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Convergence of the approximate free boundary for the multidimensional one-phase Stefan problem

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Abstract. The variational inequality arising from the one-phase multidimensional Stefan problem is discretized by piecewise-linear finite elements in space and by backward-differences in time. Error estimates for the discrete free boundary at each time-step are proved.

1 Introduction

The aim of this work is to study the rate of convergence of the discrete free boundaries to the continuous one for the multidimensional one-phase Stefan problem. Some results on the convergence of the approximate free boundary are known in the literature in the one-dimensional case (Berger, Ciment and Rogers 1979; Jerome 1980; Meyer 1977; Nitsche 1978). Error estimates for the discrete free boundaries are given in (Nochetto 1985).

Our results have been suggested by a similar analysis for elliptic variational inequalities, recently given in (Caffarelli 1981; Brezzi and Caffarelli 1983). The Stefan problem is written in terms of a parabolic variational inequality which is discretized by piecewise-linear finite elements in space and backward-differences in time. We prove that the error between the free boundaries at each time-step can be estimated by the square-root of the L^∞-distance between the continuous and the discrete solution. Such a result rests on the non-negativity of the time-derivative and on non-degeneracy properties of the continuous and the discrete problems.

An outline of the paper is as follows. Section 2 deals with the formulation of the continuous problem in terms of a variational inequality and with some properties of the solution. Section 3 presents the approximate problem. Monotonicity properties of the discrete solution are proved in Section 4, while Section 5 is devoted to the study of the rate of convergence of the approximate free boundaries. In the Appendix some examples of L^∞-error estimates (though non-optimal) for the solution are given.

Let us introduce some notations. μ will denote the Lebesgue measure in R^N, dist the euclidean distance in R^N, U_j the j-th component of the vector $U \in R^n$. Moreover, for any set $E \subset R^N$, we set $\text{int}(E) = E - \partial E$ and

$$S_\varepsilon(E) = \{x \in R^N: \text{dist}(x, E) < \varepsilon\}, \qquad \forall \varepsilon > 0.$$

2 The continuous problem

In this section we shall briefly describe the one-phase Stefan problem and its formulation in terms of variational inequality. We shall then recall some properties of the solution which will be useful in the following.

Let G_0 and G_1 be open bounded domains in R^N $(N \geqq 1)$, with $\bar{G}_0 \subset G_1$. Let their boundaries, Γ_0 and Γ_1 respectively, be connected C^∞ hypersurfaces. Let $B_R = \{|x| < R\}$ be a large ball containing G_1 and set $G = G_1 - \bar{G}_0$, $\Omega = B_R - \bar{G}_0$, $\Omega_e = \Omega - \bar{G}$, $Q = \Omega \times]0, T[$ $(T < +\infty)$, and $\partial_p Q = (\Gamma_0 \cup \partial B_R) \times [0, T[\cup \Omega \times \{0\}$. Suppose that initially G is filled with water and $\Omega - G$ is filled with ice at 0 °C. We denote by $\vartheta = \vartheta(x, t)$ the temperature of the water. We are given the initial temperature

$$\vartheta(x, 0) = \vartheta_0(x) \quad \text{if} \quad x \in G \tag{1.i}$$

as well as the temperature along the fixed boundary $\Gamma_0 \times]0, T[$

$$\vartheta(x, t) = \psi(x, t) \quad \text{if} \quad (x, t) \in \Gamma_0 \times]0, T[, \tag{1.ii}$$

where $\vartheta_0(x)$ and $\psi(x, t)$ are given positive functions. We define the water region

$$P(t) = \{x \in \Omega : \vartheta(x, t) > 0\}, \qquad t > 0, \qquad P_\tau = \bigcup_{0 < t < \tau} P(t) \times \{t\}, \qquad P = P_T$$

and the free boundary

$$\Gamma(0) = \Gamma_1, \qquad \Gamma(t) = \partial P(t) - \Gamma_0, \qquad t > 0, \qquad \Gamma_\tau = \bigcup_{0 \leqq t \leqq \tau} \Gamma(t) \times \{t\}, \qquad \Gamma = \Gamma_T.$$

Let us suppose that the water-ice interface is given by the equation $t = s(x)$. If ϑ, s are regular enough, then

$$\vartheta_t - \Delta\vartheta = 0 \quad \text{in } P; \tag{1.iii}$$

$$\vartheta(x, t) = 0, \qquad \nabla\vartheta(x, t) \times \nabla s(x) = -1 \quad \text{if} \quad t = s(x), \qquad x \in \Omega_e; \tag{1.iv}$$

$$s(x) > t, \qquad \vartheta(x, t) = 0 \quad \text{if} \quad (x, t) \in Q - \bar{P}; \tag{1.v}$$

$$s(x) = 0 \quad \text{if} \quad x \in \bar{G}. \tag{1.vi}$$

The problem of finding, under suitable assumptions on ϑ_0 and ψ, a regular solution ϑ, s of (1) is called the classical one-phase Stefan problem. The existence of a classical solution for a sufficiently small time has been proved by Meirmanov (1981). In order to focus on the difficulties of our approximation problem only, we shall assume from now on that

$$\begin{aligned}
&\vartheta_0 \in C^\infty(\bar{G}), \quad \vartheta_0(x) > 0 && \text{if} \quad x \in G, \quad \vartheta_0 = 0 \text{ on } \Gamma_1, \\
&\psi \in C^\infty(\Gamma_0 \times [0, T]), \quad \psi(x, t) > 0 && \text{if} \quad (x, t) \in \Gamma_0 \times [0, T], \\
&\vartheta_0(x) = \psi(x, 0) && \text{if} \quad x \in \Gamma_0
\end{aligned} \tag{2}$$

and

$$\exists\, m_0,\ M_0 > 0 : m_0 \leqq |\nabla\vartheta_0(x)| \leqq M_0 \quad \text{if} \quad x \in \Gamma_1. \tag{3}$$

Under these assumptions, in (Meirmanov 1981) it is shown that there exists t^*, with $0 < t^* \leqq T$, depending only on m_0 and M_0, such that the problem (1) has a unique solution $\vartheta \in C^{2,1}(\bar{P}_{t^*})$, $s \in C^1(\bar{\Omega})$, $\forall t \in [0, t^*]$.

If we choose the freezing index function

$$u(x, t) = \int_0^t \vartheta(x, \tau)\, d\tau \quad \text{if} \quad (x, t) \in Q \tag{4}$$

as unknown function, we can consider a formulation of the problem in terms of a variational inequality. Setting

$$g(x, t) = \begin{cases} \int_0^t \psi(x, \tau)\, d\tau & \text{if} \quad x \in \Gamma_0, \quad t > 0 \\ 0 & \text{if} \quad t = 0 \quad \text{or} \quad |x| = R, \end{cases} \tag{5}$$

$$f(x)=\begin{cases}\vartheta_0(x) & \text{if} \quad x\in G \\ -1 & \text{if} \quad x\in \Omega-G,\end{cases} \tag{6}$$

if ϑ, s is the classical solution, then (see Duvaut 1973; Friedman and Kinderlehrer 1975) u fulfils the following inequality

$$\begin{gathered}\int_\Omega u_t(v-u)\mathrm{d}x+\int_\Omega \nabla u\times\nabla(v-u)\mathrm{d}x\geqq\int_\Omega f(v-u)\mathrm{d}x \quad \text{a.e.} \quad t\in]0,T[\\ \forall v\in W=\{v\in H^1(Q), \quad v\geqq 0 \text{ in } Q, \quad v=g \text{ on } \partial_p Q\}.\end{gathered} \tag{7}$$

Moreover $\{x\in\Omega : u(x,t)>0\}\equiv P(t)$ for every $t>0$. This suggests the following definition of weak solution:

(P) Find $u\in W$ such that (7) holds.

It is well known (see Friedman and Kinderlehrer 1975) that, say, under the assumptions (2), problem (P) has a unique solution u, such that

$$u_t\geqq 0 \tag{8}$$

and with the further regularity $u\in L^\infty(0,T;W^{2,p}(\Omega))$, $\forall p<+\infty$; $u_t\in L^\infty(Q)\cap L^\infty(0,T;H^1(\Omega))$; moreover u_t is a continuous function (Caffarelli and Friedman 1979). A consequence of the non-negativity of u_t is the embedding

$$P(t)\subset P(t') \qquad \forall t'>t>0. \tag{9}$$

Furthermore (see Friedman and Kinderlehrer 1975) we have that

$$\bar{G}-\Gamma_0\subset P(t) \qquad \forall t>0, \tag{10}$$

which easily implies the following property

$$\operatorname{dist}(\Gamma(t),\Gamma_1)=c(t)>0 \qquad \forall t>0. \tag{11}$$

We can also prove the following result.

Proposition 1. The set $P(t)$ is connected for every $t>0$.

Proof. Let us suppose that there exists a connected component D of the set $P(t)$ not intersecting $\bar{G}$. Since $u(x,t)=0$ on the boundary of D and the equality $u_t-\Delta u=-1$ is satisfied in D, hence $D=\phi$. □

Finally, Caffarelli (1977) proves that the free boundary $\Gamma(t)$ has zero Lebesgue measure, for any $t>0$. Next, we will state some non-degeneracy properties near the free boundary. They follow from similar results for elliptic variational inequalities (Caffarelli 1981), so that they will be proven only in the set where $\Delta u=1+u_t\geqq 1$. However, this will not be restrictive because of the embedding (10). By setting

$$E(t)=P(t)-\bar{G}, \qquad t>0, \qquad E=\bigcup_{0<t<T} E(t)\times\{t\}, \tag{12}$$

$$A_\varepsilon(t)=\{x\in\Omega_e : 0<u(x,t)\leqq\varepsilon^2\}, \qquad t>0, \tag{13}$$

we recall the following properties (Friedman 1982):

P1. For any $(x,t)\in E\cup\Gamma$ and for any $r>0$, if $B_r(x)\cap\Gamma_1=\phi$, then

$$\sup_{x'\in E(t)\cap B_r(x)} u(x',t)\geqq\frac{r^2}{2N}. \tag{14}$$

P2. For any compact domain $K\subset\Omega_e$ with regular boundary ∂K, there exist two positive constants $\varepsilon(K)$ and $C(K)$ such that

$$\mu(A_\varepsilon(t)\cap K)\leqq C(K)\varepsilon \qquad \forall\varepsilon\leqq\varepsilon(K), \qquad \forall t>0. \tag{15}$$

Remark 1. For any compact domain $K \subset \Omega_e$, there exists a time $t(K) > 0$, such that $A_\varepsilon(t) \cap K \subseteqq A_\varepsilon(t)$, $\forall 0 < t \leqq t(K)$. So (15) does not give any useful information for small time. Luckily, due to the existence of the classical solution over a small time interval, we can prove a result similar to the estimate (15), but without dependence on the compact set K. We shall do that in the following lemma.

Lemma 1. There exist two positive constants ε_0 and C such that, for any $\varepsilon \leqq \varepsilon_0$ and for any $t > 0$, we have

$$\mu(A_\varepsilon(t)) \leqq C\varepsilon. \tag{16}$$

Proof. First we shall prove that assertion (16) holds for any $t \leqq t^*$. In fact, since $u, u_t \in C^{2,1}(\bar{P}_{t^*})$ and Γ_{t^*} is regular, we have that

$$\exists \varepsilon_1 > 0, \quad \exists C_1 > 0 \colon \mu(S_\varepsilon(\Gamma(t))) \leqq C_1 \varepsilon \quad \forall \varepsilon \leqq \varepsilon_1, \quad \forall t \leqq t^* \tag{17}$$

and, by the non-degeneracy property P1, we can prove that

$$\exists \varepsilon_2 > 0, \quad \exists C_2 > 0 \colon A_\varepsilon(t) \subset S_{C_2\varepsilon}(\Gamma(t)) \quad \forall \varepsilon \leqq \varepsilon_2, \quad \forall t \leqq t^*. \tag{18}$$

Properties (17) and (18) imply the existence of two positive constants C' and ε' such that

$$\mu(A_\varepsilon(t)) \leqq C'\varepsilon \quad \forall \varepsilon \leqq \varepsilon', \quad \forall t \leqq t^*. \tag{19}$$

Now we show that (16) holds for any $t > t^*$. Setting $\varepsilon_3 = c(t^*)/2$ [where $c(t^*)$ is the constant in (11)], we have $u(x, t^*) > 0$ in $\overline{S_{\varepsilon_3}(G)}$. Let $u_{\min}$ be the minimum of $u(x, t^*)$ on $\overline{S_{\varepsilon_3}(G)}$. By setting $K = \bar{\Omega} - S_{\varepsilon_3}(G)$ and $\varepsilon_4 = \sqrt{u_{\min}/2}$, since $u_t \geqq 0$, we obtain

$$A_\varepsilon(t) \subset K \quad \forall \varepsilon \leqq \varepsilon_4, \quad \forall t > t^*. \tag{20}$$

Then (20) and P2 imply that there exist two positive constants C'' and ε'' [with $\varepsilon'' = \min(\varepsilon_3, \varepsilon_4)$] such that

$$\mu(A_\varepsilon(t)) \leqq C''\varepsilon \quad \forall \varepsilon \leqq \varepsilon'', \quad \forall t > t^*. \tag{21}$$

Hence (16) follows from (19) and (21) with $\varepsilon_0 = \min(\varepsilon', \varepsilon'')$ and $C = \max(C', C'')$.

□

In (Friedman 1982), it is shown that for any compact domain $K \subset \Omega_e$, the $(N-1)$-dimensional Hausdorff measure of $\Gamma(t) \cap K$ is bounded by a constant, for each $t > 0$. Hence using again (17) and a suitable choice of the compact set K in the quoted result, we can prove the following lemma.

Lemma 2. There exist two positive constants ε_0 and C such that, for any $\varepsilon \leqq \varepsilon_0$ and for any $t > 0$, we have

$$\mu(S_\varepsilon(\Gamma(t))) \leqq C\varepsilon. \tag{22}$$

□

Moreover if $\Gamma(t)$ is locally Lipschitz continuous for any $t > 0$, then there exist two positive constants ε_0 and C such that

$$A_\varepsilon(t) \subset S_{C\varepsilon}(\Gamma(t)) \quad \forall \varepsilon \leqq \varepsilon_0, \quad \forall t > 0. \tag{23}$$

For instance, such a result holds if G_1 is star-shaped (Friedman and Kinderlehrer 1975).

3 The discrete problem

We discretize problem (P) by backward-differences in time and by piecewise-linear finite elements in space.

Fix an integer $m \geqq 1$; $k = T/m$ will be the time-step; $t^i = ik$, $i = 1, \ldots, m$.

Let τ_h be a collection of simplices τ, whose diameters are bounded by h and let $\{\tau_h\}_h$ be a family of decompositions of Ω by simplices τ. We suppose that $\{\tau_h\}_h$ is regular and quasi-uniform in the sense of Ciarlet (1978). The discrete problem is solved in $\Omega_h = \bigcup_{\tau \in \tau_h} \tau$. For convenience, we suppose that

$\Omega_h = \bar{\Omega}$ (namely, we are considering curvilinear elements at the boundary). Such assumption is not restrictive, since we are interested in the behaviour of the free boundaries. Let now the nodes of τ_h, $\{x_j\}_{j=1}^p$, be numbered as follows:

$\{x_j\}_{j=1}^{n_G}$ the nodes lying in G,

$\{x_j\}_{j=n_G+1}^{n_1}$ the nodes lying on Γ_1,

$\{x_j\}_{j=n_1+1}^{n}$ the nodes lying in Ω_e,

$\{x_j\}_{j=n+1}^{p}$ the nodes lying on $\Gamma_0 \cup \partial B_R$.

Moreover let us suppose that if a simplex τ has a node in G, then all of its nodes belong to $\bar{G}$. Let us set

$$V_h = \{\chi \in C^0(\bar{\Omega}),\ \chi_{|\tau} \text{ is a polynomial of degree} \leqq 1\ \forall \tau \in \tau_h\}$$

and denote by $\{\varphi_j\}_{j=1}^p$ the usual Lagrange basis of V_h. In order to get a diagonal mass-matrix, we use a suitable quadrature formula, such as, for instance

$$I_\tau(f) = \frac{\mu(\tau)}{N+1} \sum_{x_j \in \tau} f(x_j), \qquad \tau \in \tau_h, \qquad x_j \text{ nodes of } \tau. \tag{24}$$

Setting $(u, v)_h = \sum_{\tau \in \tau_h} I_\tau(uv)$, we introduce the matrices

$$\begin{aligned} &\boldsymbol{M} = \{(\varphi_j, \varphi_l)_h\}_{j,l=1}^n = \{m_{jl}\}, \\ &\hat{\boldsymbol{A}} = \{\int_\Omega \nabla\varphi_j(x) \cdot \nabla\varphi_l(x)\, dx\}_{j=1,l=1}^{n,\ p} = \{a_{jl}\}, \qquad \boldsymbol{A} = \{a_{jl}\}_{j,l=1}^n, \\ &\boldsymbol{B} = \boldsymbol{M} + k\boldsymbol{A} = \{b_{jl}\}. \end{aligned} \tag{25}$$

For every $i \geqq 1$, set

$$\boldsymbol{F}^i = \{\int_\Omega f(x)\varphi_j(x)\, dx - G_j^i\}_{j=1}^n, \tag{26}$$

where

$$G_j^i = \sum_{l=n+1}^{p} a_{jl}\, g(x_l, t^i), \qquad \text{for} \qquad j = 1, \ldots, n. \tag{27}$$

The discrete problem we consider reads as follows

$(\mathrm{P}_{h,k})$ $\quad \boldsymbol{U}^0 = \boldsymbol{0}$

for $i = 1, \ldots, m$, find $\boldsymbol{U}^i \in \mathrm{R}^n$ such that

$$\boldsymbol{U}^i \geqq \boldsymbol{0} \tag{28i}$$

$$\boldsymbol{B}\boldsymbol{U}^i \geqq k\boldsymbol{F}^i + \boldsymbol{M}\boldsymbol{U}^{i-1} \tag{28ii}$$

$$(\boldsymbol{B}\boldsymbol{U}^i - k\boldsymbol{F}^i - \boldsymbol{M}\boldsymbol{U}^{i-1})\boldsymbol{U}^i = \boldsymbol{0} \tag{28iii}$$

It is well known that this problem has a unique solution (see Glowinski, Lions and Tremolières 1981). The piecewise-linear function

$$u_{h,k}^i(x) = \sum_{j=1}^{n} U_j^i \varphi_j(x) + \sum_{j=n+1}^{p} g(x_j, t^i)\, \varphi_j(x) \tag{29}$$

is the approximation of the exact solution $u(\cdot, t^i)$.

The assumptions on the finite element space and on the quadrature formula allow one to obtain these formulas

$$\sum_{l=1}^{p} a_{jl} = 0 \qquad \text{for} \qquad j = 1, \ldots, n, \tag{30}$$

$$m_{jj}=\frac{\mu(\text{supp}\,\varphi_j)}{N+1} \quad \text{for} \quad j=1,\ldots,n, \quad m_{jl}=0 \quad \forall j\neq l. \tag{31}$$

From now on, we shall assume that for every h the decomposition τ_h is of *acute-type*, that is

for each $\tau\in\tau_h$ and for each vertex $x_j\in\tau$ the projection of x_j on the opposite hyperplane falls in the closure of the opposite face (32)

It is well known that assumption (32) yields

$$a_{jj}>0 \quad \text{for} \quad j=1,\ldots,n, \quad a_{jl}\leqq 0 \quad \forall j\neq l \tag{33}$$

so that the following discrete maximum principle holds (see Ciarlet and Raviart 1973).

D.M.P. Let τ_h be a decomposition of *acute-type* and D be a connected union of simplices of τ_h. Let $W\in\mathrm{R}^p$ be such that

$$(\hat{A}W)_j<0 \quad \text{for} \quad x_j\in\text{int}(D); \tag{34}$$

then

$$\max_{x_j\in\partial D} W_j>\max_{x_j\in\text{int}(D)} W_j. \tag{35}$$

□

Remark 2. Since $g(x,t)$ is a positive non-decreasing function in t, by (33) we have $G_j^i\leqq 0$ for $j=1,\ldots,n$, and

$$F_j^{i+1}\geqq F_j^i \quad \text{for} \quad j=1,\ldots,n. \tag{36}$$

Moreover, for h small enough we have $G_j^i=0$ for $j=n_G+1,\ldots,n$, and

$$F_j^i>0 \quad \text{for} \quad j=1,\ldots,n_G, \quad F_j^i<0 \quad \text{for} \quad j=n_G+1,\ldots,n. \tag{37}$$

□

Let us define the discrete positivity set

$$P^0=\text{int}\Bigg(\bigcup_{\tau:x_j(\tau)\in\bar{G}}\tau\Bigg), \quad P^i=\{x\in\Omega: u_{h,k}^i(x)>0\} \quad \text{for} \quad i=1,\ldots,m \tag{38}$$

and the discrete free boundary

$$\Gamma^i=\partial P^i-\Gamma_0 \quad \text{for} \quad i=0,1,\ldots,m. \tag{39}$$

Also set

$$E^i=P^i-\bar{P}^0 \quad \text{for} \quad i=1,\ldots,m. \tag{40}$$

Remark 3. There exist two positive constants h_0 and C such that, for any $h\leqq h_0$, one has

$$\mu(G\div P^0)\leqq Ch, \tag{41}$$

$$\text{dist}(\Gamma_1,\Gamma^0)\leqq h. \tag{42}$$

□

In Section 5, we shall bound the distance between the free boundaries by means of the L^∞-error between u and $u_{h,k}$. In order to do that, we will assume that the time-step k is chosen as a prescribed function of h, $k=k(h)$; moreover we assume that there exists a function $\eta(h)$ such that

$$\sup_{1\leqq i\leqq m}\|u(t^i)-u_{h,k}^i\|_{L^\infty(\Omega)}\leqq\eta^2(h)$$
$$\text{with} \quad \lim_{h\to 0}\eta(h)=0, \quad \frac{\eta(h)}{h}\geqq 2\sqrt{\gamma_0} \quad \text{for} \quad h \text{ small enough} \tag{43}$$

[where, here and in the following, γ_0 will be given by (59)].

The error in the free boundary will be estimated precisely in terms of $\eta(h)$. Some remarks on the behaviour of $\eta(h)$ itself will be given in the Appendix.

4 Monotonicity properties of the discrete solution

For the discrete problem, as well as for the continuous problem, the positivity set has monotonicity properties. In fact we shall prove the following theorem.

Theorem 1. For $i=1,\ldots,m$, we have

$$U_j^i \geqq U_j^{i-1} \quad \text{for} \quad j=1,\ldots,n \tag{44}$$

and

$$U_j^1 > 0 \quad \text{for} \quad j=1,\ldots,n_G. \tag{45}$$

Proof. The proof is carried out by induction on the time levels.

Step 1. $i=1$. (28.ii) becomes $(\boldsymbol{B}\boldsymbol{U}^1)_j \geqq kF_j^1$ for $j=1,\ldots,n$; then the inequality

$$b_{jj}U_j^1 > -k\sum_{l\neq j} a_{jl}U_l^1 \geqq 0 \quad \text{for} \quad j=1,\ldots,n_G \tag{46}$$

follows from (37) and (33); hence (45) holds.

Step 2. Suppose the property is satisfied for i. Let $\{x_j\}_{j\in N^i}$ be the nodes internal to P^i. By setting $\boldsymbol{D}^{i+1}=\boldsymbol{U}^{i+1}-\boldsymbol{U}^i$ and by subtracting $\boldsymbol{B}\boldsymbol{U}^i$, (28.ii) yields

$$(\boldsymbol{B}\boldsymbol{D}^{i+1})_j \geqq kF_j^{i+1} - k(\boldsymbol{A}\boldsymbol{U}^i)_j \quad \text{for} \quad j=1,\ldots,n. \tag{47}$$

On the other hand one has by (28.iii)

$$(\boldsymbol{M}\boldsymbol{U}^i + k\boldsymbol{A}\boldsymbol{U}^i)_j = kF_j^i + (\boldsymbol{M}\boldsymbol{U}^{i-1})_j \quad \text{for} \quad j\in N^i. \tag{48}$$

Then, (36) and the induction assumption yield

$$(\boldsymbol{B}\boldsymbol{D}^{i+1})_j \geqq k(F_j^{i+1}-F_j^i) + (\boldsymbol{M}\boldsymbol{D}^i)_j \geqq 0 \quad \text{for} \quad j\in N^i, \tag{49}$$

hence the assertion. □

Remark 4. It is easy to show that $U_j^i > U_j^{i-1}$ for $j\in N^i$, for $i=1,\ldots,m$.

So, by Theorem 1, $P^{i-1}\subset P^i$ and, in particular, $P^0\subset P^i$, for $i=1,\ldots,m$. This result guarantees the monotonicity of the discrete positivity sets, but does not ensure that, at the first time-step t^1, the discrete free boundary leaves ∂P^0. Such a property is proved in the following proposition.

Proposition 2. For every k, there exists a positive constant h_0 such that, for any $h\leqq h_0$, we have $U_j^1>0$ for $j=n_G+1,\ldots,n_1$.

Proof. Setting $c=c(k)/2$ [where $c(k)$ is the constant in (11) and $t=k$ is the first time level], we have $u(x,k)\geqq u_{\min}>0$ $\forall x\in\overline{S_c(G)}$. Thus, there exists $h_0>0$ such that, for every $h\leqq h_0$, the error estimates (43) yield $\|u(k)-u_{h,k}^1\|_{L^\infty(\Omega)}<u_{\min}$, which implies $u_{h,k}^1(x_j)>0$ $\forall x_j\in\overline{S_c(G)}$. □

So, by Theorem 1 and Proposition 2, $\bar{G}-\Gamma_0\subset P^i$ for $i=1,\ldots,m$.
Finally, we also prove the following property.

Proposition 3. For any $i\geqq 1$, the set P^i is connected.

Proof. Let D be a connected component of P^i which does not intersect P^0. We note that $\bar{D}$ is a connected union of simplices of τ_h which satisfies the property $U_j^i=0$ $\forall x_j\in\partial D$. In each internal node x_j to $\bar{D}$, the inequality

$$(\boldsymbol{A}\boldsymbol{U}^i)_j<0 \tag{50}$$

holds [in fact, if $U_j^i>0$, (50) follows from (28.iii), (37) and (44); if $U_j^i=0$, i.e. $x_j\in\partial D-\partial\bar{D}$ is an isolated point of the discrete free boundary Γ^i, (50) follows from (33)]. Then, by the D.M.P., $U_j^i=0$ $\forall x_j\in D$, i.e. $D=\phi$. □

5 Approximation of the free boundary

Now we shall prove a non-degeneracy property for the discrete solution which is similar to P1. Due to Theorem 1 it will be enough to prove the property outside P^0 (as in the continuous case).

Theorem 2. There exist two positive constants γ_0 and h_0 such that, for any $i \geqq 1$, for any $h \leqq h_0$, for any $r \geqq 2h$ and for any node $x_* \in E^i$ with $B_r(x_*) \cap \Gamma_1 = \phi$, we have

$$\sup_{x_j \in \overline{B_r(x_*) \cap E^i}} U_j^i > \gamma_0 r^2. \tag{51}$$

Proof. Introduce the function $\sigma^*(x) = |x - x_*|^2$ and the vectors of its nodal values $\boldsymbol{\Theta}^* = \{\sigma^*(x_j)\}_{j=1}^n$, $\hat{\boldsymbol{\Theta}}^* = \{\sigma^*(x_j)\}_{j=1}^p$. Note that (see Brezzi and Caffarelli 1983) there exist two positive constants δ_0, δ_1 such that

$$-\delta_0 \int_\Omega \varphi_j(x)\,dx \leqq (\hat{\boldsymbol{A}}\hat{\boldsymbol{\Theta}}^*)_j \leqq -\delta_1 \int_\Omega \varphi_j(x)\,dx \quad \text{for} \quad j = 1, \ldots, n. \tag{52}$$

Let $i \geqq 1$ and consider the vector

$$\boldsymbol{W}^i = \boldsymbol{U}^i - \frac{1}{2\delta_0} \boldsymbol{\Theta}^*. \tag{53}$$

Let D be the largest connected union of N-simplices such that $x_* \in \text{int}(D)$ and $\text{int}(D) \subset B_r(x_*) \cap E^i$. In each internal node x_j, the equality

$$(\boldsymbol{B}\boldsymbol{U}^i)_j = kF_j^i + (\boldsymbol{M}\boldsymbol{U}^{i-1})_j \tag{54}$$

holds, hence

$$(\boldsymbol{A}\boldsymbol{W}^i)_j = F_j^i + \left(\boldsymbol{M}\,\frac{\boldsymbol{U}^{i-1} - \boldsymbol{U}^i}{k}\right)_j - \frac{1}{2\delta_0} (\boldsymbol{A}\boldsymbol{\Theta}^*)_j. \tag{55}$$

Then, from (44) and (52), for h sufficiently small we obtain

$$(\boldsymbol{A}\boldsymbol{W}^i)_j \leqq -\frac{3}{4} \int_\Omega \varphi_j(x)\,dx + \frac{1}{2} \int_\Omega \varphi_j(x)\,dx < 0. \tag{56}$$

So, by the D.M.P., $\boldsymbol{W}^i$ takes its maximum in D on a node x_l of the boundary of D. Since $W_*^i = U_*^i > 0$, then $W_l^i > 0$, so that $U_l^i > 0$ and hence $x_l \notin \Gamma^i$. Since (56) holds in each internal node to E^i, hence there is at least one neighbouring node x_j to x_l where $W_j^i > W_l^i$. Therefore it follows that

$$\text{dist}(x_l, \partial B_r(x_*)) < h. \tag{57}$$

On the other hand the positivity of W_l^i also implies

$$U_l^i > \frac{1}{2\delta_0} \Theta_l^*. \tag{58}$$

By means of (57) we have $\Theta_l^* = |x_l - x_*|^2 > (r-h)^2$, so that (58) gives $U_l^i > \frac{1}{2\delta_0} \frac{r^2}{4}$ and (51) follows with

$$\gamma_0 = \frac{1}{8\delta_0}. \tag{59}$$

□

We will now bound the distance between the free boundaries by means of the L^∞-error estimates between u and $u_{h,k}$. We shall first prove the following lemma.

Lemma 3. Set $\varepsilon_1(h) = 3\eta(h)/2\sqrt{\gamma_0}$. There exists a positive constant h_0 such that, for any $i \geqq 1$ and for any $h \leqq h_0$, we have

$$\Omega - P^i \supset (\Omega - P(t^i)) - S_{\varepsilon_1(h)}(\Gamma(t^i)). \tag{60}$$

Proof. Set $\varepsilon_2(h) = \eta(h)/\sqrt{\gamma_0}$ and $S^i = (\Omega - P(t^i)) - S_{\varepsilon_2(h)}(\Gamma(t^i))$. We shall prove now that $U_*^i = 0$ for any node $x_* \in S^i$. Obviously we have $u(x, t^i) = 0$ if $x \in B_{\varepsilon_2(h)}(x_*) \cap \Omega$, hence from (10) it follows that

$B_{\varepsilon_2(h)}(x_*) \cap \Gamma_1 = \phi$; finally (43) implies that $\varepsilon_2(h) \geqq 2h$ for h small enough. If we suppose that $U_*^i > 0$, then Theorem 2 yields

$$\sup_{x_j \in B_{\varepsilon_2(h)}(x_*) \cap \Omega} U_j^i > \gamma_0 \varepsilon_2^2(h) = \eta^2(h) \tag{61}$$

which contradicts the error estimates (43).

We have proved that each node of S^i belongs to $\Omega - P^i$. But a simplex τ could exist such that $\operatorname{int}(\tau) \subset P^i$ with a vertex in S^i. Hence, it is sufficient to remove a further strip of width h from S^i. Globally we remove a strip of width $\varepsilon_1(h)$ from $\Omega - P(t^i)$, so that (60) holds.

□

Under the regularity assumptions (2), (3) on the data, we can estimate the measure of the symmetric difference $P(t^i) \div P^i$ between the continuous and the discrete positivity set, at each time level.

Theorem 3. There exist two positive constants h_0 and C_1 such that, for any $i \geqq 1$ and for any $h \leqq h_0$, we have

$$\mu(P(t^i) \div P^i) \leqq C_1 \eta(h). \tag{62}$$

Proof. By (60) we have

$$P^i - P(t^i) = (\Omega - P(t^i)) - (\Omega - P^i) \subset S_{\varepsilon_1(h)}(\Gamma(t^i)). \tag{63}$$

On the other hand, if $x \in P(t^i) - P^i$ then $u_{h,k}^i(x) = 0$, thus the inequality $0 < u(x, t^i) \leqq \eta^2(h)$ follows from the error estimates (43). Therefore

$$P(t^i) - P^i \subset \{x \in \Omega - \bar{P}^0 : 0 < u(x, t^i) \leqq \eta^2(h)\}. \tag{64}$$

Since (22) implies

$$\mu(S_{\varepsilon_1(h)}(\Gamma(t^i))) \leqq C\varepsilon_1(h), \tag{65}$$

then (16) and (41) imply

$$\mu\{x \in \Omega - \bar{P}^0 : 0 < u(x, t^i) \leqq \eta^2(h)\} \leqq C\eta(h), \tag{66}$$

the thesis follows due to (63–66). □

Under the stronger condition that $\Gamma(t)$ is locally Lipschitz continuous for any $t > 0$, we shall prove that, at each time-step the discrete free boundary lies in a $\eta(h)$-neighborhood of the continuous free boundary.

Theorem 4. There exist two positive constants h_0 and C_2 such that, for any $i \geqq 1$ and for any $h \leqq h_0$, we have

$$\Gamma^i \subset S_{C_2\eta(h)}(\Gamma(t^i)). \tag{67}$$

Proof. By (64), due to (23) and (42) we have

$$P(t^i) - P^i \subset \{x \in \Omega - \bar{P}^0 : 0 < u(x, t^i) \leqq \eta^2(h)\} \subset S_{\varepsilon_3(h)}(\Gamma(t^i)) \tag{68}$$

with $\varepsilon_3(h) = (1/(2\sqrt{\gamma_0}) + C)\eta(h)$. The thesis follows due to (63) and (68) with $C_2\eta(h) = \varepsilon_1(h) + \varepsilon_3(h)$.

□

Remark 5. Estimates (62) and (67) could be derived by using Proposition 2 instead of properties (41), (42). In this case (64) must be replaced by

$$P(t^i) - P^i \subset A_{\eta(h)}(t^i)$$

Setting, for $i=1,\ldots,m$, $u_h^i - k\Delta_h u_h^i \geqq kf - u_h^{i-1}$ and $\bar{u}_h^i - k\Delta_h \bar{u}_h^i \geqq kf - u^{i-1}$, it can be proved (Nitsche 1977) that for any $i \geqq 1$

$$\|u^i - \bar{u}_h^i\|_{L^\infty(\Omega)} \leqq Ck^{-1}h^2|\lg h|^\mu, \tag{75}$$

with C independent of i, h, k; hence

$$\|u^i - u_h^i\|_{L^\infty(\Omega)} \leqq Ck^{-1}h^2|\lg h|^\mu + \|u^{i-1} - u_h^{i-1}\|_{L^\infty(\Omega)} \leqq Ck^{-1}h^2 i|\lg h|^\mu \leqq Ck^{-2}h^2|\lg h|^\mu. \tag{76}$$

Concluding, we obtain the estimate

$$\sup_{1\leqq i \leqq m} \|u(t^i) - u_h^i\|_{L^\infty(\Omega)} \leqq C(k^{1/2} + h^2k^{-2}|\lg h|^\mu) \tag{77}$$

which gives $\eta^2(h) \approx h^{2/5}|\lg h|^\mu$, if $k \approx h^{4/5}$.

A similar error estimate can be obtained when the boundary data is positive [and constant in x].

Clearly both (71) and (77) are far from being optimal. Their improvement would immediately produce a better estimate for the free boundary error [in terms of powers of h and k] through (43) and (62), (67).

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