|U|^2\,dx\,dt &\leqq \frac{C}{R}\int_0^T |U_{(n)}(t)|_3\,|U(t)|_3^2\,dt \leqq \frac{C}{R}\int_0^T |U(t)|_3^3\,dt \\ &\leqq \frac{C}{R}\int_0^T |\nabla U(t)|_2^{3/2}\,|U(t)|_2^{3/2}\,dt < \frac{C}{R}T^{1/4}|U_{0n}|_2^3 \end{aligned}$$

$$\text{(b)}\quad \int_0^T \||\Delta g_R|^{1/2}\,U(t)|_2^2\,dt \leqq \frac{C}{R^2}T\,|U_{0n}|_2^2.$$

In the preceding inequalities we employed the following facts: $|\nabla g_R| \leqq C/R$, $|\Delta g_R| \leqq C/R^2$, the embedding $W^{1,2}(\Omega) \hookrightarrow L^3(\Omega)$ (*cf.* MIRANDA (1978)) and the relation $(2.3)_1$. We wish now to estimate the last integral on the right-hand side of (2.7). Since $\nabla\tau \in L^{5/4}(\Omega_T)$, because of lemma 1.3 we can redefine τ (without loss of generality) by adding a suitable function of t, in such a way that $\tau \in L^{4/5}(0, T; L^{15/7}(\Omega))$. We thus obtain

$$\text{(c)}\quad \begin{aligned} \int_{\Omega_T} \nabla g_R\, U\tau\,dx\,dt &\leqq \frac{C}{R}\left(\int_0^T |\tau(t)|_{5/7}^{5/4}\,dt\right)^{4/5}\left(\int_0^T |U(t)|_{15/8}^5\,dt\right)^{1/5} \\ &\leqq \frac{C}{R}T^{1/5}\{|U_{0n}|^2_{9/10,5/4} + |U_{0n}|_{9/10,5/4}\}\{U_{0n}|^{11/9}_{9/10,5/4} + |U_{0n}|_{9/10,5/4}\} \end{aligned}$$

where in the last step we have used (2.6), L^p interpolation and the embedding $H^{9/10,5/4}(\Omega) \hookrightarrow L^2(\Omega)$ (*cf.* MIRANDA (1978)). Integrating (2.7) with respect to time, recalling that $\int_{|x|\geqq R} |U_{0n}(x)|^2\,dx$ and $|U_{0n}|_{9/10,5/4}$ can be increased by $\int_{|x|\geqq R} |u_0(x)|^2 dx$ and $|u_0|_{9/10,5/4}$, respectively, and using (a)–(c), we deduce

$$\text{(2.8)}\quad \int_{|x|\geqq R+1} |U(n; x, t)|^2\,dx \leqq \int_{|x|\geqq R} |u_0(x)|^2\,dx + D/R,$$

where D depends only on $|U_0|_{9/10,5/4}$.

Thanks to the estimate (2.6) it is now routine to show that from the sequence $\{U(n; x, t)\}$ it is possible to select a subsequence $\{U(n_k; x, t)\}$ converging to a function $u(x, t)$ verifying the statement (i) of theorem 2.1. In particular one shows that $U \to u$ weakly in $L^2(0, T; \overset{\circ}{J}{}^{1,2})$ and weakly in $\overset{\circ}{J}(\Omega)$, uniformly with respect to t. This last fact in turn implies that u is weakly continuous in time in $\overset{\circ}{J}(\Omega)$. Moreover, since U satisfies (2.1) it also follows that u and π satisfy statement (ii). For the details, we refer the reader to LERAY (1934), HOPF (1951), LADYZHENSKAYA (1969). We shall, however, briefly sketch the deduction of the energy inequality (iii). To this end, we shall again follow an argument of LERAY (1934). First of all, employing, for example, FRIEDRICHS' inequality (*cf.* MIRANDA (1978)), one proves that $U(n_k; x, t)$ converges strongly in $L^2(0, T; L^2(\Omega'))$ for any compact subdomain $\Omega' \subset \Omega$. This implies that we can select another subsequence, again labelled n_k, which converges in $L^2(\Omega')$ for almost all $s \in [0, T]$. Now, multiplying (2.3) by $U = U(n_k; x, t)$ and integrating over $\Omega \times (s, t)$, we obtain

$$(2.9)\qquad 2\int_s^t |\nabla U(\xi)|_2^2\, d\xi + |U(t)|_2^2 = \int_{\Omega_R} |U(x, s)|^2\, dx + \int_{\Omega - \Omega_R} |U(x, s)|^2\, dx$$

where $\Omega_R = \Omega \cap \{|x| \leqq R\}$ $(R > R_0)$. Furthermore letting $n_k \to \infty$ in (2.9) by the weak convergence of U to u in $L^2(0, T; \overset{\circ}{J}{}^{1,2}(\Omega))$ and in $L^2(\Omega)$, we obtain

$$(2.10)\ 2\int_s^t |\nabla u(\xi)|_2^2\, d\xi + |u(t)|_2^2 \leqq \overline{\lim_{n_k \to \infty}} \left[\int_{\Omega_R} |U(x, s)|^2\, dx + \int_{\Omega - \Omega_R} |U(x, s)|^2\, dx\right].$$

But, by (2.8) and the strong convergence of U to u in L^2 on compact sets, the right-hand side of (2.10) tends for almost all s to $|u(s)|_2^2$ which, implies the validity of (iii). Finally, the validity of (iv) is a consequence of (iii) and of the weak continuity with respect to time of u in $\overset{\circ}{J}(\Omega)$. The theorem is therefore completely proved.

3. Decay of the kinetic energy

In this section we will prove the following theorem.

Theorem 3.1. *Assume that $u(x, t)$ is a suitable weak solution of* (1.1)–(1.2)*: namely, it enjoys the following conditions for all $T > 0$:*

(a) $$u \in L^2(0, T; \overset{\circ}{J}{}^{1,2}(\Omega)) \cap L^\infty(0, T; \overset{\circ}{J}(\Omega));$$

(b)

$$\int_{\Omega_T} \left[-u\left(\frac{\partial \phi}{\partial t} + \Delta\phi\right) + u \cdot \nabla u \cdot \phi\right] dx\, dt = \int_\Omega u \cdot \phi(x, 0)\, dx - \int_\Omega u(x, T)\, \phi(x, T)\, dx$$

for all $\phi \in L^2(0, T; \overset{\circ}{J}{}^{1,2}(\Omega) \cap W^{2,2}(\Omega))$ with $\dfrac{\partial \phi}{\partial t} \in L^2(\Omega_T)$.

(c) *The energy inequality* (iii) *of theorem 2.1 holds.*

Then, if $u_0 \in \overset{\circ}{J}(\Omega) \cap L^q(\Omega)$ *for some* $1 \leqq q \leqq 3/2$, *the kinetic energy* $E(t) = \frac{1}{2}|u(t)|_2^2$ (*is non-increasing and*) *decays in time according to the following law*:

$$E(t) = O(t^{-2\beta}) \tag{3.1}$$

where $\beta = (2-q)/4q$. *In particular, if* $\Omega \equiv \mathbb{R}^3$, $\beta = 3(2-q)/4q$ *for* $q \in [4/3, 3/2]$ *and* $\beta = 3/8$ *if* $q \in [1, 4/3]$. *Finally if* $q = 2$ *we have* $E(t) = o(1)$.

Before giving a proof of the theorem we make some remarks. To this end we denote by $\mathscr{I}$ the class of weak solutions satisfying the assumptions of the theorem.

(a) The hypothesis $\mathscr{I} \neq \emptyset$ is basic to several results on behavior asymptotic in time in exterior domains: GALDI (1975), MASUDA (1975), HEYWOOD (1980), MAREMONTI (1984a). However, as already noticed in section 2, remark (a), $\mathscr{I} \neq \emptyset$ was known to happen, so far, either when $\Omega = \mathbb{R}^3$ [LERAY (1934)] or for (smooth) arbitrary Ω but for initial data whose amplitude is suitably small: LADYZHENSKAYA (1969), MASUDA (1975), HEYWOOD (1980). On the other hand, theorem 2.1 proves $\mathscr{I} \neq 0$ for a C^2-exterior domain without further restrictions; and so it allows us to extend the results of GALDI (1975), MASUDA (1975), HEYWOOD (1980), MAREMONTI (1984) to such a case.

(b) As observed in the previous remark, LERAY proves $\mathscr{I} \neq \emptyset$ with data $u \in \overset{\circ}{J}(\mathbb{R}^3)$ only. Thus, if $u \in \overset{\circ}{J}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ theorem 3.1 furnishes, as a particular case, for LERAY's solutions $E(t) = O(t^{-3/4})$ which improves on the estimate given in SCHOENBEK (1985), which shows that $E(t) = O(t^{-1/2})$.

To prove theorem 3.1 we need some preliminary lemmas.

Lemma 3.1. *Assume* $a_0 \in \overset{\circ}{J}{}^{1,2}(\Omega) \cap L^q(\Omega)$, $q \in [1, 2)$ *with* $|a_0|_2 \, |\nabla a_0(a_0)|_2 \leqq M$, *for a suitable numerical constant* M. *Then there is a unique regular solution* $a(x, t)$, $\varphi(x, t)$ *of* (1.1)-[1.2) *with* $a(x, 0) = a_0$ *such that* $|a(t)|_2^2 = O(t^{-\beta})$ *where* $\beta = (2-q)/4q$. *In particular if* $\Omega \equiv \mathbb{R}^3$ *we can choose* $\beta = 3(2-q)/4q$ *for* $q \in [4/3, 2)$ *and* $\beta = 3/8$ *if* $a \in [1, 4/3]$. *Finally, if* $q = 2$, $|a(t)|_2^2 = o(1)$.

Proof. See MAREMONTI (1986), Teorema 1.2 and n.5.

The next result is interesting in itself and concerns an estimate of the L^q-norm of *any* weak solution satisfying (a) and (b) of theorem *3.1*. This estimate becomes trivial when Ω is a bounded domain.

Lemma 3.2. *Let* u, p *be two functions verifying* (a) *and* (b) *of theorem* 3.1. *Then if* $u_0 \in L^q(\Omega)$, *we have* $u \in L^\infty(0, T; L^q(\Omega))$ *for all* $T > 0$ *and for all* $q \in (1, 3/2]$.

Proof. Let $\phi(x, t) \equiv \psi(x, \tau - t)$, where ψ is a solution of problem (1.3) with $f(x, t) \equiv 0$ and $\psi_0(x) = \psi(x, 0) \in \mathscr{C}_0^\infty(\Omega)$. On the strength of lemma 1.2, such a ϕ can be substituted into (b) of Theorem 3.1 to obtain

$$\int_\Omega u(x, \tau), \psi_0(x) \, dx = \int_{\Omega_\tau} u \cdot \nabla u \cdot \phi \, dx \, dt + \int_\Omega u_0(x) \cdot \psi(x, \tau) \, dx. \tag{3.2}$$

From a result of SOLONNIKOV (1977, Theorem 5.1, relation (5.6)), it follows that the solution ψ obeys the estimate

$$\sup_{s\in[0,\tau]} |\psi(s)|_r \leqq C(\tau)\,|\psi_0|_r, \quad \text{for all } r > 1, \tag{3.3}$$

where $C(\tau)$ is an increasing function of τ only, which is bounded on every compact set. Now, by the Hölder inequality we have

$$\int_{\Omega_\tau} u \cdot \nabla u \cdot \phi \, dx\, dt \leqq \int_0^T |u(t)|_{\frac{2q'}{q'-2}} |\nabla u(t)|_2 \, |\phi(t)|_{q'}\, dt \tag{3.4}$$

where $q' = q/(q-1)$. On the other hand, since $q \in (1, 3/2]$, we know that $2q'/(q'-2) \in (2, 6]$ and thus from the Sobolev inequality we have (*cf.* MIRANDA (1978)),

$$|u|_{\frac{2q'}{q'-2}} \leqq C\,|\nabla u|_2^a\,|u|_2^{1-a}\; a = 3/q'. \tag{3.5}$$

Substituting (3.5) into (3.4), recalling that $u \in (L^\infty(0, T; \mathring{J}(\Omega))$ and employing (3.3), we obtain

$$\int_{\Omega_\tau} |u \cdot \nabla u \cdot \phi|\, dx\, dt \leqq C(T) \sup_{t\in[0,T]} |u(t)|_2^{1-a}\,|\psi_0|_{q'} \int_0^T |\nabla u|_2^{1+a}\, dt, \tag{3.6}$$

where $C(T)$ depends only on T. Moreover, using (3.3) again we deduce

$$\int_\Omega u_0(x) \cdot \psi(x, \tau)\, dx \leqq C(T)\,|u_0|_q\,|\psi_0|_{q'}. \tag{3.7}$$

Therefore, collecting (3.6) and (3.7) and recalling that $a < 1$, from (3.2), we obtain

$$\int_\Omega u(x, \tau) \cdot \psi_0(x)\, dx \leqq C\,|\psi_0|_{q'} \tag{3.8}$$

where C depends on T and on the norms of u in $L^\infty(0, T; \mathring{J}(\Omega))$ and $L^2(0, T; \mathring{J}^{1,2}(\Omega))$. The arbitrariness of ψ_0 in $\mathscr{C}_0^\infty(\Omega)$ and the Helmholtz decomposition theorem of $L^q(\Omega)$ [see SOLONNIKOV (1977)] allow us to conclude from (3.8) that $u \in L^\infty(0, T; L^q(\Omega))$, which completes the proof of the lemma.

We are now in a *position to prove theorem* 3.1. By assumption (c) it is easy to prove the existence of a time $\bar{t} > 0$ such that $|u(\bar{t})|_2\,|\nabla u(\bar{t})|_2 \leqq M$ and

$$|u(t)|_2^2 \leqq |u(\bar{t})|_2^2 - 2\int_{\bar{t}}^{t} |\nabla u(s)|_2^2\, ds \quad \text{for all } t \geqq \bar{t}. \tag{3.9}$$

Moreover, since $u_0 \in L^q(\Omega)$, by lemma 3.2 we also know that $|u(\bar{t})|_q < \infty$. Therefore, if we take $\bar{t}$ as the initial time, the hypotheses of lemma 3.1 are satisfied and, consequently, there is a regular solution $u^*(x, t)$, $p^*(x, t)$ having $u(x, \bar{t})$ as initial data and satisfying the estimate (3.1). On the other hand, since the weak solution $u(x, t)$ satisfies (3.9), by a well known uniqueness theorem of SATHER and SERRIN [SERRIN (1962)], we conclude that $u^*(x, t) = u(x, t)$ for all $t \geqq \bar{t}$. The theorem is therefore completely proved.

Acknowledgments. It is a pleasure to thank Professor V. A. SOLONNIKOV for discussions on linearized Navier-Stokes equations. This work was done under the auspices of GNFM of Italian C.N.R. and an MPI 40% Contract at the University of Napoli.

References

L. CAFFARELLI, R. KOHN & L. NIRENBERG, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. on Pure and Applied Math., **35** (1982) pp. 771–831.

G. P. GALDI, *Sulla stabilità asintotica in media dei moti stazionari idrodinamici e magneto-idrodinamici in domini limitati o meno*, Ricerche di Matem., **24** (1975) pp. 137–151.

G. P. GALDI & P. MAREMONTI, *A uniqueness theorem for viscous flows in exterior domains with summability assumptions on the gradient of pressure*, Rend. Acc. Naz. dei Lincei, **76** (1985) pp. 28–32.

G. P. GALDI & P. MAREMONTI, *A uniqueness theorem for viscous fluid motions in exterior domains*, Arch. Rational Mech. and Anal., **91** (1985) pp. 375–384.

G. P. GALDI & S. RIONERO, *Weighted energy methods in fluid dynamics and elasticity*, Lecture Notes in Mathematics, No. 1134, Springer-Verlag, 1985.

J. G. HEYWOOD, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J., **29** (1980) pp. 639–681.

E. HOPF, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr., **4** (1951) pp. 213–231.

T. KATO, *Strong L^p-solutions of the Navier-Stokes equations in R^m, with applications to weak solutions*, Math. Z., **187** (1984) pp. 471–480.

O. A. LADYZHENSKAYA, *The mathematical theory of viscous incompressible flow*, 2nd ed., Gordon and Breach, New York, 1969.

J. LERAY, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., **63** (1934) pp. 193–248.

P. MAREMONTI, *Asymptotic stability theorems for viscous fluid motions in exterior domains*, Rend. Sem. Matem. dell'Univ. di Padova, **71** (1984a) pp. 35–72.

P. MAREMONTI, *On the energy asymptotic decay of viscous incompressible fluids in exterior domains*, Proceedings of the workshop on mathematical aspects of fluid and plasma dynamics, Trieste, May 30–June 2, (1984b).

P. MAREMONTI, *Stabilità asintotica in media per moti fluidi viscosi in domini esterni*, Ann. Mat. Pura ed Appl., (1986 to appear).

K. MASUDA, *On the stability of incompressible viscous fluid motions past objects*, J. Math. Soc. Japan, **27** (1975) pp. 294–327.

K. MASUDA, *Weak solutions of Navier-Stokes equations*, Tohoku Math. J., **36** (1984) pp. 623–646.

C. MIRANDA, *Istituzioni di analisi funzionale lineare*, vol. I–II, Unione Matem. Italiana and C. N. R., Tip. Oderisi Ed., Gubbio (Italy), 1978.

L. E. PAYNE & H. S. WEINBERGER, *Note on a lemma of Finn and Gilbarg*, Acta Math., **98** (1957) pp. 297–299.

V. SCHEFFER, "*Turbulence and Hausdorff dimension*", in *Turbulence and the Navier-Stokes equations*, Lecture Notes in Math., No. 565, Springer-Verlag, 1976, pp. 94–112.

V. SCHEFFER, *The Navier-Stokes equation son a bounded domain* Comm. Math. Physics, **73** (1980) pp. 1–42.
M. E. SCHONBEK, *L^2-Decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. and Anal., **89** (1985) pp. 209–222.
J. SERRIN, *The initial value problem for the Navier-Stokes equations*, R. E. Langer Ed., Univ. of Wisconsin Press, 1963, pp. 69–98.
V. A. SOLONNIKOV, *Estimates for solutions of nonstationary Navier-Stokes equations*, J. of Soviet Math., **8** (1977) pp. 467–528.
V. A. SOLONNIKOV & V. E. SCADILOV, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Trudy Math. Inst. Steklov, **125** (1973) pp. 186–199.

Istituto Matematico
Università di Ferrara
and
Dipartimento di Matematica
"R. Caccioppoli"
Università di Napoli

(Received October 29, 1985)