On the Equation $\Delta u + \frac{1}{2}x \cdot \nabla u + f(u) = 0$

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1. Introduction

In this paper we shall derive some properties of solutions of the equation

$$\Delta u + \tfrac{1}{2}\, x \cdot \nabla u + f(u) = 0. \tag{1.1}$$

This equation arises in the study of similarity solutions of the parabolic equation

$$u_t = \Delta u + \lambda\, |u|^{p-1}\, u \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+$$

where $N \geqq 1$, $\lambda \in \mathbb{R}$ and $p > 1$. In that context, we find

$$f(u) = \frac{k}{2}\, u + \lambda\, |u|^{p-1}\, u \tag{1.2}$$

with $k = 2/(p-1)$. More information about the background of equation (1.1) can be found in [1, 4, 5].

We shall be mainly interested in radial solutions of equation (1.1) in $\mathbb{R}^N$, with f given by (1.2), leaving $k \in \mathbb{R}^+$ arbitrary, but setting $\lambda = 1$. Thus we are led to the problem

$$\text{(I)} \quad \begin{cases} u'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) u' + \dfrac{k}{2}\, u + |u|^{p-1} u = 0, \quad r > 0 \\ u(0) = a, \quad u'(0) = 0 \end{cases} \tag{1.3}$$

in which $a \in \mathbb{R} \setminus \{0\}$, and $r = |x|$ denotes the radial variable.

Problem I has been analyzed extensively in [4, 8, 9]. We recall (Propositions 3.1, 3.4 and 3.5 in [4]) that for every $a \in \mathbb{R}$ there exists a unique solution $u \in C^2([0, \infty))$ of Problem I, and that u has the properties

(i) $|u(r)| \leqq a$ for all $r \geqq 0$; (1.4)

(ii) $L(a) \stackrel{\text{def}}{=} \lim\limits_{r \to \infty} r^k u(r)$ exists and is finite; (1.5)

(iii) if $L(a) = 0$, then

$$\lim_{r \to \infty} r^m u(r) = 0 \quad \text{for every } m > 0. \tag{1.6}$$

(By a solution we shall always mean a solution of class $C^2([0, \infty))$).

Solutions for which $L(a) = 0$ will be of particular interest. In view of (1.5) we shall refer to them as *rapidly decaying* solution whilst the other solutions will sometimes be called *slowly decaying*. One would like to know for which values of the parameters N, k and p, rapidly decaying solutions exist, and—more specifically—when a *positive* rapidly decaying solution exists.

Also, given a solution of Problem I, one would like to have a more precise estimate of its asymptotic behaviour as $r \to \infty$ than that given by (1.5) and (1.6).

Finally one is interested in the question of uniqueness particularly of positive rapidly decaying solutions.

Some answers to these questions have been given in [4, 8, 9]. In terms of the set $\mathscr{R}$ of rapidly decaying solutions it was found that

(a) *if* $N/2 < (p+1)/(p-1)$ *there are infinitely many solutions* $u \in \mathscr{R}$;

(b) *if* $N/2 < (p+1)/(p-1)$ *and* $k < N$ *there exists at least one positive solution* $u \in \mathscr{R}$;

(c) *if* $k \geqq N$ *there exist no positive solutions of Problem I*;

(d) *if* $N = 1$ *and* $k < 1$ *there exists only one positive solution* $u \in \mathscr{R}$;

(e) *if* $\dfrac{N}{2} < \dfrac{p+1}{p-1}$, *there exists an infinite family of solutions* $u \in \mathscr{R}$ *such that* $u(r) = O(r^{(N-1)/2} e^{-r^2/8})$ *as* $r \to \infty$. *If* $k < N$ *this family includes a positive solution.*

Properties (a), (b) and (c) were proved in [8, Theorem 1] and [9, Theorem 1], (d) was proved in [8, Theorem 2] and (e) was proved in [9, Theorem 1].

The method used in [4] and [8] to prove the existence of a solution $u \in \mathscr{R}$ is based on a shooting argument, using the fact that $L(a)$ is a continuous function of a. In [9] variational methods are used.

This paper has several objectives. We shall begin by deriving precise asymptotic estimates for all solutions of Problem I. Then we derive two Pohozaev-type inequalities for the partial differential equation, which can claim some interest in their own right. As a corollary to one of these inequalities, we prove that for a certain range of values of k, N and p, solutions of Problem I can have but one sign, and must be slowly decaying.

Finally, more as a curiosity, we present an explicit solution of Problem I.

The motivation for obtaining more precise asymptotic estimates comes from the analysis in [1] of the problem

$$\text{(II)} \begin{cases} u'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) u' + \dfrac{k}{2} u - |u|^{p-1} u = 0, & r > 0 \\ u(0) = a > 0, \quad u'(0) = 0 \end{cases} \tag{1.7}$$

derived from (1.1) and (1.2) with $\lambda = -1$. Again, rapidly, decaying solutions, *i.e.*, solutions for which $x^k u(x) \to 0$ as $x \to \infty$ were of particular interest. It was shown in [1] that

If $k > N$, *there exists a unique positive rapidly decaying solution of Problem II.*

Recall that we must have $k < N$ for Problem I to have a positive rapidly decaying solution.

The basic method used in [1] to analyze equation (1.7) is geometrical, regarding the set $\{(u(r), u'(r)) : r \geqq 0\}$ as an orbit in the phase plane, and we shall adopt this approach in our analysis of equation (1.3). Many of the arguments used in [1] apply very easily to (1.3) and the asymptotic behaviour in the two cases turns out to be strikingly similar.

Theorem 1. *Let* $u(r)$ *be a solution of equation* (1.3).

(i) *If* $\lim_{r\to\infty} r^k u(r) = 0$, *there exists a constant* $A \neq 0$ *such that*

$$u(r) = Ae^{-r^2/4} r^{k-N} [1 + (N-k)(k-2) r^{-2} + o(r^{-2})] \quad \text{as } r \to \infty.$$

(ii) *If* $\lim_{r\to\infty} r^k u(r) = L \neq 0$, *then, as* $r \to \infty$,

$$u(r) = \begin{cases} Lr^{-k}[1 + k(k-N+2) r^{-2} + o(r^{-2})] & \text{if } (p-1)k > 2 \\ Lr^{-k}[1 + k(k-N+2) r^{-2} + |L|^{p-1} r^{-2} + o(r^{-2})] & \text{if } (p-1)k = 2 \\ Lr^{-k}[1 + \{2|L|^{p-1}/(p-1)k\} r^{-(p-1)k} + o(r^{-(p-1)k})] & \text{if } (p-1)k < 2. \end{cases}$$

Theorem 1 will be proved in Section 2.

In Section 3 we derive the two Pohozaev-type inequalities for solutions u of equation (1.1) in star-shaped domains $\Omega \subset \mathbb{R}^N$, which vanish on the boundary $\partial\Omega$. The first of these inequalities is a direct generalization of POHOZAEV's inequality for the equation $\Delta u + f(u) = 0$ [6]. The second one differs from the first, in that weight functions $e^{+r^2/4}$ and $r^2 e^{+r^2/4}$ are introduced in the integrands. This identity was motivated by an analogous identity proved in [3], where the equation

$$\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{k}{2} u + |u|^{p-1} u = 0$$

was studied.

By use of the asymptotic estimates of Theorem 1, these inequalities can be generalized to solutions of radially symmetric, rapidly decaying solutions of equation (1.1) in $\mathbb{R}^N$. From the second inequality thus obtained we can draw the following conclusion.

Theorem 2. *Suppose* $k \leqq N/2$ *and* $N/2 \geqq (p+1)/p - 1)$. *Then any solution of Problem I with* $a > 0$ *has the properties*

$$u(r) > 0 \quad \text{for } r \in [0, \infty) \quad \text{and } L(a) > 0.$$

Remarks 1. Theorem 2 generalizes Theorem 5(c) of [4] which gives the same result, but under more restrictive conditions. They are (i) N should be an *integer* such that $N/2 \geqq (p+1)/(p-1)$ and (ii) $k = 2/(p-1)$, and reflect the fact that the proof of Theorem 5(c) uses results of the corresponding parabolic equation. Note that $2/(p-1) < N/2$ in view of the first condition.

2. Since by property (c) there exist no positive solutions if $k \geqq N$, and by Theorem 2 there are no rapidly decaying solutions if $k \leqq N/2$ and $N/2 \geqq (p+1)/(p-1)$, the question as to the existence of positive rapidly decaying solutions when

$$\frac{N}{2} < k < N \quad \text{and} \quad \frac{N}{2} \geqq \frac{p+1}{p-1}$$

remains still unanswered. (See Note added in proof).

In the final section we give an explicit solution ϕ of Problem I when

$$k = \frac{4}{p-1} \quad \text{and} \quad \frac{N}{2} > \frac{p+1}{p-1}.$$

This solution proves to be slowly decaying.

In [4, Proposition 3.6] it was shown that if $k < N$, and $p > 1$ arbitrary, any solution of Problem I is positive and slowly decaying provided that

$$0 < a < a^* := \{\tfrac{1}{2}(N-k)\}^{\frac{1}{p-1}}.$$

It is interesting to observe that if $k \in (N/2, N)$, $\phi(0) > a^*$.

2. Proof of Theorem 1

We shall prove Theorem 1 by analyzing the solution $u(r)$ of Problem I in the phase plane. Thus we write Problem I as a first order system:

$$u' = v, \tag{2.1.a}$$

$$v' = -\left(\frac{N-1}{r} + \frac{r}{2}\right)v - \frac{k}{2}u - |u|^{p-1}u \tag{2.1.b}$$

together with the initial condition

$$(u(0), v(0)) = (a, 0).$$

We proceed in two steps. First we show that there exists a number $\zeta(a) > 0$ such that u does not change sign on $(\zeta(a), \infty)$ and then we establish the asymptotic estimates in a manner similar to the one used in [1].

For each $\lambda > 0$ we define the following sectors in the phase plane

$$\mathscr{L}_\lambda^+ = \{(u, v) : u > 0, v < 0, v \geqq -\lambda u\}$$

$$\mathscr{L}_\lambda^- = \{(u, v) : u < 0, v > 0, v \leqq -\lambda u\}$$

and we introduce the number

$$\xi(a, \lambda) = \frac{k}{\lambda} + 2\frac{|a|^{p-1}}{\lambda} + 2\lambda. \tag{2.2}$$

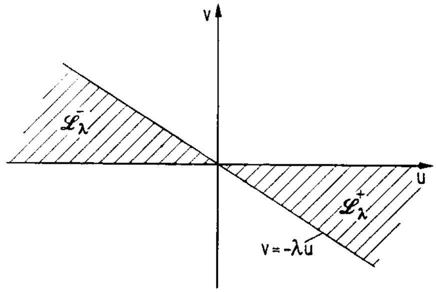

Fig. 1. The sets $\mathscr{L}_\lambda^+$ and $\mathscr{L}_\lambda^-$.

Proposition 2.1. *For any $\lambda > 0$, the sets $\mathscr{L}_\lambda^+$ and $\mathscr{L}_\lambda^-$ are positively invariant for $r_0 \geqq \xi(a, \lambda)$. That is, if $r_0 \geqq \xi(a, \lambda)$ and $(u(r_0), v(r_0)) \in \mathscr{L}_\lambda^+(\mathscr{L}_\lambda^-)$, then $(u(r), (vr)) \in \mathscr{L}_\lambda^+(\mathscr{L}_\lambda^-)$ for all $r \geqq r_0$.*

Proof. We give only the proof for $\mathscr{L}_\lambda^+$ since for $\mathscr{L}_\lambda^-$ it is similar.

It suffices to prove that if $x > \xi(a, \lambda)$, then on the boundary of $\mathscr{L}_\lambda^+$ the vector field determined by (2.1.a, b) points into $\mathscr{L}_\lambda^+$, except at the critical point (0, 0).

On the top $(u > 0, v = 0)$:

$$v' = -\left(\frac{N-1}{r} + \frac{r}{2}\right) v - \frac{k}{2} u - |u|^{p-1} u < 0$$

for all $r > 0$, while on the line

$$l_\lambda^+ = \{(u, v) : u > 0, v = -\lambda u\},$$

$$\begin{aligned} \frac{v'}{u'} &= -\left(\frac{N-1}{r} + \frac{r}{2}\right) - \frac{(k/2) u + |u|^{p-1} u}{v} \\ &= -\left(\frac{N-1}{r} + \frac{r}{2}\right) + \frac{k}{2\lambda} + \frac{|u|^{p-1}}{\lambda} \\ &< -\frac{r}{2} + \frac{k}{2\lambda} + \frac{|a|^{p-1}}{\lambda} \end{aligned}$$

because $|u(r)| \leqq a$. Therefore, if $r > \xi(a, \lambda)$ we obtain

$$\frac{u'}{v'} < -\lambda$$

which is what we wanted to show.

Proposition 2.2. *Fix $\lambda > 0$. Then $u(r)$ can have at most one zero on $(\xi(a, \lambda), \infty)$.*

Proof. If $u(r)$ has no zeros on (ξ, ∞) the proposition is plainly true, so suppose

$$r_0 = \inf\{r > \xi : u(r) = 0\}$$

exists. Then, by a uniqueness argument we may conclude that either $v(r_0) > 0$ or $v(r_0) < 0$. We assume that $v(r_0) > 0$; the other case is similar.

Note that on the coordinate half line

$$l_1 = \{(u, v): u = 0, v > 0\}.$$

we have $u' = v > 0$. This implies that after crossing l_1 at $r = r_0$, the orbit $(u(r), v(r))$ must enter the first quadrant

$$Q_I = \{(u, v): u > 0, v > 0\}.$$

In this quadrant $u' > 0$ by (2.1.a). Hence, in order to return to the v-axis, the orbit $(u(r), v(r))$ must first leave Q_I. It can do so only through the half line

$$l_2 = \{(u, v): u > 0, v = 0\},$$

on which

$$v' = -\frac{k}{2}u - |u|^{p-1}u < 0.$$

Therefore, $(u(r), v(r))$ must enter the fourth quadrant

$$Q_{IV} = \{(u, v): u > 0, v < 0\}$$

for some $r_1 > r_0$. But upon entering Q_{IV}, $(u(r), v(r))$ enters $\mathscr{L}_\lambda^+$. Since $r_1 > r_0 > \xi$, $(u(r), v(r))$ must remain in $\mathscr{L}_\lambda^+$ for all $r > r_1$ by Proposition 2.1.

Remark 2.3. Since $u(r)$ cannot have infinitely many zeros in $(0, \xi(a, \lambda))$ it follows from the previous Proposition that $u(r)$ can have at most finitely many zeros.

For the solution $u(r)$ of Problem I we define the number $\zeta(a) \in \mathbb{R}^+$ as follows. Let

$$\xi_0 = \inf\{r > 0: u \text{ has one sign on } (r, \infty)\}.$$

If $\xi_0 = 0$, we set $\zeta = 0$.
If $\xi_0 > 0$, we choose ζ so that

$$\zeta > \xi_0 \quad \text{and} \quad v(\zeta) = u'(\xi) = 0.$$

It follows from the proof of Proposition 2.2 that this number exists, and is unique. Plainly we have:

Proposition 2.4. *On* (ζ, ∞), *either* $u > 0$ *and* $v < 0$ *or* $u < 0$ *and* $v > 0$.

Having proved that $u(r)$ is eventually of one sign, we now proceed in a manner similar to [1], where only positive solutions were considered.

Proposition 2.5. *Either* $\lim_{r\to\infty} v(r)/u(r) = -\infty$ *or* $\lim_{r\to\infty} v(r)/u(r) = 0$.

Proof. Because of Propositions 2.1 and 2.4 the proofs of Lemmas 6 and 7 in [1] are valid with only minor modifications.

Proposition 2.6. *If* $\lim_{r\to\infty} v(r)/u(r) = 0$, *then*

$$\lim_{r\to\infty} \frac{rv(r)}{u(r)} = -k.$$

Proof. Define for $r > \zeta$ the function $z(r) = v(r)/u(r)$. Then

$$z'(r) + \tfrac{1}{2} rz(r) = -\frac{k}{2} - \varrho(r)$$

where

$$\varrho(r) = |u(r)|^{p-1} + \frac{N-1}{r} z(r) + z^2(r). \tag{2.3}$$

Since $z'(\zeta) = \zeta v(\zeta)/u(\zeta) = 0$, multiplying (2.3) by $e^{r^2/4}$ and integrating from ζ to $r > \zeta$ yields

$$z(r) = -e^{-r^2/4} \int_\zeta^r \left\{\frac{k}{2} + \varrho(t)\right\} e^{t^2/4}\, dt.$$

Consequently, for $r > \zeta$,

$$\frac{rv(r)}{u(r)} = \frac{-\int_\zeta^r \left\{\frac{k}{2} + \varrho(t)\right\} e^{r^2/4}\, dt}{r^{-1} e^{r^2/4}}. \tag{2.4}$$

The desired result now follows from l'Hôpital's rule applied to the right hand side of (2.4). Note that $\varrho(r) \to 0$ as $r \to \infty$ since $u(r) \to 0$ and, by assumption, $z(r) \to 0$ as $r \to \infty$.

Proposition 2.7. *Let $u(r, a)$ be a solution of Problem I. Then*

$$\lim_{r\to\infty} \frac{v(r)}{u(r)} = -\infty \Rightarrow L(a) = 0,$$

$$\lim_{r\to\infty} \frac{v(r)}{u(r)} = 0 \Rightarrow L(a) \neq 0.$$

Proof. If $\lim_{r\to\infty} v(r)/u(r) = -\infty$, then $u(r)$ decays faster than any decreasing exponential $e^{-\lambda r}$, $\lambda > 0$. Hence $L(a) = \lim_{r\to\infty} r^k u(r) = 0$.

On the other hand, if $\lim_{r\to\infty} v(r)/u(r) = 0$, then by the previous Proposition, given $\varepsilon > 0$,

$$\frac{u'(r)}{u(r)} \geqq -\frac{k+\varepsilon}{r}$$

for r sufficiently large. Thus $|u(r)| \geqq cr^{-k-\varepsilon}$ for r sufficiently large and some $c > 0$. It now follows from (1.5) and (1.6) that $L(a) \neq 0$.

Following the terminology in [1] we call orbits which enter the origin along the u axis $\left(\lim_{r\to\infty} v(r)/u(r) = 0\right)$ *slow orbits* and orbits which enter the origin along the v-axis $\left(\lim_{r\to\infty} v(r)/u(r) = -\infty\right)$ *fast orbits*. We have shown that fast orbits correspond to rapidly decaying solutions of equation (1.3) and slow orbits to slowly decaying solutions $L((a) \neq 0)$.

We are now in a position to prove Theorem 1 by the methods used in [1]. For fast orbits the proofs of Lemmas 13, 14 and 15 and Theorem 2 apply to solutions of (1.3) with only trivial modifications. Similarly, the proofs of Lemmas 18, 19, 20 and 21, and Theorems 3 and 4 can easily be adapted to the present situation.

It is interesting to observe that the asymptotic behaviour of slow orbits $(L(a) \neq 0)$ can also be derived by extending the methods of [4]. We shall do this below.

Let $w(r) = r^k u(r)$. Then (3.16) in [4] states that for $0 < r < s$,

$$w(s) - w(r) = w'(r)\, r^\alpha e^{r^2/4} \int_r^s \tau^{-\alpha} e^{-\tau^2/4}\, d\tau + \int_r^s t^\alpha e^{t^2/4} \left(\int_t^s \tau^{-\alpha} e^{-\tau^2/4}\, d\tau\right) J(t, w(t))\, dt, \tag{2.5}$$

where

$$J(t, w) = \beta t^{-2} w - t^{-k(p-1)} |w|^{p-1} w,$$
$$\alpha = N - 1 - 2k,$$
$$\beta = k(N - k - 2).$$

Letting $s \to \infty$ in (2.5), we obtain

$$L - w(r) = r^{-1} w'(r)\, g(r) + \int_r^\infty g(t)\, t^{-1} J(t, w(t))\, dt, \tag{2.6}$$

where

$$g(r) = r^{\alpha+1} e^{r^2/4} \int_r^\infty t^{-\alpha} e^{-t^2/4}\, dt.$$

By l'Hôpital's rule it follows that

$$\lim_{r\to\infty} g(r) = 2.$$

Moreover, since $r^k u(r) \to L$ and $rv(r)/u(r) \to -k$ as $r \to \infty$, by Propositions 2.6 and 2.7, we conclude that

$$rw'(r) = r^{k+1} u'(r) + k r^k u(r)$$
$$= \left(\frac{rv(r)}{u(r)} + k\right) r^k u(r) \to 0 \quad \text{as } r \to \infty.$$

Therefore

$$\lim_{r\to\infty} rw'(r)\, g(r) = 0. \tag{2.7}$$

Finally, consider for $m > 0$

$$\lim_{r\to\infty} r^m \int_r^\infty g(t)\, t^{-1} J(t, w(t))\, dt. \tag{2.8}$$

By l'Hôpital's rule (2.8) is equal to

$$\lim_{r\to\infty} \frac{g(r)\, r^{-1}\{\beta r^{-2} w(r) - r^{-k(p-1)}\,|w(r)|^{p-1}\, w(r)\}}{m r^{-m-1}} \tag{2.9}$$

if this latter limit exists. Recall that $g(r) \to 2$ as $r \to \infty$. Therefore

(i) if $k > 2/(p-1)$, we set $m = 2$ in (2.9) to conclude that

$$\lim_{r\to\infty} r^2 \int_r^\infty g(t)\, t^{-1}\, J(t, w(t))\, dt = \beta L; \tag{2.10}$$

(ii) if $k = 2/(p-1)$, we set $m = 2$ in (2.9) again to deduce that

$$\lim_{r\to\infty} r^2 \int_r^\infty g(t)\, t^{-1}\, J(t, w(t))\, dt = \beta L - |L|^{p-1}\, L; \tag{2.11}$$

(iii) if $k < 2/(p-1)$, we set $m = k(p-1)$ in (2.9) to obtain

$$\lim_{r\to\infty} r^{k(p-1)} \int_r^\infty g(t)\, t^{-1}\, J(t, w(t))\, ds = -\frac{2}{k(p-1)}\, |L|^{p-1} L. \tag{2.12}$$

Combining (2.6), (2.7), (2.10), (2.11) and (2.12) we obtain

$$\begin{aligned}
&\lim_{r\to\infty} r^2\{w(r) - L\} = -\beta L && \text{if } k > 2/(p-1),\\
&\lim_{r\to\infty} r^2\{w(r) - L\} = -\beta L + |L|^{p-1} L && \text{if } k = 2/(p-1),\\
&\lim_{r\to\infty} r^{k(p-1)}\{w(r) - L\} = \frac{2}{k(p-1)}\, |L|^{p-1} L && \text{if } k < 2/(p-1).
\end{aligned}$$

These are precisely the desired asymptotic estimates.

One feature of the second derivation of the asymptotic estimates is that the three cases emerge in a very straightforward fashion.

3. Pohožaev Formulae

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$.

We begin by establishing two identities for solutions of the problem

$$\text{(III)} \quad \begin{cases} \Delta u + \frac{1}{2} x \cdot \nabla u + f(u) = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We shall denote by ν the outward normal on $\partial\Omega$, and by F the primitive of f:

$$F(s) = \int_0^s f(t)\, dt,$$

where $f: \mathbb{R} \to \mathbb{R}$ is continuous. Also, by B we denote the differential operator

$$Bu := \frac{N}{2} u + x \cdot \nabla u. \tag{3.1}$$

In deriving the formulae it turns out to be both computationally and conceptually easier to use Bu rather than the more obvious term $x \cdot \nabla u$.

Proposition 3.1. *Let* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *be a solution of Problem III. Then*

$$\text{(A)} \qquad N \int_\Omega F(u) - \frac{N-2}{2} \int_\Omega uf(u) - \frac{N}{4} \int_\Omega u^2 = \tfrac{1}{2} \int_\Omega (Bu)^2 + \tfrac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu),$$

$$\text{(B)} \qquad N \int_\Omega \varrho F(u) - \frac{N-2}{2} \int_\Omega \varrho uf(u) - \frac{N}{4} \int_\Omega \varrho u^2 + \tfrac{1}{2} \int_\Omega r^2 \varrho F(u) - \tfrac{1}{4} \int_\Omega r^2 \varrho uf(u) - \tfrac{1}{8} \int_\Omega r^2 \varrho u^2 = \tfrac{1}{2} \int_{\partial\Omega} \varrho |\nabla u|^2 (x \cdot \nu),$$

where $\varrho(x) = \exp(r^2/4)$, $r = |x|$.

Remark. Recall that POHOŽAEV proved in [6] that if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of the problem

$$\text{(IV)} \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then

$$\text{(C)} \qquad N \int_\Omega F(u) - \frac{N-2}{2} \int_\Omega uf(u) = \tfrac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu).$$

Formulas (A) and (B) can both be considered as analogues of (C) for Problem III. The relationship between (A) and (B) is not completely clear.

We shall see below that for

$$f(u) = \frac{k}{2} u + |u|^{p-1} u, \tag{3.2}$$

both (A) and (B) yield the same non-existence result for rapidly decaying solutions of Problem III (Theorem 2).

We shall begin by proving (A), this being the simpler one to establish, and then prove (B), mimicking the proof of (A) as much as possible. Interestingly, as an intermediate step in the derivation of (B), we shall arrive at a formula which includes (C) as a special case.

Before proving the formulae (A) and (B), we collect some information about the operator B.

Lemma 3.2. *If* $u \in C^1(\Omega)$, *then*

$$Bu = -\frac{N}{2} u + \nabla \cdot (xu), \tag{3.3}$$

$$uBu = \tfrac{1}{2} \nabla \cdot (xu^2), \tag{3.4}$$

$$BF(u) = \frac{N}{2} \{F(u) - uf(u)\} + f(u)\, Bu. \tag{3.5}$$

If $u \in C^2(\Omega)$, *then*

$$\nabla Bu = \nabla u + B(\nabla u), \tag{3.6}$$

where $B(\nabla u)$ *is the vector whose* j^{th} *component is* $B(u_{x_j})$.

Proof. These are all straightforward exercises in calculus.

Lemma 3.3. *If* $u \in C^1(\Omega) \cap C(\bar{\Omega})$, *then*

$$\int_\Omega Bu = -\frac{N}{2} \int_\Omega u + \int_{\partial\Omega} u(x \cdot \nu), \tag{3.7}$$

$$\int_\Omega uBu = \frac{1}{2} \int_{\partial\Omega} u^2(x \cdot \nu), \tag{3.8}$$

$$\int_\Omega f(u)\, Bu = \frac{N}{2} \int_\Omega \{uf(u) - 2F(u)\} + \int_{\partial\Omega} F(u)\,(x \cdot \nu). \tag{3.9}$$

If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *and* $u = 0$ *on* $\partial\Omega$, *then*

$$\int_\Omega (\Delta u)\, Bu = -\int_\Omega |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2\,(x \cdot \nu). \tag{3.10}$$

Proof. Integration of (3.3) and (3.4) immediately yields, respectively, (3.7) and (3.8).

To prove (3.9), we integrate (3.5) and then use (3.7) with u replaced by $F(u)$.

The proof of (3.10) takes a little longer. We start by using (3.6) as follows:

$$\begin{aligned}\int_\Omega (\Delta u)\, Bu &= -\int_\Omega \nabla u \cdot \nabla(Bu) + \int_{\partial\Omega} (Bu)\,(\nabla u \cdot \nu)\\ &= -\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u \cdot B(\nabla u) + \int_{\partial\Omega} (Bu)\,(\nabla u \cdot \nu).\end{aligned}$$

We evaluate the middle integral on the right hand side by means of (3.8) in which we replace u by u_{x_j}, $j = 1, \ldots, N$. This yields

$$\int_\Omega (\Delta u)\, Bu = -\int_\Omega |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2\,(x \cdot \nu) + \int_{\partial\Omega} (Bu)\,(\nabla u \cdot \nu).$$

Finally we observe that on $\partial\Omega$, $u = 0$ and so $Bu = x \cdot \nabla u$ and $\nu = \pm\nabla u/|\nabla u|$ whenever $|\nabla u| \neq 0$. Thus on $\partial\Omega$

$$(Bu)\,(\nabla u \cdot \nu) = (x \cdot \nabla u)\,(\nabla u \cdot \nu) = (x \cdot \nu)\,(\nabla u \cdot \nabla u).$$

Substituting this above, we arrive at (3.10) completing the proof of Lemma 3.3.

Proof of (A). The equation for u can be rewritten as

$$\Delta u + \frac{1}{2} Bu - \frac{N}{4} u + f(u) = 0. \tag{3.11}$$

We successively multiply (3.11) by u and by Bu, integrate over Ω and use the results of Lemma 3.3, keeping in mind that $u = 0$ on $\partial\Omega$. Thus, multiplying

(3.11) by u yields

$$-\int_\Omega |\nabla u|^2 - \frac{N}{4}\int_\Omega u^2 + \int_\Omega uf(u) = 0. \tag{3.12}$$

and by Bu:

$$-\int_\Omega |\nabla u|^2 + \tfrac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x\cdot\nu) + \tfrac{1}{2}\int_\Omega (Bu)^2 + \frac{N}{2}\int_\Omega \{uf(u) - 2F(u)\} = 0. \tag{3.13}$$

Subtracting (3.13) from (3.12) we obtain formula (A).

Remark. The technical convenience of using the operator B stems from the fact that, except for a boundary integral, $\int_\Omega (\Delta u)\, Bu$ and $\int_\Omega (\Delta u)\, u$ are equal.

Conceptually, B is the generator on $L^2(\mathbb{R}^N)$ of the unitary dilation group given by $u_\lambda(x) = \lambda^{N/2}u(\lambda x)$, $\lambda > 0$:

$$\frac{d}{d\lambda} u_\lambda(x)|_{\lambda=1} = (Bu)(x).$$

Indeed, formula (A) was really derived by multiplying (3.11) by $Bu - u$, which corresponds to the scaling used in the proof of Theorem 1 in [1]. We are grateful to J. GINIBRE for pointing out the utility of the operator B.

Before turning to the proof of formula (B), we derive a generalization of Lemma 3.3 involving a weight function ϱ in the integrals.

Lemma 3.4. *If* $\varrho, u \in C^1(\Omega) \cap C(\bar\Omega)$, *then*

$$\int_\Omega \varrho Bu = -\frac{N}{2}\int_\Omega \varrho u - \int_\Omega u(x\cdot\nabla\varrho) + \int_{\partial\Omega} \varrho u(x\cdot\nu), \tag{3.14}$$

$$\int_\Omega \varrho uBu = -\tfrac{1}{2}\int_\Omega u^2(x\cdot\nabla\varrho) + \tfrac{1}{2}\int_{\partial\Omega} \varrho u^2(x\cdot\nu), \tag{3.15}$$

$$\int_\Omega \varrho f(u)\, Bu = \frac{N}{2}\int_\Omega \varrho\{uf(u) - 2F(u)\} - \int_\Omega F(u)\,(x\cdot\nabla\varrho) + \int_{\partial\Omega} \varrho F(u)\,(x\cdot\nu). \tag{3.16}$$

If $\varrho \in C^1(\Omega) \cap C(\bar\Omega)$ *and* $u \in C^2(\Omega) \cap C^1(\bar\Omega)$ *with* $u = 0$ *on* $\partial\Omega$, *then*

$$\int_\Omega \nabla\cdot(\varrho\nabla u)\, Bu = -\int_\Omega \varrho|\nabla u|^2 + \tfrac{1}{2}\int_\Omega |\nabla u|^2(x\cdot\nabla\varrho) + \tfrac{1}{2}\int_{\partial\Omega} \varrho|\nabla u|^2(x\cdot\nu). \tag{3.17}$$

Proof. To obtain (3.14) we multiply (3.3) by ϱ, integrate over Ω and use the divergence theorem.

Likewise, if we multiply (3.4) by ϱ, and use the identity

$$\nabla\cdot(\varrho xu^2) = \varrho\nabla\cdot(xu^2) + u^2(x\cdot\nabla\varrho)$$

we obtain (3.15).

To prove (3.16) we multiply (3.5) by ϱ, integrate over Ω and then use (3.14) with u replaced by $F(u)$.

Finally, the proof of (3.17) follows the same steps as the proof of (3.10), with (3.15) playing the rôle of (3.8).

Proof of (B). The equation for u can be rewritten as

$$\nabla \cdot (\varrho \nabla u) + \varrho f(u) = 0, \tag{3.18}$$

where

$$\varrho(x) = \exp(|x|^2/4). \tag{3.19}$$

For the moment, however, we think of u as satisfying (3.18) and only assume that $\varrho \in C^1(\Omega) \cap C(\bar{\Omega})$ is an arbitrary weight function.

We proceed exactly as in the proof of (A). Multiply (3.18) respectively by u and Bu, integrate over Ω, use the results of Lemma 3.4 keeping in mind that $u = 0$ on $\partial\Omega$, and subtract the two resulting equations, one from the other. Since Lemma 3.4 already contains all the technicalities, this calculation is now completely straightforward. The result is

$$N \int_\Omega \varrho F(u) - \frac{N-2}{2} \int_\Omega \varrho u f(u) + \int_\Omega \{-\tfrac{1}{2} |\nabla u|^2 + F(u)\} (x \cdot \nabla \varrho) = \tfrac{1}{2} \int_{\partial\Omega} \varrho |\nabla u|^2 (x \cdot \nu). \tag{3.20}$$

Note that if $\varrho \equiv 1$ on Ω, (3.20) reduces to formula (C).

At this point it is necessary to use the fact that ϱ is given by (3.19) and hence

$$x \cdot \nabla \varrho = \tfrac{1}{2} r^2 \varrho, \tag{3.21}$$

where $r = |x|$. Thus (3.20) contains only one undesirable term, the one on the left involving ∇u. To eliminate it we multiply (3.18) by $r^2 u$ and integrate over Ω. Remembering that $u = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned} 0 &= -\int_\Omega \varrho \nabla u \cdot \nabla(r^2 u) + \int_\Omega r^2 \varrho u f(u) \\ &= -\int_\Omega r^2 \varrho |\nabla u|^2 - 2 \int_\Omega \varrho u (x \cdot \nabla u) + \int_\Omega r^2 \varrho u f(u) \\ &= -\int_\Omega r^2 \varrho |\nabla u|^2 + N \int_\Omega \varrho u^2 - 2 \int_\Omega \varrho u B u + \int_\Omega r^2 \varrho u f(u). \end{aligned}$$

Therefore, using (3.15) and (3.21) we find that

$$\int_\Omega r^2 \varrho |\nabla u|^2 = N \int_\Omega \varrho u^2 + \tfrac{1}{2} \int_\Omega r^2 \varrho u^2 + \int_\Omega r^2 \varrho u f(u).$$

Substituting this into (3.20), again using (3.21) we arrive at formula (B).

Remark. In the case of radially symmetric solutions of Problem III, equations (3.11) and (3.18) become ordinary differential equations involving the dimension N as a parameter. Using these equations, one can derive analogues of formulas (A) and (B). Note that N now need not be an integer.

Proof of Theorem 2. Let u be a solution of Problem I with $a > 0$. We show first that $u(r) > 0$ for all $r \geq 0$.

Suppose to the contrary that $u(r) = 0$ for some $r > 0$, and set

$$R = \sup\{r > 0 : u > 0 \text{ on } [0, r)\}.$$

We regard u as a radial solution of Problem III, choosing for Ω the centered ball B_R of radius R in $\mathbb{R}^N$, and for f the function given by (3.2). Then

$$F(u) = \frac{k}{4} u^2 + \frac{|u|^{p+1}}{p+1}. \tag{3.22}$$

Formula (A) now yields, because $|u'(R)| = |\nabla u(R)| \neq 0$:

$$\left(\frac{N}{p+1} - \frac{N-2}{2}\right) \int_\Omega |u|^{p+1} + \left(\frac{k}{2} - \frac{N}{4}\right) \int_\Omega u^2 > 0 \tag{3.23}$$

which is impossible if

$$k \leqq \frac{N}{2} \quad \text{and} \quad \frac{N}{2} \geqq \frac{p+1}{p-1}.$$

Next, suppose u is a positive, rapidly decaying solution of Problem I. Then, because of (1.6) and Lemma 3.2 of [4], u satisfies formula (A) in which $\Omega = \mathbb{R}^N$ and the boundary integrals have been set equal to zero. Substitution of f and F yields (3.23) again, and, as before, a contradiction.

Thus u can only be a positive, slowly decaying solution.

Remark. One can carry out the above proof, using (B) instead of (A). However, because of the rapidly growing weight function $\varrho(x) = \exp(|x|^2/4)$, passing from $\Omega = B_R$ to $\mathbb{R}^N$ requires the more precise decay estimates for u of Theorem 1.

Finally we wish to compare the results of this section with some results of [3] where the equation

$$\Delta u - \tfrac{1}{2} x \cdot \nabla u - \frac{k}{2} u + |u|^{p-1} u = 0 \tag{3.24}$$

is studied. Using the function

$$\varrho(x) = \exp(-|x|^2/4), \tag{3.25}$$

we can also write (3.24) in divergence form:

$$\nabla \cdot (\varrho \nabla u) + \varrho \left(-\frac{k}{2} u + |u|^{p-1} u\right) = 0. \tag{3.26}$$

In Section 3 of [3] the following formula was derived for bounded solutions u of (3.24):

$$\text{(D)} \quad \left(\frac{N}{p+1} - \frac{N-2}{2}\right) \int |\nabla u|^2 \varrho \, dx + \tfrac{1}{2}\left(\tfrac{1}{2} - \frac{1}{p+1}\right) \int |x|^2 |\nabla u|^2 \varrho \, dx$$
$$+ \frac{k(p-1)-2}{8(p+1)} \left(\int |x|^2 |u|^2 \varrho \, dx - 2N \int u^2 \varrho \, dx\right) = 0,$$

the integrals being taken over $\mathbb{R}^N$. (Indeed, it is this formula in [3] which motivated the work in this section.)

By following the procedure we used to prove (A) and (B) we can derive analogous formulae (A*) and (B*) for (3.24). We obtain, when $u = 0$ on $\partial\Omega$:

(A*)
$$N \int_\Omega F(u) - \frac{N-2}{2} \int_\Omega uf(u) + \frac{N}{4} \int_\Omega u^2 + \frac{1}{2} \int_\Omega (Bu)^2 = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu);$$

(B*)
$$N \int_\Omega \varrho F(u) - \frac{N-2}{2} \int_\Omega \varrho uf(u) + \frac{N}{4} \int_\Omega \varrho u^2 - \frac{1}{2} \int_\Omega r^2 \varrho F(u) + \frac{1}{4} \int_\Omega r^2 \varrho uf(u) - \frac{1}{8} \int_\Omega r^2 \varrho u^2 = \frac{1}{2} \int_{\partial\Omega} \varrho |\nabla u|^2 (x \cdot \nu),$$

where $\varrho(x) = \exp(-|x|^2/4)$.

If we consider bounded solutions of (3.24) in $\mathbb{R}^N$, we can take $\Omega = B_R$ in (D) and (B*) and let $R \to \infty$. In this case $\varrho(x) \to 0$ as $|x| \to \infty$ very rapidly, whence the boundary terms now disappear.

As in Theorem 2, one can try to use (D), (A*) and (B*) to deduce nonexistence theorems for solutions of (3.24), *i.e.* when

$$f(u) = -\frac{k}{2} u + |u|^{p-1} u.$$

Thus, one can deduce from (A*) that if

$$k \leqq \frac{N}{2} \quad \text{and} \quad \frac{N}{2} \leqq \frac{p+1}{p-1},$$

equation (3.24) cannot have any nontrivial solutions in $\mathbb{R}^N$, which vanish rapidly enough at ∞. On the other hand, (B*) gives no result, but from (D) one can conclude [3] that if

$$k = \frac{2}{p-1} \quad \text{and} \quad \frac{N}{2} \leqq \frac{p+1}{p-1}$$

then the only bounded solutions of (3.24) are the three constant solutions

$$u = 0 \quad \text{and} \quad u = \pm (k/2)^{\frac{1}{p-1}}.$$

4. An Explicit Solution

In this section we give a family of explicit solutions of Problem I, and use it to illustrate the results in [3, 4, 5] and the previous sections.

Proposition 4.1. *Suppose* $N/2 > (p+1)/p - 1)$. *Then the function*

$$\phi(r) = A(p, N) \{1 + B(p, N) r^2\}^{-\frac{2}{p-1}} \tag{5.1}$$

in which

$$A(p, N) = \left\{\frac{2(p+1)}{(p-1)^2}\left(\frac{N}{2} - \frac{p+1}{p-1}\right)^{-1}\right\}^{\frac{1}{p-1}} \tag{5.2}$$

$$B(p, N) = \tfrac{1}{4}\left(\frac{N}{2} - \frac{p+1}{p-1}\right)^{-1} \tag{5.3}$$

is an exact solutions of Problem I if

$$k = \frac{4}{p-1} \quad \textit{and} \quad a = A(p, N).$$

The proof of Proposition 4.1 consists of a lengthy but elementary computation. We shall omit it.

It is interesting to recall here that if $N/2 = (p+1)/(p-1)$, the equation

$$\Delta u + |u|^{p-1} u = 0 \tag{5.4}$$

also has a family of explicit radial solutions defined on $\mathbb{R}^N$. It is given in [2] by

$$\psi(r) = \left[\frac{\{N(N-2)\lambda\}^{\frac{1}{2}}}{1+\lambda r^2}\right]^{\frac{N-2}{2}}, \quad \lambda \in \mathbb{R}^+.$$

Note that, whereas p is completely determined in this case, $\psi(0)$ can be chosen entirely arbitrarily by adjusting the parameter λ. This is a consequence of the homogenity of equation (5.4) in x and u.

Observe that if $k = 4/(p-1)$, then

$$L = \lim_{r\to\infty} r^k \phi(r) = \left\{\frac{32(p+1)}{(p-1)^2}\left(\frac{N}{2} - \frac{p+1}{p-1}\right)\right\}^{\frac{1}{p-1}} > 0.$$

Therefore, the family of explicit solutions ϕ given in Proposition 5.1 consists of slowly decaying solutions. Their asymptotic behaviour is given by Theorem 1(ii), where the case $(p-1)k = 4 > 2$ applies.

In Theorem 2 we have seen that when $N/2 \geqq (p+1)/(p-1)$, then if $k \leqq N/2$, *any* solution of Problem I with $a > 0$ is positive and by property (c) we know that if $k \geqq N$, then *none* is positive. The family of explicit solutions obtained above reveals that for any $k < N-2$, there exists a positive solution, provided we choose $p = 1 + (4/k)$.

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Note added in proof. The existence of positive rapidly decaying solutions when

$$\frac{N}{2} < k < N \quad \text{and} \quad \frac{N}{2} \geqq \frac{p+1}{p-1}$$

has been proved in [10] and [11]. It was found that

(i) if $N \geqq 4$, rapidly decaying solutions exist when $N/2 < k < N$ [11];

(ii) if $N = 3$, rapidly decaying solutions exist when $2 < k < 3$ [11] and they do not exist when $3/2 < k \leqq 2$ [10].

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