Solutions in L_r of the Navier-Stokes Initial Value Problem

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Introduction

In this paper we discuss the existence and uniqueness of strong solutions in L_r spaces $(1 < r < \infty)$ of the initial value problem for the Navier-Stokes equations

$$
\text{(I)} \qquad \begin{aligned}
\partial u/\partial t + (u, \nabla)\, u - \Delta u &= f - \nabla p && \text{in } D \times (0, T),\\
\operatorname{div} u &= 0 && \text{in } D \times (0, T),\\
u &= 0 && \text{on } S \times (0, T),\\
u(x, 0) &= a(x) && \text{in } D.
\end{aligned}
$$

Here D is a bounded domain in $\boldsymbol{R}^n$ $(n \geqq 2)$ with smooth boundary S; $u = \{u^j(x, t)\}_{j=1}^n$ and $p = p(x, t)$ denote, respectively, the velocity and pressure, while $f = \{f^j(x, t)\}_{j=1}^n$ and $a = \{a^j(x)\}_{j=1}^n$ denote the given external force and initial velocity.

There is an extensive literature on the solvability of the initial value problem for the Navier-Stokes equation in L_2 spaces; see LADYZHENSKAYA'S monograph [18] and further papers cited there. HOPF [13] proved the existence of a global weak solution, using the Faedo-Galerkin approximation and an energy inequality, but the uniqueness and regularity of his solution are still open problems for $n \geqq 3$.

Another approach to problem (I) is to use semigroup theory. KATO and FUJITA [5], [16] and SOBOLEVSKII [24] transformed equation (I) into an evolution equation in the Hilbert space L_2. They proved the existence of a unique global strong solution for any square-summable initial velocity when $n = 2$. On the other hand, when $n = 3$ they proved the existence of a unique local strong solution if the initial velocity has some regularity (roughly speaking, they assumed that the initial velocity has square-summable half derivatives). Other contributions to the problem also have assumed some regularity of the initial velocity, see for example SOLONNIKOV [27] and HEYWOOD [12].

Our aim in this paper is to prove the existence of a unique strong solution *without* assuming that the initial velocity is regular. To do this, we develop an L_r

theory generalizing the corresponding L_2 theory of KATO and FUJITA. The L_n case is the most important because we need only to assume that the initial velocity has an integrable n-th power. In addition, SERRIN [23] has raised the question of existence of strong solutions if $n > 4$. Our present work also answers his question.

Recently VON WAHL [28] and MIYAKAWA [21] have discussed L_r theory ($r > n$). Their treatments depend on the fact that the Stokes operator generates a bounded holomorphic semigroup in L_r spaces. This result was derived by SOLONNIKOV [27], VON WAHL [29], and MIYAKAWA [21] from the estimate of SOLONNIKOV [26]. Moreover, a completely different proof and a more precise result were obtained in [9]. Our present work indeed relies on this more precise conclusion. Finally we improve their L_r theory by using the results of [10]. (We note that Weissler's treatment [30] of L_r theory is weaker than ours, and additionally assumes that D is a half space of $\mathbf{R}^n$.)

The essential step in our work is to estimate the nonlinear term $(u, \nabla)\, u$ in (I). In [9] GIGA constructed the resolvent of the Stokes operator, applying the calculus of pseudodifferential operators. Moreover, the paper [10] characterizes explicitly the domains of fractional powers of the Stokes operator; see also [11]. Using this result, we shall estimate the term $(u, \nabla)\, u$ (Lemma 2.2 in Section 2). This estimate generalizes that of KATO and FUJITA ([5], [16]) and SOBOLEVSKII [24]. Finally in Section 3 we show that our solutions are smooth up to the boundary if the external force is smooth.

In [4], FABES, LEWIS and RIVIERE discussed the Navier-Stokes initial value problem by using parabolic singular integral operators. They proved the existence and uniqueness of a local weak solution in L_r ($r > n$). We show in Section 2 that their solutions exist globally if the data are small.

In [25] SOBOLEVSKII announced results similar to ours, but without complete details.

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1. Preliminaries

In this section we recall some results on the Stokes operator. Let D be a bounded domain in $\mathbf{R}^n$ ($n \geqq 2$) with smooth boundary S. Let $W_r^s(D)$ ($s \in \mathbb{R}$, $1 < r < \infty$) be the Sobolev space of order s such that $W_r^0(D) = L_r(D)$ (see [19]), and let $\|w\|_{s,r}$ be the norm of w in $W_r^s(D)$. Set

$$X_r = \text{closure in } (L_r(D))^n \text{ of } \{u \in (C_0^\infty(D))^n;\ \operatorname{div} u = 0\},$$

$$G_n = \{\nabla p;\ p \in W_r^1(D)\}.$$

We then have the following Helmholtz decomposition (see FUJIWARA and MORIMOTO [8])

$$(L_r(D))^n = X_r \oplus G_r \text{ (direct sum)}. \tag{1.1}$$

Let P_r be the continuous projection from $(L_r(D))^n$ to X_r associated with this decomposition, and let B_r be the Laplace operator with zero boundary condition, i.e.,

$$B_r = -\Delta \text{ with } D(B_r) = \{v \in (W_r^2(D))^n;\, v|_S = 0\}.$$

We define the Stokes operator A_r in X_r by $A_r = -P_r\Delta$, with domain $D(A_r) = X_r \cap D(B_r)$.

CATTABRIGA [3] showed that A_r has a bounded inverse and that A_r is a close linear operator in X_r; see also [9]. Concerning the semigroup $\{e^{-tA_r};\, t \geqq 0\}$ we have

Lemma 1.1 ([9], [21], [27], [29]). *The operator $-A_r$ generates (in X_r) a bounded holomorphic semigroup of class C_0.*

Concerning the dual operators A_r^*, P_r^* and the dual space X_r^*, we have (see [8])

$$A_r^* = A_{r'},\, P_r^* = P_{r'},\, X_r^* = X_{r'} \qquad (1/r' = 1 - 1/r). \tag{1.2}$$

Lemma 1.1 allows us to define the fractional powers A_r^α ($\alpha \in \boldsymbol{R}$) in the usual way. We introduce a scale of Banach spaces $\{D(A_r^\alpha);\, \alpha \in \boldsymbol{R}\}$ by defining

$$D(A_r^\alpha) = D(A_{r'}^{-\alpha})^* \text{ for } \alpha < 0. \tag{1.3}$$

Since $(A_r^\alpha)^* = A_{r'}^\alpha$, $(\alpha \in R)$ by (1.2), we see that $D(A_r^\beta)$ $(\beta < 0)$ is the completion of X_r under the norm $\|A_r^\beta v\|_{0,r'}$, namely the space $H^{\beta,r}$ of WEISSLER. On the other hand, Lemma 1.1 implies

Proposition 1.2. $\|A_r^\alpha e^{-tA_r}\| \leqq C_\alpha t^{-\alpha}$ for $\alpha \geqq 0$, $t > 0$.
It follows that $\{e^{-tA_r};\, t \geqq 0\}$ extends uniquely to a bounded holomorphic semigroup in $D(A_r^\alpha)$ for all $\alpha \in \boldsymbol{R}$. Concerning $D(A_r^\alpha)$, $\alpha \geqq 0$, we have

Lemma 1.3. ([10], [11]). *For any α, $0 \leqq \alpha \leqq 1$, the domain $D(A_r^\alpha)$ is the complex interpolation space $[X_r, D(A_r)]_\alpha$.*

Lemma 1.3 shows that $D(A_r^\alpha) = X_r \cap D(B_r^\alpha)$ (see [10]), a fact proved earlier by FUJITA and MORIMOTO [6] for $r = 2$. Moreover FUJIWARA [7] showed that $D(B_r^\alpha)$ $(\alpha \geqq 0)$ is continuously embedded in the space of Bessel potentials $(H_r^{2\alpha}(D))^n$ Thus we get

Proposition 1.4. *For any $\alpha \geqq 0$, the domain $D(A_r^\alpha)$ is continuously embedded in $X_r \cap (H_r^{2\alpha}(D))^n$.*

2. Existence and Uniqueness in X_r

In this section, we study the Navier-Stokes initial value problem (I) in the space X_r. Applying P_r to (I), we get

$$du/dt + A_r u = Fu + P_r f, \quad t > 0; \quad u(0) = a, \tag{II}$$

where $Fu = -P_r(u, \nabla)\, u$. We consider this equation in integral form

$$\text{(III)} \qquad u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}\{Fu(s) + Pf(s)\}\, ds, \qquad t > 0.$$

(Here and in the sequel we drop the subscript r attached to A and P.) To estimate Fu we need

Lemma 2.1. *For each j, $1 \leqq j \leqq n$, the operator $A^{-1/2}P(\partial/\partial x_j)$ extends uniquely to a bounded linear operator from $(L_r(D))^n$ to X_r.*

Proof. According to Proposition 1.4, $(\partial/\partial x_j)\, IA^{-1/2}: X_r \to (L_r(D))^n$ is continuous for each r, $1 < r < \infty$, where I denotes the injection $X_r \subset (L_r(D))^n$. Since $I^* = P_{r'}$, we get the desired result from duality and (1.2).

Lemma 2.2. *Let* $0 \leqq \delta < 1/2 + n(1 - r^{-1})/2$. *Then*

$$\|A^{-\delta}P(u, \nabla)\, v\|_{0,r} \leqq M\, \|A^{\theta}u\|_{0,r}\, \|A^{\varrho}v\|_{0,r}$$

with some constant $M = M(\delta, \theta, \varrho, r)$, *provided that* $\delta + \theta + \varrho \geqq n/2r + 1/2$, $\theta > 0$, $\varrho > 0$, $\varrho + \delta > 1/2$.

Proof. Assume that $0 \leqq \varepsilon < n(1 - r^{-1})/2$. Proposition 1.4 and the Sobolev embedding theorem imply that the operator

$$A_{r'}^{-\varepsilon}: X_{r'} \to D(A_{r'}^{\varepsilon}) \to X_{s'},\ 1/s' = 1/r' - 2\varepsilon/n,\ 1/r + 1/r' = 1,$$

is bounded, since $(L_{s'}(D))^n \cap X_{r'}$ is the same as $X_{s'}$. By duality, this implies that $A_r^{-\varepsilon}$ extends uniquely to a bounded operator from X_s to X_r, where $1/s = 1 - 1/s' = 1/r + 2\varepsilon/n$.

Consider first the case $\delta \geqq 1/2$. Since $P(u, \nabla)\, v$ is bilinear in u, v, it suffices to prove the estimate on a dense subspace. From now on we may therefore assume that u and v are smooth. Since $\operatorname{div} u = 0$, we get

$$(u, \nabla)\, v = \sum_{j=1}^{n} \partial(u^j v)/\partial x_j.$$

Take $\varepsilon = \delta - 1/2$ and recall that $A_r^{-\varepsilon}$ is a bounded operator from X_s to X_r. Lemma 2.1 implies that

$$\|A^{-\delta}P(u, \nabla)\, v\|_{0,r} = \Big\|\sum_{j=1}^{n} A^{-1/2-\varepsilon}P\, \partial(u^j v)/\partial x_j\Big\|_{0,r} \leqq C\, \|\, |u| \cdot |v|\, \|_{0,s}.$$

By assumption we can take p and q such that

$$1/p \geqq 1/r - 2\theta/n, \qquad 1/q \geqq 1/r - 2\varrho/n, \qquad 1/p + 1/q = 1/s, \qquad 1 < p, \quad q < \infty.$$

Proposition 1.4 and the Sobolev embedding theorem yield

$$\|\, |u| \cdot |v|\, \|_{0,s} \leqq \|u\|_{0,p}\, \|v\|_{0,q} \leqq C\, \|A^{\theta}u\|_{0,r}\, \|A^{\varrho}v\|_{0,r}$$

which is the required result for $\delta \geqq 1/2$. In particular,

$$\|A^{-1/2}P(u, \nabla)\, v\|_{0,r} \leqq M\, \|A^{\theta}u\|_{0,r}\, \|A^{\beta}v\|_{0,r}, \qquad \theta + \beta \geqq n/2r,\ \theta > 0,\ \beta > 0.$$

Similarly we find

$$\| P(u, \nabla) v \|_{0,r} \leqq C \| u \|_{0,p} \| \nabla v \|_{0,q} \leqq M \| A^\theta u \|_{0,r} \| A^{\beta+1/2} v \|_{0,r'},$$

where $1/p + 1/q = 1/r$, the required result for $\delta = 0$.

The two foregoing estimates show that the map $v \to P(u, \nabla) v$ can be regarded as a bounded operator from $D(A^\beta)$ to $D(A^{-1/2})$ and from $D(A^{\beta+1/2})$ to X_r. Lemma 1.3 and (1.3) allow us to apply complex interpolation theory to obtain

$$\| A^{-\delta} P(u, \nabla) v \|_{0,r} \leqq M \| A^\theta u \|_{0,r} \| A^\varrho v \|_{0,r}, \quad \varrho = \beta + 1/2 + - \delta,$$

for $0 \leqq \delta \leqq 1/2$. This completes the proof.

Remark. Lemma 2.2 generalizes the results given by a number of writers for the case $r = 2$. We list these results here for the reader's convenience.

	n	δ	θ	ϱ
Kato-Fujita [16]				
Sobolevskii [24]	2	1/4	1/4	1/2
Kato-Fujita [16]	3	0	1/2	3/4
Fujita-Kato [5]				
Sobolevskii [24]	3	1/4	1/2	1/2
Inoue-Wakimoto [14]	4, 5	0	$\theta > 0$, $\varrho > 1/2$, $\theta + \varrho = (n+2)/4$	

We will use Lemma 2.2 to prove an existence theorem for the integral equation (III) in X_r $(1 < r < \infty)$. To this end we introduce the iteration scheme

$$(2.1) \qquad \begin{aligned} u_0(t) &= e^{-tA} a + \int_0^t e^{-(t-s)A} Pf(s)\, ds, \\ u_{m+1}(t) &= u_0(t) + \int_0^t e^{-(t-s)A} F u_m(s)\, ds, \qquad m \geqq 0. \end{aligned}$$

The following argument is similar to that of Kato and Fujita [5], [16]. We denote the norm in X_r by $\|\cdot\|$. Moreover, for a Banach space Y, $C(I; Y)$ denotes the space of Y-valued continuous functions defined on an interval I.

Theorem 2.3. (i) (*Existence*). *Fix* γ *and choose* $\delta \geqq 0$ *such that*

$$n/2r - 1/2 \leqq \gamma < 1, \qquad -\gamma < \delta < 1 - |\gamma|.$$

Assume that a is in $D(A^\gamma)$, *and that* $\| A^{-\delta} Pf(t) \|$ *is continuous on* $(0, T)$ *the initial data and satisfies*

$$\| A^{-\delta} Pf(t) \| = o(t^{\gamma+\delta-1}) \text{ as } t \to 0.$$

Then there is a local solution $u(t)$ of (III) *such that*

(a) $u \in C([0, T_*]; D(A^\gamma))$, $u(0) = a$,

(b) $u \in C((0, T_*]; D(A^\alpha))$ *for some* $T_* > 0$,

(c) $\|A^\alpha u(t)\| = o(t^{\gamma-\alpha})$ *as* $t \to 0$ *for all* α, $\gamma < \alpha < 1 - \delta$.

(ii) (*Uniqueness*). *Any solution of* (III) *satisfying* (a) *and*

(b′) $u \in C((0, T_*]; D(A^\beta))$,

(c′) $\|A^\beta u(t)\| = o(t^{\gamma-\beta})$ *for some* β, $|\gamma| < \beta$

is unique.

Remarks. Theorem 2.3 improves the result of WEISSLER [30]. He considered equation (III) with $Pf = 0$ when D is a half-space and proved the local existence of solutions when $n/2r - 1/2 < \gamma$, $-1/3 < \gamma < 1$, a more restrictive assumption on γ than ours. We will show that our solution exists globally if a and Pf are small.

The case $r = 2$ of Theorem 2.3 was treated by KATO and FUJITA [16], who proved (i) for $n = 2$, $\delta = 1/4$ and $n = 3$, $\delta = 0$, and by INOUE and WAKIMOTO [14], who treated the cases $n = 4, 5$.

Proof of Theorem 2.3. (i) We begin by estimating the term $u_0(t)$ in (2.1). Proposition 1.2 gives

$$\begin{aligned}\|A^\alpha u_0(t)\| &\leqq \|A^\alpha e^{-tA}a\| + \int_0^t \|A^{\alpha+\delta}e^{-(t-s)A}\| \, \|A^{-\delta}Pf(s)\| \, ds \\ &\leqq \|A^\alpha e^{-tA}a\| + C_{\alpha+\delta}\int_0^t (t-s)^{-\alpha-\delta}\|A^{-\delta}Pf(s)\| \, ds \\ &\leqq K_{\alpha 0}t^{\gamma-\alpha} \text{ for } \gamma \leqq \alpha < 1 - \delta,\end{aligned}$$

with

$$K_{\alpha 0} = \sup_{0<t\leqq T} t^{\alpha-\gamma}\|A^\alpha e^{-tA}\|a + C_{\alpha+\delta}NB(1-\delta-\alpha, \gamma+\alpha), \tag{2.2}$$

$$N = \sup_{0<t\leqq T} t^{1-\gamma-\delta}\|A^{-\delta}Pf(t)\|,$$

and where $B(a, b)$ is the beta function. Here we have used the assumption $\gamma + \delta > 0$.

Suppose that for some $m \geqq 0$, $u_m(t)$ is defined and satisfies

$$\|A^\alpha u_m(t)\| \leqq K_{\alpha m}t^{\gamma-\alpha} \quad (\gamma \leqq \alpha < 1 - \delta). \tag{2.3}$$

We shall estimate $\|A^\alpha u_{m+1}(t)\|$ by using (2.1). To estimate the term $\|A^{-\delta}Fu_m(s)\|$ we choose θ and ϱ so that

$$\theta + \varrho + \delta = 1 + \gamma, \quad \gamma < \theta < 1 - \delta, \quad \gamma < \varrho < 1 - \delta,$$
$$\theta > 0, \quad \varrho > 0, \quad \delta + \varrho > 1/2.$$

This is possible because $\delta < 1 - |\gamma|$. Since $\gamma \geqq n/2r - 1/2$, $n \geqq 2$, and $\delta < 1 - \gamma$ we easily find that θ, ϱ, δ satisfy the assumptions of Lemma 2.2. Using Lemma 2.2 and (2.3), we get $\| A^{-\delta} F u_m(s) \| \leqq M K_{\theta m} K_{\varrho m} s^{\gamma+\delta-1}$. Therefore (2.1) implies that

$$\| A^{\alpha} u_{m+1}(t) \| \leqq K_{\alpha 0} t^{\gamma-\alpha} + C_{\alpha+\delta} \int_0^t (t-s)^{-\alpha-\delta} \| A^{-\delta} F u_m(s) \| \, ds$$

$$\leqq K_{\alpha,m+1} t^{\gamma-\alpha} \qquad (\gamma \leqq \alpha < 1 - \delta)$$

with

$$K_{\alpha,m+1} = K_{\alpha 0} + C_{\alpha+\delta} M B(1-\delta-\alpha, \gamma+\delta) \, K_{\theta m} K_{\varrho m}. \tag{2.4}$$

Thus $u_m(t)$ is well-defined for each $m \geqq 0$ as an element of $C([0, T]; D(A^{\gamma})) \cap C((0, T]; D(A^{\alpha}))$ for all α such that $\gamma < \alpha < 1 - \delta$; moreover $u_m(t)$ satisfies (2.3) with $K_{\alpha m}$ defined recursively by (2.2) and (2.4).

Put

$$k_m = \max\{K_{\theta m}, K_{\varrho m}\}, \qquad C_1 = \max\{C_{\theta+\delta}, C_{\varrho+\delta}\},$$

$$B_1 = \max\{B(1-\delta-\theta, \gamma+\delta), \quad B(1-\delta-\varrho, \gamma+\delta)\}.$$

Then (2.4) implies that

$$k_{m+1} \leqq k_0 + C_1 M B_1 k_m^2.$$

An elementary calculation shows that if

$$k_0 < 1/4 C_1 M B_1 \tag{2.5}$$

then for each $m \geqq 1$,

$$k_m < K \equiv \{1 - (1 - 4C_1 M B_1 k_0)\}/2 C_1 M B_1 < 1/2 C_1 M B_1,$$

$$K_{\alpha,m+1} \leqq K_{\alpha 0} + C_{\alpha+\delta} M B(1-\delta-\alpha, \gamma+\delta) \, K^2 \equiv K_{\alpha}, \tag{2.6}$$

$$\| A^{\alpha} \mathrm{u}_{m+1}(t) \| \leqq K_{\alpha} t^{\gamma-\alpha}.$$

Assuming (2.5), we prove that the sequence $\{u_m(t)\}$ converges. Set

$$w_m(t) = u_{m+1}(t) - u_m(t) = \int_0^t e^{-(t-s)A} \{F u_m(s) - F u_{m-1}(s)\} \, ds \qquad (m \geqq 1).$$

The estimate

$$\| A^{-\delta} \{F u_m - F u_{m-1}\} \| \leqq M(\| A^{\theta} w_{m-1} \| \, \| A^{\varrho} u_m \| + \| A^{\theta} u_{m-1} \| \, \| A^{\varrho} w_{m-1} \|),$$

together with an induction on m, gives

$$\| A^{\alpha} w_m(t) \| \leqq 2 K C_{\alpha+\delta} (2 C_1 M B_1 K)^{m-1} \, B(1-\delta-\alpha, \gamma+\delta) \, t^{\gamma-\alpha}$$

for each α such that $\gamma \leqq \alpha < 1 - \delta$. Since $2 C_1 M B_1 K < 1$ by (2.6), this implies the existence of

$$u(t) \in C([0, T]; D(A^{\gamma})) \cap C((0, T]; D(A^{\alpha})) \qquad (\gamma < \alpha < 1 - \delta)$$

such that

$$\|A^{\gamma}\{u_m(t) - u(t)\}\| \to 0 \text{ uniformly on } [0, T],$$

$$\|A^{\alpha}[\{u_m(t) - u(t)\}\| \to 0 \text{ uniformly on every } [\varepsilon, T] \quad (0 < \varepsilon < T)$$

as $m \to \infty$. Moreover

$$\|A^{\alpha}u(t)\| \leqq K_{\alpha}t^{\gamma-\alpha} \quad (\gamma < \alpha < 1 - \delta).$$

This in turn yields for $s \in (0, T)$

$$\|A^{-\delta}\{Fu_m(s) - Fu(s)\}\| \leqq M(\|A^{\theta}(u_m - u)\| \, \|A^{\varrho}u_m\| + \|A^{\theta}u\| \, \|A^{\varrho}(u_m - u)\|) \to 0$$

as $m \to \infty$, and

$$\|A^{-\delta}Fu_m(s)\| \leqq Cs^{\gamma+\delta-1},$$

with a constant $C > 0$ independent of m.

Applying the dominated convergence theorem to the scheme (2.1), we see that $u(t)$ is a solution on $[0, T]$. Thus we have proved the existence of a solution under assumption (2.5). Since $a \in D(A^{\gamma})$, Proposition 1.2 implies that $t^{\alpha-\gamma}\|A^{\alpha}e^{-tA}a\| \to 0$ as $t \to 0$ for $\alpha > \gamma$. If $T > 0$ is chosen sufficiently small then $K_{\alpha 0}$ $(\gamma < \alpha < 1 - \delta)$ becomes small and k_0 satisfies (2.5). This shows the existence of $T_* > 0$ with the desired properties (b) and (c).

We now prove (ii). First note that for any simple of numbers λ, μ, ν with $1 < \lambda < \mu < \nu$ the estimate

$$\|A^{\mu}v\| \leqq C\,\|A^{\lambda}v\|^{\,\theta}\,\|A^{\nu}v\|^{1-\theta}, \quad \theta = (\nu - \mu)/(\nu - \mu)$$

holds with a constant C independent of $v \in D(A^{\nu})$. This follows immediately from Lemma 1.3, (1.3), and the convexity property of norms on interpolation spaces ([2]).

Let u, v be two solutions corresponding to the same quantities Pf and a. By the above estimate, we may assume without loss of generality that there exists $\varkappa$, $|\gamma| < \varkappa < 1 - \delta$, such that

$$\|A^{\alpha}u(t)\| = o(t^{\gamma-\alpha}), \quad \|A^{\alpha}v(t)\| = o(t^{\gamma-\alpha}) \ (t \to 0)$$

for all α, $\gamma < \alpha \leqq \varkappa$. Set

$$w(t) \equiv u(t) - v(t) = \int_0^t e^{-(t-s)A}\{Fu(s) - Fv(s)\}\,ds$$

and $\delta' = 1 - \varkappa$; then $|\gamma| < \varkappa$ implies $\delta' < 1 - |\gamma|$. Note that $\gamma + \delta' > \gamma + \delta > 0$. As in the proof of (i), we can choose θ and ϱ, $|\gamma| < \theta$, $\varrho < \varkappa$ such that

$$\text{(2.7)} \qquad \|A^{\alpha}w(t)\| \leqq C_{\alpha+\delta'}M\int_0^t (t - s)^{-\alpha-\delta'}\{\|A^{\theta}w(s)\|\,\|A^{\varrho}u(s)\| + \|A^{\theta}v(s)\|\,\|A^{\varrho}w(s)\|\}\,ds$$

for $\gamma < \alpha < \varkappa$.

Let $K(t_0)$ $(t_0 \in (0, T_*])$ be a constant such that, for $\alpha = \theta$ and $\alpha = \varrho$ we have

$$\|A^\alpha u(t)\| \leqq K(t_0)\, t^{\gamma-\alpha}, \|A^\alpha v(t)\| \leqq K(t_0)\, t^{\gamma-\alpha}, (t \in (0, t_0]),$$

where

$$K(t_0) \to 0 \text{ as } t_0 \to 0.$$

By induction, we see that (2.7) implies

$$\|A^\alpha w(t)\| \leqq 2K(t_0)\,(2K(t_0)\, C_1 M B_1)^m\, t^{\gamma-\alpha} \qquad (t \in (0, T_0])$$

for each $m \geqq 1$ and for $\alpha = \theta, \varrho$. Choosing $t_0 > 0$ sufficiently small, we may assume $2K(t_0)\, C_1 M B_1 < 1$. Hence $w(t) \equiv 0$ on $[0, t_0]$. Repeating this argument on $[t_0, T_*]$, one finds a sequence $t_0 < t_1 < t_2 < \dots$, such that $w(t) \equiv 0$ on $[0, t_j]$ for any j. Since $A^\alpha u(t)$ and $A^\alpha v(t)$ $(\alpha = \theta, \varrho)$ are continuous on the closed interval $[t_0, T_*]$, it is easily seen that the sequence $\{t_j - t_{j-1}\}$ is bounded away from 0. Thus we have $w \equiv 0$ on $[0, T_*]$, which completes the proof.

Proposition 2.4. *Let u be the solution given by Theorem 2.3. Then* $A^\alpha u$ $(\gamma < \alpha < 1 - \delta)$ *is Hölder continuous on every interval* $[\varepsilon, T_*]$ $(0 < \varepsilon < T_*)$.

Proof. It suffices to prove the Hölder continuity of $A^\alpha v$, where

$$v(t) = \int_0^t e^{-(t-s)A} \{Fu(s) + Pf(s)\}\, ds.$$

Using the estimate $\|(e^{-hA} - I)\, A^{-\mu}\| \leqq C_\mu h^\mu$ $(0 \leqq \mu \leqq 1)$, which was obtained in [5], we have for $h > 0$,

$$\begin{aligned}
&\|A^\alpha v(t+h) - A^\alpha v(t)\| \\
&\leqq \int_0^t \|A^{\alpha+\delta}(e^{-hA} - I)\, e^{-(t-s)A}\| \{\|A^{-\delta}Fu(s)\| + \|A^{-\delta}Pf(s)\|\}\, ds \\
&\quad + \int_t^{t+h} \|A^{\alpha+\delta} e^{-(t+h-s)A}\| \{\|A^{-\delta}Fu(s)\| + \|A^{-\delta}Pf(s)\|\}\, ds \\
&\leqq C_\mu \|(e^{-hA} - I)\, A^{-\mu}\| \int_0^t (t-s)^{-\alpha-\delta-\mu} s^{\gamma+\delta-1}\, ds + C_\varepsilon \int_t^{t+h} (t+h-s)^{-\alpha-\delta}\, ds \\
&\leqq C_\mu h^\mu B(1-\delta-\alpha-\mu, \gamma+\delta)\, \varepsilon^{\gamma-\alpha-\mu} + C_\varepsilon h^{1-\delta-\alpha}/(1-\delta-\alpha), \\
&\qquad (0 < \mu < 1 - \delta - \alpha),
\end{aligned}$$

where $C_\varepsilon = \sup_{\varepsilon \leqq t \leqq T_*} \{\|A^{-\delta}Fu(s)\| + \|A^{-\delta}Pf(s)\|\}$. This completes the proof.

Theorem 2.5. *If* $Pf\colon (0, T] \to {}_+X_r$ *is Hölder continuous on each subinterval* $[\varepsilon, T]$ $(0 < \varepsilon < T)$, *the solution u of* (III) *given by Theorem 2.3 satisfies equation* (II) *on* $[0, T_*]$. *Moreover we have* $u(t) \in D(A)$ *for* $t \in (0, T_*]$.

Proof. We need only show the Hölder continuity of $Fu(t)$ on each subinterval $[\varepsilon, T_*]$ $(0 < \varepsilon < T_*)$. An elementary calculation shows that $u(\varepsilon) \in X_r$ and

$$u(t) = e^{-(t-\varepsilon)A}u(\varepsilon) + \int_\varepsilon^t e^{-(t-s)A}\{Fu(s) + Pf(s)\}\,ds,$$

$t \in [\varepsilon, T_*]$. Since Pf is continuous on $[\varepsilon, T]$ we have $\|Pf(t)\| = o((t-\varepsilon)^{-\alpha})$ $(t \to \varepsilon)$ for any $\alpha > 0$. The uniqueness of $u(t)$, ensured by Theorem 2.3, shows that

$$u \in C([\varepsilon, T_*]; D(A^\eta)) \cap C((\varepsilon, T_*]; D(A^\alpha)),$$

$$\|A^\alpha u(t)\| = o((t-\varepsilon)^{\eta-\alpha}) \quad (t \to \varepsilon) \text{ for } \eta < \alpha < 1,$$

where $\eta = \max\{\gamma, 0\}$. Thus Proposition 2.4 implies that $A^\alpha u(t)$ $(\eta < \alpha < 1)$ is Hölder continuous on every subinterval $[\varepsilon, T_*]$ $(0 < \varepsilon < T_*)$. Since we can choose θ, ϱ such that

$$\theta + \varrho = 1 + \eta, \quad \eta < \theta < 1, \quad \max\{\eta, 1/2\} < \varrho < 1,$$

Lemma 2.2 (with $\delta = 0$) shows that $Fu(t)$ is Hölder continuous on every $[\varepsilon, T_*]$ $(0 < \varepsilon < T_*)$. This completes the proof.

From (2.2) and (2.5) we get the following result.

Theorem 2.6. *Let $a \in D(A^\gamma)$ and $Pf \in C((0,\infty); X_r)$ be as in Theorem 2.3. Then the solution $u(t)$ given by Theorem 2.3 exists on $[0,\infty)$ provided that*

$$C_0 \|A^\gamma a\| + C_1 \sup t^{1-\gamma-\delta} \|A^{-\delta}Pf(t)\| < 1/4C_1MB_1.$$

Here $C_0 = \max\{C_{\theta-\gamma}, C_{\varrho-\gamma}\}$, and $C_{\alpha-\gamma}$ is the constant given in Proposition 1.2.

Remark. In Theorem 2.3 we considered only the case $n/2r - 1/2 < 1$. When $n/2r - 1/2 \geqq 1$, our iteration argument fails. In the latter case, however, since Sobolev's Theorem shows $D(A^\gamma) \subset X_n$ if $\gamma \geqq n/2r - 1/2 > 0$, we can appeal to X_n-theory.

3. A Regularity Theorem

The purposes of this section is to show that the solutions given by Theorem 2.3 are smooth on $\bar{D} \times (0, T_*)$ if Pf is smooth on $\bar{D} \times (0, T)$. For simplicity, we assume $Pf = 0$; the proof when $Pf \neq 0$ is essentially the same. First we give some lemmas.

Lemma 3.1. ([3]). *Let $u \in D(A)$ and $Au = f$. If f is in $(W_r^m(D))^n$ $(1 < r < \infty)$ for some integer $m \geqq 0$, then u is in $(W_r^{m+2}(D))^n$ and satisfies*

$$\|u\|_{m+2,r} \leqq C_{m,r} \|f\|_{m,r}$$

with a constant $C_{m,r} > 0$ independent of u and f.

Let $C^\mu([0, T]; X)$ denote the space of Hölder continuous functions on $[0, T]$ with exponent μ and with values in a Banach space X. Similarly let $C^\mu((0, T]; X)$ denote the space of functions which are Hölder continuous on every subinterval $[\varepsilon, T]$ of $(0, T]$, with exponent μ. The following result is proved in [5], Lemma 2.14.

Lemma 3.2. *Let $f(t)$ be in $C^\mu([0, T]; X_r)$, for some $0 < \mu < 1$. Then the function*

$$v(t) = \int_0^t e^{-(t-s)A} f(s)\, ds$$

is in $C^\nu((0, T]; D(A)) \cap C^{1+\nu}((0, T]; X_r)$ for every ν such that $0 < \nu < \mu$.

Lemma 3.3. (i) *If $u \in (W_r^m(D))^n$ $(1 < r < \infty)$ for some integer $m \geqq 0$, then Pu belongs to $(W_r^m(D))^n \cap X_r$ and satisfies*

$$\| Pu \|_{m,r} \leqq C_{m,r} \| u \|_{m,r},$$

where $C_{m,r} > 0$ is independent of u.

(ii) *For $m > n/r$, there exists a constant $C_{m,r} > 0$ such that*

$$\| P(u, \nabla)\, v \|_{m,r} \leqq C_{m,r} \| u \|_{m,r} \| v \|_{m+1,r}$$

for every $u \in (W_r^m(D))^n$, $v \in (W_r^{m+1}(D))^n$ $(1 < r < \infty)$.

(iii) *When $r > n$, we have*

$$\| P(u, \nabla)\, v \|_{0,r} \leqq C_r \| u \|_{1,r} \| v \|_{1,r}$$

for all u, $v \in (W_r^1(D))^n$.

Proof. (i) When $m = 0$, the assertion is obvious from (1.1). When $m \geqq 1$, it is known [8] that $Pu = u - \nabla p$, where p is a solution of

$$\text{(3.1)} \qquad \Delta p = \operatorname{div} u \text{ in } D, \; \partial p/\partial \nu = \nu \cdot u \text{ on } S.$$

Here ν is the unit outward normal to S and $\nu \cdot u = \sum_{j=1}^n \nu^j u^j$. Since $\operatorname{div} u \in W_r^{m-1}(D)$ and $\nu \cdot u \in W_r^{m-1/r}(S)$, the equation (3.1) admits a solution $p \in W_r^{m+1}(D)$ which is unique up to an additive constant, such that

$$\| \nabla p \|_{m,r} \leqq C(\| \operatorname{div} u \|_{m-1,r} + \| u \|_{W_r^{m-1/r}(S)})$$

(see e.g. [20]). Since the right hand side is bounded above by $\| u \|_{m,r}$, the result follows.

(ii) It is known (see [1]) that $W_r^m(D)$ forms an algebra if $m > n/r$. Thus the assertion follows immediately from (i).

(iii) Since $r > n$, the Sobolev embedding theorem shows that $W_r^1(D) \subset C(\bar{D})$, with a continuous injection. We thus find

$$\begin{aligned} \| P(u, \nabla)\, v \|_{0,r} &\leqq C \| (u, \nabla)\, v \|_{0,r} \\ &\leqq C \| u \|_{L_\infty(D)} \| \nabla v \|_{0,r} \leqq C \| u \|_{1,r} \| v \|_{1,r}. \end{aligned}$$

Our main purpose in this section is to prove

Theorem 3.4. *If* $Pf = 0$, *the solution of equation* (III) *given by Theorem 2.3 belongs to* $(C^\infty(\bar{D} \times (0, T_*]))^n$.

In proving this result, we may restrict ourselves to the case $r > n$, $a \in X_r$. To see this, note that the solution $u(t)$ given by Theorem 2.3 satisfies

$$u(t) = e^{-(t-\varepsilon)A} u(\varepsilon) + \int_0^t e^{-(t-s)A} Fu(s)\, ds \tag{3.2}$$

on every subinterval $[\varepsilon, T_*]$ $(0 < \varepsilon < T_*)$. (See the proof of Theorem 2.5.) Theorem 2.5 shows that $u(\varepsilon) \in D(A)$. Suppose now that $1 < r \leqq n$. Since $0 \leqq n/2r - 1/2 \leqq \gamma < 1$, we have $D(A^\gamma) \subset X_n$ so that $D(A) \subset X_s$ for some $s > n$. This means, by (3.2), that we may assume $r > n$ and $a \in X_r$ for some $r > n$.

Thus we have only to prove the following

Proposition 3.5. *Let* $r > n$ *and let* $a \in X_r$. *Assume that the solution* $u(t)$ *of* (III) (*with* $Pf = 0$) *given by Theorem 2.3 exists on* $[0, T]$. *Then* $u \in (C^\infty(\bar{D} \times (0, T]))^n$.

We prove this in several steps. Let us fix $r > n$ and write $\|\cdot\|_{0,r} = \|\cdot\|$. The proof of Proposition 2.4 shows that $A^\alpha u(t)$ $(0 < \alpha < 1)$ is Hölder continuous on every subinterval $[\varepsilon, T]$, with exponent μ, $0 < \mu < 1 - \alpha$. Lemma 3.3 (iii) implies that

$$Fu \in C^\mu((0, T]; X_r) \text{ for all } \mu,\ 0 < \mu < 1/2.$$

Lemma 3.2 and Lemma 3.3 (ii) now imply

Lemma 3.6. $u \in C^\mu((0, T]; D(A))$ and $u' = du/dt \in C^\mu((0, T]; X_r)$ *for all* $\mu, 0 < \mu < 1/2$. *Moreover* Fu *belongs to* $C^\mu((0, T]; (W_r^1(D))^n)$.

Lemma 3.7. *We have* $u' \in C^\mu((0, T]; D(A^{1/2}))$ *for all* μ, $0 < \mu < 1/2$.

Proof. From (3.2) we get

$$\begin{aligned} A^{1/2}u(t) &= e^{-(t-\varepsilon)A} A^{1/2} u(\varepsilon) + \int_\varepsilon^t A e^{-(t-s)A} A^{-1/2} Fu(s)\, ds \\ &= e^{-(t-\varepsilon)A} A^{1/2} u(\varepsilon) + v(t). \end{aligned}$$

Since $e^{-(t-\varepsilon)A} A^{1/2} u(\varepsilon) \in C^\infty((\varepsilon, T]; X_r)$, we need only consider $v(t)$. Integrating by parts, we get

$$\begin{aligned} v(t) &= \int_\varepsilon^t (d/ds)\, e^{-(t-s)A} A^{-1/2} Fu(s)\, ds \\ &= A^{-1/2} Fu(t) - e^{-(t-\varepsilon)A} A^{-1/2} Fu(\varepsilon) \\ &\quad - \int_\varepsilon^t e^{-(t-s)A} A^{-1/2} (Fu)'(s)\, ds, \end{aligned} \tag{3.3}$$

where $(Fu)'(s) = (d/ds)Fu(s)$. Since $u(s) \in D(A)$ $(0 < s \leqq T)$, we have $Fu(s) = -\sum_j P(\partial/\partial x_j)\{u^j(s)u(s)\}$. Hence by Lemma 2.1

$$\begin{aligned}\|A^{-1/2}(Fu)'(s)\| &= \|\sum A^{-1/2}P(\partial/\partial x_j)(u^{j'}u + u^j u')(s)\| \\ &\leqq C\||u'(s)|\cdot|u(s)|\| \leqq C\|u(s)\|_{L_\infty(D)}\|u'(s)\| \\ &\leqq C\|A^{1/2}u(s)\|\,\|u'(s)\|.\end{aligned}$$

This relation together with Lemma 3.6 shows that $A^{-1/2}(Fu)' \in C^\mu((0,T];X_r)$. Lemma 3.2 and (3.3) now imply that $v' \in C^\mu((0,T];X_r)$. The proof is complete.

Since $D(A^{1/2}) \subset (W_r^1(D))^n$, Lemmas 3.1, 3.6, 3.7 and the identity $u = A^{-1}(Fu - u')$ show that

$$u \in C^\mu((0,T];(W_r^3(D))^n). \tag{3.4}$$

We complete the proof of Proposition 3.5 by induction. We shall say that $u(t)$ has property $(P)_m$ $(m \geqq 1)$ if

$$\begin{aligned}&u^{(m)} \in C^\mu((0,T];D(A^{1/2})),\\ &u^{(j)} \in C^\mu((0,T];(W_r^{m+1-j}(D))^n), \quad 1 \leqq j \leqq m-1,\\ &u \in C^\mu((0,T];(W_r^{m+2}(D))^n),\end{aligned}$$

for all μ, $0 < \mu < 1/2$. Here $u^{(j)} = (d/dt)^j u$. Lemma 3.7 and (3.4) show that $u(t)$ has property $(P)_1$. Proposition 3.5 follows immediately from

Lemma 3.8. *$(P)_m$ implies $(P)_{m+1}$.*

Proof. By Leibniz's rule we get

$$(Fu)^{(m)} = \sum_{j=1}^m \binom{m}{j} P(u^{(j)}, \nabla)\, u^{(m-j)},$$

so the assumption $(P)_m$ and Lemma 3.3 imply that

$$(Fu)^{(m)} \in C^\mu((0,T];X_r) \text{ for all } \mu,\ 0 < \mu < 1/2.$$

This allows us to differentiate the equation $u' + Au = Fu$ m times with respect to t, giving

$$u^{(m)}(t) = e^{-(t-\varepsilon)A}u^{(m)}(\varepsilon) + \int_\varepsilon^t e^{-(t-s)A}(Fu)^{(m)}(s)\,ds$$

on all subintervals $[\varepsilon, T]$ $(0 < \varepsilon < T)$. Thus Lemma 3.2 implies that

$$\begin{aligned}&u^{(m)} \in C^\mu((0,T];D(A))\\ &u^{(m+1)} \in C^\mu((0,T];X_v)\end{aligned} \tag{3.5}$$

for all μ, $0 < \mu < 1/2$. As in the proof of Lemma 3.7, the conditions (3.5) give

$$u^{(m+1)} \in C^\mu((0,T];D(A^{1/2})). \tag{3.6}$$

Since $u^{(j)} = A^{-1}((Fu)^{(j)} - u^{(j+1)})$, $1 \leqq j \leqq m-1$, we obtain from (3.5) and Lemma 3.3 (ii) the relation

$$u^{(j)} \in C^{\mu}((0, T]; (W_r^{m+2-j}(D))^n), \quad 1 \leqq j \leqq m-1. \tag{3.7}$$

Now the property $(P)_m$ implies, by Lemma 3.3 (ii), that

$$Fu \in C^{\mu}((0, T]; (W_r^{m+1}(D))^n).$$

Again using the identify $u = A^{-1}(Fu - u')$, the relation (3.7) with $j = 1$ gives

$$u \in C^{\mu}((0, T]; (W_r^{m+3}(D))^n). \tag{3.8}$$

Property $(P)_{m+1}$ follows from (3.5), (3.6), (3.7) and (3.8). This completes the proof.
The following result is proved similarly.

Theorem 3.9. *Let $f \in (C^{\infty}(\bar{D} \times (0, T]))^n$. Suppose a and Pf satisfy the assumption of Theorem 2.3. Then u belongs to $(C^{\infty}(\bar{D} \times (0, T_*]))^n$.*

Remarks. In [22] Serrin proved that a weak solution of the Navier-Stokes equations is smooth in $x \in D$ if it belongs to $L_{r,q} = L_q(0, T; (L_r(D))^n)$ for some q, r such that $n/r + 2/q < 1$. This result was later improved by KANIEL and SHINBROT [15], who proved that the above assumption for weak solutions implies smoothness in $(x, t) \in \bar{D} \times [0, T]$ when $n = 2, 3$ if the initial data are smooth on $\bar{D}$. LADYZHENSKAYA [17] discusses interior regularity in (x, t). The solutions given by Theorem 2.3 belong to the above $L_{r,q}$ space if $n/2r - 1/2 < \gamma$, but not necessarily if $n/2r - 1/2 = \gamma$. It should be noticed that our solutions do not in general belong to the class of weak solutions if $n/2r - 1/2 \leqq \gamma < 0$. Even in this case, Theorem 3.4 guarantees the smoothness of our solutions in $\bar{D} \times (0, T]$.

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