Inequalities between Dirichlet and Neumann Eigenvalues

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To James Serrin on his sixtieth birthday

1. Introduction

The purpose of this paper is to derive some inequalities of the form

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{1.1}$$

between the eigenvalues $\lambda_1 < \lambda_2 \leqq \ldots$ of the Dirichlet problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } D \subset R^N, \\ u &= 0 \quad \text{on } \partial D \end{aligned} \tag{1.2}$$

and the eigenvalues $0 = \mu_1 < \mu_2 < \ldots$ of the Neumann problem

$$\begin{aligned} \Delta v + \mu v &= 0 \quad \text{in } D, \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial D \end{aligned} \tag{1.3}$$

for some classes of N-dimensional domains D. Here $\partial/\partial\nu$ denotes the outward normal derivative.

It is an immediate consequence of the variational formulation of the eigenvalue problems that

$$\mu_k \leqq \lambda_k \text{ for } k = 1, 2, \ldots$$

PÓLYA [6] proved that

$$\mu_2 < \lambda_1 .$$

PAYNE [5] showed that when $N = 2$ and D is a convex domain whose boundary ∂D is twice continuously differentiable,

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

Convexity in two dimensions is characterized by the fact that the curvature, which is a scalar defined on the boundary ∂D, is nonnegative. In higher dimensions there are $N - 1$ principal curvatures at each point of ∂D, and hence there are a number of possible generalizations of PAYNE's result.

In Section 2 we shall derive a generalization of the form (1.1) for each integer R on the interval $[1, N]$ under conditions which depend upon the principal curvatures of ∂D. The most important of these results is the fact that the inequality

$$\mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

is valid for all smooth bounded convex domains.

We also obtain the inequality

$$\mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

when ∂D has nonnegative mean curvature, which is a recent result of AVILES [1].

In Section 3 we show that the conditions on the principal curvatures can be stated in terms of their symmetric functions, which are the coefficients of the characteristic polynomial of the curvature matrix. A simpler but more restrictive set of such conditions is also presented.

The above results are derived under the hypothesis that the boundary ∂D has Hölder continuous second derivatives. In Section 4 we establish a semi-continuity result for the Neumann eigenvalues, which allows us to obtain the nonstrict version of the inequality (1.1) for some classes of domains whose boundary is not smooth.

In particular, we shall establish the inequality

$$\mu_{k+N} \leqq \lambda_k \quad \text{for } k = 1, 2, \ldots$$

for all bounded convex domains.

2. Inequalities for smooth domains

If D is a bounded domain in R^N with smooth boundary, the curvature matrix of a point P of the boundary ∂D is defined as follows: Let $\nu(x)$ be a continuously differentiable vector field defined in a neighborhood of P with the property that when x lies on ∂D, ν is an outward unit normal vector to ∂D at x. The matrix

$$K_{ij} = \nu_{i,j} - \sum_{k=1}^{N} \nu_{i,k}\nu_k\nu_j \tag{2.1}$$

evaluated at x in called the *curvature matrix* of of ∂D at x. It has the eigenvalue 0 with the eigenvector ν. The $N-1$ other eigenvalues of K_{ij} are called the *principal curvatures*

$$\varkappa_1 \leqq \varkappa_2 \leqq \ldots \leqq \varkappa_{N-1}$$

of ∂D at x. (The symbol $_{,j}$ denotes differentiation with respect to x_j.)

It is easy to see that the matrix K_{ij} is symmetric and that the ij entry is a tangential derivative of ν_i.

Our results are based on the following proposition:

Proposition 2.1. *Let D be a bounded domain whose boundary ∂D is of class $C^{2,\alpha}$ for some $\alpha \in (0, 1)$. If for some integer $R \in [1, N]$ the principal curvatures* *of ∂D at each of its points have the property that the sum of any $N-R+1$ of the set of N numbers*

$$\left\{\varkappa_1, \varkappa_2, \ldots, \varkappa_{N-1}, \sum_{j=1}^{N-1} \varkappa_j\right\} \tag{2.2}$$

is nonnegative, then

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.3}$$

Proof. We recall that [3, p. 399] if ϕ is continuously differentiable and

$$\int_D \phi v_i \, dx = 0 \quad \text{for } i = 1, 2, \ldots, k + R - 1, \tag{2.4}$$

where $v_1, v_2, \ldots,$ is an orthonormal set of eigenfunctions of (1.3) corresponding to the Neumann eigenvalues $0 = \mu_1 < \mu_2 \leqq \ldots,$ then

$$\int_D |\nabla\phi|^2 \, dx \geqq \mu_{k+R} \int_D \phi^2 \, dx. \tag{2.5}$$

We consider the set of functions

$$\phi = \sum_{i=1}^{k} a_i u_i + \sum_{p=1}^{N} b_p u_{k,p} \tag{2.6}$$

where $\{u_1, u_2, \ldots\}$ is an orthonormal set of eigenfunctions of (1.2) corresponding to the Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leqq \ldots$ Because the u_i are orthonormal and satisfy (1.2), integration by parts shows that

$$\int_D |\nabla\phi|^2 \, dx - \lambda_k \int_D \phi^2 \, dx = \sum_{i=1}^{k} a_i^2(\lambda_i - \lambda_k) + \oint_{\partial D} w \frac{\partial w}{\partial \nu} \, dS, \tag{2.7}$$

where we have defined

$$w = \sum_{p=1}^{N} b_p u_{k,p}.$$

The sum on the right of (2.7) is clearly nonpositive. In order to see whether the integral is also nonpositive, we note that

$$\begin{aligned} \frac{\partial w}{\partial \nu} &= \sum_{q,r=1}^{N} b_q u_{k,qr} \nu_r \\ &= \Sigma\, b_q \left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right) u_{k,r} - \lambda_k u_k \, \Sigma\, b_q \nu_q. \end{aligned} \tag{2.8}$$

Since u_k vanishes on the boundary, the last term is zero and

$$u_{k,r} = \nu_r \frac{\partial u_k}{\partial \nu} \quad \text{for } l = 1, \ldots, N. \tag{2.9}$$

Because the directional derivatives which appear in (2.8) are in directions tangent to ∂D, we can apply them to both sides of (2.9). Then (2.8) yields

$$w\frac{\partial w}{\partial \nu} = \sum_{p,q,r=1}^{N} b_p \nu_p \frac{\partial u_k}{\partial \nu} b_q \left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)\left(\nu_r \frac{\partial u_k}{\partial \nu}\right)$$

or

$$\begin{aligned} w\frac{\partial w}{\partial \nu} &= \tfrac{1}{2}\Sigma\, b_p b_q \left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)(\nu_p \nu_r\, |\nabla u_k|^2) \\ &\quad - \tfrac{1}{2}\Sigma\, b_p b_q\, |\nabla u_k|^2 \left[\nu_r \left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)\nu_p \right. \\ &\quad \left. - \nu_p \left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)\nu_r\right]. \end{aligned} \tag{2.10}$$

We see from the definition (2.1) of the curvature matrix K_{pq} that since $|\nu| = 1$ on ∂D,

$$\sum_{r=1}^{N}\left[\nu_r\left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)\nu_p - \nu_p\left(\nu_r \frac{\partial}{\partial x_q} - \nu_q \frac{\partial}{\partial x_r}\right)\nu_r\right] = K_{pq} + \Sigma\, K_{rr}\nu_p\nu_q.$$

The divergence theorem shows that the integral over ∂D of the first sum in (2.10) is zero. Thus

$$\int_{\partial D} w\frac{\partial w}{\partial \nu}\,dS = -\tfrac{1}{2}\sum_{p,q=1}^{N} b_p b_q \int_{\partial D} J_{pq}(x)\,dS \tag{2.11}$$

where

$$J_{pq} = \left(\frac{\partial u_k}{\partial \nu}\right)^2 [K_{pq} + (\Sigma\, K_{rr})\,\nu_p\nu_q]. \tag{2.12}$$

We recall that the vector ν lies in the null space of the curvature matrix. It follows that J_{pq} has the eigenvalues

$$\left\{\left(\frac{\partial u_k}{\partial \nu}\right)^2 \varkappa_1, \ldots, \left(\frac{\partial u_k}{\partial \nu}\right)^2 \varkappa_{N-1}, \left(\frac{\partial u_k}{\partial \nu}\right)^2 \sum_{r=1}^{N-1} \varkappa_r\right\}. \tag{2.13}$$

We rotate the x-coordinates so that the matrix

$$\int_{\partial D} J_{pq}\,dS \tag{2.14}$$

on the right of (2.8) is diagonal, with its diagonal elements arranged in non-decreasing order. By the first montonicity principle [8, p. 58] the ordered eigenvalues of the $(N-R+1)\times(N-R+1)$ matrix $\bar{J}(x)$ defined by

$$\bar{J}_{\alpha\beta}(x) = J_{\alpha\beta}(x) \quad \text{for } \alpha, \beta = 1, \ldots, N-R+1$$

are upper bounds for the $N-R+1$ lowest eigenvalues of the matrix J. We add these inequalities and use the hypothesis that the sum of any $N-R+1$ of the numbers (2.2) is nonnegative to see that

$$\operatorname{tr}(\bar{J}) = \sum_{p=1}^{N-R+1} J_{pp}(x) \geqq 0. \tag{2.15}$$

Since this is true at each boundary point, we have

$$\sum_{p=1}^{N-R+1} \oint_{\partial D} J_{pp}\, dS \geqq 0. \tag{2.16}$$

To show that this inequality is strict, we apply the Gauss-Bonnet formula

$$\oint_{\partial D^*} \varkappa\, dS = \omega_{N-1}$$

to the convex hull D^* of D. Here

$$\varkappa = \varkappa_1 \ldots \varkappa_{N-1}$$

is the Gaussian curvature of ∂D^*.

It is easily seen that $\varkappa = 0$ on $\partial D^* \setminus \partial D$. Since the area ω_{N-1} of the unit sphere is positive, there must be an open subset S of $\partial D^* \cap \partial D$ on which $\varkappa > 0$. Since D^* is convex, all the $\varkappa_j$ must be positive on S. We therefore see from (2.12) and (2.16) that the inequality (2.16) is strict unless $\partial u_k/\partial \nu \equiv 0$ on S. In this case, the Green's function representation shows that the solution of $\Delta u + \lambda_k u = 0$ in a sufficiently small ball B centered at a point of S with boundary values u_k in $D \cap \partial B$ and zero outside D coincides with u_k in $B \cap D$ and vanishes outside D. Because solutions of $\Delta u + \lambda_k u = 0$ are analytic, this would imply that $u_k \equiv 0$ in D. Since this would contradict the definition of an eigenfunction, we conclude that equality cannot hold in (2.16). That is,

$$\sum_{p=1}^{N-R+1} \oint_{\partial D} J_{pp}\, dS > 0.$$

This clearly implies that the largest of the terms in this sum if positive. Because the elements of the diagonal matrix (2.14) appear in nondecreasing order, we conclude that

$$\int_{\partial D} J_{pp}\, dS > 0 \quad \text{for } p = N - R + 1, N - R + 2, \ldots, N. \tag{2.17}$$

We now choose the $k + N$ numbers a_i and b_p so that they are not all zero and that they satisfy the $k + R - 1$ conditions (2.4) and the $N - R$ conditions $b_1 = \ldots = b_{N-R} = 0$. We then see from (2.11) and (2.17) that the integral on the right of (2.7) is nonpositive. The sum on the right is also nonpositive. Thus we have

$$\int_D |\nabla \phi|^2\, dx \leqq \lambda_k \int_D \phi^2\, dx. \tag{2.18}$$

If $\phi \equiv 0$, equality holds, so that the integral in (2.7) must vanish, which, in turn, implies that $b = 0$. Since the u_i are orthonormal, $\phi = \Sigma\, a_i u_i \equiv 0$ would imply that a as well as b is zero, contrary to our construction. Therefore $\phi \not\equiv 0$, and (2.5) and (2.18) imply that

$$\mu_{k+R} \leqq \lambda_k. \tag{2.19}$$

Equality holds if and only if both (2.5) and (2.16) are equalities. Equality in (2.5) is valid only if ϕ is an eigenfunction of the Neumann problem, so that $\Delta\phi + \mu_{k+R}\phi = 0$ in D and $\partial\phi/\partial\nu = 0$ on ∂D. As we have already seen, equality in (2.18) implies that $b = 0$ so that $\phi = 0$ on ∂D. As we showed in the derivation of (2.17), these conditions imply that $\phi \equiv 0$, which contradicts what we proved above. Therefore equality cannot hold in (2.18), and the statement (2.3) of the Proposition is proved.

There are two important cases in which the conditions of Proposition 2.1 are easily verified.

Theorem 2.1. *Let D be a convex domain in R^N whose boundary has Hölder-continuous second derivatives. Then the Neumann and Dirichlet eigenvalues satisfy the inequalities*

$$\mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.20}$$

Proof. Since D is convex, all the principal curvatures $\varkappa_j$ are nonnegative at each boundary point. Hence each of the numbers in the set (2.2) is nonnegative, so that the hypotheses of Proposition 2.1 hold with $R = N$. Thus Proposition 2.1 implies (2.20).

The sum of the N numbers in (2.2) is

$$2 \sum_{j=1}^{N-1} \varkappa_j = 2(N-1) H,$$

where H is called the mean curvature. Hence for $R = 1$ Proposition 2.1 yields a result which is contained in a recent paper of AVILES [1].

Theorem 2.2 (AVILES). *Let the boundary ∂D of the domain D be of class $C^{2,\alpha}$ and let its mean curvature be nonnegative at all its points. Then*

$$\mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.21}$$

The conditions of Proposition 2.1 can also be simplified for other values of T.

Theorem 2.3. *If the inequalities*

$$2 \sum_{i=1}^{N-1} \varkappa_i - \varkappa_j \geqq 0 \quad \text{for } j = 1, \ldots, N-1 \tag{2.22}$$

are satisfied and $\partial D \in C^{2,\alpha}$, *then*

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.23}$$

Proof. By adding the inequalities (2.22) we find that

$$\sum_{i=1}^{N-1} \varkappa_i \geqq 0. \tag{2.24}$$

The left-hand side of the latter inequality is the sum of the first $N-1$ elements of the set (2.2). The left-hand side of (2.22) is the sum of all the elements of (2.2) except for $\varkappa_j$. Thus (2.22) implies the hypotheses of Proposition 2.1 with $R = 2$, and the Theorem is proved.

Theorem 2.4. *If* $R > 2$, *if every sum of* $N - R + 1$ *members of the set*

$$\{\varkappa_1, \varkappa_2, \ldots, \varkappa_{N-1}\} \tag{2.25}$$

is nonnegative, and if $\partial D \in C^{2,\alpha}$, *then*

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.26}$$

Proof. Since $\varkappa_1 \leqq \varkappa_2 \leqq \ldots \leqq \varkappa_{N-1}$, the above condition is equivalent to

$$\sum_{i=1}^{N-R+1} \varkappa_i \geqq 0. \tag{2.27}$$

At least one of the terms in the sum must be nonnegative, so that

$$0 \leqq \varkappa_{N-R+1} \leqq \varkappa_{N-R+2} \leqq \cdots \leqq \varkappa_{N-1}.$$

Therefore

$$\sum_{i=1}^{N-1} \varkappa_i \geqq \varkappa_{N-1}$$

and the set (2.2) is ordered. Thus the hypotheses of Proposition 2.1 follow from (2.27). This proves the Theorem.

Remark. It is easily seen that each of the Theorems 2.1 through 2.4 implies Proposition 2.1 for the corresponding values of R.

3. Conditions involving curvature invariants

The eigenvalues of the curvature matrix K_{ij} are the zeros of the characteristic polynomial

$$\det(\lambda\, \delta_{ij} - K_{ij}) = \sum_{k=0}^{N-1} (-1)^k\, S_k(\varkappa_1, \ldots, \varkappa_{N-1})\, \lambda^{N-k}. \tag{3.1}$$

$S_k(\varkappa_1, \ldots, \varkappa_{N-1})$ is the k^{th} elementary symmetric function, which is defined as the sum of all products of k distinct elements of the set $\{\varkappa_1, \ldots, \varkappa_{N-1}\}$. By definition $S_0(\varkappa_1, \ldots, \varkappa_{n-1}) = 1$.

At least in principle, these symmetric functions can be found from the equation 3.1 without computing the eigenvalues $\varkappa_j$. The hypotheses of Theorem 2.1 are easily seen to be equivalent to

$$S_l(\varkappa_1, \ldots, \varkappa_{N-1}) \geqq 0 \quad \text{for } l = 1, \ldots, N-1. \tag{3.2}$$

The condition of Theorem 2.2 is, of course, just

$$S_1(\varkappa_1, \ldots, \varkappa_{N-1}) \geqq 0. \tag{3.3}$$

We may restate Theorem 2.3 in a similar fashion.

Theorem 3.1. *Let* $\partial D \in C^{2,\alpha}$. *If* (3.2) *is valid then*

(3.4) $$\mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

If (3.3) *is valid, then*

(3.5) $$\mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

If

(3.6)
$$\sum_{l=0}^{N-1-j} (-1)^l \binom{N-1-l}{j} S_l(\varkappa_1, \ldots, \varkappa_{N-1}) \, [2S_1(\varkappa_1, \ldots, \varkappa_{N-1})]^{N-1-j-l} \geqq 0,$$
$$\text{for } j = 0, \ldots, N-2,$$

then

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

Proof. We have already shown that (3.2) and (3.3) imply (3.4) and (3.5), respectively.

The conditions of Theorem 2.3 are that the $(N-1)\times(N-1)$ diagonal matrix whose eigenvalues are $2S_1(\varkappa, \ldots, \varkappa_{N-1}) - \varkappa_j$ has nonnegative eigenvalues. This is again equivalent to the condition that the coefficients of its characteristic polynomial have alternating signs. It is easily seen that this characteristic polynomial can be obtained from the characteristic polynomial of the $(N-1)\times(N-1)$ matrix with eigenvalues $\varkappa_j$ by replacing the variable λ by $2S_1(\varkappa_1, \ldots, \varkappa_{N-1}) - \lambda$. The characteristic polynomial of the latter matrix is just (3.1) divided by λ. In this way we obtain the conditions (3.6) and the Theorem is proved.

For other values of R we can use similar reasoning. The conditions of Theorem 2.4 are equivalent to the nonnegativity of the symmetric functions of the $\binom{N-1}{R-2}$ sums of $N-R+1$ distinct elements of $\{\varkappa_1, \ldots, \varkappa_{N-1}\}$.

Theorem 3.2. *If* $R \geqq 3$, $\partial D \in C^{2,\alpha}$, *and*

(3.7) $$S_j\left(\left\{\sum_{i=1}^{N-R+1} \varkappa_{l_i} : l_1 < l_2 < \ldots < l_{N-R+1}\right\}\right) \geqq 0 \quad \text{for } j = 1, \ldots, \binom{N-1}{R-2},$$

then

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

We remark that because the left-hand sides of (3.7) are symmetric in the $\varkappa_i$, they can be written as polynomials in the elementary symmetric functions $S_i(\varkappa_1, \ldots, \varkappa_{N-1})$. When $R = 3$, the sums are all of the form $S_1(\varkappa_1, \ldots, \varkappa_{N-1}) - \varkappa_j$ and there are $N-1$ of them. By a derivation like that of (3.6) we find the following result:

Theorem 3.3. *If*

(3.8)
$$\sum_{l=0}^{N-1-j} (-1)^l \binom{N-1-l}{j} S_l(\varkappa_1, \ldots, \varkappa_{N-1}) \, [S_1(\varkappa_1, \ldots, \varkappa_{N-1})]^{N-1-l-j} \geqq 0$$
$$\text{for } j = 1, 2, \ldots, N-2,$$

then

$$\mu_{k+3} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

We can now write down the conditions for the important case of 3 dimensions in terms of the mean curvature $H = (\varkappa_1 + \varkappa_2)/2$ and the Gaussian curvature $\varkappa = \varkappa_1 \varkappa_2$. We see that

(3.9)
$$H \geqq 0$$

implies

(3.10)
$$\mu_{k+1} < \lambda_k.$$

The conditions

(3.11)
$$H \geqq 0, \quad \varkappa \geqq 0$$

imply that

(3.12)
$$\mu_{k+3} < \lambda_k.$$

Finally, the intermediate conditions

(3.13)
$$H \geqq 0, \quad 8H^2 + \varkappa \geqq 0$$

imply that

(3.14)
$$\mu_{k+2} < \lambda_k.$$

It is easily seen from the proofs that for a fixed R, Theorem 3.1 or 3.2 is equivalent to Proposition 2.1. While it is therefore clear that the conditions of Theorem 3.2 for a larger R imply those for a smaller R, the number

$$\binom{N-1}{R-2}$$

of these conditions is not monotone in R.

We shall provide a partial remedy for this situation by establishing a slightly weaker result in which the number of conditions increases with R. We begin with the following algebraic lemma.

Lemma 3.1. *Suppose that for some (not necessarily ordered) set of numbers* $\mu_1, \ldots, \mu_\nu$ *and some positive integer* $l \leqq \nu$

(3.15)
$$S_j(\mu_1, \ldots, \mu_\nu) \geqq 0 \quad \text{for } j = 1, \ldots, l.$$

Then

(3.16)
$$S_j(\mu_1, \ldots, \mu_{\nu-1}) \geqq 0 \quad \text{for } j = 1, \ldots, l-1.$$

Proof. It is easily seen that

$$S_j(\mu_1, \ldots, \mu_\nu) = S_j(\mu_1, \ldots, \mu_{\nu-1}) + \mu_\nu S_{j-1}(\mu_1, \ldots, \mu_{\nu-1}) \tag{3.17}$$

where, as always, we define $S_0(\mu_1, \ldots, \mu_{\nu-1}) = 1$. By using this identity, we find that

$$\begin{aligned} S_j(\mu_1, \ldots, \mu_\nu)\, S_j(\mu_1, \ldots, \mu_{\nu-1}) &- S_{j+1}(\mu_1, \ldots, \mu_\nu)\, S_{j-1}(\mu_1, \ldots, \mu_{\nu-1}) \\ &= S_j(\mu_1, \ldots, \mu_{\nu-1})^2 - S_{j+1}(\mu_1, \ldots, \mu_{\nu-1})\, S_{j-1}(\mu_1, \ldots, \mu_{\nu-1}) \\ &= \frac{\nu}{(j+1)(\nu-j)} S_j(\mu_1, \ldots, \mu_{\nu-1})^2 \\ &\quad + \left\{ \frac{j(\nu-j-1)}{(j+1)(\nu-j)} S_j(\mu_1, \ldots, \mu_{\nu-1})^2 \right. \\ &\quad \left. - S_{j+1}(\mu_1, \ldots, \mu_{\nu-1})\, S_{j-1}(\mu_1, \ldots, \mu_\nu) \right\}. \end{aligned}$$

Newton's inequality [4, p. 104, Theo. 144] states that the term in braces is non-negative. Therefore

$$\begin{aligned} S_j(\mu_1, \ldots, \mu_\nu)\, S_j(\mu_1, \ldots, \mu_{\nu-1}) \geqq S_{j+1}(\mu_1, \ldots, \mu_\nu)\, S_{j-1}(\mu_1, \ldots, \mu_{\nu-1}) \\ + \frac{\nu}{(j+1)(\nu-j)} S_j(\mu_1, \ldots, \mu_{\nu-1})^2. \end{aligned}$$

We now see from the hypothesis (3.15) that if $j \leqq l-1$ and if $S_{j-1}(\mu_1 \ldots, \mu_{\nu-1}) \geqq 0$, then $S_j(\mu_1, \ldots, \mu_{\nu-1}) \geqq 0$. (Note that $S_j(\mu_1, \ldots, \mu_\nu) = 0$ implies that $S_j(\mu_1, \ldots, \mu_{\nu-1}) = 0$.) Since $S_0(\mu_1, \ldots, \mu_{\nu-1}) = 1 > 0$, the Lemma is proved by induction.

Theorem 3.4. *If* $\partial D \in C^{2,\alpha}$, $R \geqq 3$, *and at each point of* ∂D

$$S_j(\varkappa_1, \ldots, \varkappa_{N-1}) \geqq 0 \quad \text{for } j = 1, \ldots, R-1, \tag{3.18}$$

then

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \ldots \tag{3.19}$$

Proof. Apply Lemma 3.1 $R-2$ times to find that

$$S_1(\varkappa_1, \ldots, \mu_{N-R+1}) \geqq 0.$$

Since $\varkappa_1 \leqq \varkappa_2 \leqq \ldots \leqq \varkappa_{N-1}$, this give the conditions of Theorem 2.4, and the Theorem is proved.

When $R = 2$ we find the following simpler result.

Theorem 3.5. *If* $\partial D \in C^{2,\alpha}$ *and if*

$$\begin{gathered} S_1(\varkappa_1, \ldots, \varkappa_{N-1}) \geqq 0, \\ S_2(\varkappa_1, \ldots, \varkappa_{N-1}) + S_1(\varkappa_1, \ldots, \varkappa_{N-1})^2 \geqq 0, \end{gathered} \tag{3.20}$$

then

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \ldots$$

Proof. Clearly

$$S_1\left((\varkappa_1, \ldots, \varkappa_{N-2}, \sum_{i=1}^{N-1} \varkappa_i, \varkappa_{N-1}\right) 2S_1(\varkappa_1, \ldots, \varkappa_{N-1}) \geqq 0$$

while by (3.17) and (3.20)

$$S_2(\varkappa_1, \ldots, \varkappa_{N-2}, \Sigma\,\varkappa_i, \varkappa_{N-1}) = S_2(\varkappa_1, \ldots, \varkappa_{N-1}) + S_1(\varkappa_1, \ldots, \varkappa_{N-1})^2 \geqq 0.$$

Thus Lemma 3.1 shows that

$$S_1(\varkappa_1, \ldots, \varkappa_{N-2}, \Sigma\,\varkappa_i) \geqq 0,$$

and the result follows from Theorem 2.3.

We note that when $\varkappa_1 = -2, \varkappa_2 = \ldots = \varkappa_{n-2} = 0, \varkappa_{N-1} = 5$, the conditions (3.6) are satisfied while the conditions (3.20) are not, which shows that Theorem 3.5 is strictly weaker than Theorem 3.1. Similarly, the example $\varkappa_1 = -2, \varkappa_2 = \ldots \ldots = \varkappa_{N-R} = 0, \varkappa_{N-R+1} = \ldots = \varkappa_{N-1} = 3$ shows that when $3 \leqq R \leqq N-1$, Theorem 3.4 is strictly weaker than Theorem 3.2.

4. Extension to nonsmooth domains

We shall extend the inequalities (2.3) to more general domains by means of a limiting process. We will have to give up the strictness of such inequalities in the process.

We begin with an elementary lemma.

Lemma 4.1. *Let* $a, b > 0$. *If* v *is a* C^1 *function on* $[0, a+b]$, *then*

$$\int_a^{a+b} v^2\,dt \leqq 2b \coth a \int_0^{a+b} (v^2 + v'^2)\,dt. \tag{4.1}$$

Proof. A standard variational argument shows that the maximum μ of the ratio of the integral on the left to the integral on the right is the largest root of the equation

$$(\mu^{-1} - 1)^{1/2} \tan (\mu^{-1} - 1)^{1/2}\, b = \tanh a. \tag{4.2}$$

The well known estimates

$$\sin \varrho \leqq \varrho, \qquad \cos \varrho \geqq 1 - \tfrac{1}{2}\varrho^2$$

show that

$$\mu \leqq \frac{b\,(1 + \frac{1}{2}\,b \tanh a)}{\tanh a + b\,(1 + \frac{1}{2}\,b \tanh a)} \leqq b \coth a$$

which yields (4.1) of the Lemma.

As usual, we define the Minkowski distance $d(A, B)$ between the two point sets A and B to be the infimmum of numbers δ such that every point of A is within distance δ of B and every point of B is within distance δ of A.

We shall impose the following conditions on the domain D:

Condition $\mathcal{N}$. The boundary ∂D can be covered by a finite collection of open sets Z_ν such that

a) For some $\theta > 0$

$$\bigcup_\nu (Z_\nu) \supset \{x: d(x, \partial D) \leqq \theta\}.$$

b) Each Z_ν is the image of the cylinder $B_{N-1} \times (0, 1)$, where B_{N-1} is the open unit ball in R^{N-1}, under an invertible differentiable mapping $x = \gamma^\nu(\eta, t)$ whose Jacobian is bounded above and below by positive numbers. (Here $\eta \in B_{N-1}$, $t \in (0, 1)$).

c) There is a continuous function g^ν on B_{N-1} such that

$$D \cap Z_\nu = \{x = \gamma^\nu(\eta, t): \eta \in B_{N-1}, 0 < t < g^\nu(\eta)\},$$
$$\partial D \cap Z_\nu = \{x = \gamma^\nu(\eta, t): \eta \in B_{N-1}, t = g^\nu(\eta)\},$$
$$\alpha \leqq g^\nu(\eta) \leqq \beta,$$
$$0 < \alpha \leqq \beta < 1 \tag{4.3}$$

and

d) There is a positive constant s such that if $x \in Z_\nu \setminus \bar{D}$, $d(x, \bar{D})$ is the Euclidean distance from x to $\bar{D}$, and $x = \gamma^\nu(\eta, t)$, then

$$t - g^\nu(\eta) \leqq s\, d(x, \bar{D}).$$

Our conclusions on convergence will be based on the following semicontinuity lemma.

Lemma 4.2. *Let the domain D satisfy the conditions $\mathcal{N}$. Let D_n $n = 1, 2, \ldots$, be a sequence of domains, each of which satisfies a cone condition, and such that*

$$D_n \supset D \tag{4.4}$$

and

$$\lim_{n\to\infty} d(D_n, D) = 0. \tag{4.5}$$

If $\mu_l(D_n)$ is the l^{th} Neumann eigenvalue of D_n and $\mu_l(D)$ is the corresponding eigenvalue of D, then

$$\liminf_{n\to\infty} \mu_l(D_n) \geqq \mu_l(D). \tag{4.6}$$

Proof. We see from (4.3a) and (4.5) that when n is sufficiently large, D_n lies in the union of D and the Z_ν. For such an n we shall derive a bound for the difference in the integrals of ψ^2 over D and over D_n for any smooth function defined on D_n in terms of the integral of $\psi^2 + |\nabla\psi|^2$ over D_n.

In order to do this we note that

$$\int_{(D_n-D)\cap Z_\nu} \psi^2\,dx = \int_{\gamma^{-1}[(D_n-D)\cap Z_\nu]} [\psi(\gamma^\nu(t,\eta))]^2\,J\,dt\,d\eta$$

$$(4.7)\qquad \leqq \int_{B_{N-1}} \int_{g^\nu(\eta)}^{g^\nu(\eta)+r(\eta)} [\psi(\gamma^\nu(t,\eta))]^2\,J\,dt\,d\eta.$$

Since J is bounded above and $r(\eta) \leqq s\,d(D_n, D)$ by (4.3d), Lemma 4.1 shows that there is a constant c_1 such that

$$\int_{g^\nu(\eta)}^{g^\nu(\eta)+r(\eta)} \psi^2 J\,dt \leqq c_1\,d(D_n, D) \int_0^{g^\nu+r} (\psi^2 + \psi_t^2)\,dt.$$

We substitute this in (4.7), transform back to the x-coordinates, and use the fact that J has a positive lower bound to see that there is a constant c_2 such that

$$\int_{(D_n-D)\cap Z_\nu} \psi^2\,dx \leqq c_2\,d(D_n, D) \int_{D_n\cap Z_\nu} (\psi^2 + |\nabla\psi|^2)\,dx.$$

We add these inequalities to find that

$$(4.8)\qquad \int_{D_n} \psi^2\,dx - \int_D \psi^2\,dx \leqq c\,d(D_n, D) \int_{D_n} (\psi^2 + |\nabla\psi|^2)\,dx,$$

where $c = Lc_2$ with L the number of sets Z_ν.

Let $w_1, \ldots, w_l$ be a set of infinitely differentiable functions which are orthogonal on D_n and which approximate the first l Neumann eigenfunctions of D_n so well that any linear combination

$$(4.9)\qquad \psi(x) = \sum_{i=1}^{l} c_i w_i(x)$$

satisfies

$$(4.10)\qquad \int_{D_n} |\nabla\psi|^2\,dx < (\mu_l(D_n) + \varepsilon) \int_{D_n} \psi^2\,dx.$$

Select the constants in (4.9) so that they are not all zero and

$$\int_D \psi v_i\,dx = 0, \qquad i = 1, \ldots, l-1.$$

Then

$$(4.11)\qquad \mu_l(D) \int_D \psi^2\,dx \leqq \int_D |\nabla\psi|^2\,dx.$$

We observe that

$$(4.12)\qquad \int_D |\nabla\psi|^2\,dx \leqq \int_{D_n} |\nabla\psi|^2\,dx \leqq (\mu_l(D_n) + \varepsilon) \int_{D_n} \psi^2\,dx,$$

while by (4.8) and (4.10)

$$(4.13)\qquad \int_D \psi^2\,dx \geqq (1 - c\,d(D_n, D)\,[1 + \mu_l(D_n) + \varepsilon]) \int_{D_n} \psi^2\,dx.$$

Since the Dirichlet eigenvalue $\lambda_l(D)$ decreases as D grows [3, p. 409, Theorem 3], we see that

$$\mu_l(D_n) \leqq \lambda_l(D_n) \leqq \lambda_l(D).$$

Thus, if n is so large that $c\, d(D_n, D)\,[1 + \lambda_l(D) + \varepsilon] < 1$, we see from (4.11), (4.12), and (4.13) that

$$\mu_l(D) \leqq \frac{\mu_l(D_n) + \varepsilon}{1 - c\, d(D_n, SD)\,[1 + \lambda_l(D) + \varepsilon]}.$$

Since this is true for any positive ε, we have

$$\mu_l(D) \leqq \frac{\mu_l(D_n)}{1 - c\, d(D_n, D)\,[1 + \lambda_l(D)]}.$$

We now let $n \to \infty$ through a sequence to obtain the statement (4.6) of the Lemma.

Remarks 1. If D is bounded and convex and 0 is any of its points, then a closed ball of some radius R_1 centered at 0 lies in D and an open ball of radius R_2 centered at 0 contains $\bar{D}$. If (r, ω) are polar coordinates centered at 0, we can divide the annular region between the two balls into finitely many "cylinders" with the coordinates $t = (r - R_1)/(R_2 - R_1)$ and $\eta = \eta(\omega)$. Thus *any bounded convex domain satisfies condition* $\mathcal{N}$.

2. If D and D_n are any convex domains with $d(D_n, D)$ small, there are dilations of D with constants near 1, one of which takes $\bar{D}$ inside D_n while the other makes $\bar{D}$ contain D_n. The above proof then shows that *for bounded convex domains* $d(D_n, D) \to 0$ *implies that*

$$\mu_l(D_n) \to \mu_l(D).$$

3. We recall [3, p. 423, Theorem 11] that the Dirichlet eigenvalues are continuous in the sense that $d(D_n, D) \to 0$ implies that $\lambda_l(D_n) \to \lambda_l(D)$.

We can now obtain limiting forms of Proposition 2.1.

Theorem 4.1. *Let the domain* D *satisfy the condition* $\mathcal{N}$ *and suppose that there is a sequence of domains* $D_n \supset D$ *such that each* D_n *satisfies the hypotheses of Proposition 2.1, and let*

$$\lim_{n\to\infty} d(D_n, D) = 0.$$

Then the eigenvalues of D *satisfy the inequalities*

$$\mu_{k+R} \leqq \lambda_k \quad \text{for } k = 1, 2, \ldots$$

Proof. Apply lemma 4.2 to the inequalities

$$\mu_{k+R}(D_n) < \lambda_k(D_n) \leqq \lambda_k(D).$$

A theorem of MINKOWSKI [2, p. 35ff.] states that a convex domain D can be approximated in the sense of set distance by smooth convex domains $D_n \supset D$. This and Remark 1 after Lemma 4.2 yield the following result:

Theorem 4.2. *If D is any nonempty bounded convex domain, then*

$$\mu_{k+N} \leqq \lambda_k \text{ for } k = 1, 2, \ldots \tag{4.14}$$

It is not known whether there is a convex domain in $N > 1$ dimensions for which equality holds in (4.14).

It is easy to find nonconvex domains for which the condition $\mathcal{N}$ can be verified, but it is difficult to determine whether a nonconvex domain is the limit of a sequence of larger domains which satisfy the conditions of Proposition 2.1 for some $R < N$. We present one class of domains where this can be done.

Let $N = 3$ and let D be obtained by rotating a two-dimensional domain D_0 about a coplanar line which does not intersect the closure of D_0. Suppose that the boundary of D_0 is of class $C^{2,\alpha}$ with the exception of a subset of the interior of $\partial D \cap \partial D^*$ where D^* is the convex hull of D. By the Minkowski construction we construct a smooth convex domain $D_0' \supset D_0$ which is arbitrarily close to the convex hull D_0^* of D_0. We then construct a convex domain D_0'' with smooth boundary such that $\partial D_0''$ coincides with ∂D_0 on the part of $\partial D_0 \cap \partial D^*$ near the closure of $\partial D_0 \setminus \partial D^*$, and with $\partial D_0'$ near the set where ∂D_0 is not smooth. Let D_0''' be the subdomain of D_0'' whose boundary consists of $\partial D_0 \setminus \partial D_0^*$ and a part of $\partial D_0''$. Let D''' be obtained by rotating D_0'''. Then D''' contains and is arbitrarily close to D, and it has nonnegative mean curvature if this is true of $\partial D \setminus \partial D^*$. It is easily verified that D satisfies Condition $\mathcal{N}$.

The same reasoning also works when D is obtained by rotating D_0 about a line of symmetry.

We conclude from Theorem 4.1 that *if D is a domain of revolution and if $\partial D \setminus \partial D^*$ is smooth and has nonnegative mean curvature, then*

$$\mu_{k+1} \leqq \lambda_k \text{ for } k = 1, 2, \ldots$$

The same result follows for an N-dimensional domain which is obtained by rotating a two-dimensional domain about an $(N-2)$-dimensional hyperplane.

It would be interesting to find a larger class of domains to which Theorem 4.1 can be applied.

It was observed by JOSEPH HERSCH that if a domain D' is obtained from a domain D by removing a set of measure zero, then $\lambda_k(D') \geqq \lambda_k(D)$ while $\mu_k(D') \leqq \mu_k(D)$. Therefore if D satisfies the conditions of Proposition 2.1, then the eigenvalues of D' still satisfy the inequalities (2.3). Since the boundary of D' may not even have nonnegative mean curvature, this observation makes one wonder whether the inequalities (2.3) are not, in fact true for all domains.

However, numerical computation shows that $\mu_3 > \lambda_1$ for the two-dimensional annular sector

$$D = \{(r, \theta): 1 < r < 2, 0 < \theta < 3\pi/2\}.$$

By Lemma 4.2 one can find smooth domains containing this D for which the same inequality is valid. Thus the inequality (2.3) is not true for all smooth two-dimensional domains.

Computation shows that $\mu_3 < \lambda_1 < \mu_4$ for a disc and that $\mu_4 < \lambda_1 < \mu_5$ for a three-dimensional ball. Thus μ_{N+k} cannot be replaced by μ_{N+k+1} in the inequality (2.20). On the other hand, one can show that there are constants $a > 0$ and $c > 1$ such that for the N-ball

$$\mu_{ac^N} < \lambda_1 .$$

This suggests that perhaps (2.20) can be replaced by a better inequality of the form

$$\mu_{\phi(N,k)} < \lambda_k$$

for convex N-dimensional domains.

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