Stability of Static Configurations with Applications to the Theory of Capillarity

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I. Introduction

In Fig. 1 two linearly elastic strings S_0 and S_1 of unstretched lengths ℓ_0 and ℓ_1 respectively are represented, one end of each being joined at P, the other end of S_0 being tethered at P_0 and the remaining end of S_1 at P_1. The distances $d(P_0 P) = \overline{m}_0$ and $d(PP_1) = \overline{m}_1$ exceed ℓ_0 and ℓ_1 respectively. Suppose P is fixed somewhere on the straight line segment $P_0 P_1$ in such a way that there can be no interaction between the strings and let the string tensions be T_0 and T_1 respectively. Then the configurations of the two strings are in equilibrium and stable[1]; they will remain so upon freeing P, however, if and only if $T_0 = T_1$.

In Fig. 2 a horizontal tube with a closed stopcock in it is filled with liquid 1, which extends past the ends of the tube to form two equal spherical segments

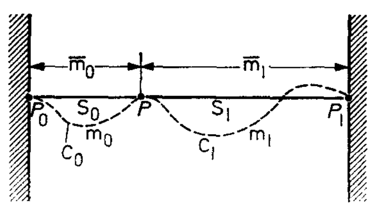

Fig. 1

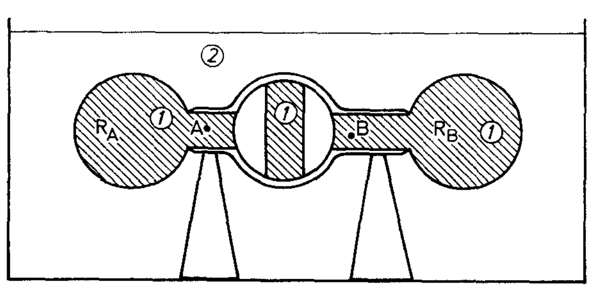

Fig. 2. Two identical balloon (spherical cap) interfaces between fluids 1 and 2, each of equal density, isolated by the closed stopcock. When the stopcock is opened, the interfaces will change.

[1] We assume that there is no external body force such as gravity acting on the strings.

which interface with liquid 2 immiscible with liquid 1 but of equal density ρ. The spherical segments are each larger than hemispheres. The barrel of the stopcock is also filled with liquid 1.

Elementary considerations show that the configurations of the fluids are stable and that the quantity $p+\rho g z$ (where p is the hydrostatic pressure and z is the height above a horizontal datum plane) is equal at points A and B; nevertheless it is well known to students of capillarity that upon opening the stopcock the configurations will alter. (A common result is that fluid will pass through the stopcock (in a direction depending on initial disturbances), until one of the free surfaces is less than a hemisphere, and of the same radius as the other surface, volume being conserved.)

We have mentioned above examples of how the stability of disjoint systems may be influenced if constraints on them are relaxed so that they may interact. In the first case the right hand end of S_0 was fixed as was the left hand end of S_1 in the disjoint case, but in the interacting case these end points were merely required to be the same point, being otherwise free. In the second case the volumes V_A and V_B of the disjoint regions R_A and R_B were both fixed, at values $\bar{V}_A$ and $\bar{V}_B$ say, but upon opening the stopcock these were merely required to satisfy the equation $V_A+V_B=\bar{V}_A+\bar{V}_B$.

Both of these examples (and many others) may be identified with members of an abstract structure which we shall develop in this paper. We shall prove a theorem of central importance concerning how the stability of an interacting collection of systems may be deduced from information about the systems it comprises, and we shall apply the theorem to a number of examples in capillarity, providing (in particular) the theory for a hitherto unexplained experimental result of PLATEAU [22]. This theorem is of wide applicability in stability theory and the fact that our examples are confined for the most part to capillarity should not be considered a limitation. The theorem will be formulated in an abstract fashion but is mathematically very simple. The advantages of abstract formulation are obvious. We shall need to introduce some primitive concepts first.

II. Primitive Concepts

A. Unit, Simple Element. A *unit* is an ordered pair $(\mathscr{C}, H)$ where $\mathscr{C}$ is a topological space called the *configuration space*, and where H is a real valued functional on $\mathscr{C}$ called the *energy functional.*[1] Each element C in $\mathscr{C}$ is called a *configuration*, and the value $H(C)$ is called the *energy* of C. A configuration $\bar{C}$ of a unit is said to be *stable* if it is locally minimizing for H, *i.e.*,

$$H(\bar{C})<H(C)\ \forall C\in\mathscr{N}(\bar{C}), \quad C\neq\bar{C}$$

where $\mathscr{N}(\bar{C})$ is a neighborhood of the configuration $\bar{C}$. If $\bar{C}$ satisfies the relation

$$H(\bar{C})\leqq H(C)\ \forall C\in\mathscr{N}(\bar{C}),$$

then it is called *weakly stable.*

[1] We use the symbol H for the energy functional because, in many examples, H turns out to be the *Helmholtz free energy* functional of the physical system considered.

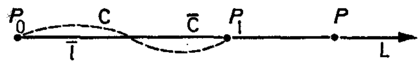

Fig. 3

Remark. The above definition is not restricted to equilibrium configurations; in fact equilibrium cannot necessarily be defined on configuration spaces in general. However, in cases for which equilibrium can be defined independently of energy considerations, such as in the theory of statics, we find that stability (based on the above definition) implies equilibrium.

For an example of a unit with a stable configuration consider a linearly elastic string of unstretched length ℓ_0, its ends fixed at the points P_0, P_1 in $\mathbb{R}^3$, the two points being at a distance $\bar{\ell}$ apart, where $\bar{\ell} > \ell_0$ (see Fig. 3). Let S be the totality of segments of smooth curves in $\mathbb{R}^3$ joining P_0 and P_1. We define a topology on S by defining convergence in S. First we define the functional[1] b taking the ordered pair (C_1, C_2) of elements of S to $\mathbb{R}$. Let x_1, x_2 be points of $\mathbb{R}^3$ belonging to the curves C_1 and C_2 respectively.

Then

$$b(C_1, C_2) \equiv \max_{x_1 \in C_1} \bar{d}(x_1, C_2) \equiv \max_{x_1 \in C_1} \left(\min_{x_2 \in C_2} \bar{d}(x_1, x_2) \right)$$

where $\bar{d}$ is the Euclidean distance function on $\mathbb{R}^3$. Given a subset $\tilde{S}$ of S, and $\hat{C} \in S$, then a sequence $C_1, C_2 \ldots C_n \ldots \in \tilde{S}$ converges to $\hat{C}$ if and only if $b(C_n, \hat{C}) \to 0$ as $n \to \infty$. $\tilde{S}$ is defined to be closed if and only if all sequences in $\tilde{S}$ which converge have limits in $\tilde{S}$. The collection of closed sets thus defined satisfies the closed set axioms for a topology on S. Let $\mathscr{C}$ be the associated topological space, and H be the strain energy functional $\frac{1}{2}k(\ell - \ell_0)^2$ where ℓ is the arc length of any curve $C \in \mathscr{C}$, k being the positive elastic constant of the string.[2] By a well known result in Euclidean geometry, the configuration $\bar{C}$ defined by the straight line $P_0 P_1$ is stable.

A *constrained unit* is an ordered trio $(\mathscr{C}, H, V)$, where $\mathscr{C}$ and H form a unit, and where V is a continuous mapping

$$\mathscr{C} \to \mathbb{R}$$

called the *constraint mapping*. We can provide an example of a constrained unit as follows: Consider a (straight) half line L in Fig. 3 starting at P_0, passing through P_1, and continuing to ∞. P is an arbitrary point of L at a distance more than ℓ_0 from P_0. $\mathscr{P}$ is the set of all such points. $\mathscr{C}$ is the totality of segments of smooth curves with one end point at P_0 and the other end point belonging to $\mathscr{P}$. Let $\tilde{P}$ be the end point in $\mathscr{P}$ of any $\tilde{C} \in \mathscr{C}$. Then $V(\tilde{C})$ will be defined as the length of the straight line $P_0 \tilde{P}$, and $H(\tilde{C})$ will be defined as before in terms of the arc length $\tilde{\ell}$ of the curve C: $H(\tilde{C}) = \frac{1}{2}k(\tilde{\ell} - \ell_0)^2$. We define the topology on $\mathscr{C}$ by

[1] The functional b is *not* a metric, being unsymmetrical and not satisfying the triangle inequality.

[2] This choice of energy functional corresponds to a physical situation for which body forces are absent, and the strings are in a state of uniform strain. The uniform strain will be discussed further in a later section on examples.

defining convergence in $\mathscr{C}$ in terms of $b(C_1, C_2)$, as in the preceding example. Then it can be verified easily that V is a continuous function on $\mathscr{C}$.

For a constrained unit suppose there exists a configuration $\bar{C}$ with constraint $\bar{V}$ (*i.e.* $V(\bar{C})=\bar{V}$) such that

$$H(\bar{C})<H(C)$$

for all C which satisfy

$$C\in\mathscr{N}(\bar{C}), \quad C\neq\bar{C}, \quad V(C)=\bar{V}; \tag{1}$$

then and only then the configuration $\bar{C}$ will be termed *stable*. In view of (1) it is convenient to introduce the notation $C(V)$ to represent any configuration with constraint V, *i.e.*

$$V(C(V))=V.$$

Then $\bar{C}(V)$ is stable if and only if

$$H(\bar{C}(V))<H(C(V)) \ \forall\, C(V)\neq\bar{C}(V), \quad C(V)\in\mathscr{N}(\bar{C}(\bar{V})).$$

We are now in a position to define a *simple element*. A simple element is a constrained unit with the following additional properties:

1) There exists a *base configuration* $\bar{C}(\bar{V})$ with the *base constraint* $\bar{V}$ such that

$$N(\bar{V})\subset V(\mathscr{N}(\bar{C}(\bar{V}))). \tag{2}$$

2) For each $V\in N(\bar{V})$ there exists a configuration $\bar{C}(V)\in\mathscr{N}(\bar{C}(\bar{V}))$ such that

$$H(\bar{C}(V))<H(C(V)) \ \forall\, C(V)\in\mathscr{N}(\bar{C}(\bar{V})), \quad C(V)\neq\bar{C}(V). \tag{3}$$

Here $N(\bar{V})$ denotes a neighborhood of $\bar{V}$ in $\mathbb{R}$, *i.e.*, an interval containing $\bar{V}$ as an interior point.

Note. The first property is essential for the proofs which we shall make later on. This property is not a consequence of the continuity of the constraint mapping!

We shall use the notation $E=(\mathscr{C}, H, V, \bar{V}, \bar{C}(V))$ for a simple element. The linearly elastic string provides an example of a simple element, $\bar{C}(V)$ being simply the straight line segment P_0P of length V for each $P\in\mathscr{P}$.

Now we shall motivate the next two important concepts by means of another simple example. These concepts are *multiple element* and *ensemble*. Consider the following elementary mechanical problem. Two linearly elastic strings of unstretched lengths, ℓ_0 and ℓ_1, and energy functionals $H_0=\frac{1}{2}k_0(m_0-\ell_0)^2$ and $H_1=\frac{1}{2}k_1(m_1-\ell_1)^2$ (where m_0, m_1 are the respective stretched lengths) are joined as shown in Fig. 1 and described in the introduction. The problem is to investigate the stability of various proposed configurations, and in particular to determine whether a given proposed location of the junction point P belongs to the stable configurations when P is free. The criterion for stability is that the configuration taken up by the strings should locally minimize the "ensemble" energy functional $H\equiv H_1+H_2$. Putting aside for the moment the fact that we know the rather obvious intuitive solution which is that the only stable configuration has each string rectilinear with P between P_0 and P_1 on the straight line joining them, and so disposed that $T_0=k_0(\overline{m}_0-\ell_0)=k_1(\overline{m}_1-\ell_1)=T_1$, where $\overline{m}_0$ and $\overline{m}_1$ are the

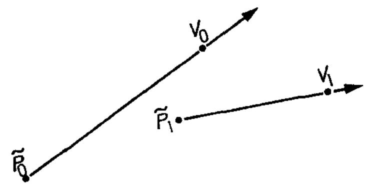

Fig. 4

(straight line) lengths of the two strings, we examine by what logic we may conclude this. First let us simplify the problem by assuming that P lies between P_0 and P_1 on the straight line joining them (we will later indicate how to remove this assumption). Next we shall show how the concept of a simple element may be used.

If we consider each string separately, it is convenient to take as origins of one dimensional coordinate systems the ends of the strings $\tilde{P}_0$ and $\tilde{P}_1$ which in the problem are attached to the points P_0 and P_1. Then we have two simple elements $E_0=(\mathscr{C}_0, H_0, V_0, \overline{V}_0, \overline{C}_0(V_0))$ and $E_1=(\mathscr{C}_1, H_1, V_1, \overline{V}_1, \overline{C}_1(V_1))$ associated with the strings (see Fig. 4). Now we put the origins $\tilde{P}_0$, $\tilde{P}_1$ at P_0, P_1, respectively, in Fig. 1 and take the coordinate axes of the two one-dimensional systems to have positive sense in the direction $\overrightarrow{P_0P_1}$. If we take $\overline{V}_0=\overline{m}_0$ and $\overline{V}_1=-\overline{m}_1$ (we also suppose that $\overline{V}_0>\ell_0$, $-\overline{V}_1>\ell_1$), then clearly, when P is *fixed*, the pair of configurations $\overline{C}_0(\overline{V}_0)$, $\overline{C}_1(\overline{V}_1)$ is in no way coupled and certainly minimizes H_0 and H_1 locally. The question to be resolved concerns whether this pair still minimizes H if the constraint $V_0=\overline{m}_0$, $V_1=-\overline{m}_1$ is relaxed to

$$V_0-V_1=\overline{m}_0+\overline{m}_1=L, \qquad \text{a constant.}$$

In considering pairs $\{C_0(V_0), C_1(V_1)\}$ of configuration functions whose Helmholtz energy is to be compared with that of $\{\overline{C}_0(\overline{V}_0), \overline{C}_1(\overline{V}_1)\}$ we note that because of properties 1) and 2) of the simple element we may ignore all except the pair of one parameter families of straight lines $\{\overline{C}_0(V_0), \overline{C}_1(V_1)\}$, so that it is sufficient to consider merely the energy *function*

$$\hat{\mathscr{H}}(V_0)\equiv\mathscr{H}(V_0, V_0-L)\equiv H(\overline{C}_0(V_0), \overline{C}_1(V_0-L))$$

instead of the energy *functional* $H(C_0, C_1)$. Then the derivatives evaluated at $V_0=\overline{V}_0$ are

$$\frac{d\hat{\mathscr{H}}}{dV_0}=k_0(\overline{V}_0-\ell_0)+k_1(\overline{V}_1+\ell_1)=T_0-T_1,$$

$$\frac{d^2\hat{\mathscr{H}}}{dV_0^2}=k_0+k_1.$$

The positivity of k_0 and k_1 ensures the convexity of $\hat{\mathscr{H}}$ which in turn guarantees the minimizing character of the equilibrium solution $T_0=T_1$.

It is with the development of the above reduction from a functional to a function that we shall be concerned in what follows. We note that we could remove the assumption that P lies on the line joining P_0 and P_1 by using pairs of three dimensional coordinate systems and corresponding three dimensional vectors V_0, and V_1. The constraint relations would then read

$$V_0 - V_1 = \bar{V}_0 - \bar{V}_1,$$

where $\bar{V}_0 - \bar{V}_1$ is the (constant) vector connecting P_0 and P_1 in Fig. 1, and we would have to use the concept of a *triple element* rather than a simple element. A *multiple element* is defined precisely below, but roughly speaking it is generalized from a simple element upon replacing the scalar V by the vector V in an n-dimensional space. Of course for a *multiple element*, $n > 1$.

B. The i^{th} Element. We shall be dealing with collections of multiple elements later, so for consistency of notation we shall define here the i^{th} element E_i, using subscripts for all quantities accordingly. We use the notation

$$E_i = (\mathscr{C}_i, H_i, V_i, \bar{V}_i, \bar{C}_i(V_i)) \tag{4}$$

for the i^{th} element. V_i is a vector in $\mathbb{R}^{k_i}$ with components $V_{1\,i}, V_{2\,i}, \ldots V_{k_i,\,i}$. The following list of entities with the properties specified comprises the i^{th} element.

1) A topological space $\mathscr{C}_i$ called the *configuration space.*
2) A continuous constraint mapping from $\mathscr{C}_i$ to $\mathbb{R}^{k_i}$, *i.e.*, a collection of *constraint functionals*

$$V_i(\alpha_i) = \{V_{1\,i}(\alpha_i), \ldots V_{k_i\,i}(\alpha_i)\}, \qquad \alpha_i \in \mathscr{C}_i.$$

3) The *energy functional* from $\mathscr{C}_i$ to $\mathbb{R}$, *i.e.*,

$$H_i(\alpha_i) \in \mathbb{R} \quad \forall \alpha_i \in \mathscr{C}_i.$$

4) A *base constraint* $\bar{V}_i$ and a *base configuration* $\bar{C}_i = \bar{C}_i(\bar{V}_i)$ which is stable, *i.e.*,

$$H_i(\bar{C}_i(\bar{V}_i)) < H_i(C_i(\bar{V}_i)) \ \ \forall C_i(\bar{V}_i) \in \mathscr{N}_i(\bar{C}_i), \qquad C_i(\bar{V}_i) \neq \bar{C}_i. \tag{5}$$

5) There exists a neighborhood $N_i(\bar{V}_i)$ of $\bar{V}_i$ in $\mathbb{R}^{k_i}$ such that

$$N(\bar{V}_i) \subset V_i(\mathscr{N}_i(\bar{C}_i(\bar{V}_i))). \tag{6}$$

6) For each $V_i \in N_i(\bar{V}_i) \ \exists \ \bar{C}_i(V_i)$ for which

$$H_i(\bar{C}_i(V_i)) < H_i(C_i(V_i)) \ \ \forall C_i(V_i) \in \mathscr{N}(\bar{C}_i(\bar{V}_i)), \qquad C_i(V_i) \neq \bar{C}_i(V_i). \tag{7}$$

Also, we define the *reduced energy functional* of the i^{th} element on $N(\bar{V}_i)$ by

$$\mathscr{H}_i = \mathscr{H}_i(V_i) \equiv H_i(\bar{C}_i(V_i)). \tag{8}$$

C. Ensemble. A collection of elements together with a certain ensemble constraint, which we shall specify precisely below, will be called an *ensemble.*

Consider a collection of elements $E_1, E_2, \ldots E_i \ldots E_r$; we define[1] the *ensemble configuration space* $\mathscr{C}$ by

$$\mathscr{C} \equiv \mathscr{C}_1 \times \mathscr{C}_2 \ldots \times \mathscr{C}_i \ldots \times \mathscr{C}_r$$

[1] We use the usual product topology on $\mathscr{C}$ and on other Cartesian products defined here.

and the *ensemble energy functional* $H(\alpha)$, $\alpha\in\mathscr{C}$, by

$$H(\alpha)\equiv\phi(H_1(\alpha_1), H_2(\alpha_2), \dots H_r(\alpha_r)), \quad \alpha_i\in\mathscr{C}_i, \tag{9}$$

where ϕ may be any strictly monotonic increasing function of r real variables. (In particular examples we shall use $H=\sum_{i=1}^{r} H_i$, but the above definition is adequate for the general theory.)

We put $n=\sum_{i=1}^{r} k_i$, and let $\{U_1 \dots U_n\}\equiv \boldsymbol{U}$ be a vector in $\mathbb{R}^n$ with coordinates $U_1, U_2 \dots U_n$ defined by arranging V_{ij} in order:

$$\{(V_{11}, V_{21}, \dots V_{k_1 1}), \dots (V_{1i}, \dots V_{k_i i}), \dots (V_{1r}, \dots V_{k_r r})\}. \tag{10}$$

$\bar{\boldsymbol{U}}$ is obtained by replacing V_{ij} in the above list with $\bar{V}_{ij}$. The *constraint functional* $\boldsymbol{U}: \mathscr{C}\to\mathbb{R}^n$ is the Cartesian product of the element constraint functionals, *i.e.*, $\boldsymbol{U}(\alpha)$ has the n coordinates $U_1(\alpha), U_2(\alpha), \dots, U_n(\alpha)$ given by the list

$$\{(V_{11}(\alpha_1) \dots V_{k_1 1}(\alpha_1)), \dots (V_{1i}(\alpha_i) \dots V_{k_i i}(\alpha_i)), \dots (V_{1r}(\alpha_r) \dots V_{k_r r}(\alpha_r))\}. \tag{11}$$

A neighborhood of α in $\mathscr{C}$ will be denoted $\mathscr{N}(\alpha)$, and a neighborhood of $\bar{\boldsymbol{U}}$ in $\mathbb{R}^n$ will be denoted $N(\bar{\boldsymbol{U}})$. The product structure together with relation (6) guarantee the existence of a neighborhood $N(\bar{\boldsymbol{U}})$ such that

$$N(\bar{\boldsymbol{U}})\subset \boldsymbol{U}(\mathscr{N}(\bar{C})),$$

where $\bar{C}$ is of course $(\bar{C}_1, \bar{C}_2, \dots \bar{C}_r)$. As before we denote by $C(\boldsymbol{U})$ any configuration in $\mathscr{C}$ having the constraint $\boldsymbol{U}$, *i.e.*,

$$C(\boldsymbol{U})=(C_1(V_1), \dots C_r(V_r)).$$

We put

$$\bar{C}(\boldsymbol{U})=(\bar{C}_1(V_1), \dots \bar{C}_r(V_r))$$

and

$$\bar{C}(\bar{\boldsymbol{U}})=(\bar{C}_1(\bar{V}_1), \dots \bar{C}_r(\bar{V}_r))=(\bar{C}_1, \bar{C}_2 \dots \bar{C}_r).$$

$\bar{C}(\bar{\boldsymbol{U}})$ will be called the *base configuration* and $\bar{\boldsymbol{U}}$ the *base constraint.*

So far we have defined a collection $(\mathscr{C}, H, \boldsymbol{U}, \bar{\boldsymbol{U}}, \bar{C}(\boldsymbol{U}))$ which is no more than a "composite" element formed by the Cartesian product of E_i, $i=1, 2 \dots r$. Suppose, however that a new ensemble constraint is introduced as follows:

i) A neighborhood $\mathscr{N}(\bar{C})$ is assigned.
ii) An ℓ dimensional $(0<\ell<n)$ manifold Γ in $\boldsymbol{U}(\mathscr{N}(\bar{C}))$ containing $\bar{\boldsymbol{U}}$ as an interior point (with respect to the relative topology of the manifold) is assigned.

Then if $\mathscr{C}^\Gamma$ is that subset of $\mathscr{C}$ whose image under the constraint mapping is Γ, the collection

$$\mathscr{E}=(\mathscr{C}^\Gamma, H, \boldsymbol{U}, \bar{\boldsymbol{U}}, \bar{C}(\boldsymbol{U}), \Gamma)$$

will be called an *ensemble*; any element $\alpha\in\mathscr{C}^\Gamma$ will be called a *configuration of the ensemble*. By $\mathscr{C}^{\bar{\boldsymbol{U}}}$, $\mathscr{C}^{\boldsymbol{U}}$, and $\mathscr{C}^\Gamma$ we denote subsets of $\mathscr{N}(\bar{C}(\bar{\boldsymbol{U}}))$ whose images under the constraint mapping are $\bar{\boldsymbol{U}}$, $\boldsymbol{U}$, and Γ, respectively. From ii) we have

$$\mathscr{C}^{\bar{\boldsymbol{U}}}\subset\mathscr{C}^\Gamma.$$

III. Stability of the Base Configuration of an Ensemble

A. General Result. From (7), (9), (10), and (11), we have

$$H(\bar{C}(\bar{U}))<H(C) \quad \forall C \neq \bar{C}(\bar{U}), \quad C \in \mathscr{C}^{\bar{U}}. \tag{12}$$

We can therefore say that the base configuration of the ensemble $\mathscr{E}$ is stable over $\mathscr{C}^{\bar{U}}$. However, we shall call the base configuration of an ensemble $\mathscr{E}$ *stable* if and only if it is stable over $\mathscr{C}^{\Gamma}$ at the point $\bar{U}$, *i.e.*,

$$H(\bar{C}(\bar{U}))<H(C) \quad \forall C \neq \bar{C}(\bar{U}), \quad C \in \mathscr{C}^{\Gamma}. \tag{13}$$

This definition shows clearly the role of the ensemble constraint which is characterized by the manifold Γ.

Remark. If some configuration $\alpha \in \mathscr{C}^{\bar{U}}$ satisfies (13) with α replacing $\bar{C}(\bar{U})$, *i.e.*, $H(\alpha)<H(C)$ $\forall C \neq \alpha$, $C \in \mathscr{C}^{\Gamma}$, then it necessarily satisfies (12) since $\mathscr{C}^{\bar{U}} \subset \mathscr{C}^{\Gamma}$. But $\bar{C}(\bar{U})$ is unique among configurations belonging to $\mathscr{C}^{\bar{U}}$ and satisfying (12). Thus $\alpha=\bar{C}(\bar{U})$. Our focus of attention on $\bar{C}(\bar{U})$ rather than on some other configuration belonging to $\mathscr{C}^{\bar{U}}$ is therefore justified.

Now consider the relation

$$H(\bar{C}(\bar{U}))<H(\bar{C}(U)) \ \forall U \neq \bar{U}, \quad U \in \Gamma. \tag{14}$$

If (13) holds, then clearly (14) also holds because $\bar{C}(U)$, for $U \in \Gamma$, certainly belongs to $\mathscr{C}^{\Gamma}$. In other words (14) is just an ℓ-dimensional special case of (13). The general result of this paper is that the converse is also true. That is to say, if (14) holds, so does (13). To facilitate the proof we introduce the notation $\mathscr{C}_i^{V_i}$, $i=1, 2, \dots r$, to mean the subsets of $\mathscr{N}(C_i(\bar{V}_i))$ for which $V_i(\mathscr{C}_i^{V_i})=V_i$ and $\prod_i \mathscr{C}_i^{V_i}$ to mean their Cartesian product. Next, we note that because of the relations (7), (10), and the monotonicity of ϕ, the following relation,

$$H(\bar{C}(U))<H(C), \tag{15}$$

holds $\forall C \neq \bar{C}(U)$, $C \in \prod_i \mathscr{C}_i^{V_i}$ and thus $\forall C \neq \bar{C}(U)$, $C \in (\prod_i \mathscr{C}_i^{V_i}, U \in \Gamma)$. Hence, in view of the product topology, (15) holds $\forall C \neq \bar{C}(U)$, $C \in \mathscr{C}^{U}$, $U \in \Gamma$. But then, using (14), we have

$$H(\bar{C}(\bar{U}))<H(C), \quad C \neq \bar{C}(\bar{U}), \quad C \in \mathscr{C}^{U}, \quad U \in \Gamma,$$

which is no more than the relation (13).

An ensemble is thus *reducible*. That is to say the variational problem associated with the stability of the ensemble can be reduced to a minimization problem over a finite dimensional manifold Γ in $\mathbb{R}^n$.

It is convenient to define the *reduced energy functional* $\mathscr{H}(U)$ on Γ by

$$\mathscr{H}(U) \equiv H(\bar{C}(U)). \tag{16a}$$

Suppose that $Z=(Z_1, Z_2, \dots Z_\ell)$ is a set of coordinates on Γ such that any point U of Γ has the representation $U=\hat{U}(Z)$, and $\hat{U}^{-1}(\Gamma)=W \subset \mathbb{R}^\ell$. Then we define the *energy function* $\hat{\mathscr{H}}(\boldsymbol{Z})$ to be the representation of $\mathscr{H}$ on W, *i.e.*

$$\hat{\mathscr{H}}(\boldsymbol{Z}) \equiv \mathscr{H}(\hat{U}(\boldsymbol{Z})) = H(\bar{C}(\hat{U}(\boldsymbol{Z}))). \tag{16b}$$

Also, we put $\bar{\boldsymbol{Z}} \equiv \hat{U}^{-1}(\bar{\boldsymbol{U}})$. By the analysis following the preceding remark, we have proved the abstract result whose representation in terms of $\boldsymbol{Z}$ is

Theorem 1. *The base configuration $\bar{C}(\bar{U})$ of an ensemble $\mathscr{E}$ is stable if an only if the function of ℓ variables, $\hat{\mathscr{H}}(Z_1, Z_2, \ldots Z_\ell)$, has at the point $\boldsymbol{Z} = \bar{\boldsymbol{Z}}$ a local minimum.*

B. Ensembles of Smooth Curvature. If $\mathscr{H}(U)$ has continuous second derivatives on Γ and $\hat{U}(\boldsymbol{Z})$ has continuous second derivatives on W, the ensemble will be said to have *smooth curvature*. Theorem 2 below follows from Theorem 1 and elementary calculus.

Theorem 2. *The base configuration of an ensemble of smooth curvature is stable if*

$$\frac{\partial \hat{\mathscr{H}}}{\partial \boldsymbol{Z}} = 0, \tag{17a}$$

and

$$(\boldsymbol{q}, \hat{\hat{\mathscr{H}}} \boldsymbol{q}) > 0 \ \ \forall \boldsymbol{q} \neq 0, \quad \boldsymbol{q} \in \mathbb{R}^\ell, \tag{17b}$$

where $\hat{\hat{\mathscr{H}}}$ is the matrix of second derivatives of $\hat{\mathscr{H}}$ and all derivatives are evaluated at the point $\boldsymbol{Z} = \bar{\boldsymbol{Z}}$. If in (17b) *the $>$ sign is weakened to $\geqq$, then equations* (17) *become necessary conditions for stability.*

Remark. Equation (17a) corresponds of course to the usual conditions of equilibrium.

C. Linear Ensembles. Suppose that for an ensemble $\mathscr{E}$, the manifold Γ is contained in an ℓ dimensional linear subspace of $\mathbb{R}^n$. Then $\mathscr{E}$ will be called a *linear ensemble*. Let $\mathscr{M}_\ell$ be the subspace containing Γ. $\mathscr{M}_\ell$ has the representation

$$\sum_{j=1}^{n} B_{ij}(U_j - \bar{U}_j) = 0; \qquad i = 1, 2 \ldots k; \ k < n, \tag{18}$$

where $k = n - \ell$. The coefficient matrix $[B_{ij}]$, where i is the row index, has full rank k.

Note. A linear ensemble is not necessarily an ensemble of smooth curvature, but if it is we have

Theorem 3. *The base configuration of a linear ensemble of smooth curvature is stable if*

$$\sum_{i=1}^{n} \frac{\partial \mathscr{H}}{\partial U_i} s_i = 0 \tag{19}$$

and

$$Q \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \mathscr{H}}{\partial U_i \partial U_j} s_i s_j, \qquad Q > 0 \tag{20}$$

where the derivatives are evaluated at $\boldsymbol{U} = \bar{\boldsymbol{U}}$ and the relations must hold for all nonzero vectors $\boldsymbol{s}$ (whose coordinates are $s_1, s_2 \ldots, s_n$) which belong to the right *hand null space of the matrix* $[B_{ij}]$. *If the base configuration of a linear ensemble of smooth curvature is stable, then* (19) *holds together with* $Q \geqq 0$.

We denote the Hessian matrix $\bar{\bar{\mathscr{H}}}$ as

$$\bar{\bar{\mathscr{H}}} \equiv \left[\frac{\partial^2 \mathscr{H}}{\partial U_i \partial U_j} \bigg|_{U_i = \bar{U}_i} \right]. \tag{21}$$

There is another form of the relations (20) which is sometimes useful. This is well treated by HANCOCK [16]. The result may be stated as follows:

Theorem 4. *If* B *is a real* $m \times n$ *matrix* $(n > m > 0)$ *of rank* m, s *is any non-null (real) column vector in its right hand null space and* $\bar{\bar{\mathscr{H}}}$ *is a real symmetric matrix, then*

$$s^T \bar{\bar{\mathscr{H}}} s \equiv (s, \bar{\bar{\mathscr{H}}} s) > 0 \tag{22}$$

if and only if the determinant

$$D(\mu) \equiv \begin{vmatrix} \bar{\bar{\mathscr{H}}} - \mu I & B^T \\ B & 0 \end{vmatrix} \tag{23}$$

vanishes for positive values of μ *and only for positive values of* μ. *In* (23) 0 *is the* $m \times m$ *null matrix and* I *the* $n \times n$ *unit matrix. It can be shown that* $D(\mu)$ *is a polynomial of degree* $n - m$,

$$D(\mu) \equiv \sum_{i=0}^{n-m} c_i \mu^i,$$

with exactly $n - m$ *real zeros. These will all be positive if and only if the coefficients* $c_0, c_1, \ldots, c_{n-m}$ *are non-zero and alternate in sign.*

IV. Linearly Elastic String Simple Element

The linearly elastic string has already been used to provide an example for a unit, a constrained unit, and a simple element; *cf.* Sections I and II. We assumed there that the configurations were uniformly strained; we show here that each uniformly strained configuration is itself the minimizing configuration of a unit whose configuration space includes all configurations of smooth non-uniform strain, thereby justifying our earlier assumption.

We start with a straight unstrained string of length ℓ_0. Suppose that one end of the string is fixed at a point P_0 and that a given material point P is at a distance x from P_0. Then let the string be strained (the material points remaining in a straight line) in such a way that the distance $P_0 P$ becomes $x + \xi(x)$. Let S be the space of real-valued C^1 functions with uniform norm whose domain is $[0, \ell_0]$, and let Ξ be the subset whose elements take on the value 0 at $x = 0$ and the particular value e at $x = \ell_0$. Then we require $\xi \in \Xi$. Thus Ξ is the configuration space. We define the Helmholtz functional

$$H \equiv \tfrac{1}{2} k \ell_0 J(\xi), \quad \text{where } \xi \in \Xi,$$

$$J(\xi) \equiv \int_0^{\ell_0} (\xi'(x))^2 \, dx, \quad \text{and} \quad \xi'(x) \equiv \frac{d\xi}{dx}.$$

Then (Ξ, H) form a unit.

To show that this unit has a stable configuration $\bar{C}$ given by the function $\bar{\xi} = \frac{ex}{\ell_0}$, we compute the difference $K(\xi) = J(\xi) - J(\bar{\xi})$ where $\xi \in \Xi$. This is facilitated by putting $\xi = \bar{\xi} + \eta$, where $\eta \in S$ with $\eta(0) = \eta(\ell_0) = 0$. We obtain

$$K(\bar{\xi}+\eta) = \int_0^{\ell_0} \left(\frac{2\eta' e}{\ell_0} + (\eta')^2 \right) dx = \int_0^{\ell_0} (\eta')^2 \, dx \geqq 0.$$

Hence $J(\bar{\xi}+\eta)$ is minimized by $\eta \equiv 0$.

This argument may be applied to a curved segment; the same formulae can be used if x is taken as the arc length from one end of the string.

V. Axisymmetric Capillary Simple Elements

The motivation for this section is the analysis of the stability of some capillary systems, *e.g.* two immiscible fluids enclosed in a rigid container maintained at constant temperature and so disposed that there is one, connected interface between the two fluids, the remaining boundary points of the fluids being also boundary points of the rigid container. We consider the fluids and solid to be mutually inert, incompressible, and not acted upon by external flields, such as gravity. Then, according to a well known principle of thermodynamics, a given configuration of the system will be stable if and only if it is locally minimizing for the Helmholtz free energy. We consider here a special case, known as the case with fixed contact line, for which it is proper to consider the Helmholtz free energy to be proportional to the area A of the liquid-fluid interface

$$H = \sigma A.$$

σ is a positive constant called the interfacial tension. We distinguish two cases.

A. Simply Connected Surface through a Fixed Contact Circle. In Fig. 5, D is the edge view of a single circular hole of radius γ in a rigid lamina L. $C(V)$ is a smooth surface segment which lies entirely on one side of L (except for its boundary points which are the edges of D) and which together with D enclose a simply connected region of volume V. $\mathscr{C}_+$ is the totality of such surfaces for $V \in (0, \infty)$. $C(0)$ is any smooth surface whose projection onto L covers D and only D, and which together with the surface D forms a collection of disjoint regions such that the sum of the volumes of the regions above L equals that of those

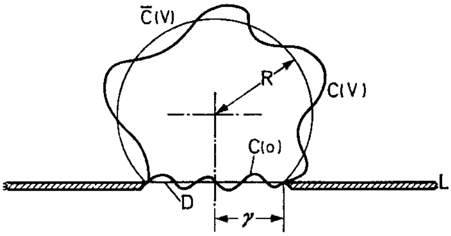

Fig. 5

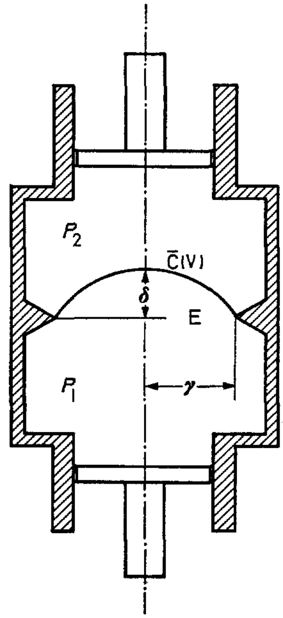

Fig. 6. Spherical cap interface $\bar{C}(V)$ of altitude δ between liquids 1 and 2, bounded by the edge of circular gap E. No body forces act, and V may be altered by displacing pistons simultaneously up, or down.

below L. $\tilde{\mathscr{C}}_0$ is the totality of such surfaces. $\mathscr{C}_0$ is $\tilde{\mathscr{C}}_0 \cup \{\text{surface } D\}$. $\mathscr{C}_-$ is formed by reflecting $\mathscr{C}_+$ in the plane of D. By convention we take $V(C_-) = -V(C)$, where $C \in \mathscr{C}_+$ and C_- is the reflection of C. $\mathscr{C}$ is $\mathscr{C}_+ \cup \mathscr{C}_0 \cup \mathscr{C}_-$, and $\mathscr{C}^V$ is the subset of $\mathscr{C}$ with volume V. $\bar{C}(V) \in \mathscr{C}^V$ is a spherical segment. We define a topology on $\mathscr{C}$ by defining convergence in $\mathscr{C}$ in the same way as in the first example. That is, first we define the functional

$$b(C_1, C_2) \equiv \max_{x_1 \in C_1} \bar{d}(x_1, C_2) \equiv \max_{x_1 \in C_1} \big(\min_{x_2 \in C_2} \bar{d}(x_1, x_2) \big)$$

where $\bar{d}$ is the Euclidean distance function on the physical space. Then we say that a sequence $C_1, C_2 \ldots C_n \in \mathscr{C}$ converges to $\hat{C} \in \mathscr{C}$ if and only if $b(C_n, \hat{C}) \to 0$ as $n \to \infty$. The topology on $\mathscr{C}$ is thus fixed.

Let A be the area of an arbitrary surface $C \in \mathscr{C}$. Then we define

$$H \equiv \sigma A.$$

If $\bar{V}$ is any real number, then from the well known fact that a sphere has minimum area for given volume we conclude that the collection $E = (\mathscr{C}, H, V, \bar{V}, \bar{C}(V))$ here described is a simple element. The family $\bar{C}(V)$ (or at any rate that part of it for which $|V|$ is conveniently modest) can be realized in practice with soap films, or with equipment such as that shown in Fig. 6. The spherical interface $\bar{C}(V)$ between two fluids of equal density is terminated at the circular gap E in the

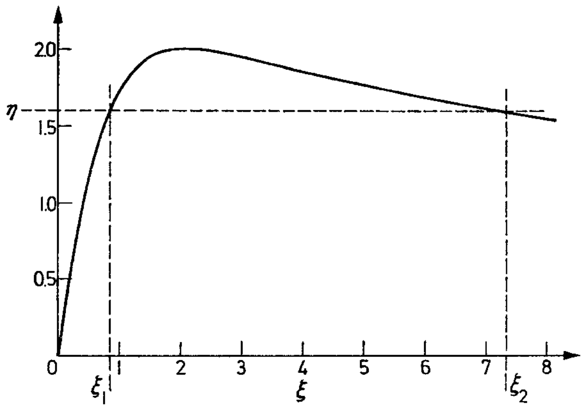

Fig. 7. $\xi - \eta$ plot for the physical system of Fig. 6

equipment shown[1]. V may be altered by moving both pistons upward or downward each sweeping out the same volume.[2] Elementary theory tells us that $\bar{C}(\bar{V})$ with fixed $\bar{V}$ minimizes the Helmholtz functional H.

Suppose that body forces are absent. Then if ΔP is the pressure difference between fluid 1 and fluid 2, R is the radius of the interface when $\delta > 0$ and minus the radius when $\delta < 0$, and $\bar{A}$ is the area of the spherical cap, we have

$$V = \frac{\pi\delta}{6}(3\gamma^2 + \delta^2), \tag{24}$$

$$\frac{d\bar{A}}{dV} = \frac{\Delta P}{\sigma} = \frac{2}{R} = \frac{4\delta}{\gamma^2 + \delta^2}, \tag{25}$$

$$\frac{d^2\bar{A}}{dV^2} = \frac{1}{\sigma}\frac{d(\Delta P)}{d\delta}\bigg/\frac{dV}{d\delta} = \frac{8(\gamma^2 - \delta^2)}{\pi(\gamma^2 + \delta^2)^3}. \tag{26}$$

In particular, when $V = 0$ the quantity given by (26) has the value

$$\frac{8}{\pi\gamma^4}. \tag{27}$$

We shall need these formulae for the treatment of the stability of some ensembles. A plot of $\xi \equiv V/\gamma^3$ against $\eta \equiv \Delta P \gamma/\sigma$ is shown in Fig. 7. This may be easily constructed (using δ/γ as a parameter) from relations (24) and (25). For values of η outside the range $[-2, 2]$ there are no corresponding values of ξ; while for values of η belonging to $(-2, 0) \cup (0, 2)$ each η corresponds to two values of ξ, for example ξ_1 and ξ_2 as shown in Fig. 7. It can be shown easily that

$$\left|\frac{d\eta}{d\xi}\right|_{\xi_2} < \left|\frac{d\eta}{d\xi}\right|_{\xi_1}. \tag{28}$$

[1] The contact line in this case is the edge of the circular gap E.

[2] By suitable preparation of the solid surfaces it is possible to ensure that the contact line remains fixed when V varies.

ξ_1 corresponds to the case $|\delta|<\gamma$ which will be called the *dish branch*, and ξ_2 to $|\delta|>\gamma$ which will be called the *balloon branch*.

B. Doubly Connected Surface Through Fixed Contact Circles.

a) Preliminary Classical Theory. First we shall discuss the case where the configuration space is limited, for purposes of simplification, to contain only axisymmetric members. This problem has a long history. (See the literature cited in the works mentioned in references 10, 13, 14 of this paper.) The classical development is far too lengthy to reproduce in detail here. Within the context of the theory developed in this paper the problem may be defined as follows.

Two points: (x_0, y_0), (x_1, y_1) are fixed in the $y>0$ region R of the $x-y$ plane, with $x_1>x_0$. We consider the totality $\mathscr{C}$ of smooth curves lying in R which join the two fixed points. We define a topology on $\mathscr{C}$ by defining convergence in $\mathscr{C}$, using the functional b defined in Section II. For each curve $C(t)\in\mathscr{C}$, $t\in[t_0, t_1]$ with $C(t_0)=(x_0, y_0)$ and $C(t_1)=(x_1, y_1)$, we define the surface area integral

$$A\equiv\int_{t_0}^{t_1} F\,dt; \qquad F=2\pi y\sqrt{\dot{x}^2+\dot{y}^2} \tag{29}$$

and the Helmholtz energy functional

$$H\equiv\sigma A.$$

The pair $(\mathscr{C}, H)$ forms a unit.

Now we consider the subset $\mathscr{C}^{\tilde{V}}$ of $\mathscr{C}$ consisting of all curves which give the integral V the fixed value $\tilde{V}$:

$$V\equiv\int_{t_0}^{t_1} G\,dt, \qquad G\equiv\pi y^2\dot{x}. \tag{30}$$

The trio $(\mathscr{C}, H, V)$ now forms a constrained unit.

Then the problem is to find, among curves in $\mathscr{C}^V$, one which locally minimizes the integral A defined by (29) if indeed such a curve exists. Because of the nature of G the problem is normal [4], and the theory proceeds by consideration of the augmented function

$$J\equiv F+\lambda G, \tag{31}$$

where λ is a Lagrange multiplier.

We note in passing a property of λ which we shall need later on: If in fact A is minimized (for a given V) by some curve C, and if δC is a variation of C which will not in general keep V constant (*i.e.*, $C+\delta C$ belongs to $\mathscr{C}$ but does not necessarily belong to $\mathscr{C}^V$), then, if δA and δV are the corresponding first variations of A and V, λ has the property that

$$\delta A=-\lambda\delta V. \tag{32}$$

(See BOLZA [5]; FORSYTHE [11] shows that this result is true even for non-axisymmetric variations.)

Weierstrass's form of the Euler equation for the above problem is

$$(\dot{x}\ddot{y}-\dot{y}\ddot{x})(\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}-(\dot{x}/y)(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}=\lambda. \tag{33}$$

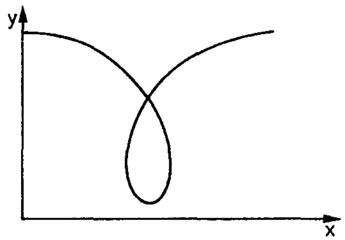

Fig. 8. One full period of nodoid integral curve of equation (33)

The strengthened Legendre condition is satisfied by all curves in $\mathscr{C}$, so that the Euler equation is regular.[1] The form of (33) implies that the integral curves correspond to the meridional curves of surfaces of constant mean curvature. The relation

$$\lambda = \frac{1}{\sigma}(P_2 - P_1) = \frac{\Delta P}{\sigma} \tag{34}$$

for the jump in pressure across the solution interface then follows.

Remark. The Euler equation is satisfied by some curves which fit given boundary conditions yet intersect themselves between the given boundary points, *e.g.* the nodoid shown in Fig. 8. Such curves are inadmissible for obvious physical reasons; non-self-intersecting portions of the nodoid are of course admissible.

If $\bar{A}(V)$ is the area function corresponding to $\bar{C}(V)$, then the result

$$\frac{d\bar{A}}{dV} = \frac{\Delta P}{\sigma} \tag{35}$$

which is due to GAUSS [12] holds and may be deduced from (32) and (34).

The solutions of (33) all correspond to curves which are traced out by one focus of a conic section when the conic section rolls along the x axis [9, 25]. These extremal curves are well described elsewhere [9, 10, 13, 14, 19, 25]. HOWE [19] and HORMANN [18] independently applied Weierstrass's theory of the isoperimetric problem (see *e.g.* BOLZA [6]) to determine under what circumstances an extremal connecting (x_0, y_0) to (x_1, y_1) and associated with a given volume actually minimizes A locally. They proved that an extremal is locally minimizing provided that the conjugate t_0^* of the point t_0 exceeds t_1. ($t_0^* \geqq t_1$ is well known to be a necessary condition for minimization. The case $t_0^* = t_1$ is both difficult and of no practical importance.) HOWE derived a non-linear equation $D(t_0, t_0^*, \theta) = 0$ to be solved to find t_0^* as a function of t_0 and a shape parameter θ for cases $\lambda \neq 0$. He solved this equation for certain pairs of values (t_0, θ). GILLETTE [13] has given an extensive tabulation of solutions.

[1] All integral curves of (33) passing through points of R lie wholly in R, except for a family of semi-circles with cusps on the x-axis. It is known that any segment of a curve of this family which includes a cusp is not minimizing of H over a set which is generalized from $\mathscr{C}^V$ upon permitting curves with discontinuous tangents. For this reason we excluded the x-axis from the region of the $y \geqq 0$ half plane at the beginning of this section.

There are two cases which present some difficulty if handled by the theory as developed by HOWE or HORMANN. The first case is the cylinder $y=a$, which satisfies equation (33) for $\lambda=-1/a$. For this case HOWE and HORMANN both came to the correct conclusion, namely, stable for $\ell/d<\pi$ and unstable for $\ell/d>\pi$. The second case is the catenoid $\lambda=0$, for which they came to no conclusion. (HOWE and HORMANN actually studied the extremization of V for given A which is equivalent to the problem above provided $\lambda\neq 0$.)

This problem for the case $\lambda=0$ has as its extremal the catenary $y=\alpha\cosh\left(\frac{x-\beta}{\alpha}\right)$, but this problem is *not* the same as the problem of the minimum surface of revolution solved by LINDELÖF [20] since for the latter problem there is no constraint on V. MAXWELL [21] recognized the former problem as distinct from the latter problem but gave the wrong solution. The correct solution of the former problem was first given in reference [10].

b) Determination of $d\Delta P/dV$. The conjugate point theory also comes up in the determination of the derivative $d\Delta P/dV$ which we shall need for the investigation of the stability of ensembles. To this end, suppose that we have fixed the manner in which we wish to parametrize an extremal curve and have found the three parameter family of extremals

$$x=x(t,\alpha,\beta,\lambda), \tag{36}$$

$$y=y(t,\alpha,\beta,\lambda). \tag{37}$$

The volume integral V then becomes a function

$$V=V(t_0,t_1,\alpha,\beta,\lambda). \tag{38}$$

Suppose that the base configuration $\bar{C}(\bar{V})$ corresponds to the variables $\bar{t}_0$, $\bar{t}_1$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\lambda}$, and that this configuration has end points $(\bar{x}_0,\bar{y}_0)$, $(\bar{x}_1,\bar{y}_1)$ and volume $\bar{V}$. Then, formally, we may find the functions $t_0(V)$, $t_1(V)$, $\alpha(V)$, $\beta(V)$, $\lambda(V)$ which will specify the family of stable configurations by solving for t_0, t_1, α, β, λ, the one parameter family of equations

$$\begin{aligned} x(t_0,\alpha,\beta,\lambda)&=\bar{x}_0,\\ x(t_1,\alpha,\beta,\lambda)&=\bar{x}_1,\\ y(t_0,\alpha,\beta,\lambda)&=\bar{y}_0,\\ y(t_1,\alpha,\beta,\lambda)&=\bar{y}_1,\\ V(t_0,t_1,\alpha,\beta,\lambda)&=V \end{aligned} \tag{39}$$

with V as parameter, $V\in N(\bar{V})$. The Jacobian of these equations is

$$\Delta(t_0,t_1,\alpha,\beta,\lambda)\equiv\left|\begin{array}{cccc:c} x_{0,t} & 0 & x_{0,\alpha} & x_{0,\beta} & x_{0,\lambda}\\ 0 & x_{1,t} & x_{1,\alpha} & x_{1,\beta} & x_{1,\lambda}\\ y_{0,t} & 0 & y_{0,\alpha} & y_{0,\beta} & y_{0,\lambda}\\ 0 & y_{1,t} & y_{1,\alpha} & y_{1,\beta} & y_{1,\lambda}\\ \hdashline V_{t_0} & V_{t_1} & V_\alpha & V_\beta & V_\lambda \end{array}\right| \tag{40}$$

where the lone suffix and the right hand suffix indicate differentiation, and where the suffices 0 and 1 indicate the left and right hand end points respectively. If $\hat{\Delta}(t_0, t_1, \alpha, \beta, \lambda)$ is the minor of this determinant formed by removing the last row and column of Δ, then formally we have

$$\frac{d\lambda}{dV} = \frac{1}{\sigma}\frac{d(\Delta P)}{dV} = \frac{\hat{\Delta}(t_0, t_1, \alpha, \beta, \lambda)}{\Delta(t_0, t_1, \alpha, \beta, \lambda)}. \tag{41}$$

Now, we suppose that the base configuration $\bar{C}(\bar{V})$ has the property that it satisfies strongly the conjugate point condition, *i.e.*, $t_0^* > t_1$; the conjugate point determinant has a form $K(t_0, t, \bar{\alpha}, \bar{\beta}, \bar{\lambda})$ due to KNESER (see, *e.g.* BOLZA [7]) which can easily be shown to be

$$K(t_0, t, \bar{\alpha}, \bar{\beta}, \bar{\lambda}) = a\Delta(t_0, t_1, \bar{\alpha}, \bar{\beta}, \bar{\lambda}),$$

where a is a non-zero constant. Since $t_0^* > t_1$ for $\bar{C}(\bar{V})$, it is clear that $\Delta(\bar{t}_0, \bar{t}_1, \bar{\alpha}, \bar{\beta}, \bar{\lambda}) \neq 0$.

Now we are in a position to show that we are dealing with a simple element and that (41) is valid.

Theorem 5. *If for given* (x_0, y_0), (x_1, y_1), *and* $V = \bar{V}$ *there exists an extremal* $\bar{C}(\bar{V})$ *for which* $t_0^* > t_1$, *then the configuration* $\bar{C}(\bar{V})$ *is imbedded in a* 1*-parameter family of stable configurations* $\bar{C}(V)$ *whose domain is a neighborhood of* $\bar{V}$. *Consequently the collection* $\{\mathscr{C}, A, V, \bar{V}, \bar{C}(V)\}$ *forms a simple element.*

For the proof we need the following:

Lemma. *Suppose* α, β *and* λ *in* (36–38) *are fixed at values* $\bar{\alpha}$, $\bar{\beta}$, $\bar{\lambda}$ *respectively so that the extremal arc is fixed by* t_0 *and* t_1. *Then*

(i) $\exists$ *a point* $\hat{t}_0 < t_0 \ni \hat{t}_0^*$ *(the conjugate of* $\hat{t}_0$*) exceeds* t_1.

(ii) *If* $D(t_0, t) = \Delta(t_0, t, \bar{\alpha}, \bar{\beta}, \bar{\lambda})$, *then we have*

$$[D(t_0, t) \neq 0 \text{ for any } t \in (t_0, t_1]] \Leftrightarrow [\exists \tilde{t}_0 < t_0 \ni D(\tilde{t}_0, t) \neq 0 \text{ for any } t \in (\tilde{t}_0, t_1]].$$

The first of these is easy to prove. The second can be established by making use of a property of $D(t_0, t)$ proved by HANCOCK [17]. The above results are true for $\alpha = \alpha(V)$, $\beta = \beta(V)$, $\lambda = \lambda(V)$ for V in the closure of some neighborhood $N(\bar{V})$ for $\hat{t}_0$, $\tilde{t}_0$ sufficiently close to t_0. Suppose them to have been chosen close enough. Let t_0^0 be the largest of these latter quantities, with $t_0^0 < t_0$, of course.

Proof of Theorem 5. $\bar{t}_0^* > \bar{t}_1 \Rightarrow \Delta(\bar{t}_0, \bar{t}_1, \bar{\alpha}, \bar{\beta}, \bar{\lambda}) \neq 0$, as noted above. But the functions on the left hand sides of (39) are differentiable arbitrarily many times with respect to their arguments. (This fact follows from the regularity of (33) and the existence theorems for differential equations [2].) Thus equations (39) have a unique one parameter family of solutions $\{t_0(V), t_1(V), \alpha(V), \beta(V), \lambda(V)\}$ for values of V on some interval containing $\bar{V}$ guaranteed by the theorem on implicit functions [3]. Further, the members of the one parameter family are C^∞ functions of V also. Now we use the lemma to show that this one parameter family corresponds to a family of stable configurations $\bar{C}(V)$. Taking $t_0 = \bar{t}_0$, we see that $\bar{t}_0^* > \bar{t}_1$ together with (ii) of the lemma implies that $\Delta(t_0^0, t, \alpha(\bar{V}), \beta(\bar{V}),$

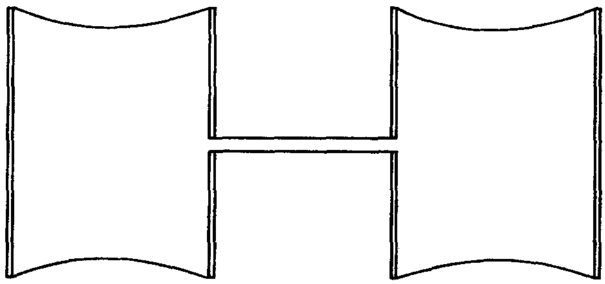

Fig. 9. An ensemble comprising two parallel connected identical capillary simple elements

$\lambda(\bar{V}))\neq 0$ for any t belonging to the *closed* interval $[t_0(\bar{V}), t_1(\bar{V})]$. Hence, by continuity, the function $\Delta(t_0^0, t, \alpha(V), \beta(V), \lambda(V))$ is non-zero $\forall t\in[t_0(V), t_1(V)]$, $V\in N(\bar{V})$, so, by (i) of the lemma, $\Delta(t_0^0, t, \alpha(V), \beta(V), \lambda(V))$ does not vanish for any $t\in(t_0^0, t_1(V)]$, $V\in N(\bar{V})$ and hence, by use of (ii) of the lemma, $\Delta(t_0(V), t, \alpha(V), \beta(V), \lambda(V))\neq 0$ for any $t\in(t_0(V), t_1(V)]$ and for any $V\in N(\bar{V})$. Hence $\bar{C}(\bar{V})$ is imbedded in the one parameter family of stable configurations $\bar{C}(V)$ characterized by the family $\{t_0(V), t_1(V), \alpha(V), \beta(V), \lambda(V)\}$, as stated.

Theorem 6. *The formal relation* (41) *is valid for a simple element for which* $t_0^*>t_1$, *and the derivative* $d\lambda/dV$ *is differentiable arbitrarily many times.*

Proof. This is an elementary consequence of the theorem on implicit functions and Theorem (5).

We may note a further result. If t_1 is replaced by t, then $\hat{\Delta}$ can be shown to be equivalent to the conjugate point determinant for the problem: minimize $\int_{t_0}^{t_1} J\,dt$ for fixed λ without any constraint. Suppose t_0' is the conjugate of t_0 for this problem. Then Bolza [6] proves that $t_0'\in(t_0, t_0^*]$ which is very plausible on intuitive grounds. There are cases where $\hat{\Delta}$ is non-zero at t_0^*, *e.g.* the constrained catenary [10] and cases where $\hat{\Delta}$ has at t_0^* a zero of the same order as Δ has [13].

Suppose t_0, α, β and λ are fixed. Then for t_1 in some neighborhood of t_0 it may be shown that $d\lambda/dV\neq 0$. Rücker [24] has studied the case where $y_0=y_1$ and has discussed how to find the first zero of $d\lambda/dV$ as a function of the parameters of the problem. His work preceded the conjugate point theory of Howe and Hormann. He established the necessity criterion $d\Delta P/dV\geqq 0$ for two equal interconnected liquid bridges between equal circular discs (Fig. 9) to be stable. In fact, because of Bolza's result mentioned above, his criterion, when trivially strengthened, is sufficient.

c) Generalization of Configuration Space. The configurations considered so far are all axisymmetric. Gillette & Dyson [15] have proved the following result, however. If a minimizing arc $x(t)$, $y(t)$ for the problem above has a representation $y(x)$ where y is a *single valued* function of x, then $y(x)$ is also minimizing over the neighboring collection of smooth surfaces $\{y(x, \theta)\}$ (where θ is an azimuthal coordinate) provided of course that these surfaces pass through the end circles and enclose the given volume.

The cylinder and the catenoid, which will be considered in detail next, both have this property.

d) Determination of $d\Delta P/dV$ for Cylinder and Catenoid. The derivatives which form the elements of Δ in these two cases are to be found in reference [10].

A cylinder of radius a and length $2\nu a$ forms a simple element provided $\nu<\pi$ [1, 10]. In this case

$$\frac{1}{\sigma}\frac{d\Delta P}{dV}=\frac{d\lambda}{dV}=\frac{\cos\nu}{4\pi a^4(\sin\nu-\nu\cos\nu)}. \tag{42}$$

Consider a catenoid interface of neck radius ζ between two impervious disks of equal diameter d separated by a distance ℓ. Let $\tau=\ell/(2\zeta)$. (The catenoid is supposed to be firmly attached to the edges of the disks.) Then for $\tau<2.23918\ldots$ the catenoid will form a simple element (d must equal $2\zeta\cosh\tau$, of course). This value of τ is roughly twice the corresponding value (1.199...) for the case when there is a perforation in the end plates [10, 20].

We find

$$\begin{aligned}\frac{1}{\sigma}\frac{d\Delta P}{dV}&=\frac{d\lambda}{dV}\\&=\frac{8\sinh\tau(\cosh\tau-\tau\sinh\tau)}{\pi\zeta^4[3\tau^2\sinh\tau-\tau\cosh\tau(1-\sinh^2\tau)(1-2\sinh^2\tau)+\sinh\tau\cosh^2\tau(1+4\sinh^2\tau)]}.\end{aligned} \tag{43}$$

We note in passing that $d\lambda/dV=0$ when $\tau=\coth\tau$. This equation has the solution (1.199...) mentioned above (LINDELÖF [20]). Thus if two identical catenoids are connected (as shown in Fig. 9), the ensemble is not stable unless each catenoid is also stable when the plates are perforated.

VI. Parallel Connected Capillary Ensembles

A. General Theory. It is convenient to represent simple elements of the type described in the above section by a box diagram (Fig. 10). This conveys the essential information—there are two fluid regions separated by an interface I_j

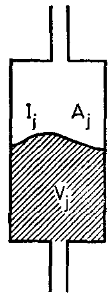

Fig. 10. Box diagram of capillary simple element

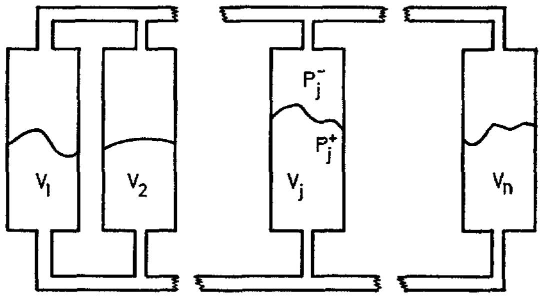

Fig. 11. Box diagram representing n parallel connected capillary simple elements

which is a two sided surface (whether the surface is simply or doubly connected is irrelevant). H_j for the unit is simply

$$H_j = \sigma_j A_j, \tag{44}$$

where A_j is the interfacial area[1]. V_j is the volume of the shaded region.

We shall discuss examples where such units are interconnected to form an ensemble. Obviously they may be connected in any way which does not lead to the creation of a new interface. This leads to a box diagram for the capillary ensemble; the one in Fig. 11 is intended to represent n simple elements connected in parallel.

The examples we shall give all belong to this class so we shall prove a theorem concerning it. Let $\bar{A}_i(V_i)$ be the area function associated with $\bar{C}_i(V_i)$ for the i^{th} element. Then

$$\mathscr{H} = \sum_{i=1}^{n} \sigma_i \bar{A}_i, \tag{45}$$

$$\frac{\partial \mathscr{H}}{\partial U_j} = \sigma_j \frac{d\bar{A}_j}{dV_j} = \Delta P_j = P_j^+ - P_j^-, \tag{46}$$

where P_j^+ is the pressure beneath the j^{th} interface and P_j^- above it. Also, $\bar{\bar{\mathscr{H}}}$ is the diagonal matrix whose i^{th} element is

$$b_i = \frac{d\Delta P_i}{dV_i}. \tag{47}$$

It follows that the relations (18–20) become

$$q_1 + q_2 + \cdots q_n = 0, \tag{48}$$

$$\sum_{j=1}^{n} \Delta P_j q_j = 0, \tag{49}$$

$$\sum_{j=1}^{n} q_j^2 b_j > 0. \tag{50}$$

[1] We consider here the case with fixed contact line; *cf.* Fig. 6. The case of a variable contact line can be treated by slight modification of (44) provided there is no hysteresis [13 Chapter 4].

From (48) and (49) we conclude that

$$\Delta P_j = \Delta P_k, \quad j, k = 1, 2 \dots n. \tag{51}$$

This has the following obvious physical interpretation. Suppose the ensemble is isolated by placing $n-1$ pistons suitably in the interconnecting tubes and fixing them so that $V_j = \overline{V}_j \forall j$. Then a requirement for the ensemble to be stable is that there be no jump in pressure across the pistons. The relations (48), (49), and (50) taken together have the physical interpretation that for a stable ensemble in order to displace the pistons distances $dX_1, dX_2, \dots dX_n$ (not all zero), positive external work must be done.

The above interpretations can be shown to be true for the stability relations for any capillary ensemble; the box diagram may be replaced by a directed multiple graph and results from graph theory used to facilitate the proof of the validity of the interpretation. We shall not pursue these matters here, however.

We shall prove a lemma concerning the relations (48, 50). Let the quantities $b_1, b_2 \dots b_n$ be arranged in non-decreasing order from left to right, with due regard to sign, and labelled $a_1, a_2, \dots a_n$.

Lemma. *$\boldsymbol{q}$ is a vector with coordinates $\{q_1, q_2, \dots q_n\}$ which belongs to $\mathbb{R}^n$, $n \geqq 2$; the coordinates $\{a_1, a_2 \dots a_i \dots a_n\}$ of the vector $\boldsymbol{a}$, $\boldsymbol{a} \in \mathbb{R}^n$ satisfy*

$$a_1 \leqq a_2 \leqq \cdots a_i \leqq \cdots a_n.$$

Q is the set of points other than the origin whose coordinates satisfy

$$q_1 + q_2 + \cdots q_i + \cdots q_n = 0. \tag{52}$$

The quadratic form

$$P(\boldsymbol{q}) = \sum_{i=1}^{n} a_i q_i^2 \tag{53}$$

is defined on Q. We use the notation $P(Q) > 0$ to mean $P(\boldsymbol{q}) > 0 \; \forall \boldsymbol{q} \in Q$. Then

$$(1) \quad \left(a_2 > 0;\, a_1 \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right) + 1 > 0 \right) \Rightarrow P(Q) > 0$$

and

$$(2) \quad P(Q) \geqq 0 \Rightarrow \textit{either } a_1 \geqq 0 \quad \textit{or} \quad \left(a_2 > 0;\, a_1 \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right) + 1 \geqq 0 \right).$$

Proof of First Part. If $q_1 = 0$, then $a_2 > 0 \Rightarrow P(Q) > 0$. Let $q_1 = -e$, a non-zero constant and $\boldsymbol{y}$ belong to a plane Y defined by

$$Y = \{\boldsymbol{y} : (y_1 = -e;\, y_2 + y_3 \cdots + y_n = e)\};$$

then $Y \subset Q$ so P is defined on Y. Further, because $a_i > 0$ for $i = 2, 3, \dots n$, P can easily be shown to attain on Y its minimum value at the point whose coordinates are

$$y_1 = -e, \quad y_j = \frac{e}{a_j \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right)}, \quad j = 2, 3, \dots n.$$

The minimum value is

$$\min_{y \in Y} P(y) = e^2 \left(a_1 + \frac{1}{\sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right)} \right).$$

This minimum will be positive if

$$a_1 \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right) + 1 > 0;$$

since e is an arbitrary non-zero number, this proves the first part. The reader may easily prove the second part by exhausting the various cases.

Theorem 7. I) *An ensemble comprising n parallel connected simple capillary elements* (Fig. 11) *is stable if the following conditions are satisfied:*

1) $\Delta P_1 = \Delta P_2 = \cdots \Delta P_n$. (54)

2) *With* $b_j = \frac{d\Delta P_j}{dV_j}$ *and the set* $\{a_j\}$ *formed from* $\{b_j\}$ *by ordering the elements of* $\{b_j\}$ *as in the lemma,*

$$a_2 > 0, \quad a_1 \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right) + 1 > 0. \tag{55}$$

II) *If the ensemble in part* I *of this theorem is stable, then the relation* 1) *is satisfied together with*

$$\textit{either } a_1 \geqq 0 \quad \textit{or} \quad \left(a_2 > 0, \sum_{k=2}^{k=n} \left(\frac{1}{a_k} \right) + 1 \geqq 0 \right). \tag{56}$$

Proof. Part I of this theorem is an immediate consequence of the lemma together with the sufficiency conditions of Theorem 3. Part II follows from the lemma together with the necessity conditions of Theorem 3.

Corollary. For $n = 2$, we have the relations

$$\Delta P_1 = \Delta P_2, \tag{57}$$

$$\frac{d\Delta P_1}{dV_1} + \frac{d\Delta P_2}{dV_2} \geqq 0, \tag{58}$$

$$\frac{d\Delta P_1}{dV_1} + \frac{d\Delta P_2}{dV_2} > 0. \tag{59}$$

Relations (57) and (58) together are necessary, and relations (57) and (59) together are sufficient for stability of the parallel connected capillary ensemble.

In what follows we shall regard (54) as always satisfied unless otherwise stated.

B. Examples. We shall apply Theorem 7 (or the corollary) together with the data obtained for the various capillary simple elements to deduce regions of stability in the ensemble parameter spaces.

a) The Ensemble of Fig. 2 *(with Valve Open!).* Here the ensemble will be stable provided one of the two surfaces is a dish branch. This result follows from

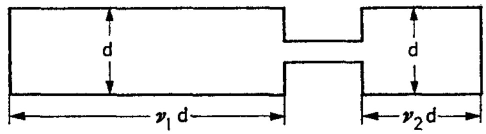

Fig. 12. Cylindrical interfaces of diameter d and lengths $\nu_1 d$, $\nu_2 d$ respectively between sharp edged parallel coaxial flat plates, interconnected in parallel to form an ensemble. Configuration shown has $\nu_1=\pi$, $\nu_2=1.35$ corresponding to supremum of stable values of $\nu_1+\nu_2$.

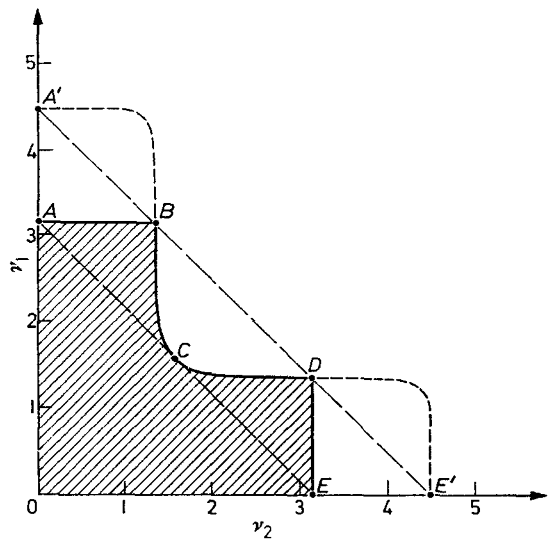

Fig. 13. $\nu_1-\nu_2$ plane for ensemble of Fig. 12. Region enclosed by the curve OABCDEO corresponds to stable ensembles.

the corollary to Theorem 7 together with (28). In the case where one of the surfaces degenerates to a hemisphere (then the other must also be a hemisphere by (57)) the ensemble can be shown by elementary methods to be stable. All other cases are unstable.

b) Consider the case of cylindrical free surfaces of equal tension σ, diameter d, and lengths $\nu_1 d$, $\nu_2 d$, as shown in Fig. 12; condition (57) is then automatically satisfied.

We may establish that all points in the triangular region OACEO in Fig. 13 correspond to a stable ensemble, since in this region $\nu_1+\nu_2<\pi$ and the system shown is certainly more restrictive than a single cylinder of length $(\nu_1+\nu_2)d$. It is interesting, then, that for $\nu_1=\nu_2$ RÜCKER [24] finds $\nu_1>\pi/2$ corresponds to an unstable ensemble. This we deduce trivially from (59).

In the general case, we consider the inequality

$$\sum_{i=1}^{2}\left(\frac{\cos\nu_i}{\sin\nu_i-\nu_i\cos\nu_i}\right)>0, \qquad \nu_1,\nu_2\in(0,\pi). \tag{60}$$

It is easily shown that provided neither v_1 nor v_2 equals $\pi/2$, the above inequality is equivalent to

$$(\tan v_1 - v_1)^{-1} + (\tan v_2 - v_2)^{-1} > 0. \tag{61}$$

The domain in which this inequality holds is easily found by considering a plot of u against $\tan u$. It then turns out that the case when v_1 or v_2 equals $\pi/2$ falls into place by the obvious limiting procedure. In this way we find this ensemble stable provided the point (v_1, v_2) lies in the region shaded *ABCDEOA* of Fig. 13. The reason for the kinks at *B* and *D* is of course that these points represent the situation where the cylindrical surfaces no longer belong to simple elements. The locus of solutions of $\tan v_1 - v_1 + \tan v_2 - v_2 = 0$ continue along the broken lines, making intercepts with the axes of length approximately equal to 4.49. This value represents the supremum of $v_1 + v_2$ since the points A', B, D & E' are easily shown to be collinear.

We also remark in passing that if one considers the problem of "making a cylinder more stable" by putting a perforated support ring somewhere between its end plates, the worst one can do is to place the ring equidistant between the plates, where it has no effect. This seems a curious result, which becomes clear in the light of the work of ALMANSI [1], who showed, by a direct method of the calculus of variations, that the "critical eigenfunction" for a cylinder of $v = \pi + 0$ has a node midway between the end plates.

RÜCKER tested his result for two equal cylinders experimentally. He failed to realize values of v_1 exceeding 1.25. This seems to be a poor result, since for the single cylinder one can obtain more than twice this easily. (BOYS [8], however, appears to have obtained 1.5.)

c) Plateau's Cylinder with Spherical End Caps. We outline here the result for the ensemble represented in Fig. 14A. Two thin parallel coaxial wire rings of common radius a are separated at a distance $2va$. The ensemble comprises a cylindrical interface and two dish interfaces of common tension σ.

We assume that the attachment of the interfaces to the wire rings is not in question. Then the application of the conditions (26, 42, 55) and

$$v \in (0, \pi) \tag{62}$$

leads to the inequality

$$\frac{2}{\tan v - v} + (2 + \sqrt{3})^2 \sqrt{3} > 0. \tag{63}$$

This is satisfied for all v in the range (62). Thus the end caps have no effect. This result has been noted experimentally by PLATEAU [22, 23].

We can also treat the case represented in Fig. 14B which is the same as that in Fig. 14A except that one dish interface has been replaced by a balloon interface. In this case we obtain, by the method already discussed, the equation

$$\tan v - v = 8 \tag{64}$$

for the critical value of v. The root of (64) is $v_0 = 1.46554\ldots$ and we have the result that the ensemble in Fig. 14B is stable for $v < v_0$.

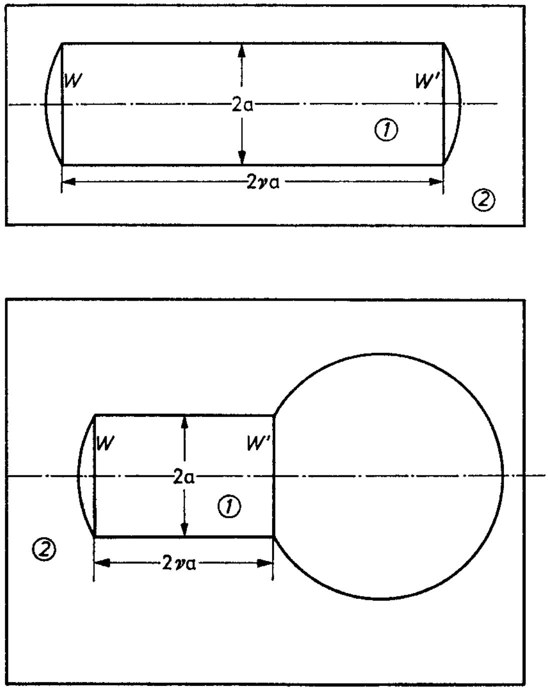

Fig. 14A and B. Two thin parallel coaxial wire rings W and W', of common radius a and separation $2\nu a$, with cylindrical interface and two spherical cap interfaces enclosing fluid 1. Fluid 2 occupies the exterior. Two dish interfaces are shown in 14A for which $\nu=\pi$, the stability supremum for the ensemble, and a dish-balloon combination in Fig. 14B, which has $\nu=1.46\ldots$ the stability supremum for the ensemble.

d) Rücker's Equal Interconnected Liquid Bridges. Consider a system of two equal axisymmetric elements such as the one shown in Fig. 9. RÜCKER [24] has noted that

$$\frac{d\Delta P}{dV} \geqq 0 \tag{65}$$

is necessary for stability. It follows from (59) that

$$\frac{d\Delta P}{dV} > 0 \tag{66}$$

is sufficient. Regions in the ℓ/d versus $4V/(\pi\ell d^2)$ plane and also in the ℓ/ζ versus ℓ/d plane, where ζ is the neck radius and where (66) holds for axisymmetric simple elements, are displayed in [13] and [14].

e) The Catenoid Film Connected to a Bubble Manometer of Equal Tension. Consider the case shown in Fig. 15 of the ensemble comprising (i) a catenoid of neck radius ζ connecting the edges of coaxial parallel plates of radius a and separation ℓ, and (ii) a (plane) bubble gauge manometer with orifice radius γ.

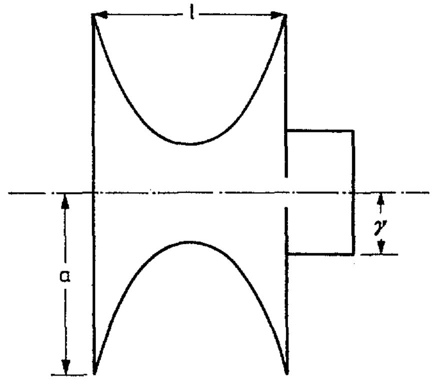

Fig. 15. Ensemble comprising catenoid interface and plane interface (across circular hole), parallel connected.

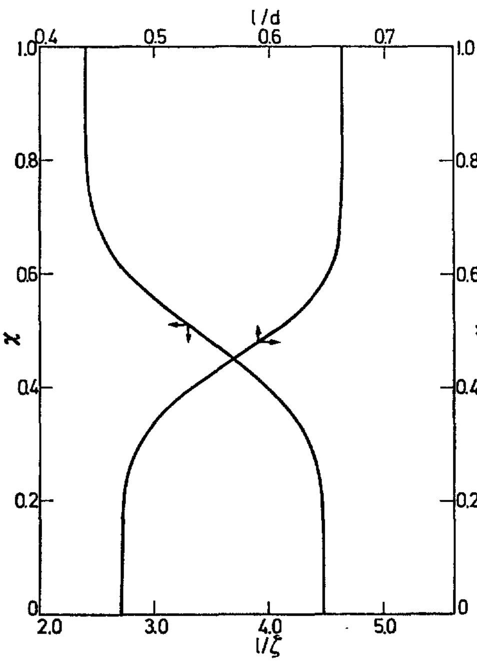

Fig. 16. Stability results for the ensemble of Fig. 15. Curves giving supremum of set of stable values of ℓ/ζ, and infimum of set of stable values of ℓ/d, for all values of χ.

We note that unless $\ell/d \leqq 0.6627...$ there is no catenoid film which can connect these plates, and if $\ell/d < 0.6627...$, then there are two. The one with the largest neck is *always* stable since it is stable in the case with no volume constraint (*cf.* LINDELÖF) [20]. Here we discuss the stability of the catenoid which is *never* stable in the case without a volume constraint: the case of a catenoid which is (under certain conditions) stable in the configuration shown in the figure but becomes unstable if the manometric film is punctured. The conditions under which this ensemble is stable may be found by use of the formulae and the stability criterion for the ensemble.

Let

$$\Phi=\frac{\gamma}{a}, \quad \chi=\frac{\Phi}{1+\Phi}.$$

Then all positive values of Φ map onto the unit interval of χ. In Fig. 16 we have plotted the results of our stability calculations: the supremum of the set of stable values of ℓ/ζ, and the infimum of the set of stable values of ℓ/d versus χ. We note that the limiting values when $\Phi\to\infty$ correspond to the limiting stable Lindelöf catenoid, while those when $\Phi\to 0$ correspond to the case of the catenoid between flat plates [10], as one might have plausibly inferred *a priori*.

VII. Conclusions

We have shown how the concept of an element and an ensemble can be used to reduce the burden of proof of stability, or lack of stability, of an interacting collection of systems. Perhaps it is worth pointing out here that stability is relative to the configuration space $\mathscr{C}$, and $\mathscr{C}$ is of course artificial. Whether a theoretical stability result found for a physical system (where $\mathscr{C}$ is some space specified by the investigator) will turn out to be realizable (more or less precisely) in the laboratory will depend of course on the choice of $\mathscr{C}$, and there is not much to be said *a priori* concerning how this choice should be made: in practice, of course, it will turn out to be some compromise arising from the investigator's conflicting objectives to handle problems that are both tractable and realistic.

Concerning the examples from capillarity theory: We confined our attention to axisymmetric simple elements which had fixed contact lines. Variable contact line problems may also be treated in the same way provided there is no contact angle hysteresis. There is no theory of non-axisymmetric elements at the present time. Nevertheless Theorem 1 would of course be applicable to such elements. All that needs to be done to use Theorem 1 is to establish that one is dealing with an element and to find the quantities ΔP_i and $\partial\Delta P_i/\partial V_j$.

All this information could be obtained experimentally for every element of interest. The stability of an ensemble comprising these elements (and perhaps others for which the information can be supplied by purely theoretical work) could then be tested by application of Theorem 1. This would be a semi-empirical approach, which is all that can be expected for arbitrary non-symmetric configurations.

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References

1. ALMANSI, E., Annali di Matematica, Series 3, **12**, 1 (1906).
2. BLISS, G. A., Lectures on the Calculus of Variations. Chicago: University of Chicago Press, 1968 (274–281).
3. *Ibid*, p. 272.
4. BOLZA, O., Lectures on the Calculus of Variations. New York: Dover 1961, footnote on p. 207.
5. *Ibid*, p. 209.
6. *Ibid*, Chapter VI.
7. *Ibid*, p. 225.
8. BOYS, C. V., Soap Bubbles and the Forces which Mould Them. Society for Promoting Christian Knowledge, London 1895 (78–79).
9. DELAUNAY, CH., Journal de Mathématiques **6**, 309 (1841).
10. ERLE, M. A., R. D. GILLETTE, & D. C. DYSON, The Chem. Eng. J. **1**, 97 (1970).
11. FORSYTHE, A. R., Calculus of Variations. Cambridge: University Press (1927), Section 328.
12. GAUSS, C. F., Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores, **7** (1830).
13. GILLETTE, R. D., Ph. D. Thesis. Houston: Rice University, November 1970.
14. GILLETTE, R. D., & D. C. DYSON, Chem. Eng. J. **2**, 44 (1971).
15. GILLETTE, R. D., & D. C. DYSON, Chem. Eng. J. **3**, 196 (1972).
16. HANCOCK, H., Theory of Maxima and Minima. New York: Dover 1960, p. 103.
17. HANCOCK, H., Lectures on the Calculus of Variations. University of Cincinnati Bulletin of Mathematics, No. 1 (1904), p. 278.
18. HORMANN, GUSTAV, Ph. D. Dissertation, George-Augustus Universität zu Göttingen, 1887.
19. HOWE, WILLY, Ph. D. Dissertation, Friedrich-Wilhelms Universität zu Berlin 1887.
20. LINDELÖF, L., Acta Societatis Scientiarum Fennicae **7**, 347 (1863).
21. MAXWELL, JAMES CLERK, Encyclopaedia Britannica, Vol. 5. Edinburgh: Adam and Charles Black, 1898, pp. 68–69 (Ninth Edition).
22. PLATEAU, J. A. F., Experimental and Theoretical Researches on the Figures of Equilibrium of a Liquid Mass. (Trans. in Annual Reports of the Smithsonian Institution, 1863–1866).
23. PLATEAU, J. A. F., Statique Expérimentale et Théorique des Liquides Soumis aux Seules Forces Moléculaires. Paris: Gauthier-Villars 1873, Volume 2, p. 231.
24. RÜCKER, A. W., Philosophical Magazine **23**, 35 (1886).
25. STURM, J. C., Journal de Mathématiques **6**, 315 (1841).

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