Existence and Stability of Necking Deformations for Nonlinearly Elastic Rods

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Introduction

In uniaxial tension tests on fibres of polymeric materials such as nylon or polyethylene, it is frequently observed that the fibre will initially elongate homogeneously until a critical value of the applied load is reached whereupon the fibre will develop regions of reduced thickness known as necks. Nonhomogeneous behaviour of this type is usually referred to as necking.

The development of a neck within a fibre is quite interesting. As the neck starts to form it both narrows and lengthens. After a certain time the narrowing stops but the lengthening continues until either another neck, which has formed in a different region of the fibre, is reached or the end of the fibre is reached. From observations, the material in the necked and unnecked regions of the bar appears to be in a state of homogeneous strain whilst the transition region between the necked and unnecked regions has large nonhomogeneous behaviour. Thus, the neck propagating along the fibre can be regarded as a mechanism for converting regions of low axial strain to high axial strain (for further details see [13]).

The phenomenon of necking has been studied extensively using plasticity and elasto-plasticity theories (see, for example, [17] and [20]). Less work, however, has been done on using finite elastostatics to predict the onset of necking. The two main attempts to show this have been quite distinct in their approaches.

The first approach, and the one we shall concentrate on in this paper, is made in [5] (see also [4] and [8]) where the equilibrium equations for a one-dimensional theory of hyperelastic rods with sufficient structure to allow axial and radial deformations are shown to possess solutions which, it is conjectured, represent necking deformations. Significantly, the elastic material considered is strongly-elliptic. A question left open in [5] is which of the non-trivial solutions satisfies the energy criterion of stability, that is, which solutions locally minimize the total stored elastic energy of the body. A brief study of the stability of the trivial solution to the equilibrium equations (which corresponds to uniform elongation) is made in [5]. It is usual to regard deformations that do not satisfy this criterion as physically unrealizable. We remark here that the relationship between the energy criterion of stability and the asymptotic behaviour of solutions to the equations of elasto-dynamics is not yet fully understood. Thus, it is not certain what is the correct choice of topology in which to look for local minimizers. For an interesting example of this problem in one-dimensional viscoelasticity see [25].

In this paper we use classical methods from the calculus of variations to give a complete description of stability of both the trivial (homogeneous) and non-trivial solutions for bars with sufficiently small undeformed radii. We show that if the total elongation of the bar is specified (a "hard loading" device) and the applied load-extension curve is non-monotone, then the homogeneous equilibrium solution will lose stability for sufficiently large elongations. Further, solutions corresponding to one or more fully developed necks can never be local minimizers (in any sense) of the energy (this latter result is independent of the initial radius). From these two results it is easy to deduce that the only stable nonhomogeneous deformation possible is a half-neck or draw.

The second approach to using elastostatics to predict the onset of localized behavior is made by ERICKSEN in [14], where the consequences of dropping strong-ellipticity are studied (see also [18]). For a one-dimensional rod theory with only axial deformations allowed, it is shown that if the stored-energy function is non-convex, the minimizers of the energy have jump discontinuities in the axial strain (the discontinuity can be identified as a transition point in the bar between two different regions of constant strain). Further, there are infinintely many minimizers of this type all with the same energy.

By setting the radius parameter formally equal to zero in the rod theory used by ANTMAN, the resulting singular problem is almost the same as the problem studied by ERICKSEN. In particular, the integrand is no longer convex (that is, strong ellipticity has been lost in the singular limit). Using the explicit form of the minimizers for the Ericksen problem we are able to show that for small undeformed radii, the half-neck solution to the Antman problem is the global minimizer of the energy and this solution takes approximately constant values of strain apart from a small region near the discontinuity of the singular minimizer where there is a rapid change in the strain.

The result is motivated by recent work [11] on a gradient theory for phase transitions in a van der Waals fluid. In [11], the problem of minimizing

$$E(u) = \int_{-L}^{L} [W(u(x)) + \tau^2 u'(x)^2] \, dx,$$

subject to the constraint

$$\int_{-L}^{L} u(x) \, dx = M,$$

is considered. Here $u(x)$ is the fluid density, $W(u)$ is the nonconvex free energy per unit volume, M is the total mass and τ is a small parameter. When $\tau = 0$ the problem reduces to that studied in [14]. For τ small but non-zero intricate phase-plane arguments are used in [11] to show that there exists a unique, monotone minimizer of $E(u)$. The integrand in $E(u)$ is similar to the stored-energy function for the Antman rod theory where τ is identified as the undeformed radius. We prove a slightly weaker result than that in [11] but our method uses variational rather phase-plane techniques and is considerably shorter.

The results in this paper rely crucially on the natural (Neumann) boundary conditions used. Physically, these imply that the bar is held in such a way as to offer no resistance to changes in thickness at the ends. Clearly, this type of support is unrealistic. If the bar is clamped at the ends, then we must impose Dirichlet boundary conditions. In this case, the variational techniques used in this paper no longer apply. However, by extending the phase-plane analysis in [11], it is shown in [24] that for sufficiently small radii the global minimizer of the energy is a single neck and there is a boundary layer at the ends of the bar.

Finally, we mention some recent work by SPECTOR [27] who looks at the stability (in the sense of being a weak local minimizer) of the homogeneous deformations for a three-dimensional cylindrical elastic bar. It is shown that for the hard-loading device the homogeneous deformation cannot become unstable until after the applied load has attained a local maximum. This result is proved using the second variation and a generalized version of the Korn inequality. An equivalent result for the one-dimensional case is proved in this paper. In addition, the result proved here suggests that the maximum extension for which the homogeneous deformation must be stable is a decreasing function of the undeformed radius.

The plan of this paper is as follows: In Section 1 we describe the reduction of the full three-dimensional problem to a one-dimensional problem and state precisely the variational problem to be considered. Section 2 contains a proof of existence and regularity to the variational problem and in Section 3 the phase-plane for the Euler-Lagrange equations is constructed. The stability results for the trivial and non-trivial solutions to the Euler-Lagrange equations are given in Section 4. Section 5 considers the convergence of minimizers to solutions of the singular problem as the radius goes to zero. Incompressible materials are considered in Section 6 and some real constitutive models are discussed.

Throughout this paper $W^{1,1}(a, b)$ will denote the usual Sobolev space of equivalence classes of integrable functions on (a, b) with integrable generalized first derivative. The canonical representation for an equivalence class is the absolutely continuous member of that class. If X is a Banach space, then $\overset{``X"}{\rightharpoonup}$ denotes weak convergence with respect to that space.

1. The one-dimensional problem

We consider a circular cylindrical rod of undeformed length 2 and radius r, and which is made of a homogeneous, isotropic, hyperelastic material. We shall be interested in deformations of the rod in a "hard loading" device, in which one end of the rod is fixed whilst the other end is extended a given distance. We ignore shearing effects (*cf.* [8]). Then the appropriate one-dimensional equilibrium problem we shall study is to find solutions of

$$(P_{r\gamma})\begin{cases} \text{Minimize } I_r(\varrho, \varepsilon) = \int_{-1}^{1} W(r\varrho'(x), \varrho(x), \varepsilon(x))\, dx \\ \text{on} \quad \mathfrak{A}_{r\gamma} = \{\varrho \in W^{1,1}(-1,1), \varepsilon \in L^1(-1,1) : \varrho > 0 \text{ a.e.}, \varepsilon > 0 \text{ a.e.}, \\ \qquad I_r(\varrho, \varepsilon) < \infty, \text{ and } \varepsilon \text{ satisfies (1.1)}\}, \end{cases}$$

where

$$\int_{-1}^{1} \varepsilon(x)\, dx = 2\gamma. \tag{1.1}$$

Here, $\varrho(x)$ is the radius of a cross-section of the rod with centre at x along the axis of the rod in the undeformed configuration, $\varepsilon(x)$ is the axial strain and $\varrho'(x) = d\varrho(x)/dx$. The function W is the one-dimensional stored-energy density function. In the admissible set, $\mathfrak{A}_{r\gamma}$, the pointwise constraints on ϱ and ε are consequences of the local invertibility of three-dimensional elasticity. The integral constraint (1.1) says that the total extension of the rod is 2γ, where $\gamma(>1)$ is given.

A detailed derivation of problem $(P_{r\gamma})$ is given in [5] (see also [4]). However, that derivation is carried out for a rod of unit undeformed radius so that the role of the radius is not made explicit. In the remainder of this section, we shall motivate why the radius r should multiply the highest-order term in the integrand of I_r by considering a class of constrained deformations. For an extensive description of more general rod theories see [3].

Suppose the rod occupies the region $\Omega = \{(X, Y, Z) : -1 \leqq X \leqq 1, Y^2 + Z^2 \leqq r^2\}$ with respect to a Cartesian coordinate system in its undeformed configuration. Consider deformations of the form $(X, Y, Z) \to (x, y, z)$ where

$$\begin{aligned} x(X, Y, Z) &= \int_{-1}^{X} \varepsilon(\tau)\, d\tau - 1, \\ y(X, Y, Z) &= \varrho(X) Y, \quad z(X, Y, Z) = \varrho(X) Z. \end{aligned} \tag{1.2}$$

Here ϱ and ε are scalar functions to be determined.

The deformation gradient for (1.2) is

$$F = \begin{pmatrix} \varepsilon(X) & 0 & 0 \\ \varrho'(X)\, Y & \varrho(X) & 0 \\ \varrho'(X)\, Z & 0 & \varrho(X) \end{pmatrix} \tag{1.3}$$

and the principal invariants of the left Cauchy-Green deformation tensor, $B = FF^T$, are

$$\begin{aligned} I_B &= 2\varrho^2 + \varepsilon^2 + (Y^2 + Z^2)(\varrho')^2, \\ II_B &= \varrho^4 + 2\varrho^2\varepsilon^2 + (Y^2 + Z^2)(\varrho')^2 \varrho^2, \\ III_B &= \varrho^4 \varepsilon^2. \end{aligned} \tag{1.4}$$

If ϱ and ε are independent of X, then (1.2) will be a homogeneous deformation. In general, (1.2) will not be a semi-inverse solution of the three-dimensional equilibrium equations nor will it satisfy the zero-traction condition on the lateral surface. (For this reason it is appropriate to regard (1.2) as a material constraint.)

Let $\Omega_r = \{(Y, Z): Y^2 + Z^2 \leqq r^2\}$ (a cross-section of the rod) and let Φ be the three-dimensional stored-energy density function. We define a *one-dimensional stored-energy density function* as

$$W^r(\varrho'(X), \varrho(X), \varepsilon(X)) = \frac{1}{\pi r^2} \int_{\Omega_r} \Phi(F)\, d\Omega_r. \tag{1.5}$$

Since the material is isotropic, there is a function h such that

$$\Phi(F) = h(I_B, II_B, III_B). \tag{1.6}$$

By changing to polar coordinates from (1.4), (1.5), (1.6) we get

$$W^r(\varrho', \varrho, \varepsilon) = \frac{2}{r^2} \int_0^r h(2\varrho^2 + \varepsilon^2 + R^2(\varrho')^2, \varrho^4 + 2\varrho^2\varepsilon^2 + R^2(\varrho')^2 \varrho^2, \varrho^4\varepsilon^2)\, R\, dR. \tag{1.7}$$

With $R = rT$, the right-hand side of (1.7) becomes

$$2 \int_0^1 h(2\varrho^2 + \varepsilon^2 + T^2(r\varrho')^2, \varrho^4 + 2\varrho^2\varepsilon^2 + T^2(r\varrho')^2 \varrho^2, \varrho^4\varepsilon^2)\, T\, dT,$$

which is of the same form as the right-hand side of (1.7) but with r replaced by 1 and ϱ' replaced by $r\varrho'$. Hence, we have

$$W^r(\varrho'(X), \varrho(X), \varepsilon(X)) = W^1(r\varrho'(X), \varrho(X), \varepsilon(X)). \tag{1.8}$$

Since we shall be interested in the behaviour of solutions to $(P_{r\gamma})$ as $r \to 0$, it is more convenient to use the right-hand side of (1.8) as the one-dimensional stored-energy density function.

The total stored-energy $E(x, y, z)$ for the deformation (1.2) is

$$E(x, y, z) = \int_{\Omega} \Phi(F)\, d\Omega = \int_{-1}^{1} \int_{\Omega_r} \Phi(F)\, d\Omega_r\, dX$$

$$= \pi r^2 \int_{-1}^{1} W^1(r\varrho'(X), \varrho(X), \varepsilon(X))\, dX.$$

2. Existence and regularity

Following [7], we assume that W satisfies the hypotheses:

H1 $W(p, u, v): \mathbb{R} \times (0, \infty) \times (0, \infty) \to [0, \infty)$ is in C^∞.

H2 $W_{pp}(p, u, v) > 0$, $W_{uu}(p, u, v) > 0$, $W_{vv}(p, u, v) > 0$ and

$$W_{pp}(p, u, v) - \frac{W_{pv}(p, u, v)^2}{W_{vv}(p, u, v)} > 0 \text{ for all } u, v \in (0, \infty) \text{ and for all } p \in \mathbb{R}.$$

H3 There are functions $h_1(\eta), h_2(\eta), 0 < \eta < \infty$, bounded below, such that

$$W(p, u, v) \geqq h_1(u) + h_2(v)$$

for every $u \in (0, \infty)$, $v \in (0, \infty)$, $p \in \mathbb{R}$, and

$$h_1(\eta) \to \infty,\ h_2(\eta) \to \infty \quad \text{as } \eta \to 0.$$

H4 There is a function $\Pi(\eta)$, $0 \leqq \eta < \infty$, bounded below, such that

$$W(p, u, v) \geqq \Pi((p^2 + v^2)^{\frac{1}{2}})$$

for all $u \in (0, \infty)$, and $\Pi(\eta)/\eta \to \infty$ as $\eta \to \infty$.

H5 Given $u > 0$, there are constants τ_0, N_0 depending on u, such that for $|y - u| < \tau_0$

$$|W_u(p, y, v)| \leqq N_0(1 + W(p, u, v))$$

for all $v \in (0, \infty)$ and $p \in \mathbb{R}$.

H6 $W(p, u, v)$ is an even function of p for every $u, v \in (0, \infty)$.

The smoothness assumption H1 can be made weaker. In H2, the first, third and fourth inequalities are consequences of the three-dimensional condition of strong ellipticity for Φ (see [7]), whilst the second inequality follows from the strong-ellipticity condition for two-dimensional plane-strain elasticity [4, p. 97]. Hypotheses H3 and H4 are one-dimensional analogues of the natural growth conditions

$$\Phi(F) \to \infty \quad \text{as } \det F \to 0,$$

$$\Phi(F) \to \infty \quad \text{as } |F| \to \infty$$

(*cf.* (1.3)). Here, $|\ |$ denotes the Euclidean norm on the space of 3×3 matrices. Hypotheses H5 is a technical requirement to ensure that the solutions of $P_{r\gamma}$ are smooth. The remaining hypothesis, H6, is an immediate consequence of isotropy and objectivity; it means the response of the bar is invariant under reversal of the x-axis [3].

Theorem 2.1. *For each $\gamma > 1$ and $r > 0$, a solution (ϱ, ε) of $(P_{r\gamma})$ exists. Furthermore, $\varrho \in C^2([-1, 1])$, $\varepsilon \in C^1([-1, 1])$, $\varrho \geqq k > 0$, $\varepsilon \geqq k > 0$ and ϱ, ε satisfy the Euler-Lagrange equations*

$$r\frac{d}{dx}[W_p(r\varrho'(x), \varrho(x), \varepsilon(x))] = W_u(r\varrho'(x), \varrho(x), \varepsilon(x)), \tag{2.1}$$

$$W_v(r\varrho'(x), \varrho(x), \varepsilon(x)) = \lambda, \tag{2.2}$$

where λ is a constant, and the boundary conditions

$$\varrho'(-1) = \varrho'(1) = 0. \tag{2.3}$$

The constant λ is just the Lagrange multiplier of classical isoperimetric problems of the calculus of variations. If $(\hat{\varrho}, \hat{\varepsilon}) \in \mathfrak{A}_{r\gamma}$ satisfies (2.1), (2.2) and (2.3), then we say $(\hat{\varrho}, \hat{\varepsilon})$ is an *extremal* of I_r.

This result is a consequence of the existence and regularity of solutions in the general one-dimensional theory of nonlinear elasticity given in [7]. The regularity part is subtle since we cannot use the standard methods [12] directly; the singular behaviour of $W(p, u, v)$ as $u \to 0$ or $v \to 0$ means that it is not obvious we can pass to the limit in

$$\frac{d}{d\tau} \int_{-1}^{1} [W(r(\varrho' + \tau\eta_1'), \varrho + \tau\eta_1, \varepsilon + \tau\eta_2) - W(r\varrho', \varrho, \varepsilon)]\, dx|_{\tau=0} = 0$$

to get

$$\int_{-1}^{1} [W_p(r\varrho', \varrho, \varepsilon)\, r\eta_1' + W_u(r\varrho', \varrho, \varepsilon)\, \eta_1 + W_v(r\varrho', \varrho, \varepsilon)\, \eta_2]\, dx = 0.$$

We shall give a proof of the regularity part of Theorem 2.1. Since we are considering a functional I_r simpler than that studied in [7] we can give a more direct proof of regularity. Our proof will also use some techniques of [10].

Proof of regularity. Without loss of generality, set $r = 1$. We show, initially, that the minimizer satisfies the weak Euler-Lagrange equations almost everywhere (*cf.* [10]).

Let $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$ be a minimizer of I_1 and define sets $\Omega_n = \left\{x \in [-1, 1] : \varrho(x) > \frac{1}{n}\right\}$, $n = 1, 2, \ldots$. Since ϱ is continuous (by the Sobolev embedding theorem) the Ω_n are open and can be written as the countable union of disjoint, open intervals $\Omega_n = \bigcup_{m=1}^{\infty} (a_{n,m}, b_{n,m})$. Let $(a_{n,k}, b_{n,k})$ be a typical component interval and suppose $a_{n,k} \to a_k$, $b_{n,k} \to b_k$ as $n \to \infty$ for $a_k, b_k \in [-1, 1]$. By continuity $\varrho(a_k) = \varrho(b_k) = 0$. Fix n and let

$$M_m = \left\{x \in (a_{n,k}, b_{n,k}) : |\varrho'(x)| < m \quad \text{or} \quad \frac{1}{m} < \varepsilon(x) < m\right\}, \quad m = 1, 2, \ldots.$$

Choose $u_m, v_m \in L^\infty(\Omega_n)$ such that $\int_{M_m} u_m(x)\, dx = \int_{M_m} v_m(x)\, dx = 0$ and $u_m(x) = 0$ for $x \in M_m^c$.

Let

$$\varrho_\tau(x) = \varrho(x) + \tau \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy,$$

$$\varepsilon_\tau(x) = \varepsilon(x) + \tau\chi_m(x)\, v_m(x)$$

where χ_m is the characteristic function for M_m. Clearly, ε_τ satisfies (1.1) and for $|\tau|$ sufficiently small $(\varrho_\tau, \varepsilon_\tau) \in \mathfrak{A}_{r\gamma}$.

By the definitions of M_m, ϱ_τ and ε_τ, we have

$$\frac{I_1(\varrho_\tau, \varepsilon_\tau) - I_1(\varrho, \varepsilon)}{\tau} = \frac{1}{\tau} \int_{M_m} \left[W\left(\varrho' + \tau u_m, \varrho + \tau \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy, \varepsilon + \tau v_m\right) \right.$$
$$\left. - W\left(\varrho', \varrho + \tau \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy, \varepsilon\right)\right] dx$$
$$+ \frac{1}{\tau} \int_{a_{n,k}}^{b_{n,k}} \left[W\left(\varrho', \varrho + \tau \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy, \varepsilon\right) \right.$$
$$\left. - W(\varrho', \varrho, \varepsilon)\right] dx. \tag{2.4}$$

Since u_m and v_m are essentially bounded, our construction of M_m implies we can use the dominated convergence theorem to show that the first integral in (2.4) approaches

$$\int_{M_r} [W_p(\varrho', \varrho, \varepsilon)\, u_m + W_v(\varrho', \varrho, \varepsilon)\, v_m]\, dx \tag{2.5}$$

as $\tau \to 0$. Similarly, by H5 and the mean value theorem we have that the second integral in (2.4) is bounded by

$$\int_{a_{n,k}}^{b_{n,k}} \left[W(\varrho', \varrho, \varepsilon) \cdot \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy \right] dx \tag{2.6}$$

for τ sufficiently small. Since $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$ and $u_m \in L^\infty$, (2.6) is finite, so the dominated convergence theorem implies that the second integral in (2.4) has the limit

$$\int_{a_{n,k}}^{b_{n,k}} \left[W_u(\varrho', \varrho, \varepsilon) \cdot \int_{a_{n,k}}^{x} \chi_m(y)\, u_m(y)\, dy \right] dx \tag{2.7}$$

as $\tau \to 0$.

From (2.6) and (2.7), it follows that $\lim_{\tau \to 0} \frac{I_1(\varrho_\tau, \varepsilon_\tau) - I_1(\varrho, \varepsilon)}{\tau}$ exists, and since (ϱ, ε) is a minimizer of I_1 this limit must be zero. Hence, by using (2.6) and integrating (2.7) by parts we conclude that

$$\int_{M_m} \left[W_p(\varrho', \varrho, \varepsilon)\, u_m + W_v(\varrho', \varrho, \varepsilon)\, v_m - \int_{a_{n,k}}^{x} W_u(\varrho'(y), \varrho(y), \varepsilon(y))\, dy \cdot u_m \right] dx = 0.$$

Since u_m, v_m are arbitrary, this equation implies that

$$W_p(\varrho'(x), \varrho(x), \varepsilon(x)) = \int_{a_{n,k}}^{x} W_u(\varrho'(y), \varrho(y), \varepsilon(y))\, dy + c_{n,m}, \tag{2.8}$$

$$W_v(\varrho'(x), \varrho(x), \varepsilon(x)) = \lambda_{n,m} \tag{2.9}$$

for almost every $x \in M_m$, where $c_{n,m}$, $\lambda_{n,m}$ are constants. Furthermore, since $M_m \subset M_{m+1}$ and $\operatorname{meas}\left((a_{n,k}, b_{n,k}) \setminus \bigcup_{m=1}^{\infty} M_m\right) = 0$, it follows that $c_{n,m} \equiv c_n$ and $\lambda_{n,m} \equiv \lambda_n$ are independent of m and (2.8), (2.9) hold for almost every $x \in (a_{n,k}, b_{n,k})$. Finally, if we let $n \to \infty$, then an argument similar to that above implies that $c_n \equiv c$ and $\lambda_n \equiv \lambda$ are independent of n and (2.8), (2.9) hold for almost every $x \in (a_k, b_k)$.

Since the right-hand side of (2.9) is uniformly bounded (trivially), H3 and H4 imply that $\operatorname{ess\,inf}_{[a_k,b_k]} \varepsilon > 0$ and $\operatorname{ess\,sup}_{[a_k,b_k]} \varepsilon < \infty$. Likewise, (2.8) shows that ϱ' is essentially bounded on every compact subset of (a_k, b_k). Note that by construction the integrand in (2.8) is singular at a_k and b_k. From the arbitrary choice of (a_k, b_k) we deduce that $\operatorname{ess\,inf}_{[-1,1]} \varepsilon > 0$ and $\varepsilon \in L^\infty(-1, 1)$. Hence, after re-defining ε on a set of measure zero we find that there is a number k such that $\varepsilon(x) \geqq k > 0$ for all $x \in [-1, 1]$.

We now show by contradiction that $\varrho(x) > 0$ for all $x \in [-1, 1]$. Suppose $\varrho(\bar{x}) = 0$ for some $\bar{x} \in [-1, 1]$. Let Λ_n be the maximal connected open interval such that $\varrho(x) < \frac{1}{n}$ on Λ_n and $\bar{x} \in \Lambda_n$. Let

$$\bar{\varrho}(x) = \begin{cases} \varrho(x) & \text{for } x \notin \Lambda_n, \\ \dfrac{1}{n} & \text{for } x \in \Lambda_n. \end{cases}$$

Then $\bar{\varrho} \in W^{1,1}(-1, 1)$, $\bar{\varrho}(x) > 0$ a.e. and $(\bar{\varrho}, \varepsilon) \in \mathfrak{A}_{1\gamma}$. Since $W(p, u, v)$ is even and convex in p,

$$I_1(\bar{\varrho}, \varepsilon) - I_1(\varrho, \varepsilon) = \int_{\Lambda_n} \left[W\left(0, \frac{1}{n}, \varepsilon(x)\right) - W(\varrho'(x), \varrho(x), \varepsilon(x)) \right] dx$$

$$\leqq \int_{\Lambda_n} \left[W\left(0, \frac{1}{n}, \varepsilon(x)\right) - W(0, \varrho(x), \varepsilon(x)) \right] dx. \qquad (2.10)$$

Since ε is bounded away from 0 and ∞, H2 and H3 imply that there is an $N_1 > 0$ such that the integrand in (2.10) is strictly negative for all $n > N_1$, contradicting the fact that (ϱ, ε) is a minimizer.

Thus, since ϱ and ε are bounded away from zero, we can now apply standard arguments [12] (*cf.* (2.3)) to show that ϱ and ε are smooth and that (2.8), (2.9) hold for every $x \in [-1, 1]$.

The natural boundary conditions ([15]) for I_1 are

$$W_p(\varrho'(-1), \varrho(-1), \varepsilon(-1)) = W_p(\varrho'(1), \varrho(1), \varepsilon(1)) = 0.$$

The boundary conditions (2.3) then follow since W is even in p. □

The test function $\bar{\varrho}$ can be thought of as lifting $\varrho(\bar{x})$ away from zero and thus lowering the energy.

3. Solutions of the Euler-Lagrange equations

The Lagrange multiplier λ in (2.2) is determined by the constraint (1.1). In this section we shall ignore (1.1), consider λ as a parameter and construct the phase-plane for (2.1), (2.2) for various values of λ. Our construction will combine the approaches in [5] and [11], and will illustrate the underlying similarities between the two works.

Consider the algebraic equation $W_u(0, u, v) = 0$. By H2, H3 and the implicit function theorem, this equation can be solved for u as a function of v. That is, there exists a C^1-function $\bar{\varrho}: (0, \infty) \to (0, \infty)$ such that

$$W_u(0, \bar{\varrho}(v), v) = 0 \tag{3.1}$$

for all $u \in (0, \infty)$. A bootstrap argument shows that $\bar{\varrho}$ is smooth.

In addition to H1–H6 we assume that W satisfies

H7 There are numbers $0 < \bar{\alpha} < \theta < \underline{\beta} < \infty$ such that the function $g(v) \overset{\text{def.}}{=} W_v(0, \bar{\varrho}(v), v) = 0$ satisfies

$$g(\bar{\alpha}) > g(\theta) > g(\underline{\beta}) > 0,$$
$$g'(v) > 0 \quad \text{for } v \in (0, \bar{\alpha}) \cup (\underline{\beta}, \infty),$$
$$g'(v) < 0 \quad \text{for } v \in (\bar{\alpha}, \underline{\beta}),$$
$$g''(v) < 0 \quad \text{for } v \in (0, \theta),$$
$$g''(v) > 0 \quad \text{for } v \in (\theta, \infty).$$

H8 $W_{uv}(p, u, v) > 0$ for all $u, v \in (0, \infty)$ and for all $p \in \mathbb{R}$.

The graph of $W_v(0, \varrho(v), v)$ against v (Figure 1) has the familiar non-monotone shape associated with many studies of phase transitions in fluids and solids (see [1], [14]). Clearly, H7 implies that $W(0, u, v)$ is not convex. In [5], $W(0, u, v)$ is assumed to be non-convex but no explicit assumption such as H7 is made.

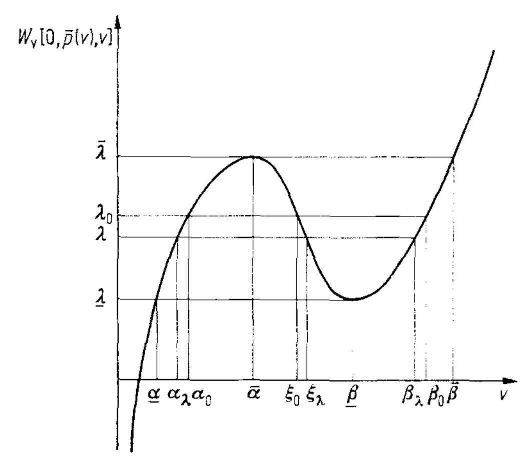

Fig. 1

From H8 and differentiation of (3.1) we see that $\bar{\varrho}(v)$ is a monotone decreasing function. We note that H8 is not a consequence of strong ellipticity. It has the physical interpretation that for a bar deforming homogeneously in uniaxial tension the radius of the bar decreases as the length increases.

Let $\underline{\lambda} = W_v(0, \bar{\varrho}(\underline{\beta}), \underline{\beta})$ and $\bar{\lambda} = W_v(0, \bar{\varrho}(\bar{\alpha}), \bar{\alpha})$ where $\bar{\alpha}, \underline{\beta}$ are given in H7. For $\lambda \in (\underline{\lambda}, \bar{\lambda})$, the equation

$$W_v(0, \bar{\varrho}(v), v) = \lambda \tag{3.2}$$

has three distinct solutions which we denote $\alpha_\lambda < \xi_\lambda < \beta_\lambda$ (*cf.* Figure 1). For $\lambda = \bar{\lambda}$, (3.2) has two solutions; one is $\bar{\alpha}$ and the other we denote $\bar{\beta}$. Likewise, for $\lambda = \underline{\lambda}$, (3.2) has solution $v = \underline{\beta}$ and another one we denote $\underline{\alpha}$. If $\lambda \notin [\underline{\lambda}, \bar{\lambda}]$, (3.2) has a unique solution c_λ.

We define the *Maxwell triplet* $(\alpha_0, \beta_0, \lambda_0)$ (*cf.* [14]) as the unique solution of the algebraic system

$$\begin{aligned} W(0, \bar{\varrho}(\beta), \beta) - W(0, \bar{\varrho}(\alpha), \alpha) &= \lambda(\beta - \alpha), \\ \lambda = W_v(0, \bar{\varrho}(\beta), \beta) &= W_v(0, \bar{\varrho}(\alpha), \alpha). \end{aligned} \tag{3.3}$$

Since $\frac{d}{dv} W(0, \bar{\varrho}(v), v) = W_v(0, \bar{\varrho}(v), v)$ (by definition of $\bar{\varrho}$), the geometrical interpretation of the Maxwell triplet is the usual equal-area rule: the area above the line $\lambda = \lambda_0$ and below the curve $W_v(0, \bar{\varrho}(v), v)$ equals the area below $\lambda = \lambda_0$ and above $W_v(0, \bar{\varrho}(v), v)$ (Figure 1).

By H2, H3 and the implicit function theorem there exists a C^1-function $\hat{\varepsilon} : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to (0, \infty)$ such that

$$\varepsilon = \hat{\varepsilon}(r\varrho', \varrho, \lambda) \tag{3.4}$$

satisfies (2.2). A bootstrap argument shows that $\hat{\varepsilon}$ is in C^∞.

Equations (2.1), (2.2) can now be written as the first-order system

$$\begin{aligned} \varrho' &= \eta, \\ \eta' &= \frac{\hat{W}_u - (\hat{W}_{pu} - \hat{W}_{pv}\hat{W}_{vu}/\hat{W}_{vv})\,\eta}{r(\hat{W}_{pp} - \hat{W}_{pv}^2/\hat{W}_{vv})} \end{aligned} \tag{3.5}$$

where $\hat{W} = \hat{W}(r\eta, \varrho, \hat{\varepsilon}(r\eta, \varrho, \lambda))$.

Proposition 3.1. (*cf.* [5])

(i) *For* $\lambda \in (-\infty, \underline{\lambda}) \cup (\bar{\lambda}, \infty)$, (3.5) *has one fixed point which is a saddle.*

(ii) *For* $\lambda \in (\underline{\lambda}, \bar{\lambda})$, (3.5) *has three distinct fixed points* $(\bar{\varrho}(\beta_\lambda), 0)$, $(\bar{\varrho}(\xi_\lambda), 0)$ *and* $(\bar{\varrho}(\alpha_\lambda), 0)$. *The first and third of these fixed points are saddles and the second is a centre.*

Proof. The fixed points of (3.5) are $(p, \eta) = (\varrho, 0)$ where ϱ satisfies $W_u(0, \varrho, \hat{\varepsilon}(0, \varrho, \lambda)) = 0$. They correspond to solutions (ϱ, ε) of

$$\begin{aligned} W_u(0, \varrho, \varepsilon) &= 0, \\ W_v(0, \varrho, \varepsilon) &= \lambda. \end{aligned} \tag{3.6}$$

The result then follows from the definitions of $\bar{\varrho}(\cdot)$, $\underline{\lambda}$ and $\bar{\lambda}$. □

We note that (3.6) are the equilibrium equations for a bar deforming homogeneously under an applied load λ. Proposition 3.1 says that for sufficiently high loads there is more than one homogeneous solution.

The system (3.5) has first integral

$$\begin{aligned} f(r\varrho', \varrho, \lambda) \stackrel{\text{def.}}{=} W_p(r\varrho', \varrho, \hat{\varepsilon}(r\varrho', \varrho, \lambda))\, r\varrho' + \lambda\hat{\varepsilon}(r\varrho', \varrho, \lambda) \\ - W(r\varrho', \varrho, \hat{\varepsilon}(r\varrho', \varrho, \lambda)) = E \end{aligned} \tag{3.7}$$

for constant E. Following the ideas in [5], we set $2\omega = (r\varrho')^2$, $V(\omega, u, v) = W(r\varrho', u, v)$, $\hat{\sigma}(\omega, u, \lambda) = \hat{\varepsilon}(r\varrho', u, \lambda)$ and $s(\omega, u, \lambda) = f(r\varrho', u, \lambda)$. Since W is even in $r\varrho'$ by (H6) it follows that f is even in $r\varrho'$, and ω, V, $\hat{\sigma}$ and s are well defined. Substituting these new variables into (3.7) gives

$$\begin{aligned} s(\omega, \varrho, \lambda) = 2\omega V_\omega(\omega, \varrho, \hat{\sigma}(\omega, \varrho, \lambda)) + \lambda\hat{\sigma}(\omega, \varrho, \lambda) \\ - V(\omega, \varrho, \hat{\sigma}(\omega, \varrho, \lambda)) = E. \end{aligned} \tag{3.8}$$

We shall show that (3.8) can be solved for ω.

The *Gibbs function*, $\mathfrak{G}(\lambda, u)$, for W (*cf.* [11]) is defined as

$$\mathfrak{G}(\lambda, u) = W(0, u, \hat{\varepsilon}(0, u, \lambda)) - \lambda\hat{\varepsilon}(0, u, \lambda). \tag{3.9}$$

Proposition 3.2. *The function* $\mathfrak{G}(\cdot, \cdot)$ *is a smooth mapping from* $\mathbb{R} \times (0, \infty) \to \mathbb{R}$. *For* $\lambda \in (-\infty, \underline{\lambda}) \cup (\bar{\lambda}, \infty)$, $\mathfrak{G}(\lambda, \cdot)$ *has exactly one critical point* $\bar{\varrho}(c_\lambda)$. *For* $\lambda \in (\underline{\lambda}, \bar{\lambda})$, $\mathfrak{G}(\lambda, \cdot)$ *has three critical points* $\bar{\varrho}(\alpha_\lambda) > \bar{\varrho}(\xi_\lambda) > \bar{\varrho}(\beta_\lambda)$ *and*

(i) $\mathfrak{G}(\lambda, \cdot)$ *is strictly decreasing on* $(0, \bar{\varrho}(\beta_\lambda)) \cup (\bar{\varrho}(\xi_\lambda), \bar{\varrho}(\alpha_\lambda))$ *and strictly increasing on* $(\bar{\varrho}(\beta_\lambda), \bar{\varrho}(\xi_\lambda)) \cup (\bar{\varrho}(\alpha_\lambda), \infty)$,

(ii) $\mathfrak{G}(\lambda, \bar{\varrho}(\beta_\lambda)) > \mathfrak{G}(\lambda, \bar{\varrho}(\alpha_\lambda))$ *for* $\lambda < \lambda_0$, $\mathfrak{G}(\lambda, \bar{\varrho}(\alpha_\lambda)) > \mathfrak{G}(\lambda, \bar{\varrho}(\beta_\lambda))$ *for* $\lambda > \lambda_0$, *and*

$$\mathfrak{G}(\lambda_0, \bar{\varrho}(\alpha_0)) = \mathfrak{G}(\lambda_0, \bar{\varrho}(\beta_0)). \tag{3.10}$$

Proof. Regularity of $\mathfrak{G}$ follows from H1, whilst the critical points are given by solutions to (3.6).

To prove (i) and (ii) we make the change of variables $u = \bar{\varrho}(v)$, so that

$$\bar{\mathfrak{G}}(\lambda, v) \stackrel{\text{def.}}{=} \mathfrak{G}(\lambda, \bar{\varrho}(v)) = W(0, \bar{\varrho}(v), v) - \lambda v.$$

Since $\bar{\varrho}$ is a decreasing function, if $\bar{\mathfrak{G}}(\lambda, \cdot)$ is strictly decreasing on $(0, \alpha_\lambda)$, then $\mathfrak{G}(\lambda, \cdot)$ is strictly increasing on $(\bar{\varrho}(\alpha_\lambda), \infty)$. From (3.1) and for $v_1, v_2 \in (0, \alpha_\lambda)$

$$\bar{\mathfrak{G}}(\lambda, v_2) - \bar{\mathfrak{G}}(\lambda, v_1) = \int_{v_1}^{v_2} [W_v(0, \bar{\varrho}(v), v) - \lambda]\, dv$$

$$= \int_{v_1}^{v_2} [W_v(0, \bar{\varrho}(v), v) - W_v(0, \bar{\varrho}(\alpha_\lambda), \alpha_\lambda)]\, dx. \tag{3.11}$$

For $v \in (0, \bar{\alpha})$, H7 implies that the integrand in (3.11) is strictly negative and hence $\bar{\mathfrak{G}}(\lambda, \cdot)$ is strictly decreasing. The remainder of (i) follows similarly.

Now let $v_1 = \alpha_\lambda$, $v_2 = \beta_\lambda$ in (3.11). For $\lambda = \lambda_0$, (3.10) is just a restatement of (3.3). For $\lambda = \bar{\lambda}$, the integrand in (3.11) is negative for $v \in (\bar{\alpha}, \bar{\beta})$ (*cf.* Figure 1) and this gives the second inequality in (ii). A similar argument holds for $\lambda < \lambda_0$. □

Differentiating (3.8) with respect to ω, changing back to the original variable $\delta\varrho'$ and using H2 shows that

$$s_\omega(\omega, \varrho, \lambda) > 0. \tag{3.12}$$

Let $\lambda \in (\underline{\lambda}, \bar{\lambda})$. Since $s(0, \varrho, \lambda) = -\mathfrak{G}(\lambda, \varrho)$, it follows that $s(0, \varrho, \lambda) \to \infty$ as $\varrho \to 0, \infty$ by H3 and H4. Proposition 3.2 then implies that the equation

$$s(0, \varrho, \lambda) = E$$

has at most four solutions $0 < \varrho_0(\lambda, E) \leqq \varrho_-(\lambda, E) < \varrho_+(\lambda, E) \leqq \varrho_1(\lambda, E) < \infty$ for λ and E satisfying

$$\left.\begin{aligned} &\underline{\lambda} < \lambda \leqq \lambda_0 \quad \text{and } \mathfrak{G}(\lambda, \bar{\varrho}(\beta_\lambda)) \leqq -E \leqq \mathfrak{G}(\lambda, \bar{\varrho}(\xi_\lambda)), \\ &\lambda_0 < \lambda \leqq \bar{\lambda} \quad \text{and } \mathfrak{G}(\lambda, \bar{\varrho}(\alpha_\lambda)) \leqq -E \leqq \mathfrak{G}(\lambda, \bar{\varrho}(\xi_\lambda)), \end{aligned}\right\} \tag{3.13}$$

(see Figure 2).

In addition,

$$s(0, \varrho, \lambda) > E \quad \text{for } \varrho \in (\varrho_0, \varrho_-) \cup (\varrho_+, \varrho_1),$$

$$s(0, \varrho, \lambda) < E \quad \text{for } \varrho \in (0, \varrho_0) \cup (\varrho_-, \varrho_+) \cup (\varrho_1, \infty).$$

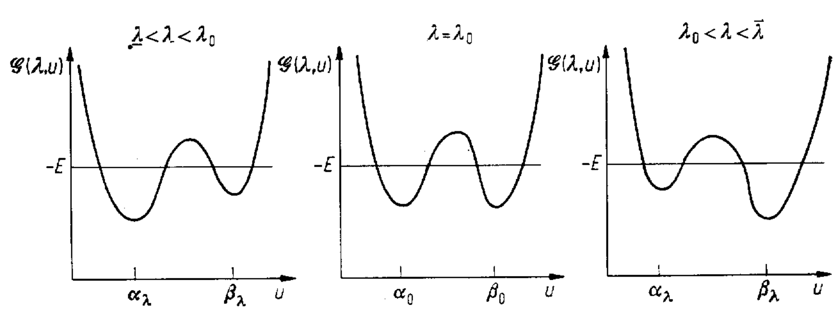

Fig. 2

Hence, by (3.12) and H4 there exists a unique $\omega \in [0, \infty)$ such that given $\varrho \in [\varrho_-, \varrho_+]$.

$$s(\omega, \varrho, \lambda) = E. \tag{3.14}$$

Let $\mathfrak{D}_{\lambda,E} = \{(\varrho, \lambda, E) : \lambda \in (\underline{\lambda}, \bar{\lambda}),\ E \text{ satisfies (3.13)}, |\varrho| \in [\varrho_-(\lambda, E), \varrho_+(\lambda, E)]\}$. Then we can define a function $H: \mathfrak{D}_{\lambda,E} \to [0, \infty)$ implicitly by

$$H(\varrho, \lambda, E) = 2\omega,$$

where ω is the unique solution of (3.14). Thus, the first integral (3.7) can now be written

$$r^2(\varrho')^2 = H(\varrho, \lambda, E) \tag{3.15}$$

for $\varrho \in [\varrho_-(\lambda, E), \varrho_+(\lambda, E)]$.

Proposition 3.3. *The function* $H: \mathfrak{D}_{\lambda,E} \to [0, \infty)$ *is smooth. Furthermore, for* $\lambda \in (\underline{\lambda}, \bar{\lambda})$ *and E satisfying* (3.13)

$$\begin{aligned} H(\varrho_-(\lambda, E), \lambda, E) &= H(\varrho_+(\lambda, E), \lambda, E) = 0, \\ H(\varrho, \lambda, E) > 0 \quad &\text{for } \varrho \in (\varrho_-(\lambda, E), \varrho_+(\lambda, E)). \end{aligned} \tag{3.16}$$

In addition, for $\lambda \in (\underline{\lambda}, \lambda_0)$

$$\varrho_+(\lambda, -\mathfrak{G}(\lambda, \bar{\varrho}(\beta_\lambda))) = \bar{\varrho}(\beta_\lambda),\ H_\varrho(\bar{\varrho}(\beta_\lambda), \lambda, -\mathfrak{G}(\lambda, \bar{\varrho}(\beta_\lambda))) = 0, \tag{3.17}$$

whilst for $\lambda \in (\lambda_0, \bar{\lambda})$

$$\varrho_+(\lambda, -\mathfrak{G}(\lambda, \bar{\varrho}(\alpha_\lambda))) = \bar{\varrho}(\alpha_\lambda),\ H_\varrho((\bar{\varrho}(\alpha_\lambda), \lambda, -\mathfrak{G}(\lambda, \bar{\varrho}(\alpha_\lambda))) = 0. \tag{3.18}$$

If $\lambda = \lambda_0$ *and* $E_0 \overset{\text{def}}{=} -\mathfrak{G}(\lambda, \bar{\varrho}(\alpha_0))$, *then*

$$\begin{aligned} \varrho_-(\lambda_0, E_0) &= \bar{\varrho}(\beta_0), \varrho_+(\lambda_0, E_0) = \bar{\varrho}(\alpha_0), \\ H_\varrho(\bar{\varrho}(\beta_0), \lambda_0, E_0) &= H_\varrho(\bar{\varrho}(\alpha_0), \lambda_0, E_0) = 0. \end{aligned} \tag{3.19}$$

Proof. The smoothness of H follows from the regularity of W and H2. The definition of H and (3.13) imply (3.16). The first part of (3.17) follows from the definitions of $\mathfrak{G}(\lambda, \varrho)$ and α_λ.

By substituting (3.15) into (3.7) and differentiating it with respect to ϱ we get

$$H_\varrho(\varrho, \lambda, E) = \frac{2W_u - 2H^{\frac{1}{2}}(\varrho, \lambda, E)\,[W_{pu} - W_{pv}W_{vu}/W_{vv}]}{W_{pp} - W_{pv}^2/W_{vv}}$$

where W is evaluated at $(H^{\frac{1}{2}}(\varrho, \lambda, E), \varrho, \hat{\varepsilon}(H^{\frac{1}{2}}(\varrho, \lambda, E), \varrho, \lambda))$. The second part of (3.17) then follows from (3.16) and (3.6).

The equalities (3.18), (3.19) are proved in the same way. □

From Propositions 3.1 and 3.2 we have enough information about H to construct the phase portrait for (3.5) from (3.15) for various values of λ, E and $\varrho \in [\varrho_-(\lambda, E), \varrho_+(\lambda, E)]$. Figure 3 (*cf.* [11]) shows the phase diagrams for $\lambda \in (\underline{\lambda}, \bar{\lambda})$

and E satisfying (3.13). The significant features of the phase diagram are the heteroclinic and homoclinic orbits enclosing a set of closed orbits. Given λ and E (satisfying (3.13)), (3.15) defines a unique orbit which crosses the $\varrho' = 0$ axis at the points $\varrho = \varrho_-(\lambda, E)$ and $\varrho = \varrho_+(\lambda, E)$.

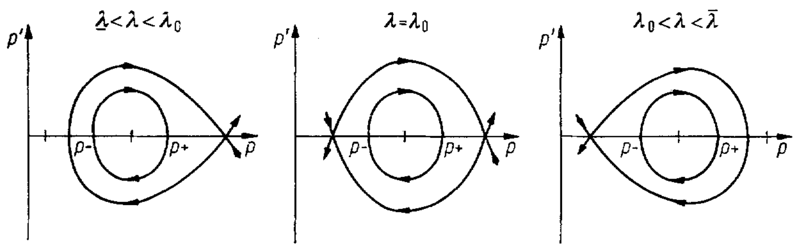

Fig. 3

To solve (3.5) and the boundary condition (2.3) we need to show that there is a trajectory starting and finishing on the η-axis with the time taken to traverse this trajectory equal to 2. (We now interpret the x variable as time.) Obviously, the constant solutions $\varrho \equiv \bar{\varrho}(\beta_\lambda)$, $\bar{\varrho}(\varrho_\lambda)$ and $\bar{\varrho}(\alpha_\lambda)$ satisfy these conditions. Hence, $(\varrho, \varepsilon) = (\bar{\varrho}(\beta_\lambda), \beta_\lambda)$, $(\bar{\varrho}(\xi_\lambda), \xi_\lambda)$ and $(\bar{\varrho}(\alpha_\lambda), \alpha_\lambda)$ are solutions of (2.1), (2.2) and (2.3).

As can be seen from figure 3, the closed orbits lying inside the separatrices connecting critical points satisfy (2.3). Consider a trajectory that starts at $\varrho_-(\lambda, E)$, traverses half an orbit and finishes at $\varrho = \varrho_+(\lambda, E)$. The time taken to traverse this half orbit is given by the time map

$$T(\varrho_-(\lambda, E)) = \int_{\varrho_-(\lambda,E)}^{\varrho_+(\lambda,E)} H(\varrho, \lambda, E)^{-\frac{1}{2}}\, d\varrho .$$

Standard results ([16]) show that $T(\varrho)$ is continuous on $(\bar{\varrho}(\beta_\lambda), \bar{\varrho}(\xi_\lambda))$ and

$$\lim_{\varrho \searrow \bar{\varrho}(\beta_\lambda)} T(\varrho) = +\infty, \tag{3.20}$$

i.e., it takes an infinite time to traverse a connecting orbit. Furthermore, a simple calculation based on Proposition 3.2 and (3.14) shows that

$$\lim_{\varrho \nearrow \bar{\varrho}(\xi_\lambda)} T(\varrho) = \pi r \left(\frac{-W_{vv} W_{pp}}{W_{uu} W_{vv} - W_{vu}^2} \right)^{\frac{1}{2}} \tag{3.21}$$

where the partial derivatives are evaluated at $(0, \bar{\varrho}(\xi_\lambda), \xi_\lambda)$. The continuity of T and the limits (3.20), (3.21) then imply that for r sufficiently small there exists at least one solution of $T(\varrho_-) = 2$, *i.e.*, there exists a non-constant solution of (3.5) and (2.3). Since the phase portrait is symmetric about the $\varrho' = 0$ axis ((3.15)) the trajectory starting at ϱ_+ and finishing at ϱ_- is also a solution. If $\varrho(x)$ is a solution of this type (*i.e.*, a monotone function), then the corresponding solution (ϱ, ε) of (2.1), (2.2) and (2.3) we shall call *monotone*. Since $I_r(\varrho(-x), \varepsilon(-x)) = I_r(\varrho(x), \varepsilon(x))$ there is no loss in generality in assuming that a monotone solution has $\varrho(x)$ decreasing.

We now consider trajectories that traverse the whole orbit one or more times. As before, consider a trajectory starting at $\varrho_-(\lambda, E)$. Let N be the number of times that the trajectory crosses the $\varrho' = 0$ axis not including the starting point (the monotone solution has $N = 1$). The time taken, $T_N(\varrho_-)$, to traverse this trajectory is just an integer multiple of $T(\varrho_-)$:

$$T_N(\varrho_-) = NT(\varrho_-) \tag{3.22}$$

(*cf.* [11]).

As above we see that if r is sufficiently small, then the equation $T(\varrho) = 2/N$ will have a solution. Hence there is a trajectory satisfying $T_N(\varrho) = 2$. The corresponding solution (ϱ, ε) of (2.1), (2.2) and (2.3) we shall call *non-monotone*.

In [5], it is conjectured that the *monotone* solutions represent drawing deformations whilst *non-monotone* solutions represent necking deformations. The constant solutions of (2.1), (2.2) and (2.3) correspond to homogeneous deformations.

The remainder of the phase portrait for $\lambda \in (\underline{\lambda}, \bar{\lambda})$ can be constructed using techniques similar to the above. The significant features of trajectories lying outside the connecting orbits are that they are unbounded and cross the $\varrho' = 0$ axis at most once. Hence, none of these trajectories satisfies (2.3).

For $\lambda \notin (\underline{\lambda}, \bar{\lambda})$ there is exactly one fixed point (Proposition 3.1) and it is straightforward to show that all other trajectories are unbounded and cross the $\varrho' = 0$ axis at most once. Thus, the only solution of (2.1), (2.2) and (2.3) is $(\varrho, \varepsilon) = (\bar{\varrho}(c_\lambda), c_\lambda)$.

If $\lambda = \underline{\lambda}, \bar{\lambda}$ the phase portrait has a structure slightly different from that described above. However, we shall see in the next section that stable non-constant solutions cannot have $\lambda = \underline{\lambda}$ or $\bar{\lambda}$.

We note that if $W_v(0, \bar{\varrho}(v), v)$ is monotone (*i.e.*, if H7 is not satisfied), then (3.5) has one fixed point for all values of λ and the only solution of (2.1), (2.2) and (2.3) is $\varrho = \text{constant}$, $\varepsilon = \text{constant}$. We conclude that in this case there is no non-homogeneous behaviour of the form (1.2).

Finally, we state a lemma which will be used in section 5. The proof of this is the same as that for the equivalent result in [11].

Lemma 3.4. *If* λ, E *satisfies* (3.13), *then*

$$\begin{gathered} \bar{\varrho}(\beta_0) < \varrho_-(\lambda, E) < \varrho_+(\lambda, E) < \bar{\varrho}(\alpha_0), \\ \alpha_0 < \hat{\varepsilon}(0, \varrho_+(\lambda, E), \lambda) < \hat{\varepsilon}(0, \varrho_-(\lambda, E), \lambda) < \beta_0. \end{gathered} \tag{3.23}$$

4. Minimizers of I_r

As stated, problem $(P_{r\gamma})$ refers only to global minimizers of I_r. We are also interested in local minimizers. To be precise, we say that $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$ is a *strong local minimizer* of I_r if there exists $\mu > 0$ such that

$$I_r(\varrho, \varepsilon) \leqq I_r(\varrho_1, \varepsilon_1) \tag{4.1}$$

whenever $(\varrho_1, \varepsilon_1) \in \mathfrak{A}_{r\gamma}$ and $\sup_{[-1,1]} |\varrho_1(x) - \varrho(x)| < \mu$. If (4.1) holds for $(\varrho_1, \varepsilon_1) \in \mathfrak{A}_{r\gamma}$ satisfying $\sup_{[-1,1]} |\varrho_1(x) - \varrho(x)| + \operatorname{ess\,sup}_{[-1,1]} |\varrho_1'(x) - \varrho'(x)| + \operatorname{ess\,sup}_{[-1,1]} |\varepsilon_1(x) - \varepsilon(x)| < \mu$, then we say that $(\varrho_1, \varepsilon_1)$ is a *weak local minimizer*. Clearly, if (ϱ, ε) is a strong local minimizer, then it is a weak local minimizer. The converse is false.

Let

$$\mathfrak{v} = \left\{(\eta_1, \eta_2) : \eta_1 \in W^{1,2}(-1, 1), \eta_2 \in L^2(-1, 1), \int_{-1}^{1} \eta_2(x)\, dx = 0\right\}.$$

For $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$, the second variation of I_r at (ϱ, ε) is

$$\begin{aligned} V_2(\eta_1, \eta_2, \varrho, \varepsilon) &\overset{\text{def.}}{=} \frac{d^2}{d\tau^2} I_r(\varrho + \tau\eta_1, \varepsilon + \tau\eta_2)|_{\tau=0} \\ &= \int_{1}^{1} [\hat{W}_{pp} r^2(\eta_1')^2 + \hat{W}_{uu}\eta_1^2 + 2\hat{W}_{uv}\eta_1\eta_2 + 2\hat{W}_{pv} r\eta_1'\eta_2 \\ &\quad + 2\hat{W}_{pu} r\eta_1'\eta_1 + \hat{W}_{vv}\eta_2^2]\, dx \end{aligned}$$

where $\hat{W} = W(r\varrho', \varrho, \varepsilon)$ and $(\eta_1, \eta_2) \in \mathfrak{v}$.

From Taylor's theorem, it is straightforward to show that a necessary and sufficient condition for an extremal (ϱ, ε) of I_r to be a weak local minimum is that

$$V_2(\eta_1, \eta_2, \varrho, \varepsilon) \geqq 0 \quad \text{for all } (\eta_1, \eta_2) \in \mathfrak{v}.$$

Surprisingly, positivity of the second variation is also a sufficient condition for a strong local minimum.

Theorem 4.1. *Suppose* $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$ *is an extremal of* I_r. *If*

$$V_2(\eta_1, \eta_2; \varrho, \varepsilon) > 0 \quad \text{for all } (\eta_1, \eta_2) \in \mathfrak{v},\ \eta_1 \not\equiv 0 \quad \text{or } \eta_2 \not\equiv 0, \tag{4.2}$$

then (ϱ, ε) *is a strong local minimum.*

The proof of this theorem, using the field theory of the calculus of variations, is given in [22].

We now consider the constant solutions of (2.1), (2.2) and (2.3). From Section 3 we know that constant solutions are of the form $(\bar{\varrho}(d_\lambda), d_\lambda)$ where d_λ satisfies (3.2). Given $\gamma \in (0, \infty)$, $(\bar{\varrho}(d_\lambda), d_\lambda) \in \mathfrak{A}_{r\gamma}$ if and only if $d_\lambda = \gamma$, i.e., d_λ satisfies (1.1) and $\lambda = W_v(0, \bar{\varrho}(\gamma), \gamma)$. Thus, $(\bar{\varrho}(\gamma), \gamma)$ is an extremal of I_r.

Theorem 4.2.
1) *For* $\gamma \leqq \alpha_0$ *or* $\gamma \geqq \beta_0$, $(\bar{\varrho}(\gamma), \gamma)$ *is the global minimizer of* I_r *on* $\mathfrak{A}_{r\gamma}$.
2) *Given r sufficiently small, there exist numbers* $\tau_1(r), \tau_2(r) > 0$ *such that if* $\gamma \in (\alpha_0, \bar{\alpha} + \tau_1(r)) \cup (\bar{\beta} - \tau_2(r), \beta_0)$, *then* $(\bar{\varrho}(\gamma), \gamma)$ *is a strong local minimizer of* I_r.
3) *Given r sufficiently small, there exist numbers* $\tau_3(r) \geqq \tau_1(r)$, $\tau_4(r) \geqq \tau_2(r)$ *such that if* $\gamma \in (\bar{\alpha} + \tau_2(r), \beta - \tau_4(r))$, *then* $(\bar{\varrho}(\gamma), \gamma)$ *is not a minimizer (in any sense) of* I_r.

Proof. 1) Suppose $\gamma \leqq \alpha_0$ and $(\varrho, \varepsilon) \in \mathfrak{A}_{r\gamma}$. Then

$$I_r(\varrho, \varepsilon) - I_r(\bar{\varrho}(\gamma), \gamma) \geqq \int_{-1}^{1} [W(0, \varrho(x), \varepsilon(x)) - W(0, \bar{\varrho}(\gamma), \gamma)]\, dx, \tag{4.3}$$

by H2 and H6. Define a smooth function $f\colon (0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$f(y, z) = W(0, y, z) - W(0, \bar{\varrho}(\gamma), \gamma)\, z.$$

By H1, H2, H3 and H4, f is bounded below and goes to ∞ as $y \to 0, \infty$ or as $z \to 0, \infty$. Thus, f attains a minimum value and the critical points of f are given by

$$W_u(0, y, z) = 0,\ W_v(0, y, z) = W_v(0, \bar{\varrho}(\gamma), \gamma) \tag{4.4}$$

(*cf.* (3.6)). For $\gamma < \underline{\alpha}$, (4.4) has a unique solution $z = \gamma$, $y = \bar{\varrho}(\gamma)$. Hence,

$$W(0, \varrho(x), \varepsilon(x)) - W_v(0, \bar{\varrho}(\gamma), \gamma)\, \varepsilon(x) \geqq W(0, \bar{\varrho}(\gamma), \gamma) - W_v(0, \bar{\varrho}(\gamma), \gamma)\, \gamma \tag{4.5}$$

for all $x \in [-1, 1]$. Positivity of the right-hand side of (4.3) follows by rearranging (4.5), integrating it and noting that $\int_{-1}^{1} [\varepsilon(x) - \gamma]\, dx = 0$.

For $\gamma \geqq \underline{\alpha}$, there is more than one solution of (4.4). A simple calculation based on H7 shows that the minimum value of f is $f(\bar{\varrho}(\gamma), \gamma)$. The result then follows as for $\gamma \leqq \underline{\alpha}$.

The proof for $\gamma \geqq \bar{\beta}$ is analogous.

2) By Theorem 4.1 we need only show that the second variation is positive for γ in the desired range. Hypothesis H6 implies that $W_p(0, u, v) = 0$ for all $u, v \in (0, \infty)$, so that the second variation at $(\bar{\varrho}(\gamma), \gamma)$ simplifies to

$$V_2(\eta_1, \eta_2; \bar{\varrho}(\gamma), \gamma) = \int_{-1}^{1} [\hat{W}_{pp} r^2 (\eta_1')^2 + \hat{W}_{uu} \eta_1^2 + 2\hat{W}_{uv} \eta_1 \eta_2 + 2\hat{W}_{vv} \eta_2^2]\, dx$$

$$= \int_{-1}^{1} \left[\hat{W}_{pp} r^2 (\eta_1')^2 + \hat{W}_{vv} \left(\eta_2 + \frac{\hat{W}_{uv}}{\hat{W}_{vv}} \eta_1 \right)^2 + \frac{\hat{W}_{uu}}{\hat{W}_{vv}} \left(\hat{W}_{vv} - \frac{\hat{W}_{uv}^2}{\hat{W}_{uu}} \right) \eta_1^2 \right] dx, \tag{4.6}$$

where now $\hat{W} = W(0, \bar{\varrho}(\gamma), \gamma)$.

By H2, V_2 is strictly positive if the last term in the integrand of (4.6) is non-negative. From the definition of $\bar{\varrho}$,

$$\frac{d^2}{dv^2} W(0, \bar{\varrho}(v), v)\,|_{v=\gamma} = \hat{W}_{vv} - \frac{\hat{W}_{uv}}{\hat{W}_{uu}}, \tag{4.7}$$

so that H7 implies the left-hand side of (4.7) is non-negative for $\gamma \geqq \bar{\alpha}$ or $\gamma \geqq \underline{\beta}$ (*cf.* Figure 1). That the last term in the integrand of (4.6) is non-negative follows from H2.

For $\tau \in [0, \underline{\beta} - \bar{\alpha}]$ let $h(\tau) = g'(\bar{\alpha} + \tau)$ where g is as defined in H7. Then

$$\begin{gathered} h(0) = 0, \quad h(\underline{\beta} - \bar{\alpha}) = 0, \quad h'(0) < 0, \quad h(\tau) < 0 \quad \text{for } \tau \in (0, \underline{\beta} - \bar{\alpha}); \\ \text{there is a } \theta_1 \in (0, \underline{\beta} - \bar{\alpha}) \text{ such that } h'(\tau) < 0 \text{ for } \tau \in [0, \theta_1]. \end{gathered} \tag{4.8}$$

Hence, h is a negative, decreasing function on $(0, \theta_1)$. For $\tau \in [0, \underline{\beta} - \bar{\alpha}]$ let $l(\tau) = W_{pp} W_{vv} / W_{uu}|_{(0, \bar{\varrho}(\bar{\alpha}+\tau), \bar{\alpha}+\tau)}$. By H1 and H2, l is a bounded, continuous function and so there are constants $0 < m \leqq M$ such that

$$m \leqq l(\tau) \leqq M \quad \text{for all } \tau \in [0, \underline{\beta} - \bar{\alpha}].$$

We define $\tau_1(r)$ as the solution of

$$h(\tau) = -r^2 \frac{\pi^2}{4} m.$$

From (4.8), this equation will have a unique solution provided r is sufficiently small.

We now show that $V_2 \geqq 0$ for all $\gamma \in (\alpha_0, \bar{\alpha} + \tau_1)$. From (4.6)

$$V_2(\eta_1, \eta_2; \bar{\varrho}(\gamma), \gamma) \geqq \int_{-1}^{1} \left[\hat{W}_{pp} r^2 (\eta_1')^2 + \frac{\hat{W}_{uu}}{\hat{W}_{vv}} \left(\hat{W}_{vv} - \frac{\hat{W}_{uv}}{\hat{W}_{uu}} \right) \eta_1^2 \right] dx,$$

with equality if and only if

$$\eta_2(x) = -\frac{\hat{W}_{uv}}{\hat{W}_{vv}} \eta_1(x) \quad \text{for a.e. } x \in [-1, 1]. \tag{4.9}$$

If (4.9) holds, then $\int_{-1}^{1} \eta_1(x)\, dx = 0$ since the partial derivatives are constant and $(\eta_1, \eta_2) \in \mathfrak{v}$. By the Poincaré inequality [21], [26]

$$\frac{4}{\pi^2} \int_{-1}^{1} (\eta_1')^2\, dx \geqq \int_{-1}^{1} \left| \eta_1 - \tfrac{1}{2} \int_{-1}^{1} \eta_1(\tau)\, d\tau \right|^2 dx = \int_{-1}^{1} \eta_1^2\, dx,$$

so that

$$\begin{aligned} V_2(\eta_1, \eta_2; \bar{\varrho}(\gamma), \gamma) &\geqq \int_{-1}^{1} \left[r^2 \frac{\pi^2}{4} \hat{W}_{pp} + \frac{\hat{W}_{uu}}{\hat{W}_{vv}} \left(\hat{W}_{vv} - \frac{\hat{W}_{uv}}{\hat{W}_{uu}} \right) \right] \eta_1^2\, dx \\ &\geqq 0 \end{aligned} \tag{4.10}$$

provided

$$h(\gamma) = \hat{W}_{vv} - \frac{\hat{W}_{uv}}{\hat{W}_{uu}} \geqq -r^2 \frac{\pi^2}{4} \hat{W}_{pp} \frac{\hat{W}_{vv}}{\hat{W}_{uu}}. \tag{4.11}$$

Since h is decreasing, the definition of τ_1 ensures that (4.11) holds for $\gamma \leqq \bar{\alpha} + \tau_1(r)$.

A similar argument holds for $\gamma \in (\underline{\beta} - \tau_2(r), \underline{\beta})$ where $\tau_2(r)$ is defined in the obvious way.

3) This result is proved in [5] but without giving an explicit definition of $\tau_3(r)$ and $\tau_4(r)$. We shall construct a variation (η_1, η_2) such that $V_2(\eta_1, \eta_2; \bar{\varrho}(\gamma), \gamma) < 0$.

For r sufficiently small, let $\tau_3(r)$ be the unique solution of

$$h(\tau) = -r^2 \frac{\pi^2}{4} M.$$

Clearly, $\tau_3(r) \geqq \tau_1(r)$ and we can define the number $\tau_4(r)$ in a similar way. Let $\eta_1(x) = \sin\left(\frac{\pi}{2}x\right)$ and $\eta_2(x) = -(\hat{W}_{uu}/\hat{W}_{vv}) \sin\left(\frac{\pi}{2}x\right)$. Then $(\eta_1, \eta_2) \in \mathfrak{v}$ and satisfy (4.9), so that

$$V_2(\eta_1, \eta_2; \bar{\varrho}(\gamma), \gamma) = \int_{-1}^{1} \left[\hat{W}_{pp} r^2 \frac{\pi^2}{4} \cos^2\left(\frac{\pi}{2}x\right) + \frac{\hat{W}_{uu}}{\hat{W}_{vv}}\left(\hat{W}_{vv} - \frac{\hat{W}_{uv}^2}{\hat{W}_{uu}}\right) \sin^2\left(\frac{\pi}{2}x\right)\right] dx$$

$$= \left[\hat{W}_{pp} r^2 \frac{\pi^2}{4} + \frac{\hat{W}_{uu}}{\hat{W}_{vv}} h(\gamma - \bar{\alpha})\right] \int_{-1}^{1} \sin^2\left(\frac{\pi}{2}x\right) dx,$$

which is negative for $\bar{\alpha} + \tau_3(r) < \gamma < \underline{\beta} - \tau_4(r)$. □

From the construction of the function h, it is straightforward to show that there is a $K > 0$ such that for $r > K$ the constant solution $(\bar{\varrho}(\gamma), \gamma)$ is a strong local minimizer for all $\gamma > 1$. Furthermore, from the definition of τ_i,

$$\tau_i(r) \to 0 \quad \text{as } r \to 0, \quad i = 1, 2, 3, 4.$$

The quantity $W_v(0, \bar{\varrho}(\gamma), \gamma)$ is the applied uniaxial tension required to produce a homogeneous extension γ (*cf.* [5]). The graph of $W_v(0, \bar{\varrho}(\gamma), \gamma)$ against γ can then be identified as the measured tension-extension curve (at least up to the onset of necking).

Theorem 4.2 shows that the homogeneous deformation cannot lose stability until after the first local maximum of the tension-extension curve is passed. Moreover, there is a finite range of deformations after this maximum point for which the homogeneous deformation is stable and this range is determined by the undeformed radius of the bar. In particular, the thinner the bar, the less extension is required before stability is lost.

In [27], it is shown that the second variation for the full three-dimensional theory for an elastic bar in a hard device is positive beyond the first maximum of the tension-extension curve. However, the bar considered is of arbitrary cross-section so that the role of the undeformed radius is not considered.

Finally, we note that for sufficiently large initial radii the homogeneous deformation will never lose stability.

We now consider the monotone and non-monotone solutions of (2.1), (2.2) and (2.3). To show directly which of these solutions satisfy (1.1) is difficult (*cf.* [11]). However, the following results shows that we do not have to consider the non-monotone solutions.

Theorem 4.3. *Suppose (ϱ, ε) is a non-monotone solution of* (2.1), (2.2), (2.3) *and* (1.1) *(that is, an extremal of I_r). Then (ϱ, ε) cannot be a local minimizer of I_r on $\mathfrak{A}_{r\gamma}$ for any $\gamma > 1$, $r > 0$.*

The proof of this surprising result is a straightforward extension of the corresponding result in [11].

Theorem 4.3 implies that the extremals corresponding to one or more necks are not "physically realizable". From Theorems 4.2 and 4.3 we conclude that the only localized behaviour possible is a half-neck or "drawing" deformation:

Theorem 4.4. *Given* r *sufficiently small such that* $\tau_3(r) < \tau_4(r)$ *and given* $\gamma \in (\overline{\alpha} + \tau_3(r), \overline{\beta} - \tau_4(r))$ *there exists* $(\overline{\varrho}, \overline{\varepsilon}) \in A_{r\gamma}$ *with* $(\overline{\varrho}, \overline{\varepsilon})$ *monotone, such that* $(\overline{\varrho}, \overline{\varepsilon})$ *is a solution of* $(P_{r\gamma})$, *i.e., a global minimizer of* I_r.

5. Convergence to minimizers of I_0

In the first part of this section we discuss the singular L^1-minimization problem produced by setting $r = 0$ in the integrand of I_r;

$$(P_{0\gamma}) \begin{cases} \text{Minimize } I_0(\varrho, \varepsilon) = \int_{-1}^{1} W(0, \varrho, \varepsilon)\, dx \\ \text{on } \mathfrak{B}_\gamma = \{\varrho \in L^1(-1, 1), \varepsilon \in L^1(-1, 1) : \varrho, \varepsilon > 0 \text{ a.e., } \varepsilon \text{ satisfies (1,1) and } I_0(\varrho, \varepsilon) < \infty\}. \end{cases}$$

Henceforth, we assume that W satisfies a hypothesis slightly stronger than H5:

H5′ There is a continuous function $\theta(\eta)$, $0 \leqq \eta < \infty$, bounded below, such that

$$W(p, u, v) \geqq \theta((p^2 + u^2 + v^2)^{\frac{1}{2}})$$

and $\dfrac{\theta(\eta)}{\eta} \to \infty$ as $\eta \to 0$.

H7 implies the integrand in I_0 is no longer convex so that we cannot expect smooth minimizers. The solutions of $(P_{0\gamma})$ are well known [14]; they satisfy the Euler-Lagrange equations (*cf.* Proposition 3.1)

$$W_u(0, \varrho, \varepsilon) = 0,$$

$$W_v(0, \varrho, \varepsilon) = \lambda$$

(λ is the Lagrange multiplier) and the Weierstrass-Erdmann corner condition: $W(0, \varrho(x), \varepsilon(x)) - \varrho(x)\, W_u(0, \varrho(x), \varepsilon(x)) - \varepsilon(x)\, W_v(0, \varrho(x), \varepsilon(x))$ is continuous for all $x \in [-1, 1]$.

For $\gamma \leqq \alpha_0$ or $\gamma \geqq \beta_0$ the only possible solution is $\varrho \equiv \overline{\varrho}(\gamma)$, $\varepsilon \equiv \gamma$. For $\gamma \in (\alpha_0, \beta_0)$ the set of solutions is far more interesting. Let S_1, S_2 be any disjoint, measurable sets whose union is $[-1, 1]$ and such that

$$\text{meas}\,(S_1) = \frac{2(\beta_0 - \gamma)}{\beta_0 - \alpha_0}, \quad \text{meas}\,(S_2) = \frac{2(\gamma - \alpha_0)}{\beta_0 - \alpha_0}. \tag{5.1}$$

Then

$$\varrho_0(x) = \begin{cases} \bar{\varrho}(\alpha_0) & \text{for } x \in S_1 \\ \bar{\varrho}(\beta_0) & \text{for } x \in S_2 \end{cases}, \qquad \varepsilon_0(x) = \begin{cases} \alpha_0 & \text{for } x \in S_1 \\ \beta_0 & \text{for } x \in S_2 \end{cases} \tag{5.2}$$

is a solution. Since S_1 and S_2 are only determined up to their measures, there are infinitely many minimizers of the form (5.2), all with the same energy

$$I_0(\varrho_0, \varepsilon_0) = \frac{2(\beta_0 - \gamma)}{\beta_0 - \alpha_0} W(0, \bar{\varrho}(\alpha_0), \alpha_0) + \frac{2(\gamma - \alpha_0)}{\beta_0 - \alpha_0} W(0, \bar{\varrho}(\beta_0), \beta_0).$$

Furthermore,

$$I_0(\varrho_0, \varepsilon_0) < I_0(\bar{\varrho}(\gamma), \gamma). \tag{5.3}$$

We call minimizers of the form (5.2) the *Maxwell solutions of* (P_{0_γ}). A point $\bar{x} \in [-1, 1]$ where $(\varrho_0, \varepsilon_0)$ has a jump discontinuity is called a *transition point*.

The main result of this section is the following:

Theorem 5.1. *Let* $(\varrho_r, \varepsilon_r)$, *be a solution of* $(P_{r\gamma})$ *for* $\gamma \in (\alpha_0, \beta_0)$ *and* $r > 0$. *There is a subsequence* $(\varrho_s, \varepsilon_s)$ *of* $(\varrho_r, \varepsilon_r)$ *such that*

$$\varrho_s(x) \to \hat{\varrho}_0(x) \quad \forall x \in [0, 1] \quad \text{and } \varepsilon_s \overset{L^1}{\rightharpoonup} \hat{\varepsilon}_0 \quad \text{as } s \to 0,$$

where $(\hat{\varrho}_0, \hat{\varepsilon}_0)$ *is the Maxwell solution with one transition and* $\hat{\varrho}_0$ *monotone decreasing.*

A result of this type is proved in [11] using a detailed phase-plane analysis. Our approach will be to use variational methods and a theorem from the theory of functions of bounded variation.

We first show that $(\varrho_r, \varepsilon_r)$ is a minimizing sequence of I_0.

Lemma 5.2.

$$I_0(\varrho_r, \varepsilon_r) \to I_0(\varrho_0, \varepsilon_0) \quad \text{as } r \to 0.$$

Proof. Let $l_1 = -1 + \text{meas}(S_1)$ and let

$$\tilde{\varepsilon}_r(x) = \begin{cases} \alpha_0 & \text{for } \quad -1 \leq x < l_1 - r, \\ \dfrac{(\beta_0 - \alpha_0)}{2r}(x - l_1 + r) + \alpha_0 & \text{for } l_1 - r \leq x < l_1 + r, \\ \beta_0 & \text{for } l_1 + r \leq x \leq 1 \end{cases}$$

and

$$\tilde{\varrho}_r(x) = \begin{cases} \bar{\varrho}(\alpha_0) & \text{for } \quad -1 \leq x < l_1 - r, \\ \dfrac{(\bar{\varrho}(\beta_0) - \bar{\varrho}(\alpha_0))}{2r}(x - l_1 + r) + \bar{\varrho}(\beta_0) & \text{for } l_1 - r \leq x < l_1 + r, \\ \bar{\varrho}(\beta_0) & \text{for } l_1 + r \leq x < 1. \end{cases}$$

Then $(\tilde{\varrho}_r, \tilde{\varepsilon}_r) \in \mathfrak{A}_\gamma$ for all $r > 0$ and $r\tilde{\varrho}_r'$ is bounded as $r \to 0$. A simple calculation shows that

$$I_r(\tilde{\varrho}_r, \tilde{\varepsilon}_r) \to (1 + l_1)\, W(0, \bar{\varrho}(\alpha_0), \alpha_0) + (1 - l_1)\, W(0, \bar{\varrho}(\beta_0), \beta_0) = I_0(\varrho_0, \varepsilon_0) \quad (5.4)$$

as $r \to 0$. Since $(\varrho_r, \varepsilon_r)$ is a minimizer of I_r and $W(p, u, v)$ is convex and even in p,

$$I_r(\tilde{\varrho}_r, \tilde{\varepsilon}_r) \geqq I_r(\varrho_r, \varepsilon_r) \geqq I_0(\varrho_r, \varepsilon_r) \geqq I_0(\varrho_0, \varepsilon_0). \quad (5.5)$$

The assertion then follows from (5.4) and (5.5). □

Proof of Theorem 5.1. By Lemma 5.2, H5′ and the de la Vallée Poussin criterion [12], we can extract a subsequence of $(\varrho_r, \varepsilon_r)$, which we again denote by $(\varrho_r, \varepsilon_r)$, such that

$$\varrho_r \overset{L^1}{\rightharpoonup} \varrho \quad \text{and } \varepsilon_r \overset{L^1}{\rightharpoonup} \varepsilon \quad \text{as } r \to 0 \quad (5.6)$$

for some $\varrho, \varepsilon \in L^1(-1, 1)$ with ε satisfying (1.1).

If $W(0, u, v)$ were a convex function, then I_0 would be weakly lower semicontinuous and we could conclude that (ϱ, ε) is a minimizer of I_0. Without convexity this argument fails.

Let $BV(-1, 1)$ denote the space of functions of bounded variation ([19], [28]). Since ϱ_r is monotone we know that $\varrho_r \in BV(-1, 1)$ for all $r > 0$ and its total variation is $V_{-1}^{+1}(\varrho_r) = \varrho_r(+1) - \varrho_r(-1)$. By Lemma 3.4, $V_{-1}^{+1}(\varrho_r)$ is uniformly bounded for all r. Furthermore, $\sup_{[-1,1]} |\varrho_r|$ is uniformly bounded for all $r > 0$. Therefore, by the Helly selection theorem ([19]) there is a subsequence (ϱ_s) of (ϱ_r) and a function $\varrho^* \in BV(-1, 1)$ such that

$$\varrho_s(x) \to \varrho^*(x) \quad \text{as } s \to 0$$

pointwise for every $x \in [-1, 1]$. Since ϱ_s is decreasing and satisfies (3.23) for all $s > 0$, we deduce that ϱ^* is decreasing and

$$\bar{\varrho}(\beta_0) \leqq \varrho^*(x) \leqq \bar{\varrho}(\alpha_0) \quad \forall x \in [-1, 1].$$

In addition, we have $\varrho = \varrho^*$ a.e..

Since $W(p, u, v)$ is a convex function of v for fixed p and u (by H2) and ϱ_s converges pointwise we can apply the lower semicontinuity result in [12, p. 352] to give

$$I_0(\varrho^*, \varepsilon) \leqq \liminf_{s \to 0} I_0(\varrho_s, \varepsilon_s) = I_0(\varrho_0, \varepsilon_0), \quad (5.7)$$

i.e., (ϱ^*, ε) is a minimizer of I_0. Since ϱ^* is monotone and all minimizers of I_0 are given by (5.2), the only possible choice for S_1 and S_2 is

$$S_1 = [-1, 1 + l_1), \quad S_2 = (-1 + l_1, 1].$$

Thus, there is exactly one transition point at $x = -1 + l_1$. □

Corollary 5.3. *Suppose $\gamma \in (\alpha_0, \beta_0)$. There is a $r_1(\gamma) > 0$ such that for $r < r_1(\gamma)$ the global minimizer of I_r on $\mathscr{A}_{r\gamma}$ is monotone.*

Proof. Fix $\gamma \in (\alpha_0, \beta_0)$. From (5.3),

$$I_r(\bar{\varrho}(\gamma), \gamma) = I_0(\bar{\varrho}(\gamma), \gamma) > I_0(\varrho_0, \varepsilon_0).$$

Theorem 5.2 and (5.7) then imply that

$$I_0(\varrho_r, \varepsilon_r) < I_0(\bar{\varrho}(\gamma), \gamma)$$

for r sufficiently small. Thus $(\bar{\varrho}(\gamma), \gamma)$ is not the global minimizer and by Theorem 4.3 the only choice for a minimizer is monotone. □

A method similar to the above has been used in [1], [2] for some problems in phase transitions in fluids.

6. Incompressible materials and applications to specific elastic models

If the bar is composed of an incompressible material, then ϱ and ε are no longer independent but satisfy the pointwise constraint

$$\varrho^2(x)\, \varepsilon(x) = 1;$$

cf. (1.3). Using this relation we can eliminate ϱ', ϱ from $W(r\varrho', \varrho, \varepsilon)$ to define a one-dimensional stored-energy function for an incompressible material, $\Omega(r\varepsilon', \varepsilon)$, by

$$\Omega(r\varepsilon'(x), \varepsilon(x)) = W\left(-\tfrac{1}{2} r\varepsilon(x)^{-\frac{3}{2}} \varepsilon'(x), \varepsilon(x)^{-\frac{1}{2}}, \varepsilon(x)\right). \tag{6.1}$$

Smoothness, convexity and growth assumptions on Ω are induced by hypotheses H1–H6 on W through (6.1). In particular, we note that $\Omega_{pp}(p, u) > 0$ and $\Omega(p, u)$ is an even function of p.

Since ϱ and ε are no longer independent, we must slightly alter the statement of hypothesis H7, since the function $\bar{\varrho}$ is no longer defined:

H7′ There are numbers $0 < \bar{\alpha}_1 < \theta_1 < \underline{\beta}_1 < \infty$ such that

$$\Omega_u(0, \bar{\alpha}_1) > \Omega_u(0, \theta_1) > \Omega_u(0, \underline{\beta}_1) > 0,$$

$$\Omega_{uu}(0, u) > 0 \quad \text{for } u \in (0, \bar{\alpha}_1) \cup (\underline{\beta}_1, \infty),$$

$$\Omega_{uu}(0, u) < 0 \quad \text{for } u \in (\bar{\alpha}_1, \underline{\beta}_1),$$

$$\Omega_{uuu}(0, u) < 0 \quad \text{for } u \in (0, \theta_1),$$

$$\Omega_{uuu}(0, u) > 0 \quad \text{for } u \in (\theta_1, \infty).$$

Hence, $\Omega_u(0, u)$ is a non-monotone function (*cf.* Figure 1). Clearly, we can disregard hypothesis H8.

The relevant variational problem to determine the equilibrium states is

$$(P_{r\gamma})' \begin{cases} \text{Minimize } J_r(\varepsilon) = \int_{-1}^{1} \Omega(r\varepsilon'(x), \varepsilon(x))\, dx \\ \text{on } \mathfrak{A}'_{r\gamma} = \{\varepsilon \in W^{1,1}(-1, 1) : \varepsilon > 0 \text{ a.e., } \varepsilon \text{ satisfies (1.1) and } J_r(\varepsilon) < \infty\}. \end{cases}$$

By repeating the analysis of sections 2–5 it is straightforward to show that the structure solutions of $(P_{r\gamma})'$ is the same as that of those of $(P_{r\gamma})$. Thus, for all $r > 0, \gamma > 1$ there is a smooth solution of $(P_{r\gamma})'$ with ε bounded away from zero and satisfying the Euler-Lagrange equations. Possible solutions of these equations are either constant, monotone or (periodic) non-monotone. Furthermore, the non-monotone solutions can never be minimizers of J_r, whilst for a certain range of γ (determined by the Maxwell numbers for Ω) and r sufficiently small the global minimizers of J_r will be monotone and "close" to the Maxwell solution with one transition (*cf.* Section 5).

We now consider some constitutive models for real elastic materials and determine whether or not H7 or H7′ is satisfied by them. The case when H7 is *not* satisfied is of some interest, especially if the functions $W_v(0, \bar{\varrho}(v), v)$ or $\Omega_u(0, u)$ are monotone, since then no localized behaviour such as necking is possible.

Hypotheses H7 and H7′ are just concerned with the behaviour of W for homogeneous deformations. Let v_1^2, v_2^2, v_3^2 be the eigenvalues of the (positive-definite, symmetric) Cauchy-Green deformation tensor FF^T. For an isotropic material there exists a symmetric function Ψ such that the stored-energy function Φ can be written as

$$\Phi(F) = \Psi(v_1, v_2, v_3). \tag{6.2}$$

If we set $\varrho' \equiv 0$ in the deformation gradient (1.3), then the principal stretches v_1, v_2, v_3 of the deformation are

$$v_1 = \varepsilon, \quad v_2 = v_3 = \varrho. \tag{6.3}$$

From (1.5), (1.8), (6.2) and (6.3) we deduce that

$$W(0, \varrho, \varepsilon) = \Psi(\varepsilon, \varrho, \varrho) \tag{6.4}$$

and if the material is incompressible that

$$\Omega(0, \varepsilon) = W(0, \varepsilon^{-\frac{1}{2}}, \varepsilon) = \Psi(\varepsilon, \varepsilon^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}}). \tag{6.5}$$

Thus, H7 or H7′ can be verified from the three-dimensional stored-energy function.

The well-known Ogden constitutive model ([23]) for incompressible materials such as vulcanized rubbers has the form

$$\Psi(v_1, v_2, v_3) = \sum_{j=1}^{M} a_j \left(\sum_{i=1}^{3} v_i^{\alpha_j} - 3 \right) + \sum_{j=1}^{N} b_j \left(\sum_{i=1}^{3} v_i^{-\beta_j} - 3 \right) \tag{6.6}$$

where $M \geqq 1$, $N \geqq 1$, $a_j > 1$ for $1 \leqq j \leqq M$, $b_j > 0$ for $1 \leqq j \leqq N$, $\alpha_1 \geqq \alpha_2 \geqq \dots \alpha_M \geqq 1$, $\beta_1 \geqq \beta_2 \geqq \dots \geqq \beta_N \geqq 0$. Using (6.5) shows that for an Ogden material,

$$\Omega(0, \varepsilon) = \sum_{j=1}^{M} a_j(\varepsilon^{\alpha_j} + 2\varepsilon^{-\frac{\alpha_j}{2}} - 3) + \sum_{j=1}^{N} b_j(\varepsilon^{-\beta_j} + 2\varepsilon^{\frac{\beta_j}{2}} - 3)$$

and

$$\Omega_{uu}(0, \varepsilon) = \sum_{j=1}^{M} a_j\alpha_j \left\{ (\alpha_j - 1)\, \varepsilon^{\alpha_j - 2} + \left(\frac{\alpha_j}{2} + 1 \right) \varepsilon^{-\frac{\alpha_j}{2} - 2} \right\}$$
$$+ \sum_{j=1}^{N} b_j\beta_j \left\{ (\beta_j + 1)\, \varepsilon^{-\beta_j - 2} + \left(\frac{\beta_j}{2} - 1 \right) \varepsilon^{\frac{\beta_j}{2} - 2} \right\}. \tag{6.7}$$

If

$$\beta_1 = 0 \quad \text{or } \beta_N \geqq 2, \tag{6.8}$$

then from (6.7) we deduce that the equation $\Omega_{uu}(0, \varepsilon) = 0$ will have no positive solutions. Thus, for Ogden materials satisfying (6.8), H7′ is not satisfied and the only stable equilibrium states are homogeneous. We note that the neo-Hookean ($m = 1$, $\alpha_1 = 2$, $\beta_1 = 0$) and Mooney-Rivlin ($M = N = 1$, $\alpha_1 = 2$, $\beta_1 = 2$) materials satisfy (6.8) (*cf.* [24], [30]).

If $M = N = 1$, $\alpha_1 = 2$ and $\beta_1 = 1$, then a simple calculation (*cf.* [9]) shows that if $b_1^2 > 64a_1(2a_1 + b_1)$, $F_{uu}(0, \varepsilon) = 0$ will have exactly two positive solutions. Further, it is easy to check that the remaining parts of H7′ are satisfied.

The OGDEN constitutive model for compressible materials has the form

$$\Psi(v_1, v_2, v_3) + k^{-1}h(v_1v_2v_3)$$

where Ψ is given by (6.6), $k > 0$ and the function $h(\delta)$ satisfies

$$h \in C^2(0, \infty),$$
$$h''(\delta) > 0 \quad \text{for } \delta \in (0, \infty),$$
$$(\delta h'(\delta))' > 0 \quad \text{for } \delta \in (0, \infty),$$
$$h(\delta) \to \infty \quad \text{as } \delta \to 0,$$
$$h(\delta)/\delta \to \infty \quad \text{as } \delta \to \infty.$$

A lengthy calculation shows that if Ψ satisfies H7′ then if k is sufficiently small (an "almost incompressible" material), the compressible material (6.9) satisfies H7.

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