On Exterior Boundary Value Problems in Linear Elasticity

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1. Introduction

The questions considered in this note were suggested by a paper by FINN & NOLL on the uniqueness and non-existence of Stokes flow [*1*]. The equations of linear isotropic elasticity are only slightly more complicated than the equations governing Stokes flow, and it proves possible to extend the treatment of FINN & NOLL to analogous elastic problems.

The equations for the displacement field $\boldsymbol{q}$ of a linearly elastic body at rest and under no body forces may be written (*cf.* [*2*]) in the form

$$\Delta \boldsymbol{q} = \operatorname{grad} p, \quad \operatorname{div} \boldsymbol{q} = -c p. \tag{1.1}$$

Here c is the positive constant $c = 1 - 2\sigma$, where σ is POISSON's ratio.

If we set $c = 0$ in (1.1) and interpret $\boldsymbol{q}$ as the velocity field, then (1.1) are the equations of Stokes flow. The results of [*1*] may be phrased this way: Let the region of flow be bounded internally by closed surfaces (curves) $\mathfrak{B}$. We seek solutions of (1.1) (with $c = 0$) which, in three dimensions, tend uniformly to zero at infinity or, in two dimensions, are bounded at infinity. There is at most one solution $\boldsymbol{q}$ of this type which assumes a given constant value $\boldsymbol{q} = \boldsymbol{q}_0$ at the internal boundary $\mathfrak{B}$. We note that this theorem remains valid also if the prescribed boundary value $\boldsymbol{q} = \boldsymbol{q}_0$ is not constant, although a simple physical interpretation for $\boldsymbol{q}$ as the flow past a rigid body can be given only when $\boldsymbol{q}_0$ is constant.

In this paper the corresponding theorem is proved for the case when $c > 0$ in (1.1). In this way, we get three different uniqueness theorems for boundary value problems in elasticity:

(a) Consider an infinite elastic medium bounded internally by closed surfaces $\mathfrak{B}$. There is at most one displacement $\boldsymbol{q}$ of the medium which assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at the internal boundaries $\mathfrak{B}$ and which tends uniformly to zero at infinity.

(b) Consider an infinite elastic medium bounded internally by an infinite cylinder $\mathfrak{B}$ (not necessarily circular). There is at most one plane displacement $\boldsymbol{q}$ of the medium in the plane perpendicular to the cylinder which assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at $\mathfrak{B}$ and which is bounded.

(c) Consider a thin infinite elastic plate bounded internally by closed contours $\mathfrak{B}$. There is at most one displacement $\boldsymbol{q}$ in the plane of the plate which assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at $\mathfrak{B}$ and which is bounded. All three results are special cases of Theorem 1, proved in Section 2. The proposition (b) corresponds to the case of plane strain. (c) corresponds to a state of generalized plane stress. In this case, $\boldsymbol{q}$ must be interpreted as the average displacement, the average being taken over the thickness of the plate. As is well known [3, p. 208], $\boldsymbol{q}$ then satisfies the two-dimensional form of the equations (1.1) when POISSON's ratio σ is replaced by $\sigma' = \frac{\sigma}{2-\sigma}$. Thus, in this case, the constant c must be given the positive value $c = 1 - 2\sigma' = \frac{2-3\sigma}{2-\sigma}$. We also prove a uniqueness theorem for the transverse deflection w of an infinite thin plate bounded internally by closed curves $\mathfrak{B}$. The function w is biharmonic: $\Delta\Delta w = 0$. We show that, if the plate is clamped in a prescribed way at $\mathfrak{B}$, so that w and $\partial w/\partial n$ are prescribed at $\mathfrak{B}$, then there is at most one deflection w which is of the order $O(r)$ as $r \to \infty$.

The essential difference between the results obtained here and the uniqueness theorems previously known lies in the conditions at infinity. Previously, complicated regularity conditions at infinity had to be imposed on $\boldsymbol{q}$ or w. We merely require simple limit conditions. It should not be difficult to obtain similar results when the surface tractions, rather than the displacements, are prescribed at the interior boundaries.

As in the case of Stokes flow so also in elasticity we observe the striking difference between the situations in two and three dimensions. In three dimensions we get a unique solution if we require that $\boldsymbol{q}(\infty) = 0$, *i.e.* that the "boundary at infinity" is fixed. If we impose the same condition in two dimensions there is, in general, no solution at all. For a physical interpretation consider, for example, a very large thin plate with a fixed external boundary. At first glance our theorem seems to imply that it is not possible to impose a rigid displacement $\boldsymbol{q} = \boldsymbol{q}_0 =$ const. on the internal boundaries, a result that contradicts physical experience. However, the correct interpretation is the following: The gradient of the displacement $\boldsymbol{q}$, and hence the strain, is very small everywhere in the plate. In the limit as the external boundary expands to infinity the gradient of the displacement tends to zero. The displacement $\boldsymbol{q}$ converges to its internal boundary value $\boldsymbol{q}_0$. But this convergence is not uniform, and the condition that $\boldsymbol{q} = 0$ on the external boundary is not preserved in the limit. A similar explanation must be given in the case of the transverse deflection of a large plate. If the external boundary is clamped $(w = \partial w/\partial n = 0)$ we get a unique solution. In the limit, however, the conditions $w = \partial w/\partial n = 0$ at the "boundary at infinity" get lost.

If we impose a displacement $\boldsymbol{q} = \boldsymbol{q}_0$ on the boundary of an internal cavity in a large elastic body with fixed external boundary, we shall get a displacement $\boldsymbol{q}$ which approximates the solution for the infinite medium determined uniquely by the condition $\boldsymbol{q}(\infty) = 0$. This solution will be given explicitly in Section 4 for $\boldsymbol{q}_0 =$ const. and a spherical cavity. The displacement gradient is not small, and the strains and stresses will remain finite in the limit as the body becomes infinite. In this case, the boundary condition $\boldsymbol{q} = 0$ on the external boundary is preserved in the limit.

2. The basic lemmas

Let $\boldsymbol{q}$ be a vector field defined and three times continuously differentiable in a neighborhood of infinity.

Lemma 1. In three dimensions, assume that $\boldsymbol{q}=\boldsymbol{q}_0+o(1)$. Then

$$\Delta(\operatorname{curl}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{curl}\boldsymbol{q}=O(r^{-2}), \tag{2.1}$$

and

$$\Delta(\operatorname{div}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{div}\boldsymbol{q}=O(r^{-2}). \tag{2.2}$$

Lemma 2. In two dimensions, assume that $\boldsymbol{q}$ is bounded. Then

$$\Delta(\operatorname{curl}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{curl}\boldsymbol{q}=O(r^{-1}), \tag{2.3}$$

and

$$\Delta(\operatorname{div}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{div}\boldsymbol{q}=O(r^{-1}). \tag{2.4}$$

The statements concerning the curl are Lemmas 4 and 5 in [*1*]. The propositions concerning the divergence can be proved in precisely the same manner as these lemmas.

Lemma 3. Assume that w is a scalar field defined and biharmonic in a neighborhood of infinity. Then $w=O(r)$ implies that $\operatorname{grad} w$ is bounded.

Proof. We denote the mean value of a function w over the circumference of a circle of radius r with center at Q by $m_0(w,r,Q)$. If w is biharmonic in a region containing this circle we have the following mean value theorem (*cf.* [*4*], p. 316, (11)*):

$$3w(Q)=4m_0(w,\tfrac{1}{2}r,Q)-m_0(w,r,Q). \tag{2.5}$$

We may replace w by $\operatorname{grad} w$ in (2.5) because $\operatorname{grad} w$ is biharmonic if w is. If we do so, multiply then by r, integrate between $r=0$ and r and finally apply GREEN's theorem, we get

$$3r\operatorname{grad} w(Q)=8\boldsymbol{m}_1(w,\tfrac{1}{2}r,Q)-\boldsymbol{m}_1(w,r,Q), \tag{2.6}$$

where

$$\boldsymbol{m}_1(w,r,Q)=\frac{1}{\omega}\int\limits_{\omega} w(Q+r\boldsymbol{e})\,\boldsymbol{e}\,d\omega.$$

We have

$$|\boldsymbol{m}_1(w,r,Q)|\leqq\max_P|w(P)|,$$

where P varies over the disc of radius r around Q. Hence, by (2.6)

$$r\,|\operatorname{grad} w(Q)|\leqq 3\max_P|w(P)|.$$

Applying this inequality to circles of increasing radius with centers that tend to infinity, we obtain the assertion of the lemma.

* It is proved there in the case of three dimensions, but both the proof and the result are applicable in any dimension. The lemma stated here is also valid in higher dimensions.

3. Uniqueness of the exterior boundary value problems

Theorem 1. Let $\mathfrak{B}$ be a region in space (in the plane) exterior to an internal boundary $\mathfrak{B}$ consisting of a finite number of piecewise smooth closed surfaces (curves). Then there is at most one vector field $\boldsymbol{q}$, defined and twice continuously differentiable in $\mathfrak{B}$, with the following properties:

(a) $\boldsymbol{q}$ is a solution of the equations

$$\Delta \boldsymbol{q} = \operatorname{grad} p, \quad \operatorname{div} \boldsymbol{q} = -c p, \tag{3.1}$$

where c is a given positive constant.

(b) $\boldsymbol{q}$ is continuous up to $\mathfrak{B}$ and assumes prescribed values

$$\boldsymbol{q} = \boldsymbol{q}_0 \quad \text{at } \mathfrak{B}. \tag{3.2}$$

(c) In three dimensions $\boldsymbol{q}$ converges uniformly to zero at infinity: $\boldsymbol{q}(\infty) = 0$ (in two dimensions, $\boldsymbol{q}$ is bounded).

Proof. We observe that (3.1) implies

$$\operatorname{curl}\operatorname{curl} \boldsymbol{q} = -(1 + c) \operatorname{grad} p. \tag{3.3}$$

Operating on (3.3) with curl and div yields

$$\Delta(\operatorname{curl} \boldsymbol{q}) = 0, \quad \Delta p = 0. \tag{3.4}$$

Here we have made use of the fact that $\boldsymbol{q}$ has continuous third derivatives. It is shown in [2] that this is a consequence of (a).

Since the equations (3.1) are linear and homogeneous, it is sufficient to show that $\boldsymbol{q}_0 = 0$ implies $\boldsymbol{q} \equiv 0$. We do this in the three-dimensional case.

By Lemma 1 and (3.4) we have

$$\operatorname{curl} \boldsymbol{q} = O(r^{-2}), \quad p = -\frac{1}{c} \operatorname{div} \boldsymbol{q} = O(r^{-2}). \tag{3.5}$$

We use the GREEN's identity

$$\int_{\mathfrak{R}} (\operatorname{curl} \boldsymbol{q})^2 \, dV = \int_{\mathfrak{R}} (\boldsymbol{q} \cdot \operatorname{curl}\operatorname{curl} \boldsymbol{q}) \, dV - \int_{\mathfrak{S}} \operatorname{curl} \boldsymbol{q} \cdot (\boldsymbol{q} \times \boldsymbol{n}) \, dS, \tag{3.6}$$

valid for any smooth vector field defined in the finite region $\mathfrak{R}$ with piecewise smooth boundary $\mathfrak{S}$.

By (3.3) we have

$$\begin{aligned} \boldsymbol{q} \cdot \operatorname{curl}\operatorname{curl} \boldsymbol{q} &= -(1 + c)\, \boldsymbol{q} \cdot \operatorname{grad} p \\ &= -(1 + c) \left[\operatorname{div}(p \boldsymbol{q}) - p \operatorname{div} \boldsymbol{q}\right]. \end{aligned}$$

By use of $(3.1)_2$ and the divergence theorem, the identity (3.6), applied to the region $\mathfrak{B}_r$ between the internal boundary $\mathfrak{B}$ and a sphere $\mathfrak{S}_r$ of radius r containing $\mathfrak{B}$, gives

$$\int_{\mathfrak{B}_r} \left[(\operatorname{curl} \boldsymbol{q})^2 + \frac{1+c}{c} (\operatorname{div} \boldsymbol{q})^2\right] dV = -\int_{\mathfrak{S}_r} \left[(1 + c)\, p\, \boldsymbol{q} \cdot \boldsymbol{n} + \operatorname{curl} \boldsymbol{q} \cdot (\boldsymbol{q} \times \boldsymbol{n})\right] dS. \tag{3.7}$$

Here we have used the fact that $\boldsymbol{q} = \boldsymbol{q}_0 = 0$ on $\mathfrak{B}$. It follows from (3.5) and $\boldsymbol{q} = o(1)$ that the right-hand side of (3.3) tends to zero as $r \to \infty$.

Hence
$$\int_{\mathfrak{B}} \left[(\operatorname{curl} \boldsymbol{q})^2 + \frac{1+c}{c} (\operatorname{div} \boldsymbol{q})^2\right] dV = 0.$$

Since the integrand is non-negative and continuous, it must vanish. Hence $\operatorname{curl} \boldsymbol{q} = 0$, $\operatorname{div} \boldsymbol{q} = 0$ whence $\Delta \boldsymbol{q} = 0$. Since $\boldsymbol{q} = 0$ on $\mathfrak{B}$ and $\boldsymbol{q} = o(1)$ the maximum and minimum principle for harmonic functions implies $\boldsymbol{q} \equiv 0$, *q.e.d.*

The proof for the two-dimensional case is the same as that of Theorem II in [*1*] when modified along the lines indicated above.

Theorem 2. Let $\mathfrak{R}$ be a region in the plane exterior to an internal boundary $\mathfrak{B}$ consisting of a finite number of piecewise smooth curves. Then there is at most one scalar field w with the following properties:

(a) w is defined and biharmonic in $\mathfrak{R}$: $\Delta\Delta w = 0$.

(b) w and its first partial derivatives are continuous up to $\mathfrak{B}$ and w and $\partial w / \partial n$ assume prescribed values
$$w = w_0, \qquad \frac{\partial w}{\partial n} = w_1 \quad \text{at } \mathfrak{B}.$$

(c) $w = O(r)$ as $r \to \infty$.

Proof. We may regard w as the stream function of a plane viscous flow $\boldsymbol{q}$. Then
$$\boldsymbol{q} = (\operatorname{grad} w)^{\perp},$$
where $\perp$ indicates the operation of rotating a vector counterclockwise by a right angle. The condition (b) implies that $\operatorname{grad} w$, and hence $\boldsymbol{q}$ assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at $\mathfrak{B}$. By Lemma 3 it follows from (c) that $\boldsymbol{q}$ is bounded. The results of [*1*] imply that $\boldsymbol{q}$, and hence $\operatorname{grad} w$, is uniquely determined. This implies the uniqueness of w because the boundary values $w = w_0$ at $\mathfrak{B}$ are given.

4. A representation formula

The representation theorem (Theorem III) of [*1*] may be extended to the case of the three-dimensional equation (1.1) in the following way: Let $\boldsymbol{q}$ be a solution of (1.1) defined in a neighborhood of infinity $\mathfrak{E}$ which is star-shaped relative to a point Q. Then there is a harmonic vector-field $\boldsymbol{u}$, defined in $\mathfrak{E}$ and harmonic at infinity, such that

$$\boldsymbol{q} = \boldsymbol{u} + \operatorname{grad} \psi, \tag{4.1}$$

where

$$\psi = \psi(Q + r\,\boldsymbol{e}) = \frac{1}{4(1+c)}\, r^{\frac{1}{2}} \int_0^r s^{\frac{1}{2}} \operatorname{div} \boldsymbol{u}(Q + s\,\boldsymbol{e})\, ds. \tag{4.2}$$

The proof is identical to the one given in [*1*]*. Using the representation (4.1), we may obtain the displacement $\boldsymbol{q}$ caused by a rigid translation $\boldsymbol{q} = \boldsymbol{q}_0 = \text{const.}$ of a spherical cavity in an infinite elastic medium. We try

$$\boldsymbol{u}(Q + r\,\boldsymbol{e}) = \alpha \frac{1}{r} \boldsymbol{q}_0 + \beta \frac{1}{r^3} [\boldsymbol{q}_0 - 3(\boldsymbol{q}_0 \cdot \boldsymbol{e})\, \boldsymbol{e}], \tag{4.3}$$

* The equations (4.3), (4.4) and (6.2) in [*1*] contain a mistake in sign.

which is harmonic at infinity. Let Q be the center of the sphere and a its radius. With (4.1) and (4.2) we can determine the constants α and β so that the boundary condition $\boldsymbol{q}(Q+a\boldsymbol{e})=\boldsymbol{q}_0$ is satisfied. The result of the calculation is ($\boldsymbol{r}=r\boldsymbol{e}$)

$$\boldsymbol{q}=\frac{1}{2(2+3c)}\left\{\boldsymbol{q}_0\left[3(1+2c)\frac{a}{r}+\left(\frac{a}{r}\right)^3\right]+3\boldsymbol{r}(\boldsymbol{q}_0\cdot\boldsymbol{r})\left[\frac{a}{r^3}-\frac{a^3}{r^5}\right]\right\}. \tag{4.4}$$

For $c=0$ this reduces to the classical solution of Stokes for the flow past a sphere.

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