On the Uniqueness of Flow of a Navier-Stokes Fluid Due to a Stretching Boundary

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To James Serrin, on the occasion of his sixtieth birthday, and in recognition of his fundamental contributions both to analysis and to applied mathematics in its finest sense

1. Introduction

When a sheet of polymer is extruded continuously from a die, it entrains the ambient fluid and a boundary layer develops. Such a boundary layer is markedly different from that in the Blasius flow past a flat plate in that the boundary layer grows in the direction of the motion of the sheet, starting at the die. SAKIADIS [1]–[3] was the first to study such a boundary layer flow due to a rigid flat continuous surface moving in its own plane. Later, ERICKSON, FAN & FOX [4] studied the boundary layer due to the motion of a porous flat plate when the transverse velocity at the surface is non-zero.

There are situations when the extruded polymer sheet is being stretched as it is being extruded (*cf.* MCCORMACK & CRANE [5]) and this is bound to alter significantly the boundary layer characteristics of the flow considered by SAKIADIS [1]. A detailed analysis of the boundary layer flow of a Navier-Stokes fluid due to a stretching sheet has been carried out recently by DANBERG & FANSLER [6]. They assume that the free-stream velocity of the fluid is a constant, while the sheet is being stretched by a velocity proportional to x, where x is the coordinate in the direction of stretch. The problem of the sheet velocity being a constant, while the free-stream velocity is proportional to x, had been studied earlier by ROTT [7].

Let us consider the two-dimensional flow of a Navier-Stokes fluid due to the stretching of a flat sheet coinciding with the plane $y = 0$, the fluid being confined to $y > 0$. A steady uniform stress leading to equal and opposite forces is applied along the x-axis so that the sheet is stretched keeping the origin fixed. Thus, the appropriate boundary conditions are

$$u = cx, \quad v = 0, \quad \text{at} \quad y = 0 \quad (c > 0) \tag{1.1}$$

and

$$u \to U_\infty \quad \text{as} \quad y \to \infty. \tag{1.2}$$

Here u and v denote the velocity components in the x and y directions, respectively, and U_∞ is the free-stream velocity.

We introduce the dimensionless quantities

$$\bar{x} = \frac{cx}{U_\infty}, \quad \bar{y} = \frac{cy}{U_\infty}\sqrt{R}, \quad \bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}\sqrt{R}, \tag{1.3}$$

where

$$R = \frac{U_\infty^2}{c\nu} \tag{1.4}$$

is the Reynolds number. We shall seek a solution of the form

$$\bar{u} = \bar{x}f'(\bar{y}) + g'(\bar{y}) \quad \text{and} \quad \bar{v} = -f(\bar{y}). \tag{1.5}$$

Notice that the above velocity field is isochoric or volume-preserving. Substituting (1.5) into the Navier-Stokes equations (with the pressure independent of x) leads to the following system of coupled ordinary differential equations:

$$f''' + ff'' - f'^2 = 0, \tag{1.6}$$

$$g''' + fg'' - f'g' = 0. \tag{1.7}$$

The appropriate boundary conditions are

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\bar{y}) \to 0 \quad \text{as} \quad \bar{y} \to \infty, \tag{1.8}$$

and

$$g(0) = 0, \quad g'(0) = 0, \quad g'(\bar{y}) \to 1 \quad \text{as } \bar{y} \to \infty. \tag{1.9}$$

It is interesting to note that for the assumed form of the velocity field (1.5), the boundary layer equations and the Navier-Stokes equations are the same. Thus, for the velocity field under consideration, the terms which are dropped out of the Navier-Stokes equations in deriving the boundary layer approximations happen to be identically zero.

Equation (1.6), subject to (1.8), has the following simple solution:

$$f(\bar{y}) = 1 - e^{-\bar{y}}. \tag{1.10}$$

Using this solution for f in (1.7), Danberg & Fansler were able to exhibit an explicit solution for g, by using the method of variation of parameters.

Since equation (1.6) is nonlinear, it is possible that (1.6), subject to (1.8), possesses other solutions in addition to (1.10). This, then, would imply the possible existence of a boundary layer with a structure different from that exhibited by (1.10).In this paper we prove that (1.10) is the unique solution to (1.6) and (1.8). An alternative treatment of this is given in [8].

Recently, Rajagopal, Na & Gupta [9], [10], have studied the boundary layer due to a stretching sheet, in the case of an incompressible homogeneous fluid of second grade. Using the velocity field (1.5), they show that the equations of motion are given by

$$f''' + ff'' - f'^2 + k\{2f'''f' - f''^2 - f^{iv}\} = 0, \tag{1.11}$$

and

$$g''' + fg'' - f'g' + k\{g'''f' + f'''g' - f''g'' - fg^{iv}\} = 0, \tag{1.12}$$

where k is a non-Newtonian parameter. They study the problem by means of a perturbation in k, their zero$^{\text{th}}$ order equation corresponding to the equations (1.6) and (1.7). Thus the uniqueness established here bears on their study also.

The boundary value problem under consideration also illustrates a pitfall which one encounters while solving nonlinear equations in unbounded domains numerically, namely, the specification of infinity. Fixing infinity at what seems a sufficiently large distance and solving the boundary value problem using a gears method suggests the existence of two distinct solutions, one of which is (1.10) (*cf.* SZERI [11]). Shifting the location of infinity moderately produces no change, even in the seventh decimal place in the numerical solution, and lends credence to the existence of a second solution. However, in the light of the uniqueness theorem proved in this paper, the problem was re-examined. It was found (*cf.* SZERI [11]) that, as the location of infinity is moved significantly further, the apparent second solution in fact diverges.

The underlying reason for uniqueness is that equation (1.6) is invariant under two simple transformations, first a translation of the independent variable, and secondly the basic transformation

$$f \to \alpha f, \qquad x \to \alpha^{-1} x.$$

This implies that we can in effect reduce the order of the equation by two, so that we deal with a first-order equation. It is then a matter of looking at the phase plane for this first-order equation, from which uniqueness of the solution can be deduced quite easily.

The plan of the paper is as follows. In section 2 we prove some easy preliminary lemmas on the asymptotic signs of possible solutions. In section 3 we give the transformation which reduces the equation to a form in which it is amenable to phase plane treatment, and in section 4 we discuss the phase plane. The uniqueness proof is then completed in section 5.

2. Some Preliminary Lemmas

Lemma 1. *The solution* (1.10) *is the only solution of* (1.6) *and* (1.8) *for which* $f' \geqq 0$.

Proof. First observe that a solution of (1.6) and (1.8) which satisfies $f' \geqq 0$ also satisfies $f'' \leqq 0$. For (1.6) implies that

$$\left\{f'' \exp\left(\int_0^x f(t)\, dt\right)\right\}' = f'^2 \exp\left(\int_0^x f(t)\, dt\right),$$

so that

$$f'' \exp\left(\int_0^x f(t)\,dt\right)$$

cannot deerease, and so f'' vanishes at most once and, if it does vanish, is positive after the zero. But if also $f' \geqq 0$, then because f'' is ultimately positive, f' tends to a strictly positive limit at infinity, which contradicts $f'(\infty) = 0$. Hence $f'' \leqq 0$.

Now suppose for contradiction that there are two solutions f_1, f_2, where $\phi = f_2 - f_1$. Then ϕ satisfies

$$\phi''' + f_2\phi'' - (f_1 + f_2)'\,\phi' + f_1''\phi = 0, \tag{2.1}$$

with

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi'(\infty) = 0.$$

Suppose without loss of generality that $\phi''(0) > 0$. Then initially $\phi > 0$, $\phi' > 0$, $\phi'' > 0$, and so long as these inequalities are maintained,

$$\phi'' \exp\left(\int_0^x f(t)\,dt\right)$$

is nondecreasing since $(f_1 + f_2)' \geqq 0$, $f_1'' \leqq 0$. Hence ϕ, ϕ', ϕ'' never vanish, which contradicts $\phi'(\infty) = 0$.

Lemma 2. *For any second solution of* (1.6) *and* (1.8) f', f'', f''' *all vanish precisely once, at* x_1, x_2, x_3, *say, with* $x_1 < x_2 < x_3$. *Also, ultimately,* $f > 0$, $f' < 0$, $f'' > 0$, $f''' < 0$.

Proof. Certainly, from Lemma 1, f' must have at least one zero, and so f'' must have at least one zero, since $f'(\infty) = 0$.

Hence, from the proof of Lemma 1, f'' has precisely one zero, at x_2, say, and f' is decreasing before x_2 and increasing after. Since $f'(0) = 1$ and $f'(\infty) = 0$, it is clear that f' also has precisely one zero, at x_1, say, with $x_1 < x_2$. (Note that a double zero of f' is impossible, for $f' = f'' = 0$ implies $f''' = 0$ and so $f =$ constant; similarly, f'' cannot have a double zero.)

Finally, $f(x) > 0$ for $x \leqq x_1$, and so, from (1.6), $f'''(x) > 0$ for $x \leqq x_1$. Since

$$f^{iv} + f'''f = f'f'',$$

it follows that in (x_1, x_2), where $f'f'' > 0$, we must have

$$f''' \exp\left(\int_0^x f(t)\,dt\right) \tag{2.2}$$

nondecreasing, and so $f'''(x_1) > 0$ implies $f'''(x_2) > 0$. For $x > x_2$, (2.2) is decreasing, and so f''' has at most one zero, and indeed exactly one since $f''(x_2) = 0$ and $f''(\infty) = 0$. So $f'''(x_3) = 0$ with $x_3 > x_2$.

The above argument shows that ultimately $f' < 0,\ f'' > 0,\ f''' < 0$. It is also true that $f(x) > 0$ for all $x > 0$. For f is first increasing (from zero) and then decreasing. But

$$f'' \exp\left(\int_0^x f(t)\,dt\right)$$

never decreases. Hence ultimately, when f''' is positive decreasing, wemust have $\exp\left(\int_0^x f(t)\,dt\right)$ increasing, so that $f > 0$. Hence $f > 0$ ultimately, and so always.

3. The Transformation of the Equation

As long as $f > 0,\ f' < 0$, we can make the change of variables

$$z = -\log f, \quad u = f'/f^2, \quad \delta u = \frac{du}{dz},$$

and (1.6) becomes

$$\delta^2 u + \left\{\frac{(\delta u)^2}{u} - 7\delta u - \frac{\delta u}{u}\right\} + 6\,(u + \tfrac{1}{6}) = 0. \tag{3.1}$$

The proof is routine. For

$$\delta u = -\frac{du}{d(\log f)} = -f\frac{du}{dx}\frac{dx}{df} = -\frac{f}{f'}\left(\frac{ff'' - 2f'^2}{f^3}\right), \tag{3.2}$$

$$f'' = -ff'(\delta u - 2u) = -e^{-3z}\,u(\delta u - 2u),$$

$$f''' = -\frac{d}{dz}\{e^{-3z}\,u(\delta u - 2u)\}\frac{dz}{dx}$$

$$= \frac{f'}{f}\,e^{-3z}\{-3(u\,\delta u - 2u^2) + u\,\delta^2 u + (\delta u)^2 - 4u\,\delta u\},$$

and substitution in (1.6) gives the answer.

Note that Equation (3.1) is second-order autonomous, so that it could be reduced to first order, but it is easier to work with it as it stands. Since it is autonomous, it is amenable to phase plane methods, and we study the phase plane in the next section. The fact that equation (1.6) can be reduced to a first-order equation can be observed at once from the fact that it is invariant under both the transformations

$$f \to f, \quad x \to x + c;$$

$$f \to \alpha f, \quad x \to \alpha^{-1} x.$$

4. The Phase Plane

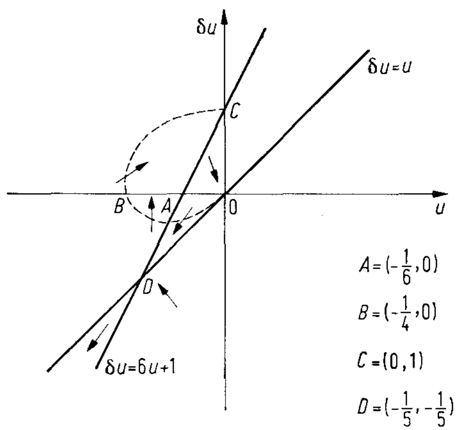

Fig. 1

The arrows denote the direction of a trajectory lying in the appropriate area of the phase plane. Thus

$$\nearrow \text{ implies both } \delta u,\ \delta^2 u > 0,$$

$$\searrow \text{ implies } \delta u > 0,\ \delta^2 u < 0$$

and so on. The appropriate areas are determined by observing that $\delta^2 u = 0$ when

$$(\delta u)^2 - 7u\,\delta u - \delta u + 6u^2 + u = 0,$$

i.e. when

$$(\delta u - u)(\delta u - 6u - 1) = 0.$$

The only rest-point ($\delta u = \delta^2 u = 0$) is the point $A(-\frac{1}{6}, 0)$ although the origin is also some sort of singular point. (The equation is singular at $u = 0$.)

We established the validity of the transformation leading to the phase plane only when $u < 0$, so that arrows are not entered in the right-hand half of the diagram.

The dotted line denotes one special trajectory. It is trivial to verify that

$$f = a + be^{-ax} \tag{4.1}$$

is a solution of (1.6) for every a, b. (Our interest will be in $a > 0$, $b > 0$). In fact, because of the invariance of the equation under the transformations previously mentioned, all solutions of the form (4.1) give the same trajectory in the phase plane, and it is given by the dotted line. Note the following points:

(i) as $x \to \infty$,

$$f \sim a, \quad f' \sim -abe^{-ax}, \quad f'' \sim a^2 be^{-ax},$$

so that

$$u \to 0, \quad \delta u \to 1;$$

(ii) as $x \to -\infty$,

$$f \sim be^{-ax}, \quad f' \sim -abe^{-ax}, \quad f'' \sim a^2be^{-ax},$$

so that, since (3.2) leads to

$$\delta u - 2u = -\frac{f''}{ff'} \sim \frac{a}{b}e^{ax} \sim -u,$$

we have $\delta u \sim u$;

(iii) when $\delta = 0$, we have

$$f'' = 2f'^2,$$

which gives

$$(a + be^{-ax})\, a^2be^{-ax} = 2a^2b^2e^{-2ax}$$

so that

$$a = be^{-ax}, \quad u = -\tfrac{1}{4}.$$

5. The Proof of Nonexistence of a Second Solution

Let C denote the closed curve consisting of the dotted curve and the portion of the δu-axis. Suppose for contradiction that we do have a second solution to (1.6) and (1.8).

Suppose first that for some value of x the trajectory of this second solution lies inside C. Then, as we decrease x (or z), we cannot get outside C. For we cannot cross the dotted portion of C, since that itself is a trajectory and trajectories do not cross without being identical. (The form (4.1) corresponding to the dotted trajectory does not give us a second solution since it cannot satisfy the inequalities of Lemma 2, which would imply $a > 0$, $b > 0$, and also $f(0) = 0$.) Also, the direction of the arrows shows that we cannot cross $\delta u = 0$. As we decrease x, therefore, the trajectory always lies inside C, so that $u < 0$, $f' < 0$, f is increasing (for x decreasing), and so $f(0) = 0$ is impossible.

The only other possibility is that the trajectory lies outside C. But then the arrows show that, as we increase x, we eventually increase δu, and so eventually we must have $\delta u > 0$, $u = 0$. But Lemma 1 says that for any second solution we must have $u < 0$ if x is sufficiently large, and this again gives a contradiction. (Note that the line $\delta u = u$ corresponds to $f''' = 0$, so that, from Lemma 1, we have to be above $\delta u = u$ when x is sufficiently large.)

This completes the proof, but there are a number of remarks that can be made.

Remark 1. We have used the fact that a trajectory, in C and going backwards in x, does not cross C. It can also be shown that a trajectory, in C and going forwards in x, does not cross C, although the argument is a little more delicate. But the final conclusion is that no trajectory can cross C, and it is interesting to ask what happens to those trajectories which lie always inside C.

It is just a matter of linearization to prove that a trajectory starting sufficiently close to the rest-point A spirals out, and although we have not carried out the calculations completely, it seems that a trajectory starting sufficiently closc to C spirals in. Hence there must be a limit-cycle inside C, which corresponds to a periodic solution of the equation (3.1), periodic, that is, in z, not in x.

Remark 2. It is possible to make a similar investigation of the equation

$$f''' + ff'' = \lambda f'^2,$$

for varying values of $\lambda\,(> 0)$. We have not carried out all the details, but in (3.1) the only change is in the constant term, and the main consequence as far as the phase plane is concerned is that, at $\lambda = 8/7$, the rest-point changes from unstable to stable and merges with the limit-cycle. For $\lambda > 8/7$, therefore, the limit-cycle disappears and we just have, inside C, spirals spiralling in to the stable rest-point. Of course, we no longer have the explicit form (1.10) for the solutions that correspond to the trajectory C, but we can establish the existence of such solutions by shooting arguments.

Remark 3. It is interesting that, with $\delta u = y$, $u = x$, the equation (3.1) becomes of the form

$$\frac{dy}{dx} = -\frac{6x^2 - 7xy + y^2 + x - y}{xy},$$

which is the type of equation that is discussed in Hilbert's sixteenth problem, with reference to the maximum number of limit-cycles that such an equation may possess.

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