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Periodic Solutions of an Epidemic Model

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Summary. The existence of periodic solutions of the equation

$$x(t) = k\left(P - \int_{-\infty}^{t} A(t-s)x(s)\,ds\right)\int_{-\infty}^{t} a(t-s)x(s)\,ds$$

is established. This equation arises in the study of the spread of a disease which does not induce permanent immunity.

Key words: Epidemic – Periodic solution.

1. Introduction

The purpose of this paper is to study the existence of periodic solutions of the equation

$$x(t) = k\left(P - \int_{-\infty}^{t} A(t-s)x(s)\,ds\right)\int_{-\infty}^{t} a(t-s)x(s)\,ds, \qquad t \in R = (-\infty, \infty). \tag{1.1}$$

This equation is related to models of the spread of a disease, that does not induce permanent immunity, in the following way: The function x is the infection rate, i.e. the rate at which individuals susceptible to the disease become infected. Then $\int_{-\infty}^{t} a(t-s)x(s)\,ds$ is approximately proportional to the "total infectivity" if the average infectivity of an individual infected at time s is proportional to $a(t-s)$ at time t, $t \geqslant s$. The constant P is the size of the population in question and then $P - \int_{-\infty}^{t} A(t-s)x(s)\,ds$ is approximately the number of susceptibles provided that the cumulative probability for the loss of immunity of an individual infected at time s is $1 - A(t-s)$. To obtain equation (1.1) one must make the assumption that the infection rate is proportional to the number of susceptibles and the "total infectivity".

For earlier results on the same kind of problems, see e.g. [1]–[4], [8]–[15] and the references mentioned there. In [11] the equation (1.1) is considered and it is shown that if the infection rate is given up to time 0, then there exists a unique, bounded, continuous and nonnegative solution of (1.1) for $t \geqslant 0$ (under appropriate assumptions on A and a). Moreover, sufficient conditions for the convergence of this solution to a limit when $t \to \infty$ are given.

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When the model considered in this paper is compared with other models that have been studied earlier, one can make the following observations: No spatial or stochastic phenomena are studied, the emphasis is completely on the effects of the delays and the assumptions concerning these delays are quite general. Thus it is for example not assumed that infected individuals have a constant recovery rate (in fact this assumption implies that $A(t) = e^{-ct}$, for some constant c, and it turns out that the assumption (2.3) in Theorem 1 below cannot be satisfied in this case), nor that the immunity is completely lost after some fixed time interval. Note also that the model we are studying is time invariant, that is we do not assume that e.g. the proportionality constant k is a periodic function of t. The point is not that such an assumption is unrealistic but that it is not needed in order to prove the existence of periodic solutions.

The proof below of the bifurcation of periodic solutions is closely related to the proof of the Hopf bifurcation theorem as it is presented in e.g. [5]. For other results concerning the bifurcation of periodic solutions of integral equations, see e.g. [6] – [8].

2. Statement of Results

We let "$\hat{\ }$" denote the Fourier transform and we extend functions defined on $R^+ = [0, \infty)$ as zero on $(-\infty, 0)$. The space of all continuous T-periodic functions on R is denoted by $C(T)$.

Theorem 1. *Assume that* $k > 0$ *and that*

A is a nonnegative, nonincreasing and absolutely continuous function on R^+, $A(0) = 1$ and $A \in L^1(R^+)$, (2.1)

a is a nonnegative, absolutely continuous function on R^+, $\int_0^\infty a(s)\,ds = 1$ and $t\dot{a}(t) \in L^1(R^+)$, ("$\dot{\ }$" $= d/dt$), (2.2)

there exist positive numbers b_0 and ω such that

$$1 - \hat{a}(\omega) + b_0 \hat{A}(\omega) = 0, \tag{2.3}$$

$$1 - \hat{a}(n\omega) + b_0 \hat{A}(n\omega) \neq 0, \qquad n = 2, 3, \ldots, \tag{2.4}$$

$$\operatorname{Im}((-\hat{a}'(\omega) + b_0 \hat{A}'(\omega))/\hat{A}(\omega)) \neq 0, \qquad (\text{"}'\text{"} = d/d\omega). \tag{2.5}$$

Then there exists a positive number η and continuously differentiable functions $(b, p, u)\colon (-\eta, \eta) \to R \times R \times C(T_0)$, $T_0 = 2\pi/\omega$, such that

$$b(0) = b_0, \qquad p(0) = 1, \qquad u(0) = 0 \tag{2.6}$$

and

$x_q(t) \overset{\text{def}}{=} u(q)(t/p(q)) + b(q)/k$, $t \in R$ is a nontrivial $p(q)T_0$-periodic solution of (1.1) when $P = (b(q)\int_0^\infty A(s)\,ds + 1)/k$, $q \in (-\eta, \eta) \backslash \{0\}$. (2.7)

In addition, there exists a positive number ε such that if y is a nontrivial pT_0-periodic solution of (1.1) *and*

$$|p-1| + \left|P - \left(b_0 \int_0^\infty A(s)\,ds + 1\right)\Big/ k\right| + \sup_{t\in R} |y(t) - b_0/k| < \varepsilon, \tag{2.8}$$

then there exists a number $q \in (0, \eta)$ *and a number* $t_0 \in [0, p(q)T_0)$ *such that*

$$P = \left(b(q) \int_0^\infty A(s)\,ds + 1\right)\Big/ k, \quad p = p(q) \quad \textit{and} \quad y(t) = x_q(t + t_0), \quad t \in R. \tag{2.9}$$

Observe that u is a function: $(-\eta, \eta) \to C(T_0)$, hence $u(q)$ is the function: $t \to u(q)(t)$ from R to R. Note also that it is easy to deduce from (2.1) and (2.2) that $\hat{a}(s)$ and $\hat{A}(s)$ are continuously differentiable and $\hat{A}(s) \neq 0$ in $\operatorname{Im} s \leqslant 0$, $s \neq 0$ so that (2.5) makes sense.

In (2.1) and (2.2) the assumptions that A and a are nonnegative and that A is nonincreasing with $A(0) = 1$ are motivated by the model under consideration and the remaining assumptions are needed for the proof to go through. Note that one can always normalize a so that $\int_0^\infty a(s)\,ds = 1$ and that it follows from (2.1) that $t\dot{A}(t) \in L^1(R^+)$.

In Theorem 1 we used P as a parameter and k was kept constant, but as one sees from the proof one can just as well use k as a parameter and keep P constant.

Next we consider the stability of the periodic solutions. The first observation one can make is that if $Pk > 1$, then $x(t) \equiv x_P = (Pk - 1)/(k \int_0^\infty A(s)\,ds)$ is a positive solution of (1.1). Moreover, if (2.1), (2.2) hold and

$$1 - \hat{a}(s) + kx_P\hat{A}(s) \neq 0, \qquad \operatorname{Im} s \leqslant 0, \tag{2.10}$$

then it is possible to show that the solution x_P is asymptotically stable in the sense that if $x(t)$ is given (and continuous) on $(-\infty, 0)$ and $\sup_{t\in(-\infty,0)} |x(t) - x_P|$ is sufficiently small, then $x(t) \to x_P$ as $t \to \infty$, provided that (1.1) holds for $t \geqslant 0$. Clearly (2.10) does not hold if (2.3) holds and the assumption (2.5) says that there exists a zero of $1 - \hat{a}(s) + kx_P\hat{A}(s)$ crossing the real axis (as P, and hence x_P, varies) when $kx_P = b_0$. If we linearize equation (1.1) around a solution x, we get the following equation

$$\begin{aligned} y(t) &- \int_{-\infty}^t a(t-s)y(s)\,ds + kx_P \int_{-\infty}^t A(t-s)y(s)\,ds \\ &+ k\int_{-\infty}^t a(t-s)(x(s) - x_P)\,ds \int_{-\infty}^t A(t-s)y(s)\,ds \\ &+ k\int_{-\infty}^t A(t-s)(x(s) - x_P)\,ds \int_{-\infty}^t a(t-s)y(s)\,ds = 0, \qquad t \in R. \end{aligned} \tag{2.11}$$

If y is given on $(-\infty, 0)$ (and is such that the integrals in (2.11) are defined), then one can solve $y(t)$ from equation (2.11) for $t \geqslant 0$ and one gets a new function $L(y)(t) \overset{\text{def}}{=} y(t + T)$ defined on $(-\infty, 0)$. In the next theorem we study the eigenvalues of this mapping L, i.e., we are looking at Floquet multipliers.

Theorem 2. *Let the assumptions* (2.1)–(2.5) *hold and assume that*

$$e^{\sigma t}\dot{a}(t), e^{\sigma t}A(t) \in L^1(R^+) \qquad \textit{for some} \qquad \sigma > 0. \tag{2.12}$$

Then there exists a positive number μ ($\mu \leqslant \eta$ *where* η *is given in Theorem 1) and a continuous function* λ: $(-\mu, \mu) \to R$ *such that* $\lambda(0) = 1$ *and whenever* $\lambda(q) \neq 1$, $q \in (-\mu, \mu)$, *there exists a nonzero function* $y_q \in C(R)$, ($y_q(t) = 0(e^{-\sigma t})$ *as* $t \to -\infty$), *that satisfies*

$$\begin{aligned} y_q(t) &- \int_{-\infty}^{t} a(t-s)y_q(s)\,ds + b(q)\int_{-\infty}^{t} A(t-s)y_q(s)\,ds \\ &+ k\int_{-\infty}^{t} a(t-s)u(q)(s/p(q))\,ds \int_{-\infty}^{t} A(t-s)y_q(s)\,ds \\ &+ k\int_{-\infty}^{t} A(t-s)u(q)(s/p(q))\,ds \int_{-\infty}^{t} a(t-s)y_q(s)\,ds = 0, \qquad t \in R \end{aligned} \tag{2.13}$$

and

$$y_q(t + p(q)T_0) = \lambda(q)y_q(t), \qquad t \in R, \tag{2.14}$$

where the functions b, p and u are given in Theorem 1. If moreover, for $j = 0$ *or* 1,

$$(-1)^j \operatorname{Im}(c(\omega)) > 0 \tag{2.15}$$

then

$$(-1)^j(\lambda(q) - 1) > 0, \qquad q \in (-\mu, \mu)\backslash\{0\} \tag{2.16}$$

and

y_q *is a continuous function of q (in the topology of* $C(R)$ *of uniform convergence on compact sets) and* $y_0(t) = u'(0)(t) + \operatorname{Re}(c(\omega))/\operatorname{Im}(c(\omega))u'(0)(t + \pi/(2\omega))$, ("$'$" $= d/dq$), (2.17)

where

$$\begin{aligned} c(\omega) = {} & (2b_0^{-2}(|\hat{a}(\omega)|^2 - \operatorname{Re}(\hat{a}(\omega)))(\hat{a}(\omega) + \hat{A}(\omega)/\hat{A}(0)) \\ & + (\hat{A}(\omega)\hat{A}(2\omega)|\hat{a}(\omega)|^2 + \hat{a}(\omega)\hat{a}(2\omega)|\hat{A}(\omega)|^2)(1 - \hat{a}(2\omega) + b_0\hat{A}(2\omega))^{-1}) \\ & \times (-\hat{a}'(\omega) + b_0\hat{A}'(\omega))^{-1}. \end{aligned} \tag{2.18}$$

It is a consequence of (2.7) that for every $q \in (-\eta, \eta)$, $q \neq 0$ the function $\dot{u}(q)(t/p(q))$ is a nonzero solution of (2.13) and $\dot{u}(q)(t/p(q) + T_0) = \dot{u}(q)(t/p(q))$. One would expect the multiplier $\lambda(q)$ to be < 1 if the solution x_q is stable and x_q to be unstable if $\lambda(q) > 1$, but it is not clear to what extent this is a valid statement. But at least the linearized equation is unstable if $\lambda(q) > 1$ and one can see when this is the case by calculating $c(\omega)$ as defined in (2.18). Using the calculations in the proof of Theorem 2 one can, however, show that the "principle of exchange of linearized stability" holds in the following sense: If $j = 1$ in (2.15) and (2.16), then the zero $s = \omega$ of $1 - \hat{a}(s) + b_0\hat{A}(s)$ becomes a zero of $1 - \hat{a}(s) - b(q)\hat{A}(s)$ in $\operatorname{Im} s < 0$ when $q \neq 0$ so that the stability condition (2.10) for the constant solution fails. If $j = 0$, then the zero $s = \omega$ moves into the upper half plane, so that it is possible that (2.10) holds (with $kx_P = b(q)$, $q \neq 0$), but there may, of course, be other zeros of $1 - \hat{a}(s) - b_0\hat{A}(s)$ in $\operatorname{Im} s \leqslant 0$.

We also remark that it is not difficult to show that if (2.1) and (2.2) hold and there exists a zero of $1 - \hat{a}(s) + b\hat{A}(s)$ in $\operatorname{Im} s < 0$ for some $b > 0$, then (2.3) holds for some positive ω and b_0. By choosing the largest zero on the real axis we see that (2.4) must be satisfied and in this case it only remains to verify the critical condition (2.5).

Finally we consider a simple example. Assume that an individual infected at time t remains infective (with constant infectivity) up to time $t + \tau$ when the infectivity drops to zero and the immunity is lost at time $t + \tau + c_1$. Here c_1 is a positive constant and τ is a random variable with exponential distribution and mean value c_2^{-1}. In this case $A(t) = 1$, $0 \leqslant t \leqslant c_1$, $A(t) = 1 - \exp(-c_2(t - c_1))$, $t > c_1$ and $a(t) = c_2 \exp(-c_2 t)$, $t \geqslant 0$. A calculation shows that the assumptions of Theorem 1 are satisfied provided that $c_1 c_2 \geqslant \inf_{\pi < t < 2\pi}(-t/\sin(t))$.

3. Proof of Theorem 1

It is clear that equation (1.1) can be rewritten as

$$x(t) - x_P - \int_{-\infty}^{t} a(t-s)(x(s) - x_P)\,ds + kx_P \int_{-\infty}^{t} A(t-s)(x(s) - x_P)\,ds$$
$$+ k\int_{-\infty}^{t} a(t-s)(x(s) - x_P)\,ds \int_{-\infty}^{t} A(t-s)(x(s) - x_P)\,ds = 0, \quad t \in R, \tag{3.1}$$

where $x_P = (kP - 1)/(k\int_0^\infty A(s)\,ds)$. We introduce the unknown period explicitely as a new parameter and hence we are going to study the mapping F: $U_1 \times V_1 \to C(T_0)$ (where U_1 is a neighbourhood of $(0, b_0, 1)$ in R^3), defined by

$$F(q,b,p,v) = \varphi_0 + v - S(p, \varphi_0 + v) + bT(p, \varphi_0 + v)$$
$$+ kqS(p, \varphi_0 + v)T(p, \varphi_0 + v), \tag{3.2}$$

where (recall that $T_0 = 2\pi/\omega$)

$$V_1 = \left\{ f \in C(T_0) \,\middle|\, \int_0^{T_0} f(t)\sin(\omega t)\,dt = \int_0^{T_0} f(t)\cos(\omega t)\,dt = 0 \right\}, \tag{3.3}$$

$$\varphi_0(t) = \cos(\omega t), \qquad t \in R, \tag{3.4}$$

$$S(p, w)(t) = \int_{-\infty}^{t} pa(p(t-s))w(s)\,ds, \qquad t \in R \tag{3.5}$$

and

$$T(p, w)(t) = \int_{-\infty}^{t} pA(p(t-s))w(s)\,ds, \qquad t \in R. \tag{3.6}$$

The product in the last term in (3.2) is the ordinary pointwise product of functions. It is a consequence of (2.1) and (2.2) that the mappings $(p, w) \to S(p, w)$ and $(p, w) \to T(p, w)$ are continuously differentiable (when $p > 0$). Therefore we conclude from (3.2) that F is continuously differentiable, too.

We want to apply the implicit function theorem and then we must first show that

$$w \to F_v(0, b_0, 1, 0)w = w - S(1, w) + b_0 T(1, w) \text{ is an isomorphism: } V_1 \to V_1. \tag{3.7}$$

Since $1 - \hat{a}(0) + b_0\hat{A}(0) = b_0 \int_0^\infty A(s)\,ds > 0$ (see (2.1), (2.2)), and (2.4) holds, we conclude that $F_v(0, b_0, 1, 0)$ is an injection. Let $f \in V_1$ be arbitrary. Define the function w by

$$w(t) = \sum_{n=-\infty,\,|n|\neq 1}^{\infty} (1 - \hat{a}(n\omega) + b_0\hat{A}(n\omega))^{-1} T_0^{-1} \int_0^{T_0} f(r) e^{-in\omega r}\,dr\, e^{in\omega t}. \tag{3.8}$$

As $\lim_{|n|\to\infty}(1 - \hat{a}(n\omega) + b_0\hat{A}(n\omega)) = 1$ and f is continuous, we see immediately from (3.8) and Plancherel's theorem that $w \in L^2(0, T_0)$. Next we note that by (2.1) and (2.2)

$$\sup_{t\geq 0}\left|\sum_{n=0}^{\infty} A(t + nT_0)\right| < \infty \qquad \text{and} \qquad \sup_{t\geq 0}\left|\sum_{n=0}^{\infty} a(t + nT_0)\right| < \infty$$

and therefore $S(1, w)$ and $T(1, w)$ belong to $C(T_0)$ since $w \in L^2(0, T_0)$ and w is T_0-periodic on R. But from (3.8) we see that $w = f + S(1, w) - b_0 T(1, w)$, that is, $w \in V_1$ and $F_v(0, b_0, 1, 0)$ is a surjection. This shows that (3.8) holds (recall that V_1 is a closed subspace of $C(T_0)$).

The second fact that must be established is that

$$\begin{aligned}(s, t) &\to sF_b(0, b_0, 1, 0) + tF_p(0, b_0, 1, 0)\\ &= sT(1, \varphi_0) + t(-S_p(1, \varphi_0)\\ &\quad + b_0 T_p(1, \varphi_0)) \qquad \text{is an isomorphism: } R \times R \to V_2,\end{aligned} \tag{3.9}$$

where $V_2 = \{c_1 \cos(t) + c_2 \sin(t) | c_1, c_2 \in R\}$. This is seen to be the case once we note that by (3.4)–(3.6)

$$T(1, \varphi_0) = 2^{-1}(\hat{A}(\omega)e^{i\omega t} + \hat{A}(-\omega)e^{-i\omega t})$$

and

$$\begin{aligned}-S_p(1, \varphi_0) + b_0 T_p(1, \varphi_0) &= 2^{-1}(\omega\hat{a}'(\omega) - b_0\omega\hat{A}'(\omega))e^{i\omega t}\\ &\quad + 2^{-1}(-\omega\hat{a}'(-\omega) + b_0\omega\hat{A}'(-\omega))e^{-i\omega t}\end{aligned}$$

because then (3.9) follows from (2.5) and the fact that

$$\det\begin{bmatrix}\hat{A}(\omega) & \omega\hat{a}'(\omega) - b_0\omega\hat{A}'(\omega)\\ \hat{A}(-\omega) & -\omega\hat{a}'(-\omega) + b_0\omega\hat{A}'(-\omega)\end{bmatrix}$$
$$= 2i\omega|\hat{A}(\omega)|^2 \operatorname{Im}((-\hat{a}'(\omega) + b_0\hat{A}'(\omega))/\hat{A}(\omega)).$$

(Note that it follows from (2.1) that $\hat{A}(\omega) \neq 0$.)

Clearly $C(T_0)$ is a direct sum of the closed subspaces V_1 and V_2 and therefore it follows from the implicit function theorem by (3.7), (3.9) and the fact that $F(0, b_0, 1, 0) = 0$ (see (2.3), (3.2), (3.4)–(3.6)) that there exists a positive number η and continuously differentiable functions (b, p, v): $(-\eta, \eta) \to R \times R \times V_1$ such that $b(0) = b_0$, $p(0) = 1$, $v(0) = 0$ and

$$F(q, b(q), p(q), v(q)) = 0, \qquad q \in (-\eta, \eta). \tag{3.10}$$

If we define the function u by $u(q) = q(\varphi_0 + v(q))$, then we see from (3.1), (3.2), (3.5), (3.6) and (3.10) that the first part of the assertion of Theorem 1 holds.

We proceed to the uniqueness part of the theorem. If y is a nontrivial pT_0-periodic solution of equation (1.1), then there exist nonnegative numbers q and t_0 so that

$$y(pt - t_0) - (Pk - 1)\bigg/\left(k\int_0^\infty A(s)\,ds\right) = q\varphi_0(t) + z(t), \qquad t \in R \tag{3.11}$$

and $z \in V_1$ satisfies

$$G(b, p, q\varphi_0 + z) = 0, \tag{3.12}$$

where

$$G(b, p, w) = w - S(p, w) + bT(p, w) + kS(p, w)T(p, w) \tag{3.13}$$

and

$$b = (Pk - 1)\bigg/\left(\int_0^\infty A(s)\,ds\right). \tag{3.14}$$

Since $\varphi_0 - S(1, \varphi_0) + b_0 T(1, \varphi_0) = 0$ it follows from (2.1), (2.2), (2.8), (3.5), (3.6) and (3.13) that there exists a constant c_1 (provided ε is sufficiently small), such that

$$\begin{aligned} &\|G(b, p, q\varphi_0 + z) - (z - S(1, z) + b_0 T(1, z)) - q(p - 1)(- S_p(1, \varphi_0) \\ &\qquad + b_0 T_p(1, \varphi_0)) - q(b - b_0)T(1, \varphi_0)\|_{C(T_0)} \\ &\quad \leqslant c_1((\|z\|_{V_1} + q + |p - 1| + |b - b_0|)\|z\|_{V_1} \\ &\qquad + q(q + |b - b_0|\,|p - 1| + |p - 1|g(p - 1))), \end{aligned} \tag{3.15}$$

where $g(t) \to 0$ as $t \to 0$. If we recall (3.7), (3.9) and (3.12) we see that (3.15) implies the existence of a constant c_2 such that

$$\begin{aligned} \|z\|_{V_1} + q|p - 1| + q|b - b_0| \leqslant c_2((&\|z\|_{V_1} + q + |p - 1| + |b - b_0|)\|z\|_{V_1} \\ &+ q(q + |b - b_0|\,|p - 1| + |p - 1|g(p - 1))). \end{aligned}$$

If ε is sufficiently small, then it follows from this inequality and (2.8), (3.11) and (3.14) that

$$\|z\|_{V_1} + q|p - 1| + q|b - b_0| \leqslant c_3 q^2 \tag{3.16}$$

for some constant c_3. From (3.16) we conclude that if $q = 0$, then $z = 0$ and then y cannot be a nontrivial periodic function. If $q > 0$, then we have by (3.2), (3.12) and (3.13)

$$F(q, b, p, z/q) = 0. \tag{3.17}$$

But since the implicit function theorem also gives the uniqueness of the solution found in (3.10) (note that q is small if ε is small), we obtain the second assertion of Theorem 1 from (3.11), (3.14) and (3.17). This completes the proof of Theorem 1.

4. Proof of Theorem 2

We use the same notation as in the proof of Theorem 1 and in addition we define

$$\varphi_1 = \dot{\varphi}_0, \tag{4.1}$$

$$H(q,\gamma,w) = w - S_0(q,\gamma,w) + b(q)T_0(q,\gamma,w) + kS(p(q),u(q))T_0(q,\gamma,w) + kT(p(q),u(q))S_0(q,\gamma,w). \tag{4.2}$$

The mappings S and T are defined in (3.5), (3.6) and

$$S_0(q,\gamma,w)(t) = \int_{-\infty}^{t} p(q)e^{-\gamma(t-s)}a(p(q)(t-s))w(s)\,ds, \qquad t\in R \tag{4.3}$$

and

$$T_0(q,\gamma,w)(t) = \int_{-\infty}^{t} p(q)e^{-\gamma(t-s)}A(p(q)(t-s))w(s)\,ds, \qquad t\in R. \tag{4.4}$$

Observe that the assumption (2.12) is essential for these definitions. It is not difficult to see from the proof of Theorem 1 that the function $v(q)$ (for fixed q) is continuously differentiable with respect to t and that $\dot{v}(q)$ is continuously differentiable with respect to q in a neighbourhood of zero. We define the mapping K: $U_2 \times V_1 \to C(T_0)$ (where U_2 is a neighbourhood of $(0,0,0)$ in R^3), by

$$K(q,\gamma,\delta,z) = H(q,\gamma,\varphi_0 + z) + \delta\gamma^{-1}H(q,\gamma,\varphi_1 + \dot{v}(q)). \tag{4.5}$$

Differentiating (3.10) with respect to t we conclude from (3.2) and the definition of $u(q)$ that $H(q,0,\varphi_1 + \dot{v}(q)) = 0$, $q\in(-\eta,\eta)$. Hence we have

$$\gamma^{-1}H(q,\gamma,\varphi_1 + \dot{v}(q)) = \gamma^{-1}\int_0^{\gamma} H_\gamma(q,r,\varphi_1 + \dot{v}(q))\,dr. \tag{4.6}$$

It is straightforward to check, using (2.1), (2.2), (2.12), the results of Theorem 1 and the fact that $\dot{v}(q)$ is a continuously differentiable function of q, that K is continuously differentiable in a neighbourhood of $(0,0,0) \times V_1$. Moreover, the derivative with respect to γ, δ, z at $(0,0,0,0)$ is given by

$$(s,t,w) \to sH_\gamma(0,0,\varphi_0) + tH_\gamma(0,0,\varphi_1) + H(0,0,w)$$

and since $H(0,0,w) = F_v(0,b_0,1,0)w$, $H_\gamma(0,0,\varphi_i) = -S_{0\gamma}(1,0,\varphi_i) + b_0T_{0\gamma}(1,0,\varphi_i)$, $i = 0,1$, it follows from (2.5), (3.4), (3.7) and (4.1) that this derivative is an isomorphism: $R \times R \times V_1 \to C(T_0)$. Hence we may apply the implicit function theorem and we find a number $\mu > 0$ and continuously differentiable functions (γ,δ,z): $(-\mu,\mu) \to R \times R \times V_1$ such that

$$K(q,\gamma(q),\delta(q),z(q)) = 0, \qquad q\in(-\mu,\mu). \tag{4.7}$$

For those values of q for which $\gamma(q) \neq 0$ we let

$$y_q(t) = e^{\gamma(q)t/p(q)}(\varphi_0(t/p(q)) + z(q)(t/p(q)) + \delta(q)\gamma(q)^{-1}(\varphi_1(t/p(q)) + \dot{v}(q)(t/p(q)))), \qquad t\in R \tag{4.8}$$

and it follows from (4.2)–(4.5), (4.7) and (4.8) that (2.13) and (2.14) hold when we define

$$\lambda(q) = e^{\gamma(q)T_0}, \qquad q \in (-\mu, \mu). \tag{4.9}$$

Next we assume that (2.15) holds. Differentiating both sides of the equation (3.10) with respect to q we obtain by (3.2)

$$\begin{aligned} v'(q) &- S(p(q), v'(q)) + b(q)T(p(q), v'(q)) + kqS(p(q), v'(q))T(p(q), \varphi_0 + v(q)) \\ &+ kqS(p(q), \varphi_0 + v(q))T(p(q), v'(q)) + p'(q)(-S_p(p(q), \varphi_0 + v(q)) \\ &+ b(q)T_p(p(q), \varphi_0 + v(q)) + kqS_p(p(q), \varphi_0 + v(q))T(p(q), \varphi_0 + v(q)) \\ &+ kqS(p(q), \varphi_0 + v(q))T_p(p(q), \varphi_0 + v(q))) + b'(q)T(p(q), \varphi_0 + v(q)) \\ &+ kS(p(q), \varphi_0 + v(q))T(p(q), \varphi_0 + v(q)) = 0, \qquad q \in (-\eta, \eta). \end{aligned} \tag{4.10}$$

If we let $q = 0$ in (4.10), then we deduce from (3.4)–(3.7) and (3.9), that

$$b'(0) = p'(0) = 0 \tag{4.11}$$

and

$$v'(0) - S(1, v'(0)) + b_0 T(1, v'(0)) + kS(1, \varphi_0)T(1, \varphi_0) = 0. \tag{4.12}$$

Now we subtract the left-hand side of the equation in (4.12) from that in (4.10), divide by $q \neq 0$ and let $q \to 0$. Then it follows from (3.7), (3.9), (4.11) and the fact that $v(0) = 0$ that $b''(0)$, $p''(0)$ and $v''(0)$ exist and satisfy the equation

$$\begin{aligned} v''(0) &- S(1, v''(0)) + b_0 T(1, v''(0)) + p''(0)(-S_p(1, \varphi_0) + b_0 T_p(1, \varphi_0)) \\ &+ b''(0)T(1, \varphi_0) + 2k(S(1, v'(0))T(1, \varphi_0) + S(1, \varphi_0)T(1, v'(0))) = 0. \end{aligned} \tag{4.13}$$

Next we go through the same calculations for equation (4.7) and if we differentiate both sides of this equation with respect to q, then we obtain by (4.2)–(4.6)

$$\begin{aligned} H_q(q, \gamma(q), \varphi_0 + z(q)) &+ \delta(q)\gamma(q)^{-1} \int_0^{\gamma(q)} H_{\gamma q}(q, r, \varphi_1 + \dot{v}(q))\, dr \\ &+ \delta(q)\gamma(q)^{-1} \int_0^{\gamma(q)} H_\gamma(q, r, \dot{v}'(q))\, dr + H(q, \gamma(q), z'(q)) \\ &+ \gamma'(q)(H_\gamma(q, \gamma(q), \varphi_0 + z(q)) + \delta(q)\gamma(q)^{-2} \int_0^{\gamma(q)} rH_{\gamma\gamma}(q, r, \varphi_1 + \dot{v}(q))\, dr) \\ &+ \delta'(q)\gamma(q)^{-1} \int_0^{\gamma(q)} H_\gamma(q, r, \varphi_1 + \dot{v}(q))\, dr = 0, \qquad q \in (-\mu, \mu). \end{aligned} \tag{4.14}$$

If we let $q = 0$ in (4.14) then we conclude that (see (4.2), (4.11) and recall that $u(q) = q(\varphi_0 + v(q))$, $\gamma(0) = \delta(0) = v(0) = z(0) = 0$)

$$\gamma'(0) = \delta'(0) = 0 \tag{4.15}$$

and

$$z'(0) - S(1, z'(0)) + b_0 T(1, z'(0)) + 2kS(1, \varphi_0)T(1, \varphi_0) = 0. \tag{4.16}$$

Now we subtract the left-hand side of equation (4.16) from that of equation (4.14), divide by $q \neq 0$ and let $q \to 0$. Again we are (by essentially the same argument as above) able to conclude that $\gamma''(0)$, $\delta''(0)$ and $z''(0)$ exist and satisfy the equation

$$\begin{aligned} z''(0) &- S(1,z''(0)) + b_0 T(1,z''(0)) + \gamma''(0)(-S_{0\gamma}(0,0,\varphi_0) + b_0 T_{0\gamma}(0,0,\varphi_0)) \\ &+ \delta''(0)(-S_{0\gamma}(0,0,\varphi_1) + b_0 T_{0\gamma}(0,0,\varphi_1)) + kS(1,z'(0))T(1,\varphi_0) \\ &+ kS(1,\varphi_0)T(1,z'(0)) + p''(0)(-S_p(1,\varphi_0) + b_0 T_p(1,\varphi_0)) + b''(0)T(1,\varphi_0) \\ &+ 2kS(1,\varphi_0)T(1,v'(0)) + 2kS(1,v'(0))T(1,\varphi_0) = 0. \end{aligned} \tag{4.17}$$

If we use (3.2)–(3.6), (4.1), (4.3), (4.4), (4.12), (4.16) and (4.17), then we conclude (after some tedious calculations) that $\gamma''(0)$ and $\delta''(0)$ satisfy

$$\gamma''(0) = -2^{-1}k^2\,\mathrm{Im}(c(\omega)),$$
$$\delta''(0) = (2\omega)^{-1}k^2\,\mathrm{Re}(c(\omega)),$$

where $c(\omega)$ is defined in (2.18). Now it follows from (2.15), (4.8), (4.9) and (4.15) that (2.16) and (2.17) hold. This completes the proof of Theorem 2.

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