Exchange of Stabilities, Symmetry, and Nonlinear Stability

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1. Introduction

The motivation for the present article is two-fold. On the one hand, the aim is to investigate the connection between the symmetric part of a linear operator and the stability-instability boundary determined by the methods of linear theory (see CHANDRASEKHAR [2]) and of nonlinear energy theory (see JOSEPH [10–14], SERRIN [25, 26], GALDI [8]). On the other, we wish to derive sufficient conditions for nonlinear energy stability in a convection-like problem involving the gravity dependent motion of a suspension of swimming micro-organisms.

In certain hydrodynamic stability problems the linear and nonlinear boundaries coincide. Such a case is the conventional Bénard problem, a result first demonstrated by JOSEPH [10–11]. However, there are many other stability problems for which little is known of the nonlinear limit, or progress is possible only by very subtle use of coupling parameters (see JOSEPH [12]). Our goal here is to show that one of the essential connections between linear and nonlinear theory is the idea of symmetry of the operator associated with the *linearized* theory. This connection was recognised by DAVIS [5, 6], although we believe the results contained herein are new and clarify the overall situation.

More exactly, it is shown here that, provided exchange of stabilities holds in a precise sense, one can probably provide a well defined link between the linear and nonlinear boundaries even in cases where the linear operator appears non-symmetric at the outset. The method of procedure employs an idea from the reformulation of energy theory due to DAVIS & VON KERCZECK [7] together with changing the energy norm by a suitable spatial weight which renders the resulting linear spatial operator symmetric. The addition of the weight has, in general, the effect of weakening, the decay of perturbations in the sense that only conditional nonlinear stability is obtained, cf. JOSEPH & HUNG [13, 15]. It is worth pointing out, however, that the present study is not simply an alternation of work of previous writers; a new approach is presented which should prove helpful in many nonlinear stability problems.

After describing a general theory for the connection between linear and nonlinear stability in § 2, we illustrate our approach by studying the Bénard problem between spherical shells and the magnetohydrodynamic Bénard problem in § 3. This is followed in § 4 by a more general theory using the concept of weighted energy. Throughout, the emphasis is on finding *quantitative* stability results e.g. critical values for the Rayleigh number in convection problems. Thus when conditional stability is derived an accurate value for the size of the allowable initial energy is presented. Our energy stability results rely on finding the maximum of a certain quotient; in § 4 sufficient conditions are given for the demonstration of existence of the maximum. This is important because a formal calculation of the necessary Euler-Lagrange equations does not take into account the fact that the maximum may *not* exist.

To complete the paper we investigate the energy stability boundary for the movement of micro-organisms in a suitable growth medium [3, 4, 17]. The relevant system of equations gives rise to a basic equilibrium solution which is nonlinear. This in turn necessitates a new approach to the study of energy stability. Since many other convection problems possess this feature we believe that the present contribution may prove of value in several other areas.

2. Symmetry, linear and nonlinear stability

The concept of stability for continuum systems is closely connected to the choice of the norm adopted for measuring the "size" of perturbations to a given basic motion. For instance, it can happen that a motion which is stable with respect to a given norm n_1 is unstable with respect to another norm n_2. In this case the appropriate choice of norm is often suggested by the physics involved. On the other hand, it is also known that if n_1 is *equivalent* to n_2 the stability properties of the basic motion remain unchanged. Therefore, when a stability result is obtained with respect to n_1 there is usually no need to investigate further stability with respect to a different but equivalent norm n_2.

Let n be a given norm and denote by $\mathscr{E}(n)$ the class of norms equivalent to n. In this paper we shall show how it is possible to obtain an immediate connection between linear and nonlinear stability by choosing n^* appropriately, within the class $\mathscr{E}(n)$. (Earlier studies which inevstigated the linear-nonlinear connection in the standard norm are due to PRODI [21] and SATTINGER [24]).

Let H be a Hilbert space endowed with a scalar product $(\, ,)$ and associated norm $|.|$. We consider in H the following initial-value problem,

$$u_t = Lu + N(u) + 0, \quad u(0) = u_0. \tag{2.1}$$

Here L represents a linear operator (possibly unbounded), and N is a non-linear operator with $N(0) = 0$ in order that (2.1) admits the null solution. We assume:

(i) L is a densely defined closed operator such that $(L - \lambda I)^{-1}$ is compact for some $\lambda \in \mathbb{C}$ (I is the identity operator in H), that is L is an operator with compact resolvent;

(ii) The bilinear form associated with L is defined (and bounded) on a space H_* which is compactly embedded in H (the norm in H_* will be denoted by $|.|_*$);

(iii) The non-linear operator N verifies the condition

(2.2) $$(N(u), u) \geqq 0, \text{ for all } u \text{ in } D(N),$$

where $D(.)$ denotes the domain of the associated operator.

Thanks to (i) the following result is true (cf. KATO [16], pp. 185–187).

Theorem 1. *The spectrum of the operator L consists entirely of an at most denumerable number of eigenvalues $\{\sigma_n\}_{n\in\mathbb{N}}$ with finite (both algebraic and geometric) multiplicities and, moreover, such eigenvalues can cluster only at infinity.*

Since the operator L is in general non-symmetric the eigenvalues are not necessarily real; they may however be ordered in the following manner:

(2.3) $$\text{re}(\sigma_1) \leqq \text{re}(\sigma_2) \leqq \ldots \leqq \text{re}(\sigma_n) \leqq \ldots$$

In accord with standard literature on stability theory we include the following definitions.

Definition 1. The null solution of (2.1) is said to be *linearly stable* (hereafter referred to by the abbreviation *LS*) if and only if

(2.4) $$\text{re}(\sigma_1) > 0.$$

Definition 2. The null solution of (2.1) is said to be *nonlinearly stable* (*NS*) if and only if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that

(i) $$|u_0| < \delta \Rightarrow |u(t)| < \varepsilon,$$

and there exists γ with $0 < \gamma \leqq \infty$, such that

(ii) $$|u_0| < \gamma \Rightarrow \lim_{t\to\infty} |u(t)| = 0.$$

If $\gamma = \infty$, we say the null solution is *unconditionally nonlinearly stable* (*UNS*).

Our purpose is to investigate the relation between *LS* and *NS*. As indicated earlier, the operator L is in general nonsymmetric, though it allows a decomposition into two parts L_1 and L_2 such that

(i) $L = L_1 + L_2$, $D(L_2) \supset D(L_1) = D(L)$;

(ii) L_1 is symmetric, with compact resolvent;*

(iii) L_2 is skew-symmetric and bounded in H_*.

From (ii) it follows that L_1 satisfies a theorem of the same type as Theorem 1. Moreover, because of the symmetry, the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ associated with L_1 are *all* real and may be ordered

$$\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n \leqq \ldots$$

* Under extra assumptions on L (which are certainly fulfilled in most practical cases) it can be *deduced* that L_1 is an operator with compact resolvent (cf. KATO [16], p. 337, Theorem 3.3).

Let $L_1[\phi, \phi]$, $\phi \in H_*$, be the bilinear form associated with the operator L_1, i.e.,

$$(L_1\phi, \phi) = L_1[\phi, \phi], \quad \forall\, \phi \in D(L_1).$$

Under the above conditions, the following lemma is standard.

Lemma 1. *Let $\bar{\phi}$ be a (normalized) eigenfunction associated with the eigenvalue λ_1. Then*

$$\lambda_1 = L_1[\bar{\phi}, \bar{\phi}] = \min_{\phi \in H_*} \frac{L_1[\phi, \phi]}{|\phi|^2}.$$

The following result establishes unconditional nonlinear stability.

Theorem 2. *Suppose*

$$\lambda_1 > 0. \tag{2.5}$$

Then the null solution of (2.1) *is unconditionally nonlinearly stable.*

Proof. Form the scalar product of (2.1) with u to obtain

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + (Lu, u) + (N(u), u) = 0. \tag{2.6}$$

Since L_2 is skew-symmetric and since by (2.2) N is non-negative, there follows from (2.6)

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \frac{L_1[u, u]}{|u|^2}|u|^2 \leqq 0.$$

With the aid of Lemma 1 we thus derive

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \lambda_1\,|u|^2 \leqq 0 \tag{2.7}$$

and so

$$|u(t)|^2 \leqq |u_0|^2 \exp(-2\lambda_1 t),$$

In the light of (2.5), the theorem follows at once.

From the above considerations it follows that while the linear stability problem is reduced to studying the eigenvalue problem associated with all of L, nonlinear stability involves the study of the eigenvalues of the *symmetric* part of L_1 only. Moreover, whenever $L_2 = 0$ the two eigenvalue problems *coincide and linear stability always implies nonlinear stability.* (It is easily seen that the converse statement also holds, that is $\lambda_1 > 0$ imples $\mathrm{re}(\sigma_1) > 0$.)

Perhaps the simplest situation where the above result applies is the homogeneous Bénard problem for which a thermally conducting fluid, between two horizontal parallel planes and under the action of a vertical gravity field g, moves due to a temperature gradient provided by heating from below. The basic flow whose stability is to be investigated is the motionless state with a linear temperature profile. For this system the eigenvalue problem for the operator L is (in the non-dimensional form of JOSEPH [10, 11, 14])

$$-\sigma \boldsymbol{v} = -\Delta p + R\theta \boldsymbol{k} + \Delta \boldsymbol{v},$$

(2.8) $$-\sigma P\theta = Rw + \Delta\theta,$$

$$\nabla \cdot \boldsymbol{v} = 0; \quad \boldsymbol{v} = \theta = 0 \text{ on the boundary}.$$

In (2.8) $\boldsymbol{v}$ is the velocity field, θ and p the perturbation temperature and pressure, $\boldsymbol{k}$ is the unit vector in the z-direction (opposite to $\boldsymbol{g}$), $w = \boldsymbol{v} \cdot \boldsymbol{k}$, P is the Prandtl number, and R is the Rayleigh number. It can be seen immediately that the operator L which operates on $(\boldsymbol{v}, \theta)$ is symmetric in the L^2-product and, therefore, $L_2 = 0$.

We remark that the coincidence of *LS* and *UNS* for the Bénard problem was originally established by JOSEPH [10, 11, 14] who employed an entirely different (and more complicated)* method of parametric differentiation. To relate the method of JOSEPH and the present one we observe that the optimum "energy" parameter is suggested naturally if a suitable symmetrization of the linear operator exists.

3. Two further examples of symmetric convection problems

A. Convection between two spherical shells. For this important geophysical problem, let $r = A, B$ $(A < B)$ denote the two spherical shells containing between them a heat conducting linear viscous fluid. If we denote by x^i the spherical coordinates r, θ, ϕ, and by $g(r)$, $b(r)$ the gravitational potential and temperature distribution in the motionless state, the equations governing the velocity perturbation u^i and temperature perturbation θ are (see JOSEPH [14], p. 83 and Ex. (59.3), p. 21) in covariant notation

$$u^i_{,t} + u^j u^i_{,j} = -g^{ij} p_{,j} + \mathscr{R}\, g(r)\, r^i \theta + g^{mn} u^i_{,mn},$$

(3.1) $$P\theta_{,t} + P u^k \theta_{,k} = u_i r^i \mathscr{R}\, b(r) + g^{kq} \theta_{,kq},$$

$$u^i_{,i} = 0,$$

where $\mathscr{R}$, P are the Rayleigh and Prandtl numbers, r_i is the contravariant vector with components $1, 0, 0$ and g_{ij} represents the metric tensor.

Since g^{ij} is symmetric and its covariant derivative is zero, it is easy to verify that when $g(r) \propto b(r)$ the linear operator corresponding to the above system is symmetric. Hence for this special case the linear and nonlinear stability boundaries coincide, a result obtained by JOSEPH [14], p. 84, using his theory of coupling parameters.

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* JOSEPH's ideas of parametric differentiation and coupling parameters apply to a very wide class of problems which moreover certainly need not be symmetric. The power of the technique may be gauged from the works of DAVIS [6] and JOSEPH [12–14].

B. The magnetohydrodynamic Bénard problem. Here we apply energy stability theory to a special case of the more complicated magnetohydrodynamic Bénard problem.

Our starting point is to assume that the electric field $\boldsymbol{E}$ is always derivable from a potential, i.e. $\boldsymbol{E} = -\nabla\phi$; thus we are considering a type of "quasi-static" approximation. The magnetic and electric fields, $\boldsymbol{H}$, $\boldsymbol{E}$, satisfy Maxwell's equations (see ROBERTS [23], pp. 7–8, for example)

$$\text{curl}\, \boldsymbol{H} = \boldsymbol{j}, \tag{3.2}$$

$$\text{curl}\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.3}$$

where $\boldsymbol{j}$ is the current and $\boldsymbol{B} = \mu \boldsymbol{H}$ is the magnetic induction. From (3.3) we immediately see that $\boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{x})$ only. Henceforth we consider only the case $\boldsymbol{B} = B_0 \boldsymbol{k}$, that is, $\boldsymbol{B}$ directed in the upward z-direction, with the fluid occupying the layer $0 < z < d$.

Here $\text{div}\, \boldsymbol{B} = 0$ and $\text{div}\, \boldsymbol{j} = 0$. Therefore since

$$\boldsymbol{j} = \sigma(-\nabla\phi + \boldsymbol{u}\times\boldsymbol{B}_0), \tag{3.4}$$

where σ is the electrical conductivity, we obtain

$$0 = \sigma(-\Delta\phi + \boldsymbol{B}_0\cdot\nabla\times\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\times\boldsymbol{B}_0). \tag{3.5}$$

Suppose now that the vertical component of vorticity in the perturbed motion is zero, i.e. $\omega_z = 0$, where $\boldsymbol{\omega} = \text{curl}\, \boldsymbol{u}$, then the second term on the right of (3.5) is zero. Moreover, from (3.2),

$$\text{curl}\, \boldsymbol{B}_0 = \frac{1}{\eta}(-\nabla\phi + \boldsymbol{u}\times\boldsymbol{B}_0),$$

where $\eta\ (= 1/\mu\sigma)$ is the resistivity. Assuming the finiteness of grad ϕ, $\boldsymbol{u}$, $\boldsymbol{B}_0$, if we allow $\eta \to \infty$ (i.e. the magnetic Prandtl number $P_m = \nu/\eta \to 0$) then $\text{curl}\, \boldsymbol{B}_0 \to 0$, whence $\boldsymbol{B}_0 = (0, 0, B_0)$, where B_0 is constant. Hence in this limit (3.5) reduces to

$$\Delta\phi = 0. \tag{3.6}$$

We require this equation to hold everywhere, and so for sufficiently strong spatial decay in ϕ we necessarily have $\phi \equiv 0$. Therefore, for this model $\boldsymbol{j} = \sigma(\boldsymbol{u}\times\boldsymbol{B}_0)$.

The above derivation shows that in the "quasi-static" electric field approximation, if $P_m \to 0$ and if we consider only motions for which $\omega_z = 0$ (this certainly includes two-dimensional rolls), then the equation for the magnetic field may be dispensed with and the Lorentz force in the momentum equation takes the appealing form

$$\boldsymbol{j}\times\boldsymbol{B} = \sigma(\boldsymbol{u}\times\boldsymbol{B}_0)\times\boldsymbol{B}_0. \tag{3.7}$$

If we introduce the non-dimensional variables

$$\boldsymbol{x} = \hat{\boldsymbol{x}}\, d, \quad t = \hat{t}\, d^2/\nu, \quad \text{Pr} = \nu/\varkappa, \quad \mathcal{R} = \sqrt{[\alpha g\beta\, d^4/\varkappa\nu]},$$
$$\boldsymbol{u} = U\hat{\boldsymbol{u}}, \quad M = B_0\, d[\sigma/\varrho\nu]^{\frac{1}{2}}, \quad T = TU[\text{Pr}\, \beta/g\alpha]^{\frac{1}{2}},$$

where $\hat{}$ denotes non-dimensional, M is the Hartmann number and T and U are a typical temperature and velocity, then the equations for the perturbation $(\boldsymbol{u}, \theta, p)$ in the layer $0 < \hat{z} < 1$ become (from here on we omit the non-dimensional sign)

$$
\begin{aligned}
u_{i,t} + u_j u_{i,j} &= -p_{,i} + \Delta u_i + \theta k_i - M^2[\boldsymbol{k}\times(\boldsymbol{u}\times\boldsymbol{k})]_i, \\
\Pr(\theta_{,t} + u_i\theta_{,i}) &= \mathscr{R}w + \Delta\theta, \\
u_{i,i} &= 0,
\end{aligned}
\tag{3.8}
$$

cf. ROBERTS [23], p. 198. On the boundaries $z = 0, 1$ we have $\theta \equiv u_i \equiv 0$. Under these conditions it is not difficult to show the linear operator in (3.8) is symmetric for perturbations $(\boldsymbol{u}, \theta, p)$ which are periodic in x and y. Thus again we may conclude that the linear instability boundary coincides with the nonlinear energy one. (In fact, it is possible to weaken the periodicity conditions, but the present ones suffice for our needs.)

For completeness, we include a linear stability analysis. Perturbations of the linear system corresponding to (3.8), of the form $e^{\sigma t}(\boldsymbol{u}(\boldsymbol{x}), \theta(\boldsymbol{x}), p(\boldsymbol{x}))$ necessarily have $\sigma \in \mathbb{R}$, due to symmetry. Hence it is sufficient to study the linear problem with $\sigma = 0$.

Standard analysis with $\boldsymbol{u}(\boldsymbol{x}) = e^{i(kx+my)}\,\boldsymbol{u}(z)$, etc., and with $D = d/dz$, yields from (3.8),

$$
\begin{aligned}
(D^2 - a^2)^2\, W &= \mathscr{R}a^2\Theta + M^2\, D^2 W, \\
(D^2 - a^2)\,\Theta &= -\mathscr{R}W,
\end{aligned}
\tag{3.9}
$$

where $(U, V, W) = \boldsymbol{u}(z)$, $\Theta = \Theta(z)$ and $a^2 = k^2 + m^2$, a being the non-dimensional wavenumber. Therefore, W satisfies,

$$
(D^2 - a^2)\,[(D^2 - a^2)^2 - M^2\, D^2]\, W = -\mathscr{R}^2 a^2 W. \tag{3.10}
$$

Equation (3.10) is just equation (135) of CHANDRASEKHAR [2], p. 165. For example, the solution for two free boundaries is given in equation (165) of the same work, namely

$$
\mathscr{R}^2 = \frac{(\pi^2 + a^2)}{a^2}\,[(\pi^2 + a^2)^2 + \pi^2 M^2]. \tag{3.11}
$$

From (3.11) *using energy theory* we recover the effect that the magnetic field *stabilizes* Bénard convection, see for example the curve of critical Rayleigh number against the square of the Hartmann number given by CHANDRASEKHAR [2], p. 171, figure 39. If the conventional energy approach is employed on the full magnetohydrodynamic system from the outset then this effect is not obtained, see RIONERO [22].

It is not difficult to combine the two subsections (A) and (B) and obtain an equivalent result for the magnetic Bénard problem between two spherical shells. Another interesting example to which the theory of § 2 may be applied is thermohaline convection, see SHIR & JOSEPH [27], p. 69. We may deduce immediately the result (6B.12) of that work, that there are no subcritical instabilities in the "heated below "salty above" situation. However, the "heated and salty below" case still necessitates the difficult analysis of JOSEPH [12].

4. Non-symmetric operators and stability

In this section we return to the abstract equation. We commence by observing that the symmetry of a given operator L in a Hilbert space H depends critically on the scalar product adopted. In other words it may be that L is non-symmetric with respect to the scalar product $(\, , \,)$ but nevertheless becomes symmetric if we replace the latter with a new product $\langle \, , \, \rangle$ whose associated norm $\|.\|$ is equivalent to $|.|$. A necessary condition for this symmetrization to hold is that the operator L have real eigenvalues. In the language of linear stability theory this means that the principle of exchange of stabilities holds. (Strictly, this principle is more general in that if the linear time dependence is $e^{\sigma t}$ for $\sigma = p + iq$, $p, q \in \mathbb{R}$, then exchange of stabilities is said to hold if $q \neq 0$ implies $p < 0$, see DAVIS [5]. However, we shall here deal with the restricted definition for which exchange of stabilities means $q = 0$.)

We assume henceforth that exchange of stabilities holds. We suppose, moreover, that we can introduce in H a new scalar product $\langle \, , \, \rangle$ with respect to which L is symmetric, i.e.

$$\langle L\phi, \psi\rangle = \langle \phi, L\psi\rangle, \qquad \forall\, \phi, \psi \in D(L). \tag{4.1}$$

Fundamental to our development is the following idea. Assume that the bilinear form $L'[\phi, \psi]$ associated with L in the new scalar product admits the decomposition:

$$L'[\phi, \psi] = I(\phi, \psi) + D(\phi, \psi) \tag{4.2}$$

where

I and A

I and D are symmetric, bounded bilinear forms in H^*,

$$\begin{aligned} I(\phi, \phi) &\leqq c_1 \, |\phi| \, |\phi|_*, \qquad \forall\, \phi \in H_*, \\ D(\phi, \phi) &\geqq c \, |\phi|_*^2, \qquad \forall\, \phi \in H_*, \end{aligned} \tag{4.3}$$

where c, c_1 are positive constants.

Because L is symmetric in $\langle \, , \, \rangle$, its eigenvalues $\{\sigma_n\}$ are real and the following lemma, analogous to lemma 1, may be established.

Lemma 2. *Let $\bar{\phi}$ be a normalized eigenfunction associated with σ_1. Then*

$$\sigma_1 = L'[\bar{\phi}, \bar{\phi}] = \min_{\phi \in H_*} \frac{L'[\phi, \phi]}{\|\phi\|^2}.$$

All energy studies we are aware of depend heavily on the existence of the functional maximum of $-I/D$; in the following lemma this existence is established provided conditions (4.3) hold (cf. GALDI [8], RIONERO [22]).

Lemma 3. *Provided the conditions* (4.3) *hold there exists a function* $\bar{\psi} \in H_*$ *such that*

$$\max_{\psi \in H_*} \left\{ -\frac{I(\psi, \psi)}{D(\psi, \psi)} \right\} = -\frac{I(\bar{\psi}, \bar{\psi})}{D(\bar{\psi}, \bar{\psi})} \left(= \frac{1}{R_E} \right).$$

Proof. From (4.3) follows the existence of a constant $\mu \in \mathbb{R}$ such that

$$-\frac{I(\psi, \psi)}{D(\psi, \psi)} \leqq \mu.$$

Thus

$$\sup_{\psi \in H_*} \left\{-\frac{I(\psi, \psi)}{D(\psi, \psi)}\right\} = l < \infty.$$

Under the normalization

$$\psi \to \psi / D^{\frac{1}{2}}(\psi, \psi),$$

the proof reduces to showing the existence of $\overline{\psi} \in H_*$ with $D(\overline{\psi}, \overline{\psi}) = 1$ such that

$$-I(\overline{\psi}, \overline{\psi}) = l.$$

To this end we note that since $l < \infty$ there exists a sequence $\{\psi_n\} \subseteq H_*$ (maximizing sequence) such that

$$\text{(4.4)} \qquad \lim_{n \to \infty} - I(\psi_n, \psi_n) = l, \quad D(\psi_n, \psi_n) = 1 \quad \forall\, n \in \mathbb{N}.$$

On the other hand, because of $(4.3)_3$ and the compact embedding $H_* \to H$, we may select from $\{\psi_n\}$ a subsequence, which we continue to denote by $\{\psi_n\}$, such that

$$\psi_n \to \overline{\psi} \text{ weakly in } H_*,$$

$$\psi_n \to \overline{\psi} \text{ strongly in } H.$$

Along this sequence

$$\text{(4.5)} \qquad |I(\psi_n, \psi_n) - I(\overline{\psi}, \overline{\psi})| = |I(\psi_n - \overline{\psi}, \psi_n - \overline{\psi}) + 2I(\overline{\psi}, \psi_n - \overline{\psi})|.$$

Since $(4.3)_2$ holds, it follows for n sufficiently large that

$$\text{(4.6)} \qquad |I(\psi_n - \overline{\psi}, \psi_n - \overline{\psi})| \leqq c_1\, |\psi_n - \overline{\psi}|\, |\psi_n - \overline{\psi}|_* < \varepsilon.$$

In addition, for fixed $\chi \in H_*$, $I(\chi, w)$ is a linear form on H_*. Hence by the Riesz representation theorem there exists $\overline{\psi} \in H_*$ such that for fixed $\overline{\psi} \in H_*$

$$I(\overline{\psi}, \psi_n - \overline{\psi}) = (\overline{\Psi}, \psi_n - \overline{\psi})_*,$$

where $(\ ,\ )_*$ is the scalar product in H^*. Since $\psi_n \to \overline{\psi}$ weakly in H_* we conclude from the preceeding relation that for n sufficiently large

$$\text{(4.7)} \qquad |I(\overline{\psi}, \psi_n - \overline{\psi})| < \varepsilon.$$

Collecting (4.5)–(4.7) we thus deduce with the aid of (4.4) that

$$\lim_{n \to \infty} - I(\psi_n, \psi_n) = -I(\overline{\psi}, \overline{\psi}) = l.$$

Obviously, $D(\overline{\psi}, \overline{\psi}) = 1$ and the theorem follows.

To associate linear stability with the energy boundary we must relate the positivity of σ_1 to the size of R_E. This is done in the next lemma.

Lemma 4. *The eigenvalue* σ_1 *and the number* R_E *are such that*

$$\sigma_1 > 0 \text{ if and only if } R_E^{-1} < 1.$$

Proof. Assume $\sigma_1 > 0$. From Lemma 2 and (4.2) we deduce

$$D(\psi, \psi)\left(-\frac{I(\psi, \psi)}{D(\psi, \psi)} - 1\right) < 0 \quad \forall\, \psi \in H_*.$$

Let $\bar{\psi}$ be the maximizing function introduced in Lemma 3. Since $D(\bar{\psi}, \bar{\psi}) \neq 0$ we obtain from the preceeding inequality

$$\frac{1}{R_E} - 1 = -\frac{I(\bar{\psi}, \bar{\psi})}{D(\bar{\psi}, \bar{\psi})} - 1 < 0.$$

Conversely, assume $R_E > 1$. Then

$$\begin{aligned} -\sigma_1 &= \left(-\frac{I(\bar{\phi}, \bar{\phi})}{D(\bar{\phi}, \bar{\phi})} - 1\right) D(\bar{\phi}, \bar{\phi}) \leqq \left(-\frac{I(\bar{\psi}, \bar{\psi})}{D(\bar{\psi}, \bar{\psi})} - 1\right) D(\bar{\phi}, \bar{\phi}) \\ &= (R_E^{-1} - 1)\, D(\bar{\phi}, \bar{\phi}) < 0, \end{aligned}$$

where in the last step $(4.3)_3$ was employed.

We must now consider the nonlinear term in equation (2.1). Although (2.2) is satisfied in the "old" scalar product there is no reason why the same relation need be true in the new product $\langle\, ,\rangle$; indeed, in general it will not. We shall assume instead that either

(4.8)
$$\begin{cases} \text{for all } \varepsilon > 0 \text{ there exists a constant } c = c(\varepsilon) \text{ such that} \\ \qquad |\langle N(u)\, u\rangle\, | \leqq \varepsilon\, D(u, u) + c(\varepsilon)\, \|u\|^\alpha \\ \text{for each } u \text{ in } D(N) \text{ and for some } \alpha > 2; \end{cases}$$

or

(4.9)
$$\begin{cases} \qquad |\langle N(u), u\rangle| \leqq k\, \|u\|^\beta\, D(u, u), \\ \text{for each } u \text{ in } D(N) \text{ and for some } k, \beta > 0. \end{cases}$$

Condition (4.9) arises naturally in the stability problem for bio-convection considered in the next section. Although we do not give any specific example of an N satisfying (4.8), an energy analysis of Burger's equation gives rise to such a condition (this example is dealt with in [9]).

We are now in a position to state our stabiity theorem correlating linear and nonlinear stability.

Theorem 3. *Suppose exchange of stabilities holds, and that* (4.1)–(4.3) *and either* (4.8) *or* (4.9) *are satisfied. Then if the null solution of* (2.1) *is linearly stable it is also nonlinearly stable. In particular, there exist constants* A, γ, $\delta > 0$ *such that*

$$\|u_0\|^2 < \gamma \text{ implies } \|u(t)\|^2 \leqq A\, \|u_0\|^2 \exp(-\delta t) \quad \forall\, t \geqq 0.$$

The proof of this result is not given explicitly, since a proof under condition (4.8) is given in reference [9] while a demonstration appropriate to (4.9) is included in our discussion of bio-convection in § 5.

5. Energy stability in bio-convection

In this section we present an energy stability analysis for the continuum mode of bio-convection derived in [4, 17].

A näive exposition of the situation is as follows. Suppose that a suspension of micro-organisms is contained in a fluid layer, say the infinite layer between the planes $z = -H, 0$, and that a gravitational field acts in the negative z direction. The organisms have a density greater than that of the containing fluid and also have a natural tendency to swim in the upward (increasing z) direction. If a sufficient number of organisms are present, eventually the situation arises where the upper layer of the fluid is dominated by micro-organisms. These in turn, being of density greater than the fluid, will tend to fall under the action of gravity. Hence an instability somewhat akin to Rayleigh-Taylor instability may develop. The striking thing about this instability is that it does not happen in a haphazard manner; rather, the organisms tend to fall in discrete "chimneys" in a somewhat ordered pattern, although several pattern types are possible, see [4, 17].

The fact that such regular patterns are commonplace in many convection problems lends credence to our belief that the energy stability theory developed here may prove valuable elsewhere. In particular, another interesting convection mechanism which exhibits a distinct "chimney" structure, albeit with rising rather than falling plumes, is that caused by one component of a mixture being frozen out of a solution; a continuum theory for this phenomenon has been derived by Loper & Roberts [18, 19], who provide further details appropriate to the earth's core in [20].

The basic model we employ for the motion of microorganisms is derived by Childress, Levandowsky & Spiegel [4]. Letting $c(\boldsymbol{x}, t)$ denote the concentration of micro-organisms in the suspension, their equations are easily written in terms of a flux $\boldsymbol{J}$ given by

$$J_i = cU(c, z)\,\delta_{i3} - D_{ij}c_{,j}, \tag{5.1}$$

where U is the upward swimming velocity of the organisms and $\boldsymbol{D}$ is the diffusion tensor given by

$$\boldsymbol{D} = \operatorname{diag}\left(\varkappa_1(c, z), \varkappa_1(c, z), \varkappa(c, z)\right). \tag{5.2}$$

Here $\varkappa$, $\varkappa_1$ are positive functions of the indicated arguments.

The required equations, based on a Newtonian fluid model, are then

$$\begin{aligned} \dot{u}_i &= -\frac{1}{\varrho}p_{,i} - g(1 + \alpha c)\,\delta_{i3} + \nu\,\Delta u_i, \\ u_{i,i} &= 0, \\ \dot{c} &= -J_{i,i}, \end{aligned} \tag{5.3}$$

where ϱ is the density (constant), $\boldsymbol{u}$ the velocity, p the pressure, ν the viscosity, while g denotes gravity and α is a positive constant which essentially expresses the ratio of density of a microorganism to that of the growth medium. A superposed dot denotes material differentiation.

The linear stability of two classes of equilibrium solutions is considered in [4], namely:
Case I.

$$U = U_0, \quad \varkappa = \varkappa_0, \quad \varkappa_1 = \delta\varkappa_0; \quad U_0, \varkappa, \delta \text{ constant.}$$

Case II. $\varkappa/U$ is not explicitly dependent on z; $\varkappa_1$ is arbitrary.

Here we are able to analyse a sub-class of the more general Class II. To describe this solution we need the following boundary conditions

$$\boldsymbol{J}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{u} = 0 \text{ when } z = 0, -H. \tag{5.4}$$

The former expresses the condition that no material flows out of the planes $z = -H, 0$, whereas the latter is the no-slip condition. To ensure uniqueness for the equilibrium solution it is also necessary to impose the following restriction on the mean concentration (cf. [4]),

$$\int_V c\,dx = c_n = \text{const}.$$

The sub-class of solutions of Case II which we study occur when $\varkappa$, U are dependent only on z, $U' \leqq 0$, and $\varkappa/U = h^{-1} = \text{constant}$. Here $\varkappa$, $U \in C^1(-H, 0)$ while $\varkappa_1 \in C^1(\mathbb{R}^2 \times (-H, 0))$, although $\varkappa_1$ is otherwise arbitrary. The basic equilibrium solution $\boldsymbol{u} \equiv \boldsymbol{0}$, $c = K(z)$ of this sub-class is

$$K(z) = c_0 \exp(U_0 z/\varkappa_0), \quad z \in [-H, 0], \tag{5.5}$$

where

$$c_0 = K(0), \tag{5.6}$$

and U_0, $\varkappa_0$ are the values of U, $\varkappa$ when $z = 0$.

One conclusion immediately evident from (5.5) is that the basic equilibrium solution is nonlinear in z, so that an energy stability analysis is likely to be different from that required for such constant gradient problems as Bénard convection (see JOSEPH [10, 11, 14]).

To study stability we let $\boldsymbol{u} = (u, v, w)$, $c = \phi(\boldsymbol{x}, t) + K(z)$ and $p = p + P(z)$, where $(\boldsymbol{u}, \phi, p)$ are perturbations of the equilibrium values $(\boldsymbol{0}, K, P)$. These perturbations satisfy, by (5.3),

$$\left\{\begin{aligned} u_{i,t} + u_j u_{i,j} &= -\frac{1}{\varrho} p_{,i} + \nu \Delta u_i - g\alpha\phi\,\delta_{i3}, \\ u_{i,i} &= 0, \\ \phi_{,t} + u_i\phi_{,i} + wK' &= (\varkappa_1\phi_x)_x + (\varkappa_1\phi_y)_y + (\varkappa\phi')' - (\phi U)', \end{aligned}\right. \tag{5.7}$$

where $' \equiv \partial/\partial z$. The boundary conditions are

$$\varkappa\phi' = \phi U, \quad u_i = 0 \text{ when } z = -H, 0, \tag{5.8}$$

and we additionally assume that

$$u_i, \phi, p \text{ are periodic functions in } x, y. \tag{5.9}$$

An inspection of $(5.7)_3$ reveals that the part of the linear operator L which is not symmetric arises from the terms

$$(\varkappa\phi')' - (\phi U)'.$$

In [4] it is shown however that the eigenvalue σ of linear theory is real; we employ their ideas to formulate a new scalar product. To begin with, we divide $(5.7)_3$ by the positive quantity K' and simultaneously introduce the non-dimensional variables

$$u_i = U_0\hat{u}_i, \quad \tilde{C} = U_0\sqrt{\frac{\sigma c_0}{g\alpha h}}, \quad R = \sqrt{\frac{\alpha g c_0 h^3}{\nu\varkappa_0}}, \quad U = \hat{U}U_0,$$

$$\hat{P} = \frac{U_0\varrho\nu}{h}, \quad p = \hat{p}\hat{P}, \quad \hat{t} = \frac{t\nu}{h^2}, \quad \phi = \hat{\phi}\tilde{C}, \quad \varkappa = \hat{\varkappa}\varkappa_0, \tag{5.10}$$

$$\varkappa_1 = \hat{\varkappa}_1\varkappa_0, \quad \sigma = \nu/\varkappa_0, \quad K = \hat{K}c_0, \quad h = \varkappa_0/U_0,$$

where σ is the Schmidt number, R is like a Rayleigh number and h is our unit of length. Furthermore we let $\lambda = H/h$ and $\hat{x}_i h = H + x_i$, $i = 1, 2, 3$, so that the layer $z \in (-H, 0)$ becomes $\hat{z} \in (0, \lambda)$.

The resulting equations from (5.7) are (we omit the non-dimensional symbol $\wedge$ for simplicity)

$$u_{i,t} + \frac{1}{\sigma}u_j u_{i,j} = -p_{,i} + \Delta u_i - R\phi\,\delta_{i3},$$

$$u_{i,i} = 0, \tag{5.11}$$

$$\frac{\sigma\phi_{,t} + u_i\phi_{,i}}{K'} = -Rw - \frac{(\phi U)'}{K'} + \frac{(\varkappa_1\phi_x)_x + (\varkappa_1\phi_y)_y}{K'} + \frac{(\varkappa\phi')'}{K'}.$$

The boundary conditions are still (5.8) and (5.9), although (5.8) now holds on $z = 0, \lambda$. We shall suppose the disturbance "cell", which we denote by V, is $2\pi r$ periodic in x and $2\pi s$ periodic in y, for positive constants r and s.

To proceed according to the theory of § 4, we need to know the spaces H and H_*. The restriction (5.8) necessitates the choice $H = (L^2(\Omega))^4$, $H_* = (H^1(\Omega))^4$, where the appropriate norms are understood to be weighted in the fourth component by $(K')^{-1}$ while and the first three components are divergence free.*

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* If we replace (5.8) by the more restrictive condition $\phi = 0$ on $z = 0, \lambda$, that is, if we are able to control the concentrations at the plates, then the analysis is much easier and we can use the space $H_* = (H_0^1(\Omega))^4$; see [9].

In the form (5.11), the linear operator L which acts on $(u, v\ w, \phi)$, is given by

$$\begin{bmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & -R \\ 0 & 0 & -R & \frac{1}{K'}[\partial_\alpha(\varkappa_1\,\partial_\alpha\,\cdot\,) + \partial_z(\varkappa\,\partial_z\,\cdot\,) - \partial_z(U\,\cdot)] \end{bmatrix}$$

where the repeated α signifies summation over $\alpha = 1, 2$. With the aid of (5.8) and (5.9) it is easily verified that L is symmetric, thanks to the weight K'.

The natural energy to use is suggested by (5.11), namely

$$E(t) = \tfrac{1}{2}\int_V \left(u_i u_i + \frac{\sigma\phi^2}{K'}\right) dx. \tag{5.12}$$

Further calculation reveals the energy equation to be

$$\dot{E} = -2R\int \phi w\, dx - D(u) - \int \frac{\varkappa_1(\phi_x^2 + \phi_y^2)}{K'}\, dx - \int \frac{\varkappa(\phi' - \phi/h)^2}{K'}\, dx - \frac{1}{2h}\int \frac{w\phi^2}{K'}\, dx, \tag{5.13}$$

where we have omitted the volume identification V from the integral and where $D(\cdot)$ denotes the Dirichlet integral.

To relate this equation to the work of § 4 we must specify the forms I and D in (4.2). To this end let ξ^α, $\alpha = 1, 2$, be the vectors $(u^\alpha, v^\alpha, w^\alpha, \phi^\alpha)$; then we choose

$$I(\xi^1, \xi^2) = R\int (\phi^1 w^2 + \phi^2 w^1)\, dx - \int \frac{U}{K'}(\phi^1\phi^{2\prime} + \phi^2\phi^{1\prime})\, dx, \tag{5.14}$$

$$D(\xi^1, \xi^2) = \int (\nabla u^1\, \nabla u^2 + \nabla v^1\, \nabla v^2 + \nabla w^1\, \nabla w^2)\, dx + \int \left\{\frac{\varkappa_1(\phi_x^1\phi_x^2 + \phi_y^1\phi_y^2) + \varkappa\phi^{1\prime}\phi^{2\prime} + U\phi^1\phi^2/h}{K'}\right\} dx. \tag{5.15}$$

Recollecting that U, $\varkappa_1$, $\varkappa$ are bounded both from above and below, it is not difficult to see that (4.3) holds and that $L' = -I + D$ is the correct form associated with L under conditions (5.8) and (5.9).

We return to (5.13) and let $\xi = (u, v, w, \phi)$ and $\mathscr{D} = D(\xi, \xi)$. The key to energy stability is Lemma 4. For, if $\sigma_1 > 0$ then $R_E^{-1} < 1$ and we have a condition necessary for nonlinear stability. Thus, it is sufficient to use the linear results of CHILDRESS, LEVANDOWSKY & SPIEGEL [4] to infer energy stability. We must, of course, prove the decay of the perturbations. To this end suppose that $\sigma_1 > 0$ and let $R_1 = 1 - R_E^{-1} > 0$. From (5.13) we derive

$$\dot{E} \leqq -R_1\mathscr{D} + \frac{1}{2h}\left|\int \frac{w\phi^2}{K'}\, dx\right|. \tag{5.16}$$

Although we have gained by symmetrizing the problem it has been necessary to include in (5.16) a nonlinear term $N = (1/2h) \, | \int K'^{-1} \, w \, \phi^2 \, dx |$. To deal with this term we note that $K = c_0 \exp(z - \lambda)$, so $c_0 e^{-\lambda} \leqq K' \leqq c_0$. By the Cauchy-Schwarz inequality,

$$N \leqq \frac{e^{\lambda}}{2hc_0} \, [\int w^2 \, dx]^{\frac{1}{2}} \, [\int \phi^4 \, dx]^{\frac{1}{2}}. \tag{5.17}$$

ADAMS ([1], p. 104) shows that $\|u\|_{L^6} \leqq 2^{\frac{1}{2}} \, 4 \, \|u\|_{H^1}$ when the domain geometry is a rectangular box, as V is here. If we combine this with a simple application of Hölder's inequality we find

$$(\int \phi^4 \, dx)^{\frac{1}{2}} \leqq 32 \cdot 2^{1/3} \, \pi^{1/3} \, (rs\lambda)^{1/6} \int (\phi^2 + \phi_{,i}\phi_{,i}) \, dx.$$

Hence from (5.17) we derive

$$N \leqq \frac{16 \cdot 2^{5/6} \, \pi^{1/3} \, (rs\lambda)^{1/6} \, e^{\lambda}}{hc_0^2} \, E^{\frac{1}{2}} \int \frac{\phi^2 + \phi_{,i}\phi_{,i}}{K'} \, dx.$$

Next, let U_*, $\varkappa_*$, $\varkappa_{1*}$ denote the lowest values for U, $\varkappa$, $\varkappa_1$ in V and set $a = \max \{1, \varkappa_{1*}^{-1}, \varkappa_*^{-1}, h/U_*\}$. We finally obtain

$$N \leqq b \, \mathscr{D} \, E^{\frac{1}{2}}, \tag{5.18}$$

where

$$b = 16 \cdot 2^{5/6} \, \pi^{1/3} \, (rs\lambda)^{1/6} \, e^{\lambda} a / hc_0^2. \tag{5.19}$$

Next, we combine (5.18) and (5.16) to deduce that

$$\dot{E} \leqq -\mathscr{D}(R_1 - bE^{\frac{1}{2}}/2h). \tag{5.20}$$

Suppose now that

$$E^{\frac{1}{2}}(0) < 2hR_1/b. \tag{5.21}$$

Then from (5.20) find that $E^{\frac{1}{2}}(t)$ satisfies (5.21) for every $t \geqq 0$. Furthermore, from Poincaré's inequality

$$\mathscr{D} \geqq 2\lambda_1 E$$

for some constant $\lambda_1 > 0$, whence (5.20) leads to

$$\dot{E} \leqq -2\lambda_1 R_1 E + \frac{b\lambda_1}{h} E^{3/2}.$$

This inequality is easily integrated, yielding

$$E^{\frac{1}{2}}(t) \leqq \frac{2hR_1 E^{\frac{1}{2}}(0)}{bE^{\frac{1}{2}}(0) + [2hR_1 - bE^{\frac{1}{2}}(0)] \exp(\lambda_1 R_1 t)}. \tag{5.22}$$

Since (5.21) holds, it follows that $E \to 0$ as $t \to \infty$. Thus we have shown that, when the initial energy satisfies (5.21) and the linear stability condition $\sigma_1 > 0$ holds, the solution $K(z)$ is also nonlinearly stable.

Several interesting, conclusions may be drawn from the above analysis. In particular, if we look closely at (4.21) we find $E(0) < ke^{-2\lambda}\lambda^{-1/3}$ where k is a constant dependent essentially on the values at the upper plate. Since $\lambda = H/h$ we see that the larger is H the smaller must be $E(0)$ before we can guarantee stability. This agrees with the findings of [5, 17] and more or less says that the greater the depth the more organisms will be present and therefore the greater the likelihood of instability. Another important point is that the energy decay is conditional and does not, therefore, preclude all subcritical instabilities; if R is near its critical value, then R_E^{-1} is close to 1 and so $E(0)$ must be very small to ensure decay. Again, this agrees with the work of CHILDRESS & SPIEGEL [3] who have constructed a two-dimensional solution which bifurcates subcritically.

In conclusion we observe that the approach in § 4, especially Lemma 4, seems different from that adopted by most fluid dynamicists. It is more usual to calculate directly the Euler-Lagrange equations for $\max(-I/D)$, and then to verify by inspection that the equations obtained agree with those of linear theory. This approach is perfectly correct and equivalent to that used in § 4, though we again point out that the maximum must exist (otherwise the calculations are purely formal and possibly incorrect). In the interests of clarity, therefore, we include a calculation of the necessary Euler-Lagrange equations for the problem at hand.

The functional to be maximised is $-I/D$, where, from (5.14), (5.15),

$$-I = -2R\int \phi w\, dx + 2\int \frac{U\phi\phi'}{K'}\, dx,$$

$$D = \int u_{i,j}u_{i,j}\, dx + \int \frac{1}{K'}\left[\varkappa(\phi_x^2 + \phi_y^2) + \varkappa(\phi') + U\phi^2/h\right] dx.$$

Following standard practice, if η_i, $i = 1, \ldots, 4$, denotes a perturbation from the maximising solution $\boldsymbol{v} = (u_i, \phi)$, then the maximum is given by

$$\int \left[\eta_i \frac{\partial I}{\partial v_i} + \eta_{i,j}\frac{\partial I}{\partial v_{i,j}}\right] dx + \Lambda \int \left[\eta_i \frac{\partial D}{\partial v_i} + \eta_{i,j}\frac{\partial D}{\partial v_{i,j}}\right] dx = 0.$$

Integrating by parts, formally, we find that

$$\text{(5.23)} \qquad \int_V \eta_i(R\phi\,\delta_{i3} - \Lambda\,\Delta u_i + p_{,i})\, dx + \Lambda \int_{\partial V} n_j\eta_i u_{i,j}\, dA$$
$$+ \int_V \eta_4\left\{Rw + \frac{(U\phi)'}{K'} - \Lambda\frac{(\varkappa_1\phi_x)_x}{K'} - \Lambda\frac{(\varkappa_1\phi_y)_y}{K'} - \Lambda\frac{(\varkappa\phi')'}{K'}\right\} dx$$
$$+ \int_{\partial V} \eta_4\left\{\left[\frac{\Lambda\varkappa\phi' - U\phi}{K'}\right]_z + \Lambda\frac{\varkappa_1\phi_x}{K'}\bigg|_x + \Lambda\frac{\varkappa_1\phi_y}{K'}\bigg|_y\right\} dA = 0.$$

Here $\Lambda = \max(-I/D)$, i, j are to be summed from 1 to 3, and in the last term the subscripts x, y, z denote evaluation on the respective boundary sections of ∂V. The stability boundary is $R_E^{-1} = 1$ and so for the critical case we take $\Lambda = 1$.

Now, selecting $\eta \in (C_0^\infty(V))^4$, we derive immediately the equations of linear theory with the time derivative terms zero. Thus we have shown that the Euler-Lagrange equations corresponding to max $(-I/D)$ coincide with the equations for marginal stability of linear theory. To derive the correct boundary conditions requires more technical analysis, but we note that they follow from what remains of (5.23) for η again to be suitably selected.

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