Remainder Estimates for the Asymptotics of Elliptic Eigenvalue Problems with Indefinite Weights

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Abstract

Let $\mathscr{A}$ be a positive self-adjoint elliptic operator of order $2m$ on a bounded open set $\Omega \subset \mathbb{R}^k$. We consider the variational eigenvalue problem

$$\text{(P)} \qquad \mathscr{A}u = \lambda r(x)\, u, \quad x \in \Omega,$$

with Dirichlet or Neumann boundary conditions; here the "weight" r is a real-valued function on Ω which is allowed to change sign in Ω or to be discontinuous. Such problems occur naturally in the study of many nonlinear elliptic equations. In an earlier work [*Trans. Amer. Math. Soc.* **295** (1986), pp. 305–324], we have determined the leading term for the asymptotics of the eigenvalues λ of (P). In the present paper, we obtain, under more stringent assumptions, the corresponding remainder estimates. More precisely, let $N^{\pm}(\lambda)$ be the number of positive (respectively, negative) eigenvalues of (P) less than $\lambda > 0$ (respectively, greater than $\lambda < 0$); set $r_{\pm} = \max(\pm r, 0)$ and $\Omega_{\pm} = \{x \in \Omega : r(x) \gtrless 0\}$. We show that

$$N^{\pm}(\lambda) = \int_{\Omega_{\pm}} (\lambda r(x))^{\frac{k}{2m}} \mu'_{\mathscr{A}}(x)\, dx + O(|\lambda|^{\frac{k-1}{2m}+\delta}) \quad \text{as } \lambda \to \pm\infty,$$

where $\delta > 0$ and $\mu'_{\mathscr{A}}(x)$ is the Browder-Gårding density associated with the principal part of $\mathscr{A}$. How small δ can be chosen depends on the "regularity" of the leading coefficients of $\mathscr{A}$, $r_{\pm}$, and of the boundary of $\Omega_{\pm}$. These results seem to be new even for positive weights.

I. Introduction

In an earlier work [FlLa], we have studied the eigenvalue distribution of elliptic boundary value problems with an indefinite weight function defined on an open set $\Omega \subset \boldsymbol{R}^k$, $k \geqq 1$:

$$\text{(P)} \qquad \begin{cases} \mathscr{A}u = \lambda r u & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $Bu = u$ or $Bu = \partial u/\partial n$ according to whether (P) is the Dirichlet or Neumann problem. Here $\mathscr{A}$ is a positive elliptic self-adjoint operator of order $2m$ $(m \geqq 1)$, and the weight function r *changes sign* in Ω (*i.e.*, is "indefinite"). Under appropriate assumptions, we have proved, in particular, that for Ω bounded [FlLa, Theorems 4.1 and 4.2, pp. 315 and 316],

$$N_i^+(\lambda) \sim \Phi(r_+, \Omega, \mathscr{A})\, \lambda^{k/2m} \quad \text{as } \lambda \to +\infty,$$

where $N_i^+(\lambda) = N_i^+(\lambda; r, \Omega, \mathscr{A})$ denotes the number of positive eigenvalues less than $\lambda > 0$ of the variational Dirichlet $(i = 0)$ or Neumann $(i = 1)$ problem (P); further,

$$\Phi(r_+, \Omega, \mathscr{A}) = \int_\Omega (r_+(x))^{k/2m}\, \mu'_{\mathscr{A}}(x)\, dx,$$

where r_+ is the positive part of r and $\mu'_{\mathscr{A}}(x)$ is the Browder-Gårding density associated with the principal symbol of $\mathscr{A}$.

In this paper, we establish remainder estimates corresponding to the above asymptotic formula in the case where Ω is a bounded open set in $\boldsymbol{R}^k$. More precisely, we show that under suitable additional hypotheses, we have, as $\lambda \to +\infty$,

$$\left| N_i^+(\lambda) - \Phi(r_+, \Omega, \mathscr{A})\, \lambda^{\frac{k}{2m}} \right| = O(\lambda^{\frac{k-1}{2m} + \delta}),$$

with $\delta \in (0, 1/2m]$; how small δ can be chosen will depend on the "regularity" of the boundary $\partial\Omega$ of Ω, of the weight function r, and of the leading coefficients of the operator $\mathscr{A}$. Naturally, we also obtain analogous results for $N_i^-(\lambda) = N_i^-(\lambda; r, \Omega, \mathscr{A})$, the number of negative eigenvalues of (P) larger than $\lambda < 0$.

The above remainder estimate is well known in the classical case where $r \equiv 1$. (Of course, $N_i^-(\lambda) = 0$ in this situation.) Early work in this setting is described in [CoHi, Section VI.5, pp. 443–445]. Many further results on this subject have since been obtained, among which we mention, in particular, those in [AgKa, Ho 1, Mt, Ph, Se 1–2, Iv 1–2, Me]. Although the methods used in these references differ widely, they are essentially of two kinds: one is based on the study of the spectral function; for instance, the use of Fourier integral operators leads to sharp estimates for operators with smooth coefficients acting on smooth domains. (See [Ho 2], Vol. III, Section XVII.5 and Vol. IV, Chapter XXIX.) The second one relies on the "maximum-minimum principle" and its consequences; this method, also known by physicists as the "Dirichlet-Neumann bracketing", enables one to study operators with nonsmooth coefficients acting on irregular domains. (See [CoHi, Chapter VI], [Mt] and [ReSi, Section XVIII.15, pp. 260–279].)

We use here an extension of the latter method to problems with indefinite weight functions. Such problems have received much attention over the last decade, due in part to their importance in the study of semilinear elliptic partial differential equations (see, *e.g.*, [HeKa], [dF], [GoLD], and the references therein). For example, once linearized about the origin, the nonlinear eigenvalue problem

$$\mathcal{A}u = \lambda f(x, u), \qquad x \in \Omega,$$

under appropriate boundary conditions and with $f(x, 0) = 0$, yields the above linear eigenvalue problem (P) with weight function $r(x) := (\partial f/\partial u)(x, 0)$; and, clearly, r need not keep a constant sign in Ω in this situation; further, r need not be continuous either.

Indefinite problems have been used in numerous areas of engineering, physics and applied mathematics (see, *e.g.*, [FlLa] and the references therein). They arise, for instance, in the study of transport theory, laser theory, crystal coloration, hydrodynamics (flow through porous media), and reaction-diffusion equations of population biology.

The first asymptotic estimates for the eigenvalues of indefinite problems in the multidimensional case were given by PLEIJEL [Pl]; he considered the Laplacian with a continuous weight on a bounded domain in $\boldsymbol{R}^2$. His work has been extended to more general elliptic operators in [BiSo 1–2, Ro, La 1–2, FeFl, Fe, FlLa, He]. Several of these references deal also with indefinite problems with discontinuous weight functions on bounded as well as unbounded open sets (operators of Schrödinger type). We now intend to sharpen some of these results by obtaining estimates for the remainder term. We note that our work seems to be of interest even in the case of positive weight functions.

We conclude this introduction by briefly outlining the content of this paper. After having introduced some basic definitions and stated our main results in Section II, we examine in Section III the relatively simple but important case of the Laplace operator on a bounded open set. We then consider a general positive elliptic operator with variable coefficients in Section IV.

II. Notation and Main Results

We use the same notation as in [FlLa], insofar as possible. Throughout this paper, Ω is a bounded open set in $\boldsymbol{R}^k$ $(k \geqq 1)$, with boundary $\partial\Omega$, closure $\overline{\Omega}$ and k-dimensional Lebesgue measure $|\Omega|$, and r is a measurable real-valued function on Ω. We let $r_+ = \max(r, 0)$ and $r_- = \max(-r, 0)$ denote the positive and negative parts of r.

Given $\alpha = (\alpha_1, \ldots, \alpha_k) \in \boldsymbol{N}^k$, D^α stands for the derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_k$; further, for $\xi = (\xi_1, \ldots, \xi_k) \in \boldsymbol{R}^k$, we set $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}$.

Let m be a positive integer. By $H^m(\Omega)$ we mean the usual Sobolev space of all complex-valued functions $u \in L^2(\Omega)$ with distributional derivatives $D^\alpha u$ also in $L^2(\Omega)$ for $|\alpha| \leqq m$, endowed with the Hilbert norm

$$\|u\|_{H^m(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leqq m} |D^\alpha u(x)|^2 \, dx\right)^{\frac{1}{2}};$$

further, $H_0^m(\Omega)$ is the subspace of $H^m(\Omega)$ obtained by completing $C_0^\infty(\Omega)$ [the space of all infinitely differentiable functions with compact support in Ω] with respect to this norm (see, *e.g.*, [Ad] or [Ag]). Similar notation will be used for functions defined on other open subsets of Ω.

If J is a finite set, we denote its cardinal by $\# J$. Moreover, if $I \subset J$, $J \setminus I$ stands for the complement of I in J.

We shall use various constants throughout the text; they will be denoted by c, c_0, c', C, *etc.* Often, the same letter will be used to represent different constants.

In order to establish our remainder estimates, we shall work locally on small cubes of $\boldsymbol{R}^k$. We thus introduce the following definitions.

Here and thereafter, given any $\eta > 0$ sufficiently small, we consider a "tessellation" of $\boldsymbol{R}^k$ by a countable family of disjoint open cubes $(Q_\zeta)_{\zeta \in \boldsymbol{Z}^k}$, with center x_ζ and sides of length η, such that

$$\boldsymbol{R}^k = \bigcup_{\zeta \in \boldsymbol{Z}^k} \overline{Q}_\zeta .$$

Definition 1. *Let D be a bounded open set in $\boldsymbol{R}^k$. Given $\beta > 0$, we say that the boundary ∂D satisfies the "β-condition" if there exist positive constants c_0 and $\eta_0 < 1$ such that for all η with $0 < \eta \leqq \eta_0$,*

$$\text{(1)} \qquad \frac{\#(J \setminus I)}{\# I} \leqq c_0 \eta^\beta ,$$

where

$$\text{(2–1)} \qquad I = I(D) := \{\zeta \in \boldsymbol{Z}^k : \overline{Q}_\zeta \subset D\}$$

and

$$\text{(2–2)} \qquad J = J(D) := \{\zeta \in \boldsymbol{Z}^k : Q_\zeta \cap \overline{D} \neq \emptyset\}.$$

Remarks 1. (a) This condition will enable us to measure the regularity of the boundary of D. In some sense, when $\beta > k$, it could be viewed as a quantitative version of the assumption that D is Jordan contented. (Roughly, a bounded subset of $\boldsymbol{R}^k$ is said to be "Jordan contented" if it is well approximated from within and without by a finite union of cubes; see [LoSt, Chapter 6, §§ 6–7] and [ReSi, p. 271].) Note that the coefficient β allows us to measure the "smoothness" of ∂D; indeed, the larger β, the "smoother" ∂D.

(b) The sets I and J defined by (2) are finite since D is bounded; we clearly have $I \subset J$. Moreover, it is easy to check that $\partial D \subset \bigcup_{\zeta \in J \setminus I} \overline{Q_\zeta}$ and $|\partial D| = 0$.

Let D be a bounded open set in $\boldsymbol{R}^k$ and let $f \in L^{k/2m}(D)$ be a nonnegative function on D.

For $\zeta \in I(D)$, we set

$$\text{(3)} \qquad f_\zeta = \left(|Q_\zeta|^{-1} \int_{Q_\zeta} f^{k/2m}\right)^{2m/k} = \eta^{-2m} \|f\|_{L^{k/2m}(Q_\zeta)}$$

and

(4) $$\varrho_\zeta(f) = \|f - f_\zeta\|^{k/2m}_{L^{k/2m}(Q_\zeta)}.$$

Definition 2. *Given $\gamma > 0$, we say that the function f satisfies the "γ-condition" on D if there exist positive constants c_1 and $\eta_1 < 1$, such that for all $\zeta \in I(D)$ and all η with $0 < \eta \leqq \eta_1$,*

(5) $$\varrho_\zeta(f) \leqq c_1 \eta^\gamma.$$

Remarks 2. (a) Since, by (3),

(6) $$f_\zeta^{k/2m} = |Q_\zeta|^{-1} \int_{Q_\zeta} f^{k/2m},$$

$f_\zeta^{k/2m}$ is nothing but the mean value of $f^{k/2m}$ over the cube Q_ζ. We mention that the above condition should be compared with the assumptions made in [EdEv].

(b) Note that the coefficient γ enables us to measure the "smoothness" of f: the larger γ, the "smoother" f.

(c) It clearly follows from Definitions 1 and 2 that, if the boundary ∂D (respectively, the function f) satisfies the "β-condition" for some $\beta > 0$ (respectively, the "γ-condition" on D for some $\gamma > 0$), then it satisfies it for any β' with $0 < \beta' \leqq \beta$ (respectively, for any γ' with $0 < \gamma' \leqq \gamma$).

Example 1. If $f \equiv 1$ on D, then it satisfies the "γ-condition" on D for any $\gamma > 0$.

More generally, we have:

Example 2. If a positive function f is Hölder continuous of order θ and is bounded away from zero on D, with $\theta > 0$, then it satisfies the "γ-condition" on D for any γ with $0 < \gamma \leqq k + (k\theta/2m)$.

Indeed, there exists a positive constant L such that for all $\zeta \in I$ and all $x \in Q_\zeta$,

$$|f(x) - f(x_\zeta)| \leqq L\,|x - x_\zeta|^\theta \leq L(\eta\sqrt{k})^\theta.$$

Since f is bounded away from zero on D, it follows that for η small enough and for all $\zeta \in I$ and all $x \in Q_\zeta$,

$$0 < f(x_\zeta) - c\eta^\theta \leqq f(x) \leqq f(x_\zeta) + c\eta^\theta.$$

Hence, by (3) or (6),

$$f(x_\zeta) - c\eta^\theta \leqq f_\zeta \leqq f(x_\zeta) + c\eta^\theta.$$

Finally, since $|Q_\zeta| = \eta^k$, we conclude from (4) that

$$\varrho_\zeta(f) = \int_{Q_\zeta} |f(x) - f_\zeta|^{k/2m} \leqq c_1 \eta^{k + k(\theta/2m)}.$$

Actually, we can obtain a weaker conclusion by assuming that f is only continuous:

Example 3. If f is positive and continuous on $\overline{D}$, then it satisfies the "γ-condition" on D for any γ with $0 < \gamma \leqq k$.

In fact, since f is uniformly continuous and bounded away from zero on the compact set $\overline{D}$, we can choose η so small that for all $\zeta \in I$ and all $x \in Q_\zeta$,

$$0 < \tfrac{1}{2} f(x_\zeta) \leqq f(x) \leqq 2 f(x_\zeta).$$

We then continue just as in Example 2 and obtain $\varrho_\zeta(f) \leqq c_1 \eta^k$.

We note that in the setting of Example 2 or 3, it would be more convenient in the following to define f_ζ by $f(x_\zeta)$, where x_ζ is the center of Q_ζ, rather than by (3).

We can now state our hypotheses and present our problem in a more precise form:

(7) Let Ω be a bounded open set in $\boldsymbol{R}^k$ with boundary $\partial\Omega$ satisfying the "β-condition" for some $\beta > 0$.

(8) Let $r \in L^\infty(\Omega)$ be such that $\Omega_+ := \{x \in \Omega : r(x) > 0\}$ is of positive Lebesgue measure and $|\Omega_+ \setminus \Omega_+^\circ| = 0$. We assume in addition that Ω_+° has the "segment property" [Ag, p. 11] (*e.g.*, $\partial\Omega_+^\circ$ is locally Lipschitz).

(8′) The same hypotheses as in (8) hold for r except with Ω_+ replaced by $\Omega_- := \{x \in \Omega : r(x) < 0\}$.

Remarks 3. (a) Here $\Omega_\pm^\circ$ denotes the interior of $\Omega_\pm$ and $\partial\Omega_\pm^\circ$ stands for the boundary of $\Omega_\pm^\circ$. If r is continuous in Ω, then $\Omega_\pm$ is open and the condition $|\Omega_\pm \setminus \Omega_\pm^\circ| = 0$ is automatically satisfied.

(b) In (8) or (8′), one could replace the segment property by the weaker condition (C′) defined in [Mt, p. 156]. (See [FlLa, Remark 1, p. 312.])

For technical reasons, we also need to introduce the following condition. (See the comment following Eq. (67) at the end of Section IV.)

($\mathscr{K}_\pm$) There exists a positive constant c_2 such that for all η sufficiently small and all $\zeta \in I(\Omega_\pm^\circ)$, we have $|r|_\zeta \geqq c_2$.

Remarks 4. (a) By definition, the weight function r is certainly positive (respectively, negative) in Ω_+° (respectively, Ω_-°); condition ($\mathscr{K}_\pm$) requires somewhat more but still allows r to vanish at boundary points of $\Omega_\pm$ and to change sign in Ω.

(b) Note that condition ($\mathscr{K}_\pm$) is automatically satisfied if r is bounded away from zero in $\Omega_\pm^\circ$; indeed, by (6), the assumption $|r| \geqq c_2 > 0$ in $\Omega_\pm^\circ$ implies that $|r_\zeta| \geqq c_2$ for all $\zeta \in I(\Omega_\pm^\circ)$. It follows, in particular, that condition ($\mathscr{K}_\pm$) holds if r is continuous in $\overline{\Omega}$ (or, more generally, in a compact neighborhood of

$\Omega_{\pm}^{\circ}$). The weight function r need not be continuous, however; for instance, r defined as in Corollary 1 of Section III (in particular, $r \equiv \pm 1$ in $\Omega_{\pm}$) satisfies both condition $(\mathscr{K}_+)$ and $(\mathscr{K}_-)$.

(9) Let $\mathscr{A}$ be a positive, uniformly elliptic and formally selfadjoint operator of order $2m$ ($m \geqq 1$) defined on Ω:

$$\mathscr{A} = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x)\, D^{\beta});$$

we suppose that $a_{\alpha\beta} = \overline{a_{\beta\alpha}} \in L^{\infty}(\Omega)$ for $|\alpha|$, $|\beta| \leqq m$, $a_0 \geqq 0$, and that the associated hermitian form

$$a(u, v) = \int_{\Omega} \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} a_{\alpha\beta}(x)\, D^{\alpha}u\overline{D^{\beta}v}$$

is coercive in $H^m(\Omega)$.

(10) The coefficients of the leading part a', $a_{\alpha\beta}$ with $|\alpha| = |\beta| = m$, when they are not constant, are Hölder continuous of order $\sigma \in [0, 1]$ on Ω, with $\sigma > 0$; moreover, there exists $d > 0$ such that for all open sets $\omega \subset \Omega$,

$$a'_{\omega}(u, u) \geqq d\, \|u\|^2_{H^m(\omega)} \quad \text{for all } u \in H^m(\omega),$$

with

$$a'_{\omega}(u, v) := \int_{\omega} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x)\, D^{\alpha}u\overline{D^{\beta}v}.$$

We consider the following variational eigenvalue problem:

$$(P) \qquad \mathscr{A}u = \lambda r(x)\, u, \quad x \in \Omega,$$

with Dirichlet or Neumann boundary conditions.

More precisely, we say that λ is an eigenvalue of the variational Dirichlet (respectively, Neumann) problem (P) if there exists a nonzero $u \in H_0^m(\Omega)$ [respectively, $u \in H^m(\Omega)$] such that

$$a(u, v) = \lambda \int_{\Omega} ru\bar{v},$$

for all $v \in H_0^m(\Omega)$ [respectively, $v \in H^m(\Omega)$], with a defined as in (9). (See, *e.g.* [FlLa, § 4, pp. 314–316]; see also [Wn] for an abstract formulation of similar variational problems.)

Since Ω is bounded and $|\Omega_{\pm}^{\circ}| = |\Omega_{\pm}| > 0$, it is known that the "spectrum" (*i.e.*, the set of eigenvalues) of (P) is discrete and consists of a double sequence of eigenvalues (one positive and one negative) of finite multiplicity:

$$\ldots \leqq \lambda_{n+1}^- \leqq \lambda_n^- \leqq \ldots \leqq \lambda_2^- \leqq \lambda_1^- > 0 < \lambda_1^+ \leqq \lambda_2^+ \leqq \ldots \leqq \lambda_n^+ \leqq \lambda_{n+1}^+ \leqq \ldots,$$

with $|\lambda_n^{\pm}|$ tending to $+\infty$ as n tends to ∞; here, each eigenvalue is repeated according to its multiplicity.

These eigenvalues are given by the "max-min formula" (see [FlLa, Eq. (4.6), p. 314, in conjunction with § 4.B, pp. 315–316]):

$$\frac{1}{\lambda_n^+} = \max_{F_n \in \mathcal{F}_n} \min_{u \in F_n} \left\{ \int_\Omega r\,|u|^2 : a(u, u) = 1 \right\}, \tag{11}$$

where $\mathcal{F}_n$ is the set of n-dimensional subspaces of $H_0^m(\Omega)$ [respectively, $H^m(\Omega)$], in the case of the Dirichlet (respectively, Neumann) problem (P). Since $\lambda r = (-\lambda)(-r)$, a similar formula holds for λ_n^-.

In the following, we shall also need to consider the analogue (P_ω) of problem (P), where Ω is replaced by an open set $\omega \subset \Omega$. Given $\lambda > 0$, we then denote by $N_i^+(\lambda; r, \omega, \mathcal{A})$ the number of positive eigenvalues λ_n^+ less than λ of the variational Dirichlet $(i = 0)$ or Neumann $(i = 1)$ problem (P_ω). Similarly, for $\lambda < 0$, we let $N_i^-(\lambda; r, \omega, \mathcal{A})$ be the number of negative eigenvalues λ_n^- larger than λ of the Dirichlet $(i = 0)$ or Neumann $(i = 1)$ problem (P_ω). The notation $N_i^\pm(\lambda; r, \omega, \mathcal{A})$ indicates the dependence on the weight function r, the open set ω and the operator $\mathcal{A}$.

Observe that, by the aforementioned symmetry, $N_i^-(\lambda; r, \omega, \mathcal{A}) = N_i^+(|\lambda|; -r, \omega, \mathcal{A})$. Further, if r is nonnegative in ω, then (P_ω) does not have any negative eigenvalues; in this case, $N_i^-(\lambda; r, \omega, \mathcal{A}) = 0$ and we simply write

$$N_i^+(\lambda; r, \omega, \mathcal{A}) = N_i(\lambda; r, \omega, \mathcal{A}).$$

When no confusion may arise, we write (P) instead of (P_Ω) and $N_i^\pm(\lambda)$ instead of $N_i^\pm(\lambda; r, \Omega, \mathcal{A})$.

In [FlLa, Theorems 4.1 and 4.2, pp. 315 and 316], we have established the following asymptotic formula:

$$N_i^\pm(\lambda; r, \Omega, \mathcal{A}) \sim \Phi(r_\pm, \Omega, \mathcal{A})\,|\lambda|^{k/2m} \quad \text{as } \lambda \to \pm\infty, \tag{12–0}$$

where $i = 0$ or 1 and

$$\Phi(r_\pm, \Omega, \mathcal{A}) := \int_\Omega (r_\pm(x))^{k/2m} \mu'_{\mathcal{A}}(x)\,dx, \tag{12–1}$$

with

$$\mu'_{\mathcal{A}}(x) := (2\pi)^{-k} \left| \left\{ \xi \in \mathbf{R}^k : \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x)\,\xi^{\alpha+\beta} < 1 \right\} \right|; \tag{12–2}$$

note that $\mu'_{\mathcal{A}}$ depends only on the leading part $\mathcal{A}'$ of $\mathcal{A}$.

$$\left(\mathcal{A}' = \sum_{|\alpha| = |\beta| = m} (-1)^{|m|} D^\alpha(a_{\alpha\beta}(x)\,D^\beta) \right).$$

Further, since by (8), $|\Omega_\pm \setminus \Omega_\pm^\circ| = 0$, we clearly have

$$\Phi(r_\pm, \Omega, \mathcal{A}) = \Phi(r, \Omega_\pm^\circ, \mathcal{A}).$$

We next state our main results, in which we obtain the remainder estimates associated with (12–0).

Theorem 1. *We suppose that* Ω, r *and* $\mathcal{A}$ *satisfy* (7), (8) *and* (9)–(10) *as well as condition* $(\mathcal{K}_+)$, *that* $\partial\Omega_+^\circ$ *satisfies the "*β*-condition" and that* r_+ *satisfies the "*γ*-condition" on* Ω, *with* $\beta > 0$ *and* $\gamma \geqq k$. *Set* $\nu = \min(\beta, \gamma - k, \sigma)$. *Then, for all* $\delta \in [1/2m(\nu+1), 1/2m]$, *there exist positive constants* C *and* λ_0 *such that for all* $\lambda \geqq \lambda_0$:

$$\left| N_i^+(\lambda; r, \Omega, \mathcal{A}) - \Phi(r_+, \Omega, \mathcal{A})\, \lambda^{\frac{k}{2m}} \right| \leqq C \lambda^{\frac{k-1}{2m}+\delta},$$

where $i = 0$ *or* 1 *according to the boundary conditions, and* $\Phi(r_+, \Omega, \mathcal{A})$ *is given by* (12–1).

Remarks 5. (a) According to our assumptions, we have $\nu \geqq 0$; further, $\nu = 0$ if and only if $\gamma = k$; in this case, the only possible choice of δ in Theorem 1 is $\delta = 1/2m$.

(b) The constant C in the statement of Theorem 1 depends on k, Ω, $\|r\|_{L^\infty(\Omega)}$, $\max_{|\alpha|,|\beta| \leqq m} \|a_{\alpha\beta}\|_{L^\infty(\Omega)}$, as well as on the constants of uniform ellipticity and uniform coercivity of $\mathcal{A}$ and on the constants c_0 and c_1 which enter in the definitions of the "β-condition" and of the "γ-condition". A similar remark holds for λ_0.

(c) It should be clear from the context whether we work with the scalar β or with the multi-index $\beta = (\beta_1, \ldots, \beta_k)$.

By changing r into $-r$ in Theorem 1 and noting that $(-r)_+ = r_-$, we obtain an analogous result for $N_i^-(\lambda) = N_i^-(\lambda; r, \Omega, \mathcal{A})$:

Theorem 1′. *We assume that the hypotheses of Theorem 1, except with* (8) *replaced by* (8′) *and the subscript "+" replaced by "−", are fulfilled. Then, for all* $\lambda \leqq -\lambda_0 < 0$, *we have for* $i = 0$ *or* 1:

$$\left| N_i^-(\lambda; r, \Omega, \mathcal{A}) - \Phi(\lambda; r_-, \Omega, \mathcal{A})\, |\lambda|^{\frac{k}{2m}} \right| \leqq C\, |\lambda|^{\frac{k-1}{2m}+\delta};$$

where δ, C *and* λ_0 *are as in Theorem 1.*

Remarks 6. (a) In order to obtain good remainder estimates, we want to choose a small δ; since $\nu = \min(\beta, \gamma - k, \sigma)$ and $\delta \geqq 1/2m(\nu+1)$, this implies that β, $\gamma - k$ and σ must be large. Consequently, this means that the boundaries $\partial\Omega$, $\partial\Omega_\pm^\circ$, the weight functions $r_\pm$ and the leading coefficients of $\mathcal{A}$ have to be "smooth". (See Remarks 1.a, 2.b and 2.c.)

(b) If the leading coefficients of $\mathcal{A}$ are constant, then we may set $\sigma = +\infty$ and hence $\nu = \min(\beta, \gamma - k)$. Recall that in the other cases we assume that $\sigma \in (0, 1]$ in (10). Further, as will follow from the proof of Theorem 2, condition $(\mathcal{K}_\pm)$ does not need to be assumed if $\mathcal{A}$ has constant coefficients.

(c) If $r_\pm$ is Hölder continuous of order θ on Ω and is continuous on $\overline{\Omega}$, it follows from Example 2 that $r_\pm$ satisfies the "γ-condition" on Ω with $\gamma - k = k\theta/2m$. Observe that $r_\pm$ is Hölder continuous of order θ on Ω provided, for example, that r is Hölder continuous of order θ on Ω and there exists a positive constant T such that, for all $x \in \Omega_\pm$, we have $r(x) \leqq T[d(x, \Omega_\mp^\sim)]^\theta$, where $\Omega_+^\sim := \{x \in \Omega : r(x) \geqq 0\}$ and $\Omega_-^\sim := \{x \in \Omega : r(x) \leqq 0\}$. Remainder estimates for the Laplacian with an indefinite weight were obtained under similar—although stronger—hypotheses in [La 2, Theorem 3.1].

(d) If $r_{\pm}$ is only continuous on $\bar{\Omega}$ (for instance, if r is continuous on $\bar{\Omega}$), it follows from Example 3 that $r_{\pm}$ satisfies the "γ-condition" on Ω with $\gamma - k = 0$; in this case and without further information on r, we must have $\delta = 1/2m$, according to Remark 5.a.

[The preceding remarks apply, with the obvious changes, to the case of the Laplace operator studied in Section III.]

We note that in [FlLa, § 4] we have proved the asymptotic formula (12–0) under much less stringent assumptions on r, Ω or $\mathscr{A}$. For the Dirichlet problem, it suffices to assume that Ω is an arbitrary bounded open set such that $|\Omega_{\pm}^{\circ}| > 0$ and $|\Omega_{\pm} \setminus \Omega_{\pm}^{\circ}| = 0$, and that $r \in L^p(\Omega)$, with $p > k/2m$ if $k \geqq 2m$ and $p = 1$ otherwise [FlLa, Theorem 4.1, p. 315]. For the Neumann problem [FlLa, Theorem 4.2, p. 316], we also suppose that $|\partial\Omega| = 0$ (which, in view of Remark 1.b, is always true if $\partial\Omega$ satisfies the "β-condition"), and that Ω has the "segment property". It is clear, however, that the additional "smoothness" assumptions that we have made in order to obtain the above remainder estimates are not just of a technical nature.

In [FlLa, § 5, Theorem 5.1, p. 318, and Proposition 5.4, p. 321], we have also determined the asymptotic distribution of the eigenvalues of Schrödinger operators of order $2m$ acting on unbounded open sets with indefinite weight functions. We intend to obtain the corresponding remainder estimates in a later work.

III. The Laplace Operator

In this section, we examine the case of the Laplacian on a bounded open set. In Section IV, we shall then study the case of a general positive elliptic selfadjoint operator of order $2m$. This division should make the arguments easier to follow. Further, the case of the Laplacian is of independent interest because of its importance in analysis and differential geometry.

III.1. Remainder estimates

We consider the variational eigenvalue problem

$$(P) \qquad -\Delta u = \lambda r(x)\, u, \qquad x \in \Omega,$$

with Dirichlet ($i = 0$) or Neumann ($i = 1$) boundary conditions (see [FlLa, § 3, pp. 309–314]); here $\Delta = \sum_{j=1}^{k} \partial^2/\partial x_j^2$ stands for the distributional Laplacian in $\boldsymbol{R}^k$.

Since $\mathscr{A} = -\Delta$ is only nonnegative, $\lambda = 0$ is an eigenvalue of the Neumann problem (P_ω) for $\omega \subset \Omega$; consequently, we interpret $N_{\mathrm{I}}^{+}(\lambda; r, \omega, -\Delta)$ [respectively, $N_{\mathrm{I}}^{-}(\lambda; r, \omega, -\Delta)$] as the number of nonnegative (respectively, nonpositive) eigenvalues less than $\lambda > 0$ (respectively, less than $\lambda < 0$) of the Neumann problem (P_ω); when r is nonnegative, a similar convention applies to $N_1(\lambda; r, \omega, -\Delta)$.

We introduce the following hypothesis:

$(\mathscr{H}_\pm)$ For $\omega = \Omega \setminus \overline{\Omega_\pm}$, the eigenvalue 0 of the Neumann problem (P_ω) has finite multiplicity.

Remarks 7. (a) Note that in Section II (and Section IV below), the operator $\mathscr{A}$ is required to be positive and hence, for all open sets $\omega \subset \Omega$, 0 is *not* an eigenvalue of the Neumann problem (P_ω). On the other hand, condition $(\mathscr{H}_+)$ or $(\mathscr{H}_-)$ will *not* need to be assumed in Theorem 2 or 2′ below, just as for operators with constant coefficients (see Remark 6.b).

(b) In [FlLa, § 3], in order to deal with the zero eigenvalue, we tacitly replaced $-\Delta$ by $-\Delta + \tau$ (with $\tau > 0$) in our derivation of the asymptotic formula [FlLa, Theorems 3.1 and 3.2, pp. 310 and 312]. (See [FlLa, Remark, p. 314].) This substitution does not affect the result since for Ω bounded, it yields a compact perturbation of $-\Delta$.

(c) Hypothesis $(\mathscr{H}_\pm)$ holds, for example, if $\Omega \setminus \overline{\Omega_\pm}$ has a finite number of connected components, each of which is simply connected and has piecewise continuously differentiable boundary.

(d) If r is positive in Ω, then $\Omega \setminus \overline{\Omega_+} = \emptyset$ and hypothesis $(\mathscr{H}_+)$ is trivially satisfied.

In the present situation, $\mathscr{A} = -\Delta$, $m = 1$, $\mu'_{-\Delta}(x) = (2\pi)^{-k}\mathscr{B}_k$ and

$$\Phi(r_\pm, \Omega, -\Delta) = (2\pi)^{-k}\mathscr{B}_k \, \|r_\pm\|^{k/2}_{L^{k/2}(\Omega)} = (2\pi)^{-k}\, \mathscr{B}_k \, \|r\|^{k/2}_{L^{k/2}(\Omega_\pm)};$$

here we have used (12–1) and (12–2) and let $\mathscr{B}_k = \pi^{k/2}/\Gamma(1 + (k/2))$ denote the volume of the unit ball in $\mathbf{R}^k$.

We can now state the analogue of Theorem 1 for the Laplace operator.

Theorem 2. *We suppose that Ω and r satisfy* (7), (8) *and hypothesis* $(\mathscr{H}_+)$, *that $\partial\Omega_\pm^\circ$ satisfies the "β-condition" and that r_+ satisfies the "γ-condition" on Ω, with $\beta > 0$ and $\gamma \geqq k$. Set $\nu = \min(\beta, \gamma - k)$. Then, for all $\delta \in [1/2(\nu + 1), 1/2]$, there exist constants C and λ_0 such that for all $\lambda \geqq \lambda_0$ and for $i = 0$ or* 1:

$$\left| N_i^+(\lambda; r, \Omega, -\Delta) - (2\pi)^{-k}\, \mathscr{B}_k \, \|r_+\|^{k/2}_{L^{k/2}(\Omega)} \lambda^{\frac{k}{2}} \right| \leqq C\lambda^{\frac{k-1}{2}+\delta}.$$

Similarly, by substituting r for $-r$, we obtain:

Theorem 2′. *We assume that the hypotheses of Theorem 2 hold, except with* (8) *replaced by* (8′) *and the subscript "+" replaced by "−", are fulfilled. Let δ, λ_0 and C be as in Theorem 2. Then, for all $\lambda \leqq -\lambda_0 < 0$, we have for $i = 0$ or* 1:

$$\left| N_i^-(\lambda; r, \Omega, -\Delta) - (2\pi)^{-k}\, \mathscr{B}_k \, \|r_-\|^{k/2}_{L^{k/2}(\Omega)} |\lambda|^{\frac{k}{2}} \right| \leqq C|\lambda|^{\frac{k-1}{2}+\delta}.$$

When $r \equiv 1$, Theorem 2 yields the remainder estimates associated with the classical HERMANN WEYL's formula. (See, *e.g.*, [CoHi, § VI.5, pp. 443–445], [Mt, § VI, pp. 194–199], [Iv] and [Ho 2, Vol. III, § XVII.5, pp. 42–62 and Vol. IV, § XXIX, pp. 249–275] for instances of such estimates.)

More generally, it is instructive to consider the following example. Let Ω be a bounded open set in $\boldsymbol{R}^k$; let $\Omega_\pm$ be measurable subsets of Ω such that $|\Omega_\pm| > 0$, $|\Omega_\pm \setminus \Omega_\pm^\circ| = 0$, and $\Omega_\pm^\circ$ has the "segment property" (for instance, $\partial\Omega_\pm^\circ$ is locally Lipschitz continuos) and satisfies hypothesis $(\mathscr{H}_\pm)$ [see Remark 7.c above for a simple sufficient condition]. Define a weight function r on Ω by $r(x) = -1$ if $x \in \Omega_-$, $+1$ if $x \in \Omega_+$, and 0 otherwise. In view of Example 1, we then obtain from Theorems 2 and 2′ the remainder estimates associated with the "signed HERMANN WEYL's formula", thereby sharpening [La 1, Corollary 2, p. 266] and [FlLa, Corollary 3.1, p. 312]:

Corollary 1. *Suppose that $\partial\Omega$, $\partial\Omega_+^\circ$ and $\partial\Omega_-^\circ$ satisfy the "β-condition" for some $\beta > 0$. Then, for any $\delta \in [1/2(\beta + 1), 1/2]$ and for $i = 0$ or 1, we have*:

$$N_i^\pm(\lambda) = (2\pi)^{-k}\, \mathscr{B}_k\, |\Omega_\pm|\, |\lambda|^{\frac{k}{2}} + O(|\lambda|^{\frac{k-1}{2}+\delta}),$$

as $\lambda \to \pm\infty$, respectively.

III.2. Proof of Theorem 2

We shall establish Theorem 2 in two steps: first, in the case of a positive weight (Section III.2.A) and then, in the case of an indefinite weight (Section III.2.B).

III.2.A. Positive weight

We suppose here that r is a (strictly) positive weight function. We shall prove the following result, which, under our assumptions, does not seem to be known for $r \not\equiv 1$.

Proposition 1. *Theorem 1 holds if r is positive on Ω.*

We note that in this case, the problem (P) does not have any negative eigenvalues; moreover, $r = r_+$, $r_- = 0$, $\Omega = \Omega_+ = \Omega_+^\circ$, $\Omega_- = \emptyset$ and $N_i^+(\lambda; r, \omega, \mathscr{A}) = N_i(\lambda; r, \omega, \mathscr{A})$, for $\omega \subset \Omega$ and $i = 0,1$.

According to Remark 7.d, hypothesis $(\mathscr{H}_+)$ does not need to be assumed in Proposition 1.

Proof of Proposition 1. It is well known (see, *e.g.*, [CoHi, Section VI.4] and [ReSi, Proposition 2, pp. 266–267]) that for a cube Q_ξ with side of length η, there exists $c_1 > 0$ such that for all $\lambda > 0$ and for $i = 0$ or 1:

$$\left| N_i(\lambda; 1, Q_\zeta, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \eta^k \lambda^{\frac{k}{2}} \right| \leqq c_1 (1 + (\lambda\eta^2)^{\frac{k-1}{2}});$$

here the constant c_1 is independent of λ, η, ζ, or the number r_ζ introduced below.

Since $N_i(\lambda r_\zeta; 1, Q_\zeta, -\Delta) = N_i(\lambda; r_\zeta, Q_\zeta, -\Delta)$, it follows that if r_ζ is a positive number, then for $i = 0$ or 1 and all $\lambda > 0$:

$$\left| N_i(\lambda; r_\zeta, Q_\zeta, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \eta^k (\lambda r_\zeta)^{\frac{k}{2}} \right| \leqq c_1 (1 + (\lambda r_\zeta \eta^2)^{\frac{k-1}{2}}). \tag{13}$$

We now consider the tessellation $\{Q_\zeta\}_{\zeta \in Z^k}$ of $\boldsymbol{R}^k$ by cubes of sides of length η, as defined at the beginning of Section II; throughout the remainder of the proof of Proposition 1, we set, as in (2),

$$I = I(\Omega) \quad \text{and} \quad J = J(\Omega). \tag{14}$$

By using the method of "Dirichlet-Neumann bracketing", we obtain:

$$\begin{aligned} \sum_{\zeta \in I} N_0(\lambda; r, Q_\zeta, -\Delta) &\leqq N_0(\lambda; r, \Omega, -\Delta) \leqq N_1(\lambda; r, \Omega, -\Delta) \\ &\leqq \sum_{\zeta \in I} N_1(\lambda; r, Q_\zeta, -\Delta) + \sum_{\zeta \in J \setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, -\Delta). \end{aligned} \tag{15}$$

We briefly pause to explain the steps leading to (15). Define two disjoint open sets by

$$\Omega_1 = \bigcup_{\zeta \in I} Q_\zeta \quad \text{and} \quad \Omega_2 = \bigcup_{\zeta \in J \setminus I} (Q_\zeta \cap \Omega). \tag{16}$$

In view of [FlLa, Lemmas 4.3 and 5.1, pp. 315 and 317], the middle inequality of (15) is immediate while that on the left results from the inclusion $\Omega_1 \subset \Omega$ and that on the right follows from the fact that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, up to a set of Lebesgue measure zero. (We mention for later use that these results in [FlLa] also hold for indefinite weight functions as well as for more general elliptic operators; for an exposition of the method of "Dirichlet-Neumann bracketing", we refer to [CoHi, Chapter VI] and, in the classical case where $r \equiv 1$, to [Mt] or [ReSi, Section XVIII.15].)

Set

$$\varphi(\lambda) = (2\pi)^{-k} \mathscr{B}_k \lambda^{k/2} \int_\Omega r^{k/2} \tag{17–1}$$

and, for $\zeta \in I$,

$$\varphi(\lambda, \zeta) = (2\pi)^{-k} \mathscr{B}_k \lambda^{k/2} \eta^k r_\zeta^{k/2}, \tag{17–2}$$

where r_ζ is defined by (3) [except with f replaced by r]; note that since r is positive (Lebesgue almost everywhere) in Ω, it follows from (3) that $r_\zeta > 0$ for every $\zeta \in I$.

According to (15), we have:

$$\begin{aligned}(18)\qquad &\sum_{\zeta\in I} N_0(\lambda; r, Q_\zeta, -\Delta) - \varphi(\lambda)\\ &\leqq N_0(\lambda; r, \Omega, -\Delta) - \varphi(\lambda) \leqq N_1(\lambda; r, \Omega, -\Delta) - \varphi(\lambda)\\ &\leqq \sum_{\zeta\in I} N_1(\lambda; r, Q_\zeta, -\Delta) - \varphi(\lambda) + \sum_{\zeta\in J\setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, -\Delta).\end{aligned}$$

We shall now first find a lower bound for $N_0(\lambda; r, \Omega, -\Delta) - \varphi(\lambda)$ by use of the left-hand side of (18); then we shall obtain an analogous upper bound for $N_1(\lambda; r, \Omega, -\Delta) - \varphi(\lambda)$ by means of the right-hand side of (18).

Step 1: *a lower bound.* We write

$$(19\text{–}0)\qquad A = \sum_{\zeta\in I} N_0(\lambda; r, Q_\zeta, -\Delta) - \varphi(\lambda) = A_1 + A_2 + A_3,$$

where

$$(19\text{–}1)\qquad A_1 = \sum_{\zeta\in I} [N_0(\lambda; r_\zeta, Q_\zeta, -\Delta) - \varphi(\lambda, \zeta)],$$

$$(19\text{–}2)\qquad A_2 = \sum_{\zeta\in I} \varphi(\lambda, \zeta) - \varphi(\lambda),$$

and

$$(19\text{–}3)\qquad A_3 = \sum_{\zeta\in I} [N_0(\lambda; r, Q_\zeta, -\Delta) - N_0(\lambda; r_\zeta, Q_\zeta, -\Delta)].$$

Although the lower bound is only concerned with the Dirichlet boundary conditions, we shall work both with Dirichlet $(i = 0)$ and Neumann $(i = 1)$ boundary conditions in order to avoid repeating an analogous calculation in the study of the upper bound.

It follows from (13) and (17–2) that there exists $c_1 > 0$ such that for all $\zeta \in I$ and $i = 0$ or 1,

$$|N_i(\lambda; r_\zeta, Q_\zeta, -\Delta) - \varphi(\lambda, \zeta)| \leqq c_1(1 + (\lambda\eta^2 r_\zeta)^{\frac{k-1}{2}}).$$

Since r is bounded on Ω, it follows from (6) that

$$(20)\qquad \sup_{\zeta\in I} r_\zeta \leqq \|r\|_{L^\infty(\Omega)} =: M < +\infty.$$

Hence

$$(21)\qquad |N_i(\lambda; r_\zeta, Q_\zeta, -\Delta) - \varphi(\lambda, \zeta)| \leqq c(1 + (\lambda\eta^2)^{\frac{k-1}{2}}),$$

where c does not depend on λ, η or $\zeta \in I$.

Then, according to (21),

$$(22)\qquad \sum_{\zeta\in I} |N_i(\lambda; r_\zeta, Q_\zeta, -\Delta) - \varphi(\lambda, \zeta)| \leqq c(\#I)\,(1 + (\lambda\eta^2)^{\frac{k-1}{2}}).$$

By (2–1) and (14) and since $|Q_\zeta| = \eta^k$,

(23) $$\#I \leqq \eta^{-k}\,|\Omega|;$$

thus, choosing, for λ large enough $(\lambda \geqq 1)$,

(24) $$\eta = \lambda^{-a} \quad \text{with } 0 < a \leqq \delta,$$

we deduce that

(25) $$(\#I)\,(1 + (\lambda\eta^2)^{\frac{k-1}{2}}) \leqq |\Omega|\,(\lambda^{\delta k} + \lambda^{\frac{k-1}{2}+\delta}).$$

Since $\delta \leqq 1/2$ (by hypothesis of Theorem 2) and $k \geqq 1$, we have:

$$\delta k = \frac{k-1}{2} + \delta - (k-1)\,(\tfrac{1}{2} - \delta) \leqq \frac{k-1}{2} + \delta;$$

and hence, by combining (19–1), (25) and (22), with $i = 0$, we obtain, for λ large enough:

(26) $$|A_1| \leqq c\lambda^{\frac{k-1}{2}+\delta}.$$

We now find an upper bound for $|A_2|$. By (14), (16), (17) and (19–2),

$$\begin{aligned} -A_2 &= \varphi(\lambda) - \sum_{\zeta\in I} \varphi(\lambda,\zeta) \\ &= (2\pi)^{-k}\,\mathscr{B}_k\lambda^{k/2}\left[\int_\Omega r^{k/2} - \sum_{\zeta\in I}\int_{Q_\zeta} r_\zeta^{k/2}\right] \\ &= (2\pi)^{-k}\,\mathscr{B}_k\lambda^{k/2}\sum_{\zeta\in I}\int_{Q_\zeta}[r^{k/2} - r_\zeta^{k/2}] + (2\pi)^{-k}\,\mathscr{B}_k\lambda^{k/2}\sum_{\zeta\in J\setminus I}\int_{Q_\zeta\cap\Omega} r^{k/2}. \end{aligned}$$

In the last equality, we have used implicitly the fact that the cubes Q_ζ are disjoint and that Ω_1 and Ω_2 in (16) form a partition of Ω, up to a Lebesgue null set.

By the very definition of r_ζ [see (3) or (6)], we have

$$\int_{Q_\zeta}[r^{k/2} - r_\zeta^{k/2}] = 0 \quad \text{for all } \zeta \in I,$$

and so,

$$|A_2| = (2\pi)^{-k}\,\mathscr{B}_k\lambda^{k/2}\sum_{\zeta\in J\setminus I}\int_{Q_\zeta\cap\Omega} r^{k/2}.$$

Since Ω and r are bounded, we deduce that

$$|A_2| \leqq c\lambda^{k/2}\eta^k\,\#(J\setminus I).$$

Since $\partial\Omega$ satisfies the "β-condition", we obtain by combining (1), (14), (23) and (24) that

(27) $$\#(J\setminus I) \leqq c_0\,|\Omega|\,\eta^{\beta-k} = c_0\,|\Omega|\,\lambda^{-a(\beta-k)},$$

for sufficiently large λ. Consequently,

$$|A_2| \leqq c\lambda^{\frac{k}{2}-a\beta} \leqq c\lambda^{\frac{k-1}{2}+\delta}, \tag{28}$$

provided that $a \geqq \frac{1}{\beta}[\frac{1}{2}-\delta]$; since $\nu := \min(\beta, \gamma-k) \leqq \beta$, this latter condition will certainly hold if we choose a positive a such that

$$\frac{1}{\nu}(\tfrac{1}{2}-\delta) \leqq a \leqq \delta. \tag{29}$$

Note that such a choice is possible since by hypothesis of Theorem 2, $\frac{1}{2(\nu+1)} \leqq \delta \leqq \frac{1}{2}$ and hence $\frac{1}{\nu}(\frac{1}{2}-\delta) \leqq \delta$. (If $\delta = \frac{1}{2}$, then we let $a = \frac{1}{2}$ in (29).)

From now on, we assume that $\eta = \lambda^{-a}$ with a satisfying (29).

Observe that (27) holds for $\eta \leqq \eta_0$, by (1), and thus, we must consider $\lambda \geqq \lambda_1$ with $\lambda_1 = (\eta_0)^{-\frac{2\nu}{1-2\delta}}$. (If $\delta = \frac{1}{2}$, we set $\lambda_1 = (\eta_0)^{-2}$.)

We now consider A_3. Since $r = (r - r_\zeta) + r_\zeta$, we deduce from the "max-min formula" (11) and the monotonicity of the eigenvalues with respect to the weight (see, *e.g.*, Eq. (3.2) and Lemma 3.2, p. 310, in [FlLa], and the proof of Lemma 2.3 in [La 2]) that, for $\zeta \in I$ and $i = 0$ or 1:

$$N_i(\lambda; r, Q_\zeta, -\Delta) \leqq N_i(\lambda; r_\zeta, Q_\zeta, -\Delta) + N_i(\lambda; |r - r_\zeta|, Q_\zeta, -\Delta). \tag{30}$$

Therefore

$$|N_i(\lambda; r, Q_\zeta, -\Delta) - N_i(\lambda; r_\zeta, Q_\zeta, -\Delta)| \leqq N_i(\lambda; |r - r_\zeta|, Q_\zeta, -\Delta). \tag{31}$$

By (4) and by [FlLa], Theorem 3.1, p. 310, or 3.2, p. 312, for $i = 0$ or 1, respectively, there exists $c' > 0$ such that for all λ large enough and all $\zeta \in I$.

$$N_i(\lambda; |r - r_\zeta|, Q_\zeta, -\Delta) \leqq c'\lambda^{k/2}\varrho_\zeta(r). \tag{32}$$

We note that in this case,

$$N_i^+(\lambda; |r - r_\zeta|, Q_\zeta, -\Delta) = N_i(\lambda; |r - r_\zeta|, Q_\zeta, -\Delta)$$

since $|r - r_\zeta|$ is nonnegative.

By hypothesis, r satisfies the "γ-condition" on Ω; thus, by combining (5), (23), (24), (29), (31) and (32), we have:

$$\sum_{\zeta \in I} |N_i(\lambda; r, Q_\zeta, -\Delta) - N_i(\lambda; r_\zeta, Q_\zeta, -\Delta)| \leqq c'\lambda^{\frac{k}{2}} \sum_{\zeta \in I} \varrho_\zeta(r) \tag{33}$$

$$\leqq c_1 c' \lambda^{\frac{k}{2}} \#(I)\, \eta^\gamma \leqq c_1 c' \, |\Omega| \, \lambda^{\frac{k-1}{2}+\frac{1}{2}-a(\gamma-k)} \leqq c\lambda^{\frac{k-1}{2}+\delta},$$

provided that $(1/2) - a(\gamma - k) \leqq \delta$ or, equivalently,

$$a(\gamma - k) \geqq \tfrac{1}{2} - \delta \geqq 0. \tag{34}$$

This requires in particular that $\gamma \geqq k$, which is the case by assumption of Theorem 2. Now, by (29) and since $\nu := \min(\beta, \gamma - k) \leqq \gamma - k$,

$$a \geqq \frac{1}{\nu}(\tfrac{1}{2} - \delta) \geqq \frac{1}{\gamma - k}(\tfrac{1}{2} - \delta);$$

hence (34) holds. (If $\gamma = k$, then $\delta = a = \frac{1}{2}$; see Remark 5.a.)

In light of (19–3) and (31) to (33), with $i = 0$, we obtain:

$$(35) \qquad |A_3| \leqq c\lambda^{\frac{k-1}{2}+\delta}.$$

According to (19), (26), (28) and (35), we now have:

$$|A| \leqq c\lambda^{\frac{k-1}{2}+\delta}.$$

Consequently, we deduce from (18) and (19–0) the following lower bound:

$$(36) \qquad N_0(\lambda; r, \Omega, -\Delta) - \varphi(\lambda) \geqq -c\lambda^{\frac{k-1}{2}+\delta},$$

for all λ large enough.

This completes Step 1 and concludes the first part of the proof of Proposition 1.

Step 2: *an upper bound.* We now find an upper bound for

$$N_1(\lambda; r, \Omega, -\Delta) - \varphi(\lambda).$$

Much as in the previous part, we write

$$(37\text{–}0) \qquad \begin{aligned} B &= \sum_{\zeta \in I} N_1(\lambda; r, Q_\zeta, -\Delta) - \varphi(\lambda) + \sum_{\zeta \in J \setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, -\Delta) \\ &= B_1 + B_2 + B_3 + B_4, \end{aligned}$$

where

$$(37\text{–}1) \qquad B_1 = \sum_{\zeta \in I} [N_1(\lambda; r_\zeta, Q_\zeta, -\Delta) - \varphi(\lambda, \zeta)],$$

$$(37\text{–}2) \qquad B_2 = \sum_{\zeta \in I} \varphi(\lambda, \zeta) - \varphi(\lambda),$$

$$(37\text{–}3) \qquad B_3 = \sum_{\zeta \in I} [N_1(\lambda; r, Q_\zeta, -\Delta) - N_1(\lambda; r_\zeta, Q_\zeta, -\Delta)],$$

and

$$(37\text{–}4) \qquad B_4 = \sum_{\zeta \in J \setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, -\Delta).$$

The first three terms can be handled exactly as in Step 1. Note, in particular, that for $j = 1$ or 3, B_j in (37–j) is defined just as A_j in (19–j), except with the subscript 0 replaced by 1.

In view of (13), with $i=1$, (17–2) and (37–1), we have:

$$|B_1| \leqq c\lambda^{\frac{k-1}{2}+\delta}. \tag{38}$$

Moreover, $B_2 = A_2$ by (19–2) and (37–2), and so, by (28):

$$|B_2| \leqq c\lambda^{\frac{k-1}{2}+\delta}. \tag{39}$$

Finally, in light of (33), with $i=1$, and (37–3), we obtain:

$$|B_3| \leqq c\lambda^{\frac{k-1}{2}+\delta}. \tag{40}$$

We now examine B_4, which is a boundary term, characteristic of Neumann boundary conditions, and without counterpart in our study of the lower bound in Step 1. By Eq. (3.2), p. 310, and Theorem 3.2, p. 312, in [FlLa], and by (20), there exists a constant c', depending only on k, such that for all $\zeta \in J \setminus I$ and all λ sufficiently large, we have:

$$N_1(\lambda; r, Q_\zeta \cap \Omega, -\Delta) \leqq c'\lambda^{k/2} \int_{Q_\zeta \cap \Omega} r^{k/2} \leqq c'(M\eta^2\lambda)^{k/2}.$$

Hence, by (1), (14), (27) and (37–4), and with the previous choice of $\eta = \lambda^{-a}$, we obtain:

$$|B_4| \leqq c\eta^k \lambda^{k/2} \,\#(J \setminus I) \leqq c\lambda^{\frac{k}{2}-a\beta}.$$

Since a satisfies (29) we find, just as in (28), that

$$|B_4| \leqq c\lambda^{\frac{k-1}{2}+\delta}. \tag{41}$$

In view of (37) and (38) to (41),

$$|B| \leqq c\lambda^{\frac{k-1}{2}+\delta}.$$

Therefore, we deduce from (18) and (37–0) the following upper bound:

$$N_1(\lambda; r, \Omega, -\Delta) - \varphi(\lambda) \leqq c\lambda^{\frac{k-1}{2}+\delta}, \quad \text{for all } \lambda \text{ large enough}. \tag{42}$$

This completes Step 2.

We now combine (17–1), (18), (36) and (42) to conclude that there exist positive constants C and λ_0 such that for all $\lambda \geqq \lambda_0$ and for $i=0$ or 1:

$$\left| N_i(\lambda; r, \Omega, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \lambda^{\frac{k}{2}} \|r\|_{L^{k/2}(\Omega)}^{k/2} \right| \leqq C\lambda^{\frac{k-1}{2}+\delta}.$$

This establishes Proposition 1. □

III.2.B. Indefinite weight

We now establish Theorem 2. So we assume that the weight function r changes sign in Ω (*i.e.*, is indefinite) and that (7), (8) and hypothesis $(\mathscr{H}_+)$ hold. We also suppose that $\partial\Omega_+^\circ$ satisfies the "β-condition" and that r_+ satisfies the "γ-condition" on Ω for $\gamma \geqq k$; according to Definition 2 and since $r = r_+$ on $\Omega_+^\circ \subset \Omega$, this implies, in particular, that r satisfies the "γ-condition" on Ω_+° for the same value of γ.

Proof of Theorem 2. We use an extension of the method of "Dirichlet-Neumann bracketing" to variational eigenvalue problems with indefinite weights [FlLa, Lemmas 4.3 and 5.1, pp. 315 and 317] and obtain, for all positive λ:

$$(43) \qquad \begin{aligned} N_0(\lambda; r, \Omega_+^\circ, -\Delta) &\leqq N_0^+(\lambda; r, \Omega, -\Delta) \leqq N_1^+(\lambda; r, \Omega, -\Delta) \\ &\leqq N_1(\lambda; r, \Omega_+^\circ, -\Delta) + N_1^+(\lambda; r, \Omega \setminus \overline{\Omega_+}, -\Delta). \end{aligned}$$

According to hypothesis $(\mathscr{H}_+)$ and since $r \leqq 0$ almost everywhere in $\Omega \setminus \overline{\Omega_+}$, we have, for every positive λ:

$$N_1(\lambda; r, \Omega \setminus \overline{\Omega_+}, -\Delta) = l < +\infty,$$

where l denotes the dimension of the kernel of the distributional Laplacian acting on $\Omega \setminus \overline{\Omega_+}$ with Neumann boundary conditions. (By $\overline{\Omega_+}$, we mean here the closure of Ω_+ in Ω.)

Since, on the open set Ω_+°, the weight function r is positive, we know from Proposition 1 that there exist positive constants C' and $\lambda_0 > 1$ such that for all $\lambda \geqq \lambda_0$ and for $i = 0$ or 1:

$$(44) \qquad \left| N_i(\lambda; r, \Omega_+^\circ, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \|r\|_{L^{k/2}(\Omega_+^\circ)}^{k/2} \lambda^{\frac{k}{2}} \right| \leqq C' \lambda^{\frac{k-1}{2}+\delta};$$

note that we use here the fact that $\partial\Omega_+^\circ$ satisfies the "β-condition" and that r satisfies the "γ-condition" on Ω_+°, with $\gamma \geqq k$; further, since $|\Omega_+ \setminus \Omega_+^\circ| = 0$, we have

$$\|r\|_{L^{k/2}(\Omega_+^\circ)}^{k/2} = \int_{\Omega_+} r^{k/2} = \int_{\Omega} r_+^{k/2} = \|r_+\|_{L^{k/2}(\Omega)}^{k/2}.$$

By combining (43) and (44), we thus see that for all $\lambda \geqq \lambda_0$:

$$\begin{aligned} -C'\lambda^{\frac{k-1}{2}+\delta} &\leqq N_0^+(\lambda; r, \Omega, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \|r_+\|_{L^{k/2}(\Omega)}^{k/2} \lambda^{\frac{k}{2}} \\ &\leqq N_1^+(\lambda; r, \Omega, -\Delta) - (2\pi)^{-k} \mathscr{B}_k \|r_+\|_{L^{k/2}(\Omega)}^{k/2} \lambda^{\frac{k}{2}} \\ &\leqq C'\lambda^{\frac{k-1}{2}+\delta} + l \leqq (C' + l)\,\lambda^{\frac{k-1}{2}+\delta}. \end{aligned}$$

It now suffices to let $C = C' + l$ in order to obtain the desired conclusion of Theorem 2. □

IV. Elliptic Operators with Variable Coefficients

We now assume that the hypotheses of Theorem 1 hold. In particular, $\mathscr{A}$ is a positive self-adjoint and uniformly elliptic operator of order $2m$ associated with a coercive form a satisfying (9) and (10).

In the following derivation of Theorem 1, we shall often refer to the proof of Theorem 2 provided in Section III, in order to shorten or clarify our exposition. Naturally, from a technical point of view, the main additional difficulty lies in the fact that $\mathscr{A}$ has variable coefficients and is not homogeneous.

Proof of Theorem 1. First, we note that we may assume that the weight function r is positive in Ω. Indeed, in view of [FlLa, Lemmas 4.3 and 5.1, pp. 315 and 317], the passage from the case of a positive weight to that of an indefinite weight can then be handled just as in the proof of Theorem 2 in Section III.2.B; actually, it is even easier since $\mathscr{A}$ being a positive operator, zero is never an eigenvalue of the Neumann problem (P_ω) for $\omega \subset \Omega$ (so that $l = 0$, with the notation of Section III.2.B).

From now on, we suppose that r is positive in Ω. Consequently, $r = r_+$, $r_- = 0$, $\Omega_+ = \Omega_+^\circ = \Omega$, $\Omega_- = \emptyset$, $N_i^-(\lambda; r, \omega, \mathscr{A}) = 0$ and $N_i^+(\lambda; r, \omega, \mathscr{A}) = N_i(\lambda; r, \omega, \mathscr{A})$ for $i = 0$ or 1 and $\omega \subset \Omega$.

We work as in Section III.2.A with the tessellation of $\boldsymbol{R}^k$ by cubes Q_ζ with center x_ζ and sides of length η. We let, throughout the proof of Theorem 1, $I = I(\Omega)$ and $J = J(\Omega)$, as in (14). By means of the "Dirichlet-Neumann bracketing" [FlLa, Lemmas 4.3 and 5.1, pp. 315 and 317], we then obtain the counterpart of (15):

$$(45) \qquad \sum_{\zeta \in I} N_0(\lambda; r, Q_\zeta, \mathscr{A}) \leqq N_0(\lambda; r, Q_\zeta, \mathscr{A}) \leqq N_1(\lambda; r, \Omega, \mathscr{A}) \\ \leqq \sum_{\zeta \in I} N_1(\lambda; r, Q_\zeta, \mathscr{A}) + \sum_{\zeta \in J \setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, \mathscr{A}).$$

We set, by analogy to (17),

$$(46\text{–}1) \qquad \varphi(\lambda) = \Phi(r, \Omega, \mathscr{A})\, \lambda^{k/2m}$$

and, for $\zeta \in I$,

$$(46\text{–}2) \qquad \varphi(\lambda, \zeta) = \Phi(r_\zeta, Q_\zeta, \mathscr{A}'_\zeta)\, \lambda^{k/2m} = \mu_\zeta \eta^k (\lambda r_\zeta)^{k/2m}.$$

Here $\Phi(r, \Omega, \mathscr{A})$ and $\Phi(r_\zeta, Q_\zeta, \mathscr{A}'_\zeta)$ are defined as in (12–1); further, $\mathscr{A}'_\zeta$ is the operator obtained by "freezing" the leading part $\mathscr{A}'$ of $\mathscr{A}$ at the center x_ζ of Q_ζ, r_ζ is given by (3) [with f replaced by r] and $\mu_\zeta := \mu'_{\mathscr{A}'_\zeta}(x) = \mu'_{\mathscr{A}}(x_\zeta)$, with $\mu'_{\mathscr{A}}$ as in (12–2). Observe that $\varphi(\lambda, \zeta)$ corresponds to the homogeneous operator of order $2m$ and with constant coefficients, $\mathscr{A}'_\zeta$, acting on the cube Q_ζ, and with constant positive weight equal to r_ζ.

Proceeding as in (18), (19) and (37) and using similar notation, we derive from (45) the following inequality:

$$
\begin{aligned}
(47) \qquad T_{0,1} + T_{0,2} + T_{0,3} &\leqq N_0(\lambda; r, \Omega, \mathscr{A}) - \varphi(\lambda) \\
&\leqq N_1(\lambda; r, \Omega, \mathscr{A}) - \varphi(\lambda) \\
&\leqq T_{1,1} + T_{1,2} + T_{1,3} + T_{1,4},
\end{aligned}
$$

where, for $i = 0$ or 1,

$$(48\text{-}1) \qquad T_{i,1} = \sum_{\zeta \in I} [N_i(\lambda; r_\zeta, Q_\zeta, \mathscr{A}) - \varphi(\lambda, \zeta)],$$

$$(48\text{-}2) \qquad T_{i,2} = \sum_{\zeta \in I} \varphi(\lambda, \zeta) - \varphi(\lambda),$$

$$(48\text{-}3) \qquad T_{i,3} = \sum_{\zeta \in I} [N_i(\lambda; r, Q_\zeta, \mathscr{A}) - N_i(\lambda; r_\zeta, Q_\zeta, \mathscr{A})],$$

and where

$$(48\text{-}4) \qquad T_{1,4} = \sum_{\zeta \in J \setminus I} N_1(\lambda; r, Q_\zeta \cap \Omega, \mathscr{A}).$$

We assume that

$$(49) \qquad \eta = \lambda^{-a} \quad \text{with} \quad 0 < a \leqq \delta.$$

The terms $T_{i,3}$ $(i = 0,1)$ and $T_{1,4}$ can be handled exactly as their counterparts in Section III.2.B:

We note that $\mu'_{\mathscr{A}}$ is bounded on Ω; this results from (12–2) and from the uniform ellipticity of $\mathscr{A}$. As in the derivation of (41), it thus follows from [FlLa, Theorem 4.2, p. 316] (see Eq. (12) above with $i = 1$) and (20), (27), and (48–4) that there exist positive constants c, c' such that for all λ large enough,

$$(50) \qquad |T_{1,4}| \leqq c' \lambda^{k/2m} \eta^k \,\#(J \setminus I) \leqq c\lambda^{\frac{k-1}{2} + \frac{1}{2m} - a\beta} \leqq c\lambda^{\frac{k-1}{2m} + \delta},$$

provided that $(1/2m) - a\beta \leqq \delta$ or, equivalently,

$$(51) \qquad \frac{1}{\beta}\left(\frac{1}{2m} - \delta\right) \leqq a.$$

As in the derivation of (35) and (40), we deduce from the "max-min formula" (11) and [FlLa, Theorems 4.1 and 4.2, pp. 315 and 316] that for all $\zeta \in I$ and $i = 0$ or 1,

$$|N_i(\lambda; r, Q_\zeta, \mathscr{A}) - N_i(\lambda; r_\zeta, Q_\zeta, \mathscr{A})| \leqq N_i^+(\lambda; |r - r_\zeta|, Q_\zeta, \mathscr{A}) \leqq c'' \lambda^{k/2m} \varrho_\zeta(r),$$

with $\varrho_\zeta(r)$ given by (4).

Consequently, since r satisfies the "γ-condition" and according to (5), (23), (48–3) and (49), we have for $i = 0$ or 1:

$$(52) \qquad |T_{i,3}| \leqq c' \lambda^{\frac{k}{2m}} (\#I)\, \eta^\gamma \leqq c\lambda^{\frac{k-1}{2m}} \lambda^{\frac{1}{2m} - a(\gamma - k)} \leqq c\lambda^{\frac{k-1}{2m} + \delta},$$

provided that $(1/2m) - a(\gamma - k) \leqq \delta$ or, equivalently,

$$\frac{1}{\gamma - k}\left(\frac{1}{2m} - \delta\right) \leqq a. \tag{53}$$

(Recall that by assumption of Theorem 1, $\gamma \geqq k$ and $\delta \leqq 1/2m$; further, according to Remark 5.a, if $\gamma = k$, then $\delta = 1/2m$ and we let $a = \delta = 1/2m$. Note that $a(\gamma - k) \geqq (1/2m) - \delta$ and hence we must have $\gamma \geqq k$ if $\delta \leqq 1/2m$.)

We now consider $T_{i,2}$. We can proceed as in the proof of (28), except that we must take into account the fact that $\mathscr{A}$ has variable coefficients. According to (12–1), (12–2), (16), (46) and (48–2), we have successively, for $i = 0$ or 1:

$$\begin{aligned} -T_{i,2} &= \varphi(\lambda) - \sum_{\zeta\in I} \varphi(\lambda, \zeta) \\ &= \lambda^{k/2m} \int_{\Omega} r(x)^{k/2m} \mu'_{\mathscr{A}}(x)\, dx - \lambda^{k/2m} \sum_{\zeta\in I} \int_{Q_\zeta} r_\zeta^{k/2m} \mu_\zeta\, dx \\ &= \lambda^{k/2m} \sum_{\zeta\in I} \int_{Q_\zeta} r(x)^{k/2m} [\mu'_{\mathscr{A}}(x) - \mu_\zeta]\, dx \\ &\quad + \lambda^{k/2m} \sum_{\zeta\in I} \int_{Q_\zeta} [r(x)^{k/2m} - r_\zeta^{k/2m}]\, \mu_\zeta\, dx \\ &\quad + \lambda^{k/2m} \sum_{\zeta\in J\setminus I} \int_{Q_\zeta\cap\Omega} r(x)^{k/2m} \mu'_{\mathscr{A}}(x)\, dx. \end{aligned} \tag{54}$$

The last two terms in the last equality of (54) can be treated as in Section III: the second term vanishes, by definition of r_ζ [see (3) or (6)]; moreover, since $\partial\Omega$ satisfies the "β-condition" and r and $\mu'_{\mathscr{A}}$ are bounded on Ω, it follows from (27) that the third term is less than $c\lambda^{\frac{k-1}{2m} + \frac{1}{2m} - a\beta}$, which is less than $c\lambda^{\frac{k-1}{2m} + \delta}$ because, by (51), $\dfrac{1}{2m} - a\beta \leqq \delta$.

We handle the last term of the last equality of (54) as follows. Since, by (10), the leading coefficients of $\mathscr{A}$ are Hölder continuous of order σ, we deduce from (12–2) that there exist positive constants c_1 and η_1 such that for all $\eta \leqq \eta_1$, all $\zeta \in I$ and all $x \in Q_\zeta$,

$$|\mu'_{\mathscr{A}}(x) - \mu_\zeta| = |\mu'_{\mathscr{A}}(x) - \mu'_{\mathscr{A}}(x_\zeta)| \leqq c_1\eta^\sigma. \tag{55}$$

Consequently, according to (20) and (23), we have for all λ large enough:

$$\begin{aligned} &\left|\lambda^{\frac{k}{2m}} \sum_{\zeta\in I} \int_{Q_\zeta} r(x)^{\frac{k}{2m}} [\mu'_{\mathscr{A}}(x) - \mu_\zeta]\, dx\right| \\ &\qquad \leqq c_1\eta^\sigma\eta^k(\#I)\,(\lambda M)^{\frac{k}{2m}} \\ &\qquad \leqq c\lambda^{\frac{k-1}{2m} + \frac{1}{2m} - a\sigma} \leqq c\lambda^{\frac{k-1}{2m} + \delta}, \end{aligned} \tag{56}$$

provided that $(1/2m) - a\sigma \leqq \delta$ or, equivalently,

$$\frac{1}{\sigma}\left(\frac{1}{2m} - \delta\right) \leqq a. \tag{57}$$

In light of (56), (54) and the discussion following it,

$$|T_{i,2}| \leqq c\lambda^{\frac{k-1}{2m}+\delta}, \tag{58}$$

for $i = 0$ or 1 and for all λ sufficiently large.

Unless explicitly stated otherwise, we assume from now on that

$$\eta = \lambda^{-a}, \tag{59–1}$$

where a is positive and satisfies the inequality:

$$\frac{1}{\nu}\left(\frac{1}{2m} - \delta\right) \leqq a \leqq \delta; \tag{59–2}$$

this choice of a is possible since by hypothesis of Theorem 1, $\delta \geqq \frac{1}{2m}(\nu + 1)$ and so $\frac{1}{\nu}\left(\frac{1}{2m} - \delta\right) \leqq \delta$. $\left(\text{If } \delta = \frac{1}{2m}, \text{ it is understood that } a = \frac{1}{2m} \text{ in (59–2).}\right)$

We note that since, by definition, $\nu = \min(\beta, \gamma - k, \sigma)$, (59) implies that (49), (51), (53), and (57) hold.

Finally, we consider the term $T_{i,1}$ given by (48–1). We thus compare the operators $\mathscr{A}$ and $\mathscr{A}'_\zeta$, acting on Q_ζ and with constant positive weight r_ζ. We work as in [Mt, pp. 162–163, pp. 178–179, ...] although we must also take into account the fact that $r_\zeta \not\equiv 1$. We shall need the following two lemmas.

Lemma 1. *There exists a positive constant d' such that for all η small enough, all $\zeta \in I$ and all $u \in H^m(Q_\zeta)$,*

$$a'_\zeta(u, u) \geqq d' \|u\|^2_{H^m(Q_\zeta)},$$

with

$$a'_\zeta(u, v) := \sum_{|\alpha|=|\beta|=m} \int_{Q_\zeta} a_{\alpha\beta}(x_\zeta)\, D^\alpha u \overline{D^\beta v}\, dx.$$

This is a simple consequence of (10) and (55) because for all $\eta \leqq \eta_1$,

$$a'_\zeta(u, u) \geqq a'_{Q_\zeta}(u, u) - c'_1(\eta_1)^\sigma \|u\|^2_{H^m(Q_\zeta)} \geqq d' \|u\|^2_{H^m(Q_\zeta)},$$

with $d' := d - c'_1(\eta_1)^\sigma$; here the positive constant c'_1 is directly related to c_1 in (55) and we choose η_1 so small that $d' > 0$. Observe, for later use, that a'_ζ is the (leading) form associated with the operator $\mathscr{A}'_\zeta$.

Our next lemma provides the exact counterpart of the equation preceding (13). Its proof will be given in an appendix to this paper; we note that the case $i = 0$ corresponds to estimate (*ii*) of [Mt, Proposition 4.1, p. 162]. In the statement of Lemma 2, the side η of Q_ζ is not necessarily assumed to depend on λ.

Lemma 2. *There exists a positive constant* C_1 *such that for all* $\lambda > 0$, *all* $\eta > 0$, *all* $\zeta \in I$ *and for* $i = 0$ *or* 1:

$$\left| N_i(\lambda; 1, Q_\zeta, \mathscr{A}'_\zeta) - \mu_\zeta \eta^k \lambda^{\frac{k}{2m}} \right| \leqq C_1 \left(1 + \eta^{k-1} \lambda^{\frac{k-1}{2m}}\right).$$

We can now continue the proof of Theorem 1. We proceed as in [Mt, pp. 178–179]. By means of (10), (55) and of the interpolation inequalities [Ag, p. 24], we see that there exist positive constants K and $\varkappa$ such that for all $\tau \in (0, 1)$ and all $\zeta \in I$, we have:

$$\begin{aligned} (60) \qquad & |a_{Q_\zeta}(u, u) - a'_\zeta(u, u)| \\ & \leqq K(\varepsilon + \tau)\, a_{Q_\zeta}(u, u) + \varkappa[\tau^{1-2m} + \eta^{1-2m}]\, \|u\|^2_{L^2(\Omega)}, \end{aligned}$$

for all $u \in H^m(Q_\zeta)$ [and hence for the restriction to Q_ζ of any $u \in H^m(\Omega)$]; here we have set

$$(61) \qquad \varepsilon = c'_1 \eta^\sigma = c'_1 \lambda^{-a\sigma}.$$

It follows from (60) and [FlLa, Lemma 4.2, p. 315] that

$$(62) \qquad N_1(\lambda; r_\zeta, Q_\zeta, \mathscr{A}) \leqq N_1(\lambda'; r_\zeta, Q_\zeta, \mathscr{A}'_\zeta) = N_1(\lambda' r_\zeta; 1, Q_\zeta, \mathscr{A}'_\zeta),$$

with

$$(63) \qquad \lambda' := [1 + K(\varepsilon + \tau)]\, \lambda + \frac{\varkappa}{r_\zeta} [\tau^{1-2m} + \eta^{1-2m}].$$

According to (46–2), (48–1) and (62),

$$\begin{aligned} (64) \qquad T_{1,1} \leqq \sum_{\zeta \in I} \{ & [N_1(\lambda' r_\zeta; 1, Q_\zeta, \mathscr{A}'_\zeta) - (\lambda')^{k/2m}\, \Phi(r_\zeta, Q_\zeta, \mathscr{A}'_\zeta)] \\ & + [(\lambda')^{k/2m} - \lambda^{k/2m}]\, \Phi(r_\zeta, Q_\zeta, \mathscr{A}'_\zeta)\}. \end{aligned}$$

By (20), (46–2), and Lemma 2, with $i = 1$,

$$\begin{aligned} (65) \qquad & \left| N_1(\lambda' r_\zeta; 1, Q_\zeta, \mathscr{A}'_\zeta) - (\lambda')^{\frac{k}{2m}}\, \Phi(r_\zeta, Q_\zeta, \mathscr{A}'_\zeta) \right| \\ & \leqq C_1 \left(1 + \eta^{k-1} (\lambda' r_\zeta)^{\frac{k-1}{2m}}\right) \leqq C_1 \left(1 + \eta^{k-1} (\lambda' M)^{\frac{k-1}{2m}}\right). \end{aligned}$$

We note that, by (63),

$$\lambda' = \lambda \left[1 + K(\varepsilon + \tau) + \frac{\varkappa}{\lambda r_\zeta} [\tau^{1-2m} + \eta^{1-2m}]\right].$$

We choose

$$(66\text{–}1) \qquad \tau = \lambda^{-u},$$

with u such that

$$(66\text{–}2) \qquad 0 < \frac{1}{2m} < u < \frac{1}{2m-1} \left[1 - \left(\frac{1}{2m} - \delta\right)\right] \leqq \frac{1}{2m-1}.$$

Observe that these inequalities are compatible and that $u > (1/2m) - \delta$ and $1 - u(2m-1) > (1/2m) - \delta$.

We deduce from (59–1) and (61) that

$$\lambda' = \lambda\left[1 + K[c_1'\lambda^{-a\sigma} + \lambda^{-u}] + \frac{\varkappa}{r_\zeta}(\lambda^{u(2m-1)-1} + \lambda^{a(2m-1)-1})\right];$$

note that according to (57) and (59–2),

$$\frac{1}{2m} - \delta \leqq a\sigma < u$$

since

$$\frac{1}{\sigma}\left[\frac{1}{2m} - \delta\right] \leqq a \leqq \delta \leqq \frac{1}{2m} < u$$

and $\sigma \in [0, 1]$. (The case when $\delta = +\infty$ is trivial; see Remark 6.b.) Moreover, since $m \geqq 1$, we have

$$\frac{1}{2m} - \delta \leqq 1 - u(2m-1) > 1 - a(2m-1).$$

Consequently, there exist positive constants c, $\tilde{c}$ and v, with $v \geqq (1/2m) - \delta$, such that for all λ sufficiently large:

$$\lambda(1 + \tilde{c}\lambda^{-v}) \leqq \lambda' \leqq \lambda(1 + c\lambda^{-v}); \tag{67}$$

we use here the fact that condition $(\mathscr{K}_+)$, introduced immediately after Eq. (8′) in Section II, holds; indeed in view of $(\mathscr{K}_+)$ and (6), we have for all $\zeta \in I$, $r_\zeta \geqq c_2 > 0$, where c_2 is a constant independent of ζ or λ; this provides the second inequality in (67). Moreover, we use (20) in order to obtain the first inequality in (67).

By combining (20), (23), (46–2), (64), (65) and (67), we obtain, for all large λ:

$$\text{(68–1)}\quad T_{1,1} \leqq c(\#I) + c(\#I)\,\eta^k\eta^{-1}\lambda^{\frac{k-1}{2m}}(1 + c\lambda^{-v})^{\frac{k-1}{2m}} + c\lambda^{\frac{k}{2m}-v}\eta^k(\#I)$$
$$\leqq c\lambda^{ak} + c\lambda^{\frac{k-1}{2m}+a} + c\lambda^{\frac{k-1}{2m}+a-v} + \lambda^{\frac{k-1}{2m}+\frac{1}{2m}-v} \leqq c\lambda^{\frac{k-1}{2m}+\delta},$$

since $(1/2m) - v \leqq \delta$ and $a \leqq \delta \leqq 1/2m$. (Note that $ka = (k-1)a + a \leqq (k-1)/2m + \delta$.)

A lower bound for $T_{0,1}$ is obtained in exactly the same manner by deducing from (60) that

$$N_0(\lambda; \lambda'' r_\zeta, Q_\zeta, \mathscr{A}_\zeta') \geqq N_0(\lambda'', r_\zeta, Q_\zeta, \mathscr{A}_\zeta') = N_0(\lambda'' r_\zeta; 1, Q_\zeta, \mathscr{A}_\zeta'),$$

with

$$\lambda'' := [1 - K(\varepsilon + \tau)]\,\lambda - \frac{\varkappa}{r_\zeta}[\tau^{1-2m} + \eta^{1-2m}].$$

We thus have for all λ large enough:

$$T_{0,1} \geqq c\lambda^{\frac{k-1}{2m}+\delta}. \tag{68-2}$$

In light of (47), the conclusion of Theorem 1 now follows from (50), (52), (58) and (68). □

Appendix

In this appendix, we establish Lemma 2 that was used in the proof of Theorem 2 towards the end of Section III.

As was noted earlier, for Dirichlet boundary conditions ($i = 0$), Lemma 2 is nothing but [Mt, Proposition 4.1, estimate (ii), p. 162].

For Neumann boundary conditions ($i = 1$), the desired conclusion follows essentially from the assumption of uniform coerciveness of a' made in (10). We now briefly describe how this is achieved. We apply [Mt, Proposition 2.7, p. 138] with $V = H^m(Q_\zeta)$, $V_0 = H_0^m(Q_\zeta)$ and $H = L^2(Q_\zeta)$. Hence, with the notation of MÉTIVIER [Mt] (we write $\mathcal{N}$ instead of N),

$$\mathcal{N}(\lambda, ; H^m, L^2, a) = \mathcal{N}(\lambda; H_0^m, L^2, a) + \mathcal{N}(\lambda; Z_\lambda, L^2, a) - \dim(Z_\lambda \cap H_0^m).$$

By means of Lemma 1 of Section III (which results from the uniform coerciveness of a'_ζ), we can replace [Mt, Eq. (4.15), p. 164] by

$$\gamma_1 \|u\|^2_{H^m(Q_\zeta)} \leqq a'_\zeta(u, u) + \|u\|^2_{L^2(Q_\zeta)}, \quad \text{for all } u \in H^m(Q_\zeta).$$

In turn, this enables us to replace [Mt, p. 165, 1.–8] by

$$\mathcal{N}(\lambda; Z_\lambda, L^2(Q_\zeta), a) \leqq \mathcal{N}(\mu; Z_\lambda, L^2(Q_\zeta)),$$

with $\mu := (\lambda + 1)/\gamma_1$; note that μ, like γ_1, is independent of $\zeta \in I$.

In view of [Mt, Eq. (4.17), p. 165], we now conclude the proof of Lemma 2 in case $i = 1$ by use of [Mt, Propositions 2.7 and 4.1, pp. 138 and 162].

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