Asymptotic Theory of Wave-Propagation

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Abstract

A general method is presented for finding asymptotic solutions of problems in wave-propagation. The method is applicable to linear symmetric-hyperbolic partial differential equations and to the integro-differential equations for the electromagnetic field in a dispersive medium. These equations may involve a large parameter λ. In the electromagnetic case λ is a characteristic frequency of the medium. The parameter may also appear in initial data or in the source terms of the equations, in a variety of different ways. This gives rise to a variety of different types of asymptotic solutions. The expansion procedure is a "ray method", *i.e.*, all the functions that appear in the expansion satisfy ordinary differential equations along certain space-time curves called rays. In general, these rays do not lie on characteristic surfaces, but may, for example, fill out the interior of a characteristic hypercone. They are associated with an appropriately defined "group velocity". In subsequent papers the *ray method* developed here will be applied to the analysis of transients, Cerenkov radiation, transition radiation, and other phenomena of wave-propagation.

An interesting by-product of the ray method is the conclusion, derived in section 6.3, that the theory of relativity imposes no restriction on the speed of energy transport in anisotropic media.

Contents

1. Introduction

In recent years asymptotic methods have been developed for the solution of certain boundary-value and intial-value problems for linear partial differential equations. These problems involve a parameter, and the methods provide one or more terms of the asymptotic expansion, say for large values of the parameter, of the solution of the problem. They are often applicable to problems for which no exact solution method is known, and even for problems which can be solved exactly it frequently happens that only the asymptotic expansion of the solution is sufficiently simple to be useful in practical applications. Furthermore, it is invariably true that the methods which yield the asymptotic expansion directly are very much simpler than the procedure which involves first finding the exact solution and then its asymptotic expansion.

An important class of asymptotic methods is characterized by the fact that certain curves, often called "rays", play a central role in the theory. The rays are of fundamental importance because all of the functions which make up the various terms of the expansion can be shown to satisfy *ordinary* differential equations along these curves. Often these equations can be solved to yield explicit formulas for the asymptotic solution of a given problem.

The "ray method" has been extensively developed, primarily by J.B. KELLER and his co-workers at New York University, for certain linear elliptic partial differential equations involving a parameter. When applied to the reduced wave equation or the time-reduced form of Maxwell's equations the method yields a "geometrical theory of diffraction" [7] which generalizes the classical theory of geometrical optics. The extensive literature on the asymptotic theory of the reduced wave equation and Maxwell's equations has recently been unified and summarized in [12].

More recently ray methods have been applied to linear hyperbolic partial differential equations [9, 10, 11, 13] such as the (time-dependent) wave-equation

$$u_{tt}-c^2(X)\Delta u=0. \tag{1}$$

This equation involves no parameter, but a parameter may appear in the data of a problem or in a source term which is not indicated in (1). Other hyperbolic equations such as

$$u_{tt} - c^2(X)\,\Delta u + \lambda^2 b^2(X)\,u = 0 \tag{2}$$

may involve a large parameter λ.

From the standpoint of the asymptotic method, (2) and (1) are the simplest representatives of two important classes of linear hyperbolic equations, namely the "dispersive" and "non-dispersive" hyperbolic equations that conserve energy. Ray methods for these two classes have several important differences, the most striking being the nature of the rays themselves. In both cases, the rays are space-time curves, but for non-dispersive equations they lie on "characteristic hypersurfaces" and are called "bicharacteristics". For dispersive equations they do not lie on characteristic hypersurfaces, and may, for example, fill out the interior of a characteristic hypercone. Applications of the ray method to various problems for (1) and (2) have recently been treated in [*11*] and [*10*].

The purpose of the present paper is to study the application of the asymptotic theory to equations more general than (1) and (2), and to develop a method of solution which is applicable to a large variety of problems in wave propagation. In order to avoid undue complications and to maintain close contact with physical problems we have not attempted to find the most general class of linear hyperbolic equations to which our method is applicable. Most of the hyperbolic systems of equations which are important in mathematical physics fall in the class of "symmetric-hyperbolic" equations [*3*]. We shall show, in section 2.3, that our method applies to "asymptotically conservative symmetric-hyperbolic equations". But our main emphasis in this paper is not on these equations.

Perhaps the most important classical problems in wave propagation are concerned with electromagnetic waves. The electric and magnetic fields satisfy Maxwell's equations which, for free space, are symmetric-hyperbolic equations. But for dispersive media the equations are not symmetric-hyperbolic, in fact they are not even differential equations. The time-dependent form of Maxwell's equations for dispersive media are integro-differential equations. They are discussed in section 2.1. In order to encompass this important class of problems we have formulated our theory in terms of a "general system of equations" which includes, as special cases, the integro-differential equations of electromagnetic theory as well as the symmetric-hyperbolic differential equations. The "general system" is described in section 2.2.

As pointed out in section 2.1, in the electromagnetic case, the large expansion parameter λ is a characteristic frequency of the medium. The true meaning of our asymptotic expansion is better understood, however, by introducing an equivalent dimensionless parameter $\lambda_0 = \lambda\, a/c \gg 1$. Here "$a$" is a characteristic dimension of the problem and c is the speed of light.

Let us indicate briefly the steps involved in the asymptotic method which is described in chapter 3. For problems which can be solved exactly (such as those discussed in chapter 4) examination of the asymptotic expansion of the solution shows that it consists of a sum of terms, each of which is an asymptotic power series in λ^{-1} involving a "phase function" and an infinite sequence of "amplitude

functions". For complex problems, we therefore assume that the solution is also given by a sum of such series. By inserting such a series into the general system of equations, we find first that the phase function satisfies a first order partial differential equation, the "dispersion equation", which can be solved by the method of characteristics. The characteristic curves are the rays which we have mentioned. The characteristic (ordinary) differential equations which define these curves will be called "ray equations". By the method of characteristics, the phase function can be obtained by integrating a simple ordinary differential equation along the rays. We also find that the amplitude functions satisfy ordinary differential equations along the rays and that, in the most important cases, these "transport equations" can be solved. In order to find the rays, and the phase and amplitude functions, initial conditions for all these ordinary differential equations are required. In some cases the required intial conditions follow directly from the data of the problem. In others the initial conditions are obtained by an "indirect method". Here one solves a "canonical problem" which has the same local features as the given problem. It is, however, sufficiently simple to be solved exactly. The required intial conditions for the given problem are obtained by examination of the asymptotic expansion of the solution of the canonical problem.

The indirect method was originated by J.B. KELLER for the solution of "diffraction" problems for the reduced wave equation. In these problems "diffracted rays" are produced when ordinary rays are incident upon edges or corners of boundary surfaces, or are tangent to smooth surfaces. Then the initial conditions for the diffracted rays are determined from a canonical problem involving a boundary with local features resembling those of the given problem. These canonical problems are often quite difficult because of the presence of the boundaries. In our work we consider systems of equations much more complex than the reduced wave equation. However we are concerned here with problems of "radiation" from sources, which are much simpler than the diffraction problems*. Thus the canonical problems we must consider are not complicated by the presence of boundaries. They are problems for the general system of equations with constant coefficients, in infinite space; whereas the original problem may involve variable coefficients and boundaries. Chapter 4 is devoted to the solution of the canonical problems which we require.

In chapter 5 we find the asymptotic solution of two representative radiation problems. These two problems are intended only to illustrate the asymptotic method. A much larger variety of problems for (1) and (2) are discussed in [*11*] and [*10*]. Other problems for more general systems of equations are under investigation by the asymptotic method. In particular, these involve intial-boundary value problems for symmetric-hyperbolic equations [*4*], and the very interesting problems of "Cerenkov radiation" [*5*] and "transition radiation" [*2*] for the electromagnetic field equations and other equations.

In chapter 6 we investigate the special features of our asymptotic solutions which appear when the general system of equations is non-dispersive. In particular we find that we may then superpose asymptotic solutions to obtain one which

* Problems involving reflection and refraction by surfaces and interfaces are considered in [*4*]. A diffraction problem for (2) is described in [*1*]. Diffraction problems for more general equations have not yet been investigated.

involves an arbitrary "wave-form" function. Such "progressing wave" solutions have been studied for a variety of hyperbolic equations [*3*, *13*]. In [*11*] progressing waves are applied to problems for the wave equation (1) involving amplitude-modulated waves, radiation from moving sources, and reflection by moving surfaces. Although specific problems are not treated in chapter 6, the discussion there illuminates the relationship between the dispersive and non-dispersive cases.

The introduction of the general system of equations in chapter 2 was largely motivated by the electromagnetic field equations for dispersive media. In chapter 7 we return to those equations in order to demonstrate that the general theory which we have developed is indeed applicable to the electromagnetic case. In section 7.2 the solution of the transport equations for inhomogeneous isotropic media is analyzed in detail. This leads to a discussion of the rotation of the electric polarization vector.

Throughout the paper certain conditions are imposed on the equations considered in order that the asymptotic method may be applicable. These conditions are listed for ready reference in appendix H. A brief summary of notation is given in appendix G.

The historical development of the asymptotic theory of ordinary differential equations has proceeded in two stages. First methods were developed for obtaining formal series solutions of certain problems. In many cases this *formal* theory has been supplemented by an *analytical* theory, wherein the asymptotic nature of the formal solution is rigorously proved. The asymptotic theory of partial differential equations is still mainly in the formal stage, and the work presented here is entirely of that nature. Nevertheless there is abundant evidence of the validity of our methods. This evidence has been obtained mainly by comparing our results with the asymptotic expansion of the solution of numerous problems which are sufficiently simple to be solved exactly. In every case that has been examined the asymptotic expansion of the exact solution agrees perfectly with the results obtained by methods such as those presented here. These investigations [*1*, *4*] yield not only confidence in the validity of our method but also an appreciation of its relative simplicity.

2. The General System of Equation

2.1. The electromagnetic field equations for dispersive media

In Gaussian units, Maxwell's equations take the form

$$\frac{\partial}{\partial t}\boldsymbol{D}-c\nabla\times\boldsymbol{H}=-4\pi\boldsymbol{J}, \quad \frac{\partial}{\partial t}\boldsymbol{B}+c\nabla\times\boldsymbol{E}=0, \tag{1}$$

$$\nabla\cdot\boldsymbol{D}=4\pi\rho, \qquad \nabla\cdot\boldsymbol{B}=0. \tag{2}$$

The source terms $\rho(t, \boldsymbol{X})$ and $\boldsymbol{J}(t, \boldsymbol{X})$ must satisfy the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}=0. \tag{3}$$

From (1) and (3) we see that

$$\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{B})=0, \quad \frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{D}-4\pi\rho)=0. \tag{4}$$

Hence the equations (2) are automatically satisfied if they are satisfied at any given time t. For a non-dispersive medium equations (1) and (2) are supplemented by the constitutive equations

$$\boldsymbol{D}=\varepsilon \boldsymbol{E}, \quad \boldsymbol{B}=\mu \boldsymbol{H}. \tag{5}$$

Here ε, μ are scalar or matrix functions of $\boldsymbol{X}=(x_1, x_2, x_3)$ which characterize the medium. It is well known that the solution of (1, 2, 5) with given source terms ρ, $\boldsymbol{J}$ satisfying (3) is uniquely determined by the initial values $\boldsymbol{E}(0, \boldsymbol{X})$, $\boldsymbol{H}(0, \boldsymbol{X})$ of the electric and magnetic fields, say at $t=0$. If the initial conditions satisfy (2), then those equations are satisfied for all $t \geqq 0$.

The fourier (time) transforms of $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \rho, \boldsymbol{J}$ satisfy the "reduced equations" which are given by (1), (2), (3) and (5) with $\partial/\partial t$ replaced by $-i\hat{\omega}$. Here $\hat{\omega}$ is the transformation variable. However, dispersive media are characterized by the fact that ε and μ are functions of $\hat{\omega}$ as well as $\boldsymbol{X}$. It is important to note that then equations (5) do not hold. For dispersive media, the constitutive equations for the time-dependent fields are convolution integral relations. If, for convenience, we replace $\boldsymbol{E}$, $\boldsymbol{H}$ by $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{D}$, $\boldsymbol{B}$ by $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and ε, μ by $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$, then for dispersive media the constituitive equations for the transformed fields are given by

$$\boldsymbol{V}_\nu(\hat{\omega}, \boldsymbol{X})=\hat{\varepsilon}_\nu(\hat{\omega}, \boldsymbol{X})\, \boldsymbol{U}_\nu(\hat{\omega}, \boldsymbol{X}), \qquad \nu=1,2. \tag{6}$$

It follows from elementary properties of fourier transforms that the time-dependent fields are related by

$$\boldsymbol{V}_\nu(t, \boldsymbol{X})=\int f_\nu(\tau, \boldsymbol{X})\, \boldsymbol{U}_\nu(t-\tau, \boldsymbol{X})\, d\tau; \qquad \nu=1,2. \tag{7}$$

The functions $f_\nu(t, \boldsymbol{X})$ are, of course, fourier transforms of the dielectric permeability $\hat{\varepsilon}_1=\varepsilon$ and the magnetic permeability $\hat{\varepsilon}_2=\mu$:

$$\hat{\varepsilon}_\nu(\hat{\omega}, \boldsymbol{X})=\int e^{i\hat{\omega}t} f_\nu(t, \boldsymbol{X})\, dt, \qquad \nu=1,2. \tag{8}$$

These equations, which are thoroughly discussed in [8], have an interesting physical interpretation: We see from (7) that at each point $\boldsymbol{X}$, the fields $\boldsymbol{V}_\nu$ depend on the fields $\boldsymbol{U}_\nu$ only at earlier times, provided we assume that the real functions f_ν satisfy the *causality condition*

$$f_\nu(t, \boldsymbol{X}) \equiv 0 \quad \text{for} \quad t<0. \tag{9}$$

This equation has important consequences for the analytical properties of the functions $\hat{\varepsilon}_\nu$ which are discussed in [8].

In Gaussian units $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are dimensionless. Since $\hat{\omega}$ is a frequency, $\hat{\varepsilon}_\nu$ must be a function of $\hat{\omega}/\lambda$ where λ is a characteristic frequency of the medium*. Thus we may set

$$\hat{\varepsilon}_\nu(\hat{\omega}, \boldsymbol{X})=\tilde{\varepsilon}_\nu(\omega, \boldsymbol{X}); \qquad \omega=\hat{\omega}/\lambda. \tag{10}$$

Then from (8) we find that

$$f_\nu(t, \boldsymbol{X})=\frac{\lambda}{2\pi}\int e^{-i\lambda\omega t}\, \tilde{\varepsilon}_\nu(\omega, \boldsymbol{X})\, d\omega; \qquad \nu=1,2. \tag{11}$$

* Specific examples of $\hat{\varepsilon}_1$ are discussed in appendix D.

The time-dependent form of the electromagnetic field equations for dispersive media are now given by

$$\frac{\partial \boldsymbol{V}_1}{\partial t} - c\nabla \times \boldsymbol{U}_2 = -4\pi \boldsymbol{J}, \quad \frac{\partial \boldsymbol{V}_2}{\partial t} + c\nabla \times \boldsymbol{U}_1 = 0; \tag{12}$$

$$\boldsymbol{V}_\nu(t, \boldsymbol{X}) = \int f_\nu(\tau, \boldsymbol{X})\, \boldsymbol{U}_\nu(t-\tau, \boldsymbol{X})\, d\tau; \qquad \nu = 1, 2. \tag{13}$$

Here we have omitted equations (2) since, as we have seen, those equations will be satisfied automatically if they are satisfied at a given time. It is clear that the initial values of the fields $\boldsymbol{E} = \boldsymbol{U}_1$ and $\boldsymbol{H} = \boldsymbol{U}_2$ do not uniquely determine the solution of this system of equations, since the function $\boldsymbol{V}_\nu$ depends on the values of the functions $\boldsymbol{U}_\nu$ for $t < 0$. In the absence of any existence and uniqueness theorems we assume that an appropriate problem for this system of equations is as follows: Source functions ρ and $\boldsymbol{J}$ satisfying (3) are given for all t and $\boldsymbol{X}$. Prior to some initial time, say $t = 0$, fields $\boldsymbol{V}_\nu$ and $\boldsymbol{U}_\nu$ satisfying (2), (12), and (13) are given. We are then required to find the values of $\boldsymbol{V}_\nu$ and $\boldsymbol{U}_\nu$ for $t > 0$. In the simplest case, the sources and fields are identically zero, for $t < 0$, and the sources are "switched on" at $t = 0$ or later.

Much of our work in the later sections will be concerned with the solution of such a problem, asymptotically for $\lambda \to \infty$. (The functions f_ν and $\tilde{\varepsilon}_\nu$ depend on this parameter.) Since λ has the dimensions of a frequency, the physical meaning of our expansion will be clarified if we can find an equivalent dimensionless expansion parameter, λ_0. In order to do this we define a characteristic length*, a. We then introduce dimensionless space and time variables $\boldsymbol{X}_0 = \boldsymbol{X}/a$, $t_0 = ct/a$, and set

$$\begin{gathered} \boldsymbol{V}_{\nu 0}(t_0, \boldsymbol{X}_0) = \boldsymbol{V}_\nu(t, \boldsymbol{X}); \quad \boldsymbol{U}_{\nu 0}(t_0, \boldsymbol{X}_0) = \boldsymbol{U}_\nu(t, \boldsymbol{X}), \quad f_{\nu 0}(t_0, \boldsymbol{X}_0) = \frac{a}{c} f_\nu(t, \boldsymbol{X}), \\ \tau_0 = c\tau/a, \qquad \tilde{\varepsilon}_{\nu 0}(\omega, \boldsymbol{X}_0) = \tilde{\varepsilon}_\nu(\omega, \boldsymbol{X}), \qquad \boldsymbol{J}_0(t_0, \boldsymbol{X}_0) = \frac{a}{c} \boldsymbol{J}(t, \boldsymbol{X}). \end{gathered} \tag{14}$$

Then (11), (12) and (13) become

$$f_{\nu 0}(t_0, \boldsymbol{X}_0) = \frac{\lambda_0}{2\pi} \int e^{-i\lambda_0 \omega t_0} \tilde{\varepsilon}_{\nu 0}(\omega, \boldsymbol{X}_0)\, d\omega; \qquad \nu = 1, 2; \tag{15}$$

$$\frac{\partial \boldsymbol{V}_{10}}{\partial t_0} - \nabla_0 \times \boldsymbol{U}_{20} = -4\pi \boldsymbol{J}_0, \quad \frac{\partial \boldsymbol{V}_{20}}{\partial t_0} + \nabla_0 \times \boldsymbol{U}_{10} = 0; \tag{16}$$

$$\boldsymbol{V}_{\nu 0}(t_0, \boldsymbol{X}_0) = \int f_{\nu 0}(\tau_0, \boldsymbol{X}_0)\, \boldsymbol{U}_{\nu 0}(t_0 - \tau_0, \boldsymbol{X}_0)\, d\tau_0; \qquad \nu = 1, 2; \tag{17}$$

where

$$\lambda_0 = \lambda a/c. \tag{18}$$

It is not difficult to see that if we find the asymptotic expansion for $\lambda_0 \to \infty$ of a solution of (16, 17), the result is equivalent, under the above transformation, to the asymptotic expansion for $\lambda \to \infty$ of the corresponding solution of (12, 13). We conclude that the correct interpretation of our expansion is that it is valid for $1 \ll \lambda_0 = \lambda a/c$.

* For problems with boundaries, we may choose a typical boundary dimension as the characteristic length, a. For problems involving inhomogeneous media we may use an average value of the functions $\left| \frac{\tilde{\varepsilon}_\nu(\omega, \boldsymbol{X})}{|\nabla \tilde{\varepsilon}_\nu(\omega, \boldsymbol{X})|} \right|$. Then $\tilde{\varepsilon}_\nu$ changes by a small fraction of itself over distances small compared to a.

Before proceeding to our discussion of the asymptotic solution of (12, 13), we shall rewrite that system of equations in matrix form. The matrix notation has several advantages. It is not only compact and suggestive, but more important, it will enable us to include in our general discussion a large set of equations, of which (12, 13) is a special case. In particular, we will show that this set includes the important class of (weakly dissipative) symmetric hyperbolic partial differential equations. The notation used is explained in appendix G. We first introduce column vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{f}$ with 6 components, defined as follows:

$$\begin{gathered}\boldsymbol{u}=(\boldsymbol{U}_1,\boldsymbol{U}_2)=(\boldsymbol{E},\boldsymbol{H})=(E_1,E_2,E_3,H_1,H_2,H_3),\\ \boldsymbol{v}=(\boldsymbol{V}_1,\boldsymbol{V}_2),\quad \boldsymbol{f}=(-4\pi\boldsymbol{J},0).\end{gathered}\tag{19}$$

Corresponding to any 3-vector $\boldsymbol{Z}=(z_1, z_2, z_3)$, we introduce a 3×3 matrix $(\boldsymbol{Z})$ given by

$$(\boldsymbol{Z})=\begin{bmatrix}0 & -z_3 & z_2\\ z_3 & 0 & -z_1\\ -z_2 & z_1 & 0\end{bmatrix}.\tag{20}$$

Then if $\boldsymbol{V}$ is any 3-vector, in column form, $(\boldsymbol{Z})\,\boldsymbol{V}=\boldsymbol{Z}\times\boldsymbol{V}$. We now set $\boldsymbol{K}=(k_1, k_2, k_3)$ and define three 6×6 matrices A^1, A^2, A^3 by the equation

$$\sum_{\nu=1}^{3} k_\nu A^\nu=\begin{bmatrix}0 & -c(\boldsymbol{K})\\ c(\boldsymbol{K}) & 0\end{bmatrix}.\tag{21}$$

Here we have used matrix block notation. (Since k_1, k_2, k_3 are arbitrary, we may, for example, choose $k_1=1$ and $k_2=k_3=0$. Then (20) and (21) give A^1, *etc.*) We now note that the matrices A^ν are constant and

$$\sum_{\nu=1}^{3} A^\nu \boldsymbol{u}_{x_\nu}=\sum_{\nu=1}^{3}\frac{\partial}{\partial x_\nu}A^\nu\boldsymbol{u}=\begin{bmatrix}0 & -c(\nabla)\\ c(\nabla) & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{bmatrix}=\begin{bmatrix}-c\,\nabla\times\boldsymbol{U}_2\\ c\,\nabla\times\boldsymbol{U}_1\end{bmatrix},\tag{22}$$

hence (12) takes the form

$$\boldsymbol{v}_t+\sum_{\nu=1}^{3} A^\nu \boldsymbol{u}_{x_\nu}=\boldsymbol{f}.\tag{23}$$

Next we set

$$F(t,\boldsymbol{X})=\begin{bmatrix}f_1(t,\boldsymbol{X}) & 0\\ 0 & f_2(t,\boldsymbol{X})\end{bmatrix},\quad \tilde{\mathscr{E}}(\omega,\boldsymbol{X})=\begin{bmatrix}\tilde{\varepsilon}_1(\omega,\boldsymbol{X}) & 0\\ 0 & \tilde{\varepsilon}_2(\omega,\boldsymbol{X})\end{bmatrix}.\tag{24}$$

For anisotropic media $\hat{\varepsilon}_\nu$ and f_ν are 3×3 matrices. In the isotropic case they are scalars, and in (24) they multiply the 3×3 identity matrix I_3 which we omit. Now (11) and (13) become

$$F(t,\boldsymbol{X})=\frac{\lambda}{2\pi}\int e^{-i\lambda\omega t}\tilde{\mathscr{E}}(\omega,\boldsymbol{X})\,d\omega,\tag{25}$$

and

$$\boldsymbol{v}(t,\boldsymbol{X})=\int F(\tau,\boldsymbol{X})\,\boldsymbol{u}(t-\tau,\boldsymbol{X})\,d\tau.\tag{26}$$

If we increase the number of space dimensions from 3 to n, then $\boldsymbol{X}=(x_1, \ldots, x_n)$ and (23) becomes

$$\boldsymbol{v}_t+\sum_{\nu=1}^{n} A^{\nu}\boldsymbol{u}_{x_\nu}=\boldsymbol{f}. \tag{27}$$

Equations (25−27) are the matrix forms of (11−13).

2.2 The general system of equations

In the succeeding sections of this paper we shall discuss systems of equations of the form (2.1.26, 7) where $\boldsymbol{u}$, $\boldsymbol{v}$ are column vectors of dimension m and A^{ν}, F and $\tilde{\mathscr{E}}$ are $m\times m$ matrices. The matrices A^{ν} are hermitian and, in general, may be functions of $\boldsymbol{X}$. In this paper we require that A^{ν} and $\tilde{\mathscr{E}}$ be *smooth** functions of $\boldsymbol{X}$ in the infinite $\boldsymbol{X}$-space. If appropriate boundary conditions are specified on *boundary* surfaces, or if A^{ν} and $\tilde{\mathscr{E}}$ are discontinuous at interior surfaces called *interfaces*, our methods must be modified by introducing *reflected*, *transmitted*, and *diffracted* waves. These features are discussed in [*1*, *2*, *4*, *7*, *11*, *12*].

In addition the function $\tilde{\mathscr{E}}=\tilde{\mathscr{E}}(\omega, \boldsymbol{X}; \lambda)$ is required to satify five "basic conditions" which are listed in appendix H. The significance of these conditions (and the other "special conditions") are best appreciated by examining the role they play in later sections. For the electromagnetic field equations in isotropic media, examples of the dielectric and magnetic permeability functions satisfying these conditions are discussed in appendix D.

2.3. Symmetric-hyperbolic partial differential equations

In this section we shall show that, for an appropriate choice of $\tilde{\mathscr{E}}$, consistent with the basic conditions, (2.1.26, 7) reduce to a system of symmetric-hyperbolic partial differential equations. We first note that (2.1.25, 6) yield

$$\boldsymbol{v}(t, \boldsymbol{X})=\int F(t-s, \boldsymbol{X})\,\boldsymbol{u}(s, \boldsymbol{X})\,ds=\frac{\lambda}{2\pi}\int e^{-i\lambda\omega(t-s)}\tilde{\mathscr{E}}(\omega, \boldsymbol{X})\,\boldsymbol{u}(s, \boldsymbol{X})\,d\omega\,ds, \tag{1}$$

and the fourier integral theorem implies that

$$\boldsymbol{u}(t, \boldsymbol{X})=\frac{\lambda}{2\pi}\int e^{-i\lambda\omega(t-s)}\,\boldsymbol{u}(s, \boldsymbol{X})\,d\omega\,ds. \tag{2}$$

We now set

$$\tilde{\mathscr{E}}=\mathscr{E}-(i\lambda)^{-1}\mathscr{D}, \quad \mathscr{E}=A(\boldsymbol{X})+\frac{iB(\boldsymbol{X})}{\omega}, \quad \mathscr{D}=\frac{C(\boldsymbol{X})}{\omega}, \tag{3}$$

where A is positive definite** and B is anti-hermitian (*i.e.* $B^*=-B$). It follows then from (1) and (2) that

$$\boldsymbol{v}_t=A\,\boldsymbol{u}_t+\lambda B\,\boldsymbol{u}+C\,\boldsymbol{u}, \tag{4}$$

* A "smooth" function posesses sufficient continuity or differentiability properties to justify the operations performed with it.

** The matrix Q is *positive definite* (or *non-negative*) *if* $(\boldsymbol{x}, Q\boldsymbol{x})>0$ [or $(\boldsymbol{x}, Q\boldsymbol{x})\geqq 0$] for every non-zero vector $\boldsymbol{x}$.

hence (2.1.27) becomes

$$A\,u_t+\sum_{\nu=1}^{n}A^\nu u_{x_\nu}+\lambda B\,u+C\,u=f. \tag{5}$$

Since A is positive definite and the A^ν are hermitian, (5) is a "symmetric hyperbolic" system [3]. The asymptotic theory of such systems is discussed in [4]. There we see that the condition that B be anti-hermitian is essential. Such systems will be called "asymptotically conservative".

Since

$$A^0=\frac{\partial}{\partial\omega}(\omega\mathscr{E})=A,$$

it is easy to see that (3) satisfies conditions 1, 2, and 4 of appendix H. To see that condition 5 is satisfied we note that $G=\mathscr{A}-\omega A$, where $\mathscr{A}=k_\nu A^\nu-iB$. $\mathscr{A}$ is hermitian because B is anti-hermitian. Thus the roots $\omega=h$ of $\det G=0$ are "principal values" of the hermitian matrix $\mathscr{A}$, with respect to the positive definite matrix A; and it follows that such roots are real. Condition 3 imposes an additional restriction on the matrices A^ν, A, and B.

3. Asymptotic Solution of the General System of Equations

3.1. The asymptotic expansion

In this section we shall develop a method for obtaining asymptotic solutions of the general system of equations (2.1.26), (2.1.27) with $f=0$. For convenience we introduce the summation convention (see appendix G), and frequently we shall not indicate explicitly the dependence of various functions on the variable X. Thus we may write the general system of equations in the form

$$v_t+A^\nu u_{x_\nu}=0, \tag{1}$$

$$v(t)=\int F(\tau)\,u(t-\tau)\,d\tau. \tag{2}$$

From (2.1.25) we see that

$$F(\tau)=\lambda\mathscr{H}(\lambda\tau),\qquad \mathscr{H}(y)=\frac{1}{2\pi}\int e^{-i\omega y}\tilde{\mathscr{E}}(\omega)\,d\omega. \tag{3}$$

If we introduce the transformation $\lambda\tau=y$, then (2) becomes

$$v(t)=\int\mathscr{H}(y)\,u(t-y/\lambda)\,dy. \tag{4}$$

We now assume that (1) and (2) have solutions given by asymptotic power series of the form

$$u(t,X)=e^{i\lambda s(t,X)}\sum_{m=0}^{\infty}(i\lambda)^{-m}z_m(t,X). \tag{5}$$

We shall determine the *phase function* s and the *amplitude function* $z=z_0$ by inserting (5) in (1) and (2)*. We introduce the convenient notation**

$$k_\nu=s_{x_\nu}(t,X),\qquad \omega=-s_t(t,X), \tag{6}$$

* Presumably, the lower order terms, $z_1, z_2, \ldots$, can be obtained by a similar procedure.

** This notation is motivated by the consideration of "plane-wave" solutions. (See the footnote to section 3.2.)

and expand $u(t-y/\lambda)$ for large λ. In order to do so we first note that

$$e^{i\lambda s(t-y/\lambda)} = e^{i\lambda s(t)+i\omega y}\left[1-\frac{y^2 s_{tt}}{2i\lambda}+O(\lambda^{-2})\right], \tag{7}$$

and

$$\sum_{m=0}^{\infty}(i\lambda)^{-m} z_m(t-y/\lambda) = z(t)+(i\lambda)^{-1}[z_1(t)-i y z_t(t)]+O(\lambda^{-2}). \tag{8}$$

Hence

$$u(t-y/\lambda) = e^{i\lambda s(t)+i\omega y}\{z(t)+(i\lambda)^{-1}[z_1(t)-i y z_t(t)-\tfrac{1}{2}y^2 s_{tt} z(t)]+O(\lambda^{-2})\}. \tag{9}$$

From (3) we see that

$$\tilde{\mathscr{E}}(\omega)=\int e^{i\omega y}\mathscr{H}(y)\,dy, \quad \tilde{\mathscr{E}}_\omega = \int i y e^{i\omega y}\mathscr{H}(y)\,dy, \quad \tilde{\mathscr{E}}_{\omega\omega} = -\int y^2 e^{i\omega y}\mathscr{H}(y)\,dy, \tag{10}$$

and we apply condition 1:

$$\tilde{\mathscr{E}}(\omega) = \mathscr{E}(\omega)-(i\lambda)^{-1}\mathscr{D}(\omega)+O(\lambda^{-2}). \tag{11}$$

Then (4), (9), (10), and (11) yield

$$v(t) = e^{i\lambda s(t)}\{\mathscr{E}(\omega)z+(i\lambda)^{-1}[\mathscr{E}(\omega)z_1-\mathscr{D}(\omega)z-\mathscr{E}_\omega z_t+\tfrac{1}{2}s_{tt}\mathscr{E}_{\omega\omega}z]+O(\lambda^{-2})\}. \tag{12}$$

By differentiating (12) and (5) we obtain

$$v_t(t) = e^{i\lambda s}\left\{-i\lambda\omega\mathscr{E}z+\left[\mathscr{E}z_t-\omega\mathscr{E}z_1+\omega\mathscr{D}z+\omega\mathscr{E}_\omega z_t-\frac{\omega}{2}s_{tt}\mathscr{E}_{\omega\omega}z+\mathscr{E}_\omega\omega_t z\right]+O(\lambda^{-1})\right\}, \tag{13}$$

and

$$A^\nu u_{x_\nu} = e^{i\lambda s}\{i\lambda k_\nu A^\nu z+[A^\nu z_{x_\nu}+k_\nu A^\nu z_1]+O(\lambda^{-1})\}. \tag{14}$$

We may now insert (13) and (14) in (1) and equate to zero the coefficients of λ and λ^0. The resulting equations can be written more simply if we introduce the matrices

$$G = k_\nu A^\nu-\omega\mathscr{E} = k_\nu A^\nu(X)-\omega\mathscr{E}(\omega,X) \tag{15}$$

and

$$A^0 = (\omega\mathscr{E})_\omega = -G_\omega. \tag{16}$$

Then

$$A^0_t = A^0_\omega\omega_t = -s_{tt}[\omega\mathscr{E}_{\omega\omega}+2\mathscr{E}_\omega], \tag{17}$$

and (1), (13), and (14) yield

$$Gz = 0 \tag{18}$$

and

$$Gz_1+A^0 z_t+A^\nu z_{x_\nu}+\tfrac{1}{2}A^0_t z+\omega\mathscr{D}z = 0. \tag{19}$$

From (18) and (19) we shall derive the equations which determine s and z. In order to do so we must first examine some properties of G.

3.2. The eigenvectors of G

From (3.1.18) we see that the vector function z can be non-trivial only if

$$\det G = \det[k_\nu A^\nu(X)-\omega\mathscr{E}(\omega,X)] = 0. \tag{1}$$

For real values of ω, $\boldsymbol{X}$, and $\boldsymbol{K}$, this equation defines a functional relation between these quantities which we call the *dispersion relation**. The same functional relation can also be expressed in the form

$$\omega = h(\boldsymbol{K}, \boldsymbol{X}). \tag{2}$$

Then, in general, h is a multiple-valued function of $\boldsymbol{K}$ and $\boldsymbol{X}$, defined by (1).

For each value $\omega = h$, the hermitian matrix G is singular, therefore there exists a positive integer $q \leqq m$, and q linearly independent *null eigenvectors* $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^q$ of G, such that

$$G\boldsymbol{r}^j = 0; \qquad j = 1, \ldots, q. \tag{3}$$

The integer q is the *nullity* (or the multiplicity of the zero eigenvalue) of G. The nullity corresponding to each value of (2) may be different. In our work we shall require that, for each value of (2), q is independent of $\boldsymbol{K}$, so that we may differentiate (3) with respect to k_ν. This requirement is a consequence of condition 3 of appendix H.

By condition 2 the matrix A^0 is positive definite for all real ω. Therefore for each value of (2) we may orthonormalize** the null eigenvectors by the condition

$$(\boldsymbol{r}^i, A^0 \boldsymbol{r}^j) = \delta_{ij}; \qquad i, j = 1, \ldots, q. \tag{4}$$

The inner product used here is defined in appendix G. δ_{ij} is the Kronecker symbol.

If we differentiate $G = k_\nu A^\nu - \omega \mathscr{E}$ with respect to σ_ν and recall that

$$A^0 = \frac{\partial}{\partial \omega}(\omega \mathscr{E}),$$

we obtain

$$G_{k_\nu} = A^\nu - g_\nu A^0; \qquad \nu = 1, \ldots, n; \tag{5}$$

where

$$g_\nu = \frac{\partial \omega}{\partial k_\nu} = \frac{\partial h}{\partial k_\nu}(\boldsymbol{K}, \boldsymbol{X}); \qquad \nu = 1, \ldots, n. \tag{6}$$

Hence differentiation of (3) yields

$$G\boldsymbol{r}^j_{k_\nu} + (A^\nu - g_\nu A^0)\boldsymbol{r}^j = 0, \tag{7}$$

and from (4)

$$(\boldsymbol{r}^i, A^\nu \boldsymbol{r}^j) = g_\nu \delta_{ij}; \qquad i, j = 1, \ldots, q; \quad \nu = 1, \ldots, n. \tag{8}$$

This *basic identity* will be used repeatedly in our further work. From (4) we see that it holds also for $\nu = 0$, if we define

$$g_0 = 1. \tag{9}$$

The eigenvectors $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^q$ are not completely determined by (3) and (4), for if we set

$$\hat{\boldsymbol{r}}^i = \sum_{\nu=1}^{q} \beta_{i\nu} \boldsymbol{r}^\nu; \qquad i = 1, \ldots, q, \tag{10}$$

---

* In case A^ν and $\mathscr{E}$ are independent of X, and $\tilde{\mathscr{E}} \equiv \mathscr{E}$, the dispersion relation can be obtained by assuming "plane wave solutions" of (3.1.1, 2) of the form $u = z \exp\{i\lambda(k_\nu x_\nu - \omega t)\}$ where z, k_ν, and ω are constants.

** This orthonormalization may, for example, be accomplished by the Gram-Schmidt method using the inner product $[x, y] = (x, A^0 y)$.

where (β_{iv}) is any q-dimensional unitary matrix, the $\hat{\boldsymbol{r}}^i$ still satisfy (3), (4), and even (8). Let us now suppose that condition 6 of appendix H is satisfied. Then if we set $\alpha_{ij}=(\boldsymbol{r}^i, \omega\mathscr{D}\boldsymbol{r}^j)$, the q-dimensional matrix (α_{ij}) is hermitian. If we should now take (β_{iv}) to be the unitary matrix which diagonalizes (α_{iv}), then we would find that the new eigenvectors $\hat{\boldsymbol{r}}^i$ would satisfy the identity

$$(\boldsymbol{r}^i, \omega\mathscr{D}\boldsymbol{r}^j)=\eta_j\delta_{ij}; \qquad i,j=1,\ldots,q. \tag{11}$$

Therefore, if condition 6 holds, we may choose eigenvectors $\boldsymbol{r}^1,\ldots,\boldsymbol{r}^q$ which satisfy (3), (4), (8), and (11).

3.3. The dispersion equation and the ray equations

The dispersion relation, introduced in the last section, is a functional relation between ω, $\boldsymbol{K}$, and $\boldsymbol{X}$. Since $\omega=-s_t$ and $k_v=s_{x_v}$ (3.2.1 or 2) is also a first order partial differential equation for the phase function $s(t, \boldsymbol{X})$. We call that equation the *disperion equation*. Some readers may find it convenient to view the dispersion equation in the form (3.2.2) as a Hamilton-Jacobi equation. In any case it may be solved by the "method of characteristics" [3]. Thus we introduce the characteristic (Hamilton's) equations

$$\frac{dx_v}{dt}=g_v=\frac{\partial h}{\partial k_v}; \qquad \frac{dk_v}{dt}=-\frac{\partial h}{\partial x_v}; \qquad v=1,\ldots,n. \tag{1}$$

An immediate consequence of these equations, and the fact that the *hamiltonian h* is independent of t, is

$$\frac{d\omega}{dt}=\frac{dh}{dt}=\frac{\partial h}{\partial k_v}\frac{dk_v}{dt}+\frac{\partial h}{\partial x_v}\frac{dx_v}{dt}=0. \tag{2}$$

From (1) we see that

$$\frac{ds}{dt}=\frac{\partial s}{\partial x_v}\frac{dx_v}{dt}+\frac{\partial s}{\partial t}=\ell, \tag{3}$$

where ℓ is the *lagrangian*

$$\ell=k_v g_v-\omega=k_v\frac{\partial h}{\partial k_v}-h. \tag{4}$$

(1) is a system of $2n$ first order ordinary differential equations for the $2n$ functions $x_v(t)$, $k_v(t)$. Each solution of this system defines a curve $[t, \boldsymbol{X}(t)]=[t, x_1, \ldots, x_n]$ in space-time which we shall call a *ray**.

If initial values for $\boldsymbol{X}$ and $\boldsymbol{K}$ are known, then, in many cases, the *ray equations* (1) may be solved to determine a ray: Along this ray we see from (2) that ω is constant and from (3) that s may be obtained by integration. In this process it is clear that we require initial values for $\boldsymbol{X}$, $\boldsymbol{K}$, ω, and s. Let us examine how such initial values are obtained:

We shall find that in typical problems the values of s are known on some manifold** $\mathscr{M}$. Thus, for example, $\mathscr{M}$ may be an n-dimensional hypersurface given

* Sometimes the term "ray" is used for the projections $\boldsymbol{X}(t)$ of these curves into $\boldsymbol{X}$-space.

** The manifold $\mathscr{M}$ is closely related to the source region. The discussion in this section will be clarified by examining the concrete examples in section 5.

parametrically in terms of n parameters $\Phi=(\varphi_1, \ldots, \varphi_n)$ by

$$X=X_0(\Phi), \quad t=\tau(\Phi). \tag{5}$$

Then s is given by

$$s[\tau(\Phi), X_0(\Phi)]=s_0(\Phi) \tag{6}$$

where s_0 is a known function. By differentiating (6) we see that

$$-\omega \frac{\partial \tau}{\partial \varphi_j}+k_\nu \frac{\partial x_{0\nu}}{\partial \varphi_j}=\frac{\partial s_0}{\partial \varphi_j}, \tag{7}$$

and then for each Φ (7) and the dispersion relation provide $n+1$ equations for the determination of the initial values $\omega_0, k_{01}, \ldots, k_{0n}$ of the $n+1$ quantities ω, $k_1, \ldots, k_n$.

But, in general, the manifold $\mathscr{M}$ may be of dimension $r \leqq n$. Then, for the purpose of our asymptotic method, we shall require the solution s of the dispersion equation which is equal to s_0 on $\mathscr{M}$, and for which the corresponding rays all emanate from $\mathscr{M}$ with increasing t. This solution is, in general, multiple-valued, not only because the dispersion relation defines ω as a multiple-valued function of K and X, but also because, for each value of ω, more than one ray emanating from $\mathscr{M}$ may pass through a given space-time point (t, X). In order to construct this solution, s we note that (7) and the dispersion relation with $X=X_0(\Phi)$ now provide $r+1$ equations relating the $n+r+1$ quantities

$$\varphi_1, \ldots, \varphi_r, \ \omega_0, \ k_{01}, \ldots, k_{0n}. \tag{8}$$

We may then introduce any n of these quantities or any n functions

$$\gamma_j=\gamma_j(\varphi_1, \ldots, \varphi_r, \omega_0, k_{01}, \ldots, k_{0n}) \tag{9}$$

of these quantities as independent parameters, provided the $n+r+1$ equations (9), (7), and the dispersion relation, can be solved to obtain the quantities (8) as functions of $\Gamma=(\gamma_1, \ldots, \gamma_n)$. The functions (9) can, in general, be so chosen that $\varphi_1, \ldots, \varphi_r, \omega_0, k_{01}, \ldots, k_{0n}$ will be single-valued functions of Γ. Then, from (5) and (6) we shall have the initial values X_0, τ, s_0, ω_0, K_0 given as single-valued functions of Γ. Furthermore, to each value of Γ, there corresponds one branch of the multiple-valued function h for which $\omega_0(\Gamma)=h[K_0(\Gamma), X_0(\Gamma)]$. With this branch of h we may solve the ray equations (1) and obtain the solution

$$X=X(t; \Gamma), \quad K=K(t; \Gamma). \tag{10}$$

This solution satisfies the initial conditions $X(\tau; \Gamma)=X_0(\Gamma)$, $K(\tau; \Gamma)=K_0(\Gamma)$.

The parameters Γ lie in a certain *parameter space* $\mathscr{P}$ which is determined as follows: In (5), the point $(t, X)=(\tau, X_0)$ ranges over the manifold $\mathscr{M}$ as $\Phi=(\varphi_1, \ldots, \varphi_r)$ ranges over some space $\mathscr{S}$. Then $\mathscr{P}$ consists of all values $\Gamma=(\gamma_1, \ldots, \gamma_n)$ assumed by the functions (9) when Φ ranges over $\mathscr{S}$ and ω_0, K_0 take on all real values that satisfy (7) and the dispersion relation.

For Γ in $\mathscr{P}$, and $t \geqq \tau(\Gamma)$, the equation $(t, X)=[t, X(t; \Gamma)]$ defines an n-parameter family of rays. On each ray, ω has the constant value

$$\omega=\omega_0(\Gamma)=h[K(t; \Gamma), X(t; \Gamma)]. \tag{11}$$

Now if we set

$$g_\nu(t;\Gamma)=h_{k_\nu}[K(t;\Gamma),X(t;\Gamma)];\qquad \nu=1,\ldots,n \tag{12}$$

and

$$\ell(t;\Gamma)=k_\nu(t;\Gamma)\,g_\nu(t;\Gamma)-\omega_0(\Gamma); \tag{13}$$

then (3) yields by integration

$$s(t;\Gamma)=s[t,X(t;\Gamma)]=s_0(\Gamma)+\int_{\tau(\Gamma)}^{t}\ell(t';\Gamma)\,dt'. \tag{14}$$

This equation provides the required multiple-valued solution of the dispersion equation which is equal to s_0 on $\mathscr{M}$, and for which the corresponding rays all emanate from $\mathscr{M}$ with increasing t.

Note. In this and succeeding sections the symbol $f(t;\Gamma)$ denotes the value of the function $f(t,X)$ on the ray $[t,X(t;\Gamma)]$ at time t; *i.e.*

$$f(t;\Gamma)=f[t,X(t;\Gamma)]. \tag{15}$$

For fixed t, $f(t,X)$ may be multiple-valued although $f(t;\Gamma)$ is single-valued because more than one ray may pass through the point (t,X).

3.4. The ray transformation

For each fixed value of t, the function $X=X(t;\Gamma)$ defines a transformation from the parameter space $\mathscr{P}$ to X-space. The jacobian $j(t;\Gamma)$ of this *ray transformation* is defined by

$$j(t;\Gamma)=\det\left[\frac{\partial x_\nu(t;\Gamma)}{\partial\gamma_\mu}\right]. \tag{1}$$

The jacobian may vanish at certain space-time points called *caustic points.* The locus of all such points is called the *caustic* of the ray family.

A complete study of $j(t;\Gamma)$ would require the solution of the ray equations to obtain the function $X(t;\Gamma)$. However, we can obtain valuable information about the behavior of j near $\mathscr{M}$ by means of an expansion of $X(t;\Gamma)$ for small $[t-\tau(\Gamma)]$. From the ray equations (3.3.1) we obtain

$$x_\nu(t;\Gamma)=x_{0\,\nu}(\Gamma)+[t-\tau(\Gamma)]\,h_{k_\nu}[K_0(\Gamma),X_0(\Gamma)]+O[(t-\tau)^2]. \tag{2}$$

Then

$$\frac{\partial x_\nu}{\partial\gamma_\mu}=\frac{\partial x_{0\,\nu}}{\partial\gamma_\mu}-\tau_{\gamma_\mu}h_{k_\nu}+O(t-\tau). \tag{3}$$

From (1) it follows that

$$j(\tau)=j[\tau(\Gamma);\Gamma]=\det\left[\frac{\partial x_{0\,\nu}}{\partial\gamma_\mu}-\tau_{\gamma_\mu}h_{k_\nu}\right]. \tag{4}$$

If $j(\tau)=0$ for some value of Γ then the point $(t,X)=[\tau,X(\tau;\Gamma)]$ is a caustic point, and we find from (3) that there is a positive integer ν such that

$$j(t;\Gamma)=(t-\tau)^\nu\tilde{j}(\tau;\Gamma)+O[(t-\tau)^{\nu+1}], \tag{5}$$

where

$$\tilde{j}(\tau;\Gamma)=\lim_{t\to\tau}(t-\tau)^{-\nu}j[t;\Gamma]. \tag{6}$$

The quantity j is non-zero and finite. It can be obtained from further terms of the expansion (3). It is of practical importance to note that ν and $\tilde{j}$ can be obtained from the ray equations (3.3.1) and the initial conditions $X_0(\Gamma)$, $K_0(\Gamma)$ without solving the ray equations explicitly. In general, the integer ν, which is the *order of the zero of* $j(t)$ at $t=\tau(\Gamma)$, is a function of Γ. If $j(\tau)\neq 0$ then the point $[\tau, X(\tau;\Gamma)]$ is not a caustic point. However (5) and (6) remain trivially valid with $\nu=0$ and $\tilde{j}(\tau;\Gamma)=j(\tau;\Gamma)$. Thus ν and $\tilde{j}$ are defined at all points on every ray, and in general ν is a non-negative integer.

It is interesting to note that, since $g_\nu=h_{k_\nu}$,

$$j(\tau)=\det\begin{bmatrix} 1 & g_1 & \cdots & g_n \\ \tau_{\gamma_1} & \dfrac{\partial x_{01}}{\partial\gamma_1} & \cdots & \dfrac{\partial x_{0n}}{\partial\gamma_1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \tau_{\gamma_n} & \dfrac{\partial x_{01}}{\partial\gamma_n} & \cdots & \dfrac{\partial x_{0n}}{\partial\gamma_n} \end{bmatrix}. \tag{7}$$

This equation follows from (4) by simple determinant operations. From the ray equations, we see that the first row-vector of (7) is tangent to the ray $[t, X(t;\Gamma)]$ at $t=\tau(\Gamma)$. The remaining row-vectors are tangent vectors of $\mathcal{M}$. If, at a given point, the n tangent vectors are linearly dependent, then either that point is a singular point of $\mathcal{M}$ or $\mathcal{M}$ is of dimension less than n. In either case $j(\tau)$ vanishes there, hence the point is a caustic point. If at a given point, the n tangent vectors are linearly independent, then we see from (7) that $j(\tau)$ vanishes if and only if the ray at that point is tangent to $\mathcal{M}$.

3.5. The transport equations

In the preceeding section we have seen that the phase function s can be obtained by solving a system of ordinary differential equations, the ray equations. In this section we shall show that the amplitude function z can also be obtained by solving a system of ordinary differential equations.

From (3.1.18) we see that the vector z lies in the null space of the matrix G, hence is a linear combination of the null eigenvectors $r^1, \ldots, r^q$ of G. Thus there exist coefficients $\sigma_1, \ldots, \sigma_q$ such that

$$z=\sum_{m=1}^{q}\sigma_m r^m. \tag{1}$$

If we take the inner product of (3.1.19) with r^ℓ and note that $(r^\ell, Gz_1)=(Gr^\ell, z_1)=(0, z_1)=0$, we obtain

$$(r^\ell, A^j z_{x_j})+\tfrac{1}{2}(r^\ell, A_t^0 z)+(r^\ell, \omega\mathscr{D} z)=0; \quad \left(\sum_{j=0}^{n}\right). \tag{2}$$

Here, for convenience, we have set

$$x_0=t. \tag{3}$$

We now insert (1) in (2) and simplify the resulting equation by using the basic identity (3.2.8). Thus we obtain

$$g_j(\sigma_\ell)_{x_j}+(\boldsymbol{r}^\ell, A^j \boldsymbol{r}^m_{x_j})\sigma_m+\tfrac{1}{2}(\boldsymbol{r}^\ell, A^0_t \boldsymbol{r}^m)\sigma_m+(\boldsymbol{r}^\ell, \omega \mathscr{D} r^m)\sigma_m=0, \quad \left(\sum_{j=0}^{n}, \sum_{m=1}^{q}\right). \tag{4}$$

But from (3.3.1) we see that

$$\sum_{j=0}^{n} g_j(\sigma_\ell)_{x_j}=(\sigma_\ell)_t+\sum_{j=1}^{n}(\sigma_\ell)_{x_j}\frac{dx_j}{dt}=\frac{d\sigma_\ell}{dt},$$

hence (4) may be written in the form

$$\frac{d\sigma_\ell}{dt}+\sum_{m=1}^{q}\tau_{\ell m}\sigma_m=0; \qquad \ell=1,\ldots,q \tag{5}$$

where

$$\tau_{\ell m}=\sum_{j=0}^{n}(\boldsymbol{r}^\ell, A^j \boldsymbol{r}^m_{x_j})+\tfrac{1}{2}(\boldsymbol{r}^\ell, A^0_t \boldsymbol{r}^m)+(\boldsymbol{r}^\ell, \omega \mathscr{D} \boldsymbol{r}^m); \qquad \ell, m=1,\ldots,q. \tag{6}$$

(5) is a system of q first order ordinary differential equations for the coefficients $\sigma_1, \ldots, \sigma_q$ which determine z. These equations are called the *transport equations*. The solution of (5) yields the values of $\sigma_1, \ldots, \sigma_q$, hence the value of z, along a ray. In the next four sections we shall obtain the solution of the transport equations in several important special cases. From (3.2.11) we see that if condition 6 holds, the last term in (6) is

$$(r^\ell, \omega \mathscr{D} r^m)=\eta_\ell \delta_{\ell m}. \tag{7}$$

3.6. Homogeneous media

We shall say that our general system of equations (2.1.26, 7) describes the propagation of waves in a *homogeneous medium* if the matrices A^ν, $\mathscr{E}$, and $\mathscr{D}$ are independent of $\boldsymbol{X}$. In this case it is possible to show that if condition 6 holds, the system of q equations (3.5.5) uncouples. In fact it follows from the theorem of appendix A that

$$\tau_{l m}=[\tfrac{1}{2}\nabla\cdot\boldsymbol{G}+\eta_\ell]\delta_{\ell m}; \qquad \nabla\cdot\boldsymbol{G}=\sum_{\nu=1}^{n}(g_\nu)_{x_\nu}; \tag{1}$$

and we obtain the uncoupled equations

$$\frac{d\sigma_\ell}{dt}+[\tfrac{1}{2}(g_\nu)_{x_\nu}+\eta_\ell]\sigma_\ell=0, \qquad \ell=1,\ldots,q. \tag{2}$$

For homogeneous media h and g_ν are independent of $\boldsymbol{X}$. It then follows from the ray equations that the functions k_ν are constant on each ray, *i.e.*

$$k_\nu(t;\boldsymbol{\Gamma})=k_{\nu 0}(\boldsymbol{\Gamma}). \tag{3}$$

Then, of course,

$$\omega(t;\boldsymbol{\Gamma})=\omega_0(\boldsymbol{\Gamma})=h[\boldsymbol{K}_0(\boldsymbol{\Gamma})], \quad g_\nu(t;\boldsymbol{\Gamma})=g_{\nu 0}(\boldsymbol{\Gamma})=h_{k_\nu}[\boldsymbol{K}_0(\boldsymbol{\Gamma})], \tag{4}$$

and the solution of the ray equations (33) is given by

$$x_\nu=x_\nu(t;\boldsymbol{\Gamma})=x_{\nu 0}(\boldsymbol{\Gamma})+[t-\tau(\boldsymbol{\Gamma})]\,g_{\nu 0}(\boldsymbol{\Gamma}). \tag{5}$$

Thus the rays form an n-parameter family of straight lines in space-time.

From (3.3.4) we see that the lagrangian has the constant value

$$\ell(t;\Gamma)=\ell_0(\Gamma)=k_{\nu 0}\, g_{\nu 0}-\omega_0 \tag{6}$$

on each ray, hence (3.3.14) becomes

$$s(t;\Gamma)=s_0(\Gamma)+[t-\tau(\Gamma)]\,\ell_0(\Gamma). \tag{7}$$

The solution of (2) is given in (12) of appendix C. There we see that

$$\sigma_\ell(t;\Gamma)=\sigma_\ell(t_0;\Gamma)\left[\frac{j(t_0;\Gamma)}{j(t;\Gamma)}\right]^{\frac{1}{2}}\exp\{-(t-t_0)\,\eta_\ell(\Gamma)\};\qquad \ell=1,\dots,q. \tag{8}$$

Here j is the jacobian defined by (3.4.1), and η_ℓ is given by

$$\begin{gathered}\eta_\ell(\Gamma)=[r^l(\Gamma),\,\omega_0\,\mathscr{D}(\omega_0)\,r^l(\Gamma)];\qquad r^l(\Gamma)=r^l(\omega_0,K_0);\\ \omega_0=\omega_0(\Gamma);\qquad K_0=K_0(\Gamma).\end{gathered} \tag{9}$$

(8) provides the value of σ_ℓ at any point on the ray $[t, X(t;\Gamma)]$ in terms of its value at a given point $t=t_0$. From (3.5.1) and (3.2.8) we see that $\sigma_m=(r^m, A^0 z)$, therefore (3.5.1) and (8) yield

$$\begin{aligned}z(t)&=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\sum_{m=1}^{q}\exp\{-(t-t_0)\,\eta_m\}\,\sigma_m(t_0)\,r^m\\ &=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\sum_{m=1}^{q}\exp\{-(t-t_0)\,\eta_m\}\,[r^m, A^0 z(t_0)]\,r^m.\end{aligned} \tag{10}$$

Here, for the sake of brevity, we do not indicate the dependence on the parameters Γ. If condition 7 of appendix H is satisfied the functions η_m; $m=1,\dots,q$; are all equal. Then (10) becomes

$$z(t)=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\{-(t-t_0)\,\eta\}\,z(t_0). \tag{11}$$

These equations provide the explicit solution of the transport equations for the case of homogeneous media.

It is interesting to note that for homogeneous media

$$\frac{\partial x_\nu(t,\Gamma)}{\partial \gamma_\mu}=(x_{\nu 0}-\tau\, g_{\nu 0})_{\gamma_\mu}+(g_{\nu 0})_{\gamma_\mu}\,t, \tag{12}$$

hence the jacobian (3.4.1) is a polynomial in t with coefficients which are functions of Γ. The degree of the polynomial is at most n. If $\tau_1(\Gamma),\dots,\tau_n(\Gamma)$ are the roots of this polynomial, then the factor $j(t_0)/j(t)$ which appears in (8), (10), and (11) is given by

$$\frac{j(t_0)}{j(t)}=\frac{(t_0-\tau_1)\dots(t_0-\tau_n)}{(t-\tau_1)\dots(t-\tau_n)}. \tag{13}$$

(If the degree of the polynomials is less than n then some of the roots τ_j in (13) are infinite.) The points $t=\tau_j(\Gamma)$ are, of course, caustic points. At those points (13) becomes infinite and our formulas for the amplitude z are invalid. In the

neighborhood of a caustic point, a different asymptotic representation of the solution is required*. If t and t_0 are separated by one or more caustic points then (13) may be negative. In such cases the factor $[j(t_0)/j(t)]^{\frac{1}{2}}$ in (8), (10), and (11) is ambiguous. The ambiguity may be removed by the "phase-shift rule" which will be discussed in appendix F.

3.7. The energy density

We turn now to a consideration of the transport equations (3.5.5) without assuming that A^ν, $\mathscr{E}$ and $\mathscr{D}$ are independent of X. We first note that the basic identity (3.2.8) implies

$$(r^l, A^0 r_t^m)+(r_t^l, A^0 r^m)+(r^l, A_t^0 r^m)=(r^l, A^0 r^m)_t=(\delta_{lm})_t=0; \tag{1}$$

hence, from (3.5.6) and (3.2.8)

$$\begin{aligned}\tau_{lm}+\tau_{ml}^* &= \sum_{\nu=1}^{n}[(r^l, A^\nu r_{x_\nu}^m)+(r_{x_\nu}^l, A^\nu r^m)]+[r^l, \omega(\mathscr{D}+\mathscr{D}^*) r^m] \\ &= \sum_{\nu=1}^{n}[(r^l, A^\nu r^m)_{x_\nu}-(r^l, A_{x_\nu}^\nu r^m)]+[r^l, \omega(\mathscr{D}+\mathscr{D}^*) r^m] \end{aligned} \tag{2}$$

$$= \cdot(\nabla\cdot \boldsymbol{G})\delta_{lm}+[r^l, \omega(\mathscr{D}+\mathscr{D}^*) r^m]-\sum_{\nu=1}^{n}(r^l, A_{x_\nu}^\nu r^m), \tag{3}$$

where

$$\nabla\cdot \boldsymbol{G}=\sum_{\nu=1}^{n}(g_\nu)_{x_\nu}. \tag{4}$$

Next we introduce the *energy density* function

$$w=(z, A^0 z)=\sigma_l^*\sigma_l\left(\sum_{l=1}^{n}\right) \tag{5}$$

which is discussed in appendix B. (The second equation in (5) follows from (3.5.1) and (3.2.8).) The transport equations (3.5.5) now yield

$$\frac{dw}{dt}=-\sigma_l^*(\tau_{lm}+\tau_{ml}^*)\sigma_m. \tag{6}$$

Comparison of (6) and (3) shows that we may obtain a single ordinary differential equation for w provided conditions 7 and 8 are satisfied. (We note that condition 8 is trivially satisfied if the matrices A^ν are independent of X. Furthermore, conditions 7 and 8 are obviously satisfied if the nullity is $q=1$.)

Under these conditions, (6) now becomes

$$\frac{dw}{dt}+[\nabla\cdot \boldsymbol{G}+2(\operatorname{Re}\eta+\beta)]w=0, \tag{7}$$

and the solution of this equation is given in appendix C. From (C.10) we see that

$$w(t; \Gamma)=w(t_0; \Gamma)\frac{j(t_0; \Gamma)}{j(t; \Gamma)}\exp\left\{-2\int_{t_0}^{t}[\operatorname{Re}\eta(t'; \Gamma)+\beta(t'; \Gamma)]\,dt'\right\}. \tag{8}$$

* See [6].

As in the preceding section we note that j vanishes at caustic points and therfore w becomes infinite at such points. Except for the exponential factor, (8) would imply that $w(t;\Gamma)j(t;\Gamma)$ is constant; *i.e.* independent of t. Let $dX(t)$ be an element of space, each point of which moves according to the solution $X=X(t;\Gamma)$ of the ray equations. Then $dX(t)=j(t;\Gamma)\,d\Gamma$, and the element of energy $w(t;\Gamma)\,dX(t)=w(t;\Gamma)j(t;\Gamma)\,d\Gamma$ is independent of t. Thus energy is conserved in $dX(t)$. Since $dX(t)$ vanishes at a caustic point, it is not surprising that the energy density becomes infinite there.

The energy density w is by definition a non-negative real quantity. However $j(t_0)/j(t)$ may be negative if t_0 and t are separated by a caustic point. In this case the derivation of (8) in appendix C is not valid. However, it follows from the discussion of appendix F that (8) is valid for all t except caustic points, provided we replace the ratio of jacobians by its absolute value.

In this section we have obtained a formula for the energy density w, but not for the amplitude function z. For many purposes this formula suffices. (We recall, *e.g.*, that often optical intensities, *i.e.* average energy densities, are measurable whereas the corresponding fields are not.) As we shall see in the next two sections, further progress can be made in the important special cases $q=1$ and $q=2$.

3.8. Nullity one

If $q=1$ then conditions 7 and 8 are trivially satisfied with

$$\eta(\omega)=[\boldsymbol{r},\omega\mathscr{D}\boldsymbol{r}] \quad \text{and} \quad \beta=-\tfrac{1}{2}\sum_{\nu=1}^{n}(\boldsymbol{r},A^{\nu}_{x_\nu}\boldsymbol{r}). \tag{1}$$

From (3.5.5, 6) we see that the transport equation becomes

$$\frac{d\sigma}{dt}+\tau_0\sigma=0 \tag{2}$$

where

$$\tau_0=\sum_{j=0}^{n}(\boldsymbol{r},A^{j}\boldsymbol{r}_{x_j})+\tfrac{1}{2}(\boldsymbol{r},A^{0}_{t}\boldsymbol{r})+\eta. \tag{3}$$

Let $\vartheta=\arg\sigma$. Then

$$\sigma=|\sigma|e^{i\vartheta}, \tag{4}$$

and (2) yields

$$\left[\frac{d|\sigma|}{dt}+i|\sigma|\frac{d\vartheta}{dt}\right]+|\sigma|\tau_0=0. \tag{5}$$

By taking the imaginary part of (5) we find that

$$\frac{d\vartheta}{dt}=-\operatorname{Im}\tau_0, \quad \text{hence} \quad \vartheta(t)=\vartheta(t_0)-\int_{t_0}^{t}\operatorname{Im}\tau_0(t')\,dt'. \tag{6}$$

From (3.7.5) we see that $|\sigma|=\sqrt{w}$, hence (3.7.8) yields

$$|\sigma(t)|=|\sigma(t_0)|\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}[\operatorname{Re}\eta(t')+\beta(t')]\,dt'\right\}. \tag{7}$$

Thus from (4), (6), and (7)

$$\sigma(t)=\sigma(t_0)\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}[\operatorname{Re}\eta(t')+\beta(t')+i\operatorname{Im}\tau_0(t')]\,dt'\right\}. \tag{8}$$

Since (3.1.17) shows that A_t^0 is hermitian, we see that $(\boldsymbol{r}, A_t^0\boldsymbol{r})$ is real. Hence, from (3),

$$\operatorname{Re}\eta+i\operatorname{Im}\tau_0=\eta+i\zeta \tag{9}$$

where

$$\zeta=\operatorname{Im}\sum_{j=0}^{n}(\boldsymbol{r}, A^j\boldsymbol{r}_{x_j}). \tag{10}$$

(If $A^1, \ldots, A^n$ and $\mathscr{E}$ are real, then the eigenvector $\boldsymbol{r}$ is real and $\zeta=0$.)

From (3.5.1) and (3.2.8) we see that

$$\boldsymbol{z}=\sigma\boldsymbol{r} \quad \text{and} \quad \sigma=(\boldsymbol{r}, A^0\boldsymbol{z}). \tag{11}$$

Therefore (8), (9), and (11) yield

$$\boldsymbol{z}(t)=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}[\eta(t')+\beta(t')+i\zeta(t')]\,dt'\right\}[\boldsymbol{r}(t_0), A^0(t_0)\boldsymbol{z}(t_0)]\boldsymbol{r}(t). \tag{12}$$

Here, for the sake of brevity we have not indicated the dependence on $\boldsymbol{\Gamma}$ of all the functions in (12). Again the ambiguity of the factor $[j(t_0)/j(t)]^{\frac{1}{2}}$ may be removed by the "phase-shift rule" of appendix F.

We note that the eigenvector $\boldsymbol{r}$ is determined only up to a unitary factor $e^{i\alpha}$, where α is an arbitrary real number. It can be shown*, however, that (12) is independent of α.

3.9. Nullity two

In important applications of our theory, such as the electromagnetic field equations for isotropic media, the nullity q of the matrix G is two. In such cases it is possible to solve the coupled transport equations, provided we impose conditions 6, 7, 8, and 9. Then we may choose the eigenvectors $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ to be real, and it follows from (3.5.6, 7) that τ_{lm} is real. In this case (3.7.2) implies that

$$\tau_{11}=\tau_{22}=\tau_1 \tag{1}$$

where

$$\tau_1=\tfrac{1}{2}\nabla\cdot\boldsymbol{G}+\eta+\beta. \tag{2}$$

Similarly, we see from (3.7.2) that $\tau_{12}+\tau_{21}=0$, hence (3.5.6, 7) yield

$$\tau_{12}=-\tau_{21}=\tau_2 \tag{3}$$

* Suppose $\hat{\boldsymbol{r}}=e^{i\alpha}\boldsymbol{r}$. Then $\hat{\eta}=\eta$, $\hat{\beta}=\beta$, and $\hat{\boldsymbol{r}}_{x_j}=i\alpha_{x_j}e^{i\alpha}\boldsymbol{r}+e^{i\alpha}\boldsymbol{r}_{x_j}$. Hence $(\hat{\boldsymbol{r}}, A^j\boldsymbol{r}_{x_j})=(\boldsymbol{r}, A^j\boldsymbol{r}_{x_j})+i\alpha_{x_j}g_j$, where from (3.2.8) $g_j=(\boldsymbol{r}, A^j\boldsymbol{r})$. It follows that

$$\hat{\zeta}=\zeta+\frac{d\alpha}{dt} \quad \text{and} \quad \int_{t_0}^{t}\hat{\zeta}(t')\,dt'=\int_{t_0}^{t}\zeta(t')\,dt'+\alpha(t)-\alpha(t_0).$$

But

$$\exp\{-i[\alpha(t)-\alpha(t_0)]\}[\hat{\boldsymbol{r}}(t_0), A^0(t_0)\boldsymbol{z}(t_0)]\hat{\boldsymbol{r}}(t)=[\boldsymbol{r}(t_0), A^0(t_0)\boldsymbol{z}(t_0)]\boldsymbol{r}(t).$$

where

$$\tau_2=\sum_{j=0}^{n}(r^1, A^j r_{x_j}^2)+\tfrac{1}{2}(r^1, A_t^0 r^2). \tag{4}$$

The transport equations (3.5.5) now take the form

$$\frac{d\sigma_1}{dt}+\tau_1\sigma_1+\tau_2\sigma_2=0, \quad \frac{d\sigma_2}{dt}-\tau_2\sigma_1+\tau_1\sigma_2=0. \tag{5}$$

These equations can be simplified by introducing the quantities

$$\rho=+\sqrt{w}, \quad \beta_1=\frac{\sigma_1}{\rho}, \quad \beta_2=\frac{\sigma_2}{\rho}. \tag{6}$$

Then, since $\rho^2=w$, we see from (3.7.7) that

$$2\rho\frac{d\rho}{dt}+[V\cdot G+2(\eta+\beta)]\rho^2=0, \tag{7}$$

hence (2) yields

$$\frac{d\rho}{dt}+\tau_1\rho=0. \tag{8}$$

By inserting (6) in (5) we obtain

$$\frac{d\beta_1}{dt}+\tau_2\beta_2=0, \quad \frac{d\beta_2}{dt}-\tau_2\beta_1=0, \tag{9}$$

and it is easily seen that the solution of (9) is given by

$$\begin{aligned}\beta_1(t)&=\beta_1(t_0)\cos\delta(t)-\beta_2(t_0)\sin\delta(t),\\ \beta_2(t)&=\beta_1(t_0)\sin\delta(t)+\beta_2(t_0)\cos\delta(t)\end{aligned} \tag{10}$$

where

$$\delta(t)=\int_{t_0}^{t}\tau_2(t')\,dt'. \tag{11}$$

Here $\beta_1(t)=\beta_1(t;\Gamma)$ is the value of β_1 at the point $[t, X(t;\Gamma)]$, *etc.*

Now, from (6) and (3.7.5) we see that

$$|\beta_1|^2+|\beta_2|^2=(w)^{-1}[|\sigma_1|^2+|\sigma_2|^2]=1, \tag{12}$$

while (3.5.1) and (6) yield

$$z(t)=\sigma_1(t)r^1(t)+\sigma_2(t)r^2(t)=\sqrt{w(t)}\,[\beta_1(t)r^1(t)+\beta_2(t)r^2(t)]. \tag{13}$$

Furthermore (6), (3.5.1), (3.2.8) and (3.7.5) imply that

$$\beta_m=\frac{(r^m, A^0 z)}{\sqrt{w}}=\frac{(r^m, A^0 z)}{\sqrt{(z, A^0 z)}}; \qquad m=1,2. \tag{14}$$

By inserting (3.7.8) in (13) and by using (3.7.5) we obtain

$$z(t)=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}[\eta(t')+\beta(t')]\,dt'\right\}[z(t_0), A^0(t_0)z(t_0)]^{\frac{1}{2}}\times \tag{15}$$

$$\times[\beta_1(t)r^1(t)+\beta_2(t)r^2(t)].$$

We now set

$$\hat{\beta}_m(t)=[z(t_0), A^0(t_0)\,z(t_0)]^{\frac{1}{2}}\,\beta_m(t)=\sqrt{w(t_0)}\,\beta_m(t).$$

Then $\hat{\beta}_1$ and $\hat{\beta}_2$ satisfy (10), while (14) and (15) yield

$$\hat{\beta}_m(t_0)=[\boldsymbol{r}^m(t_0), A^0(t_0)\,z(t_0)]; \qquad m=1,2 \tag{16}$$

and

$$z(t)=\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}[\eta(t')+\beta(t')]\,dt'\right\}[\hat{\beta}_1(t)\,\boldsymbol{r}^1(t)+\hat{\beta}_2(t)\,\boldsymbol{r}^2(t)]. \tag{17}$$

Thus if $z(t_0)$ is given, (16) and (10) yield $\hat{\beta}_1(t)$ and $\hat{\beta}_2(t)$, and then (17) determines $z(t)$.

3.10. Generalized formulas for the amplitude, z

In sections 3.6, 3.8, and 3.9 we have derived formulas for the amplitude function $z(t;\Gamma)$ in terms of its values $z(t_0;\Gamma)$ at some point $t=t_0(\Gamma)$ on each ray. We have seen that z becomes infinite at caustic points because the jacobian vanishes there. Thus our formulas are valid only if t_0 is not a caustic point.

In section 3.3 we derived the formula (3.3.14) for the phase function s in terms of its values on the manifold $\mathscr{M}$, which consists of those points on the family of rays for which $t=\tau(\Gamma)$. In general, these points may be caustic points.

For applications of our theory we require formulas for z and s in terms of their values on the *same* manifold $\mathscr{M}$. If $t=\tau(\Gamma)$ is not a caustic point, we can merely set $t_0=\tau$ in our formulas for z. However, if $t=\tau$ is a caustic point $z(\tau,\Gamma)$ is infinite and we must modify our formulas for z appropriately. That is the object of this section.

We shall make use of the analysis of the behavior of the jacobian near $\mathscr{M}$, discussed in section 3.4. In particular we shall use (3.4.6). Thus if we let $t_0-\tau$ tend to zero through positive values in (3.7.8), we find that

$$w(t;\Gamma)=\tilde{w}(\tau;\Gamma)\,\frac{\tilde{j}(\tau;\Gamma)}{j(t;\Gamma)}\exp\left\{-2\int_{\tau}^{t}[\mathrm{Re}\,\eta(t';\Gamma)+\beta(t';\Gamma)]\,dt'\right\}, \tag{1}$$

where

$$\tilde{w}(\tau;\Gamma)=\lim_{(t-\tau)\to 0+}(t-\tau)^{\nu}\,w(t;\Gamma). \tag{2}$$

(The limit is taken through positive values of $t_0-\tau$ in (3.7.8) because if $t_0<\tau<t$, then t_0 and t are separated by a caustic point and (3.7.8) is not valid.)

The positive function $\tilde{w}$ describes the asymptotic behavior of w as $t\to\tau$. If $\nu=0$, then $t=\tau$ is not a caustic point, $\tilde{w}(\tau,\Gamma)=w(\tau;\Gamma)$, $\tilde{j}=j$, and (1) reduces to (3.7.8). Equation (1) is valid under conditions 7 and 8.

If we let $(t_0-\tau)$ tend to zero through positive values in (3.9.10, 11, 16, 17), we obtain

$$z(t)=\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{\tau}^{t}[\eta(t')+\beta(t')]\,dt'\right\}[\tilde{\beta}_1(t)\,\boldsymbol{r}^1(t)+\tilde{\beta}_2(t)\,\boldsymbol{r}^2(t)]; \tag{3}$$

where

$$\tilde{\beta}_m(\tau)=\lim_{(t-\tau)\to 0+}(t-\tau)^{\frac{1}{2}\nu}\hat{\beta}_m(t)=[\boldsymbol{r}^m(\tau),A^0(\tau)\tilde{\boldsymbol{z}}(\tau)];\qquad m=1,2; \tag{4}$$

$$\tilde{\boldsymbol{z}}(\tau)=\tilde{\boldsymbol{z}}(\tau;\boldsymbol{\Gamma})=\lim_{(t-\tau)\to 0+}(t-\tau)^{\frac{1}{2}\nu}\boldsymbol{z}(t;\boldsymbol{\Gamma}); \tag{5}$$

and

$$\left.\begin{aligned}\tilde{\beta}_1(t)&=\tilde{\beta}_1(\tau)\cos\delta(t)-\tilde{\beta}_2(\tau)\sin\delta(t)\\ \tilde{\beta}_2(t)&=\tilde{\beta}_1(\tau)\sin\delta(t)-\tilde{\beta}_2(\tau)\cos\delta(t)\end{aligned}\right\};\qquad \delta(t)=\int_\tau^t\tau_2(t')\,dt'. \tag{6}$$

Equations (3), (4), and (6) are valid under conditions 6, 7, 8, and 9, for the case of nullity $q=2$. If $\tilde{\boldsymbol{z}}(\tau;\boldsymbol{\Gamma})$ is known, then (4) gives the values of $\tilde{\beta}_m(\tau)$, (6) gives the values of $\tilde{\beta}_m(t)$, and (3) determines $\boldsymbol{z}(t;\boldsymbol{\Gamma})$. If $\nu=0$ then $\tilde{\boldsymbol{z}}=\boldsymbol{z}$, $\tilde{j}=j$, and the formulas reduce to those of section 3.9, from which they were derived.

If we let $(t_0-\tau)$ tend to zero through positive values in (3.8.12), we obtain

$$\boldsymbol{z}(t)=\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_\tau^t[\eta(t')+\beta(t')+i\zeta(t')]\,dt'\right\}[\boldsymbol{r}(\tau),A^0(\tau)\tilde{\boldsymbol{z}}(\tau)]\boldsymbol{r}(t), \tag{7}$$

where $\tilde{\boldsymbol{z}}$ is again given by (5). Equation (7) is valid for the case of nullity $q=1$. It provides the value of $\boldsymbol{z}(t)$ directly in terms of $\tilde{\boldsymbol{z}}(\tau)$, and reduces to (3.8.12) if $\nu=0$.

For the case of homogenous media, we may let $(t_0-\tau)$ tend to zero through positive values in (3.6.10, 11). Then if condition 6 holds, we obtain

$$\boldsymbol{z}(t)=\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}\sum_{m=1}^{q}\exp\{-(t-\tau)\eta_m\}\,[\boldsymbol{r}^m,A^0\tilde{\boldsymbol{z}}(\tau)]\,\boldsymbol{r}^m \tag{8}$$

from (3.6.10). Here η_m, $\boldsymbol{r}^m$, and A^0 are constant on a ray and η_m is given by (3.2.11). If condition 7 is also satisfied (3.6.11) yields

$$\boldsymbol{z}(t)=\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}\exp\{-(t-\tau)\eta\}\,\tilde{\boldsymbol{z}}(\tau). \tag{9}$$

In (8) and (9) $\tilde{\boldsymbol{z}}$ is again given by (5). If $\nu=0$ (8) and (9) reduce to (3.6.10, 11).

From (3.4.6) we see that $j(t)$ and $\tilde{j}(\tau)$ have the same sign provided $\tau<t<\tau'$, where $t=\tau'(\boldsymbol{\Gamma})$ is the location of the next caustic point on the ray $\boldsymbol{X}(t;\boldsymbol{\Gamma})$. For t outside this interval $\tilde{j}(\tau)/j(t)$ may be negative. We assume then that (1) remains valid with $\tilde{j}/j$ replaced by $|\tilde{j}/j|$. We also assume that the other formulas of this section may be interpreted according to the "phase-shift rule" of appendix F.

It is interesting to note that for homogenous media ω, $\boldsymbol{K}$, $\boldsymbol{r}^m$, η_m and A^0 are constant on each ray, while $\beta=0$. Then it is easy to demonstrate that (3) reduces to (9). (The demonstration follows from (3.6.1) which implies that $\tau_2=0$, and from (6) which implies that β_1 and β_2 are constant.) It is also easy to show that (7) reduces to (9) if condition 6 is satisfied, for in this case the theorem of appendix A shows that τ_0 is real and (3.8.9) implies that $\zeta=0$.

3.11. Asymptotic Solutions

Equations (3, 4, 6), (7), (8), and (9) of the preceding section provide formulas, valid under different conditions, for the amplitude function $\boldsymbol{z}(t;\boldsymbol{\Gamma})$ in terms of $\tilde{\boldsymbol{z}}(\tau;\boldsymbol{\Gamma})$. In addition (3.3.14) provides a formula for $s(t;\boldsymbol{\Gamma})$ in terms of the value

$s_0(\Gamma)$ of s on $\mathcal{M}$. By inserting these results into (3.1.5), we obtain

$$u \sim e^{i\lambda s(t;\Gamma)} z(t;\Gamma) \tag{1}$$

for the leading term of the asymptotic expansion of u. This equation, and the equation

$$X = X(t;\Gamma), \tag{2}$$

which is obtained by solving the ray equations, provide a parametric representation of $u(t, X)$ with n parameters $\Gamma = (\gamma_1, \ldots, \gamma_n)$. For a given value of (t, X) (2) may have zero, one or more solutions Γ. It is understood that for each (t, X) (1) is to be *summed* over all values of Γ which satify (2). (In geometric terms, we sum over all rays of the n-parameter family which pass through a given space-time point.)

We have yet to answer a very important question: How do we determine the functions $s_0(\Gamma)$ and $\tilde{z}(\tau;\Gamma)$ which we need to determine s and z? It is possible to give at least a partial answer here.

Obviously s_0 and $\tilde{z}$ must be determined from the *data* of a suitable problem for our system of equations (2.1.26, 27). Often these data consist of the values of the source function f. In problems involving boundaries, the boundary values also provide data. In initial value problems, which are appropriate for systems such as those described in section 2.3, the initial conditions also are part of the data of the problem.

In some cases, the values of s_0 and $\tilde{z}$ can be determined directly from the data. This is true, *e.g.* for initial-value problems with "oscillatory initial data" discussed in [*1, 4, 9, 10*].

In other cases, such as those which we shall discuss in chapter 5, $\mathcal{M}$ is of dimension $r<n$ hence consists of caustic points. The values of s_0 on $\mathcal{M}$ can be deduced from the data of the problem, but z is infinite on $\mathcal{M}$, and the values of $\tilde{z}$ must be obtained by an *indirect method.* In that method, we complete the construction of the solution (1, 2) with the function $\tilde{z}(\tau;\Gamma)$ unspecified. Then we specialize our problem to one with constant coefficients, *i.e.* with A^ν and $\tilde{\mathscr{E}}(\omega)$ independent of X, in unbounded X-space. (The constant values chosen for A^ν and $\tilde{\mathscr{E}}(\omega)$ are the values of $A^\nu(X)$ and $\tilde{\mathscr{E}}(\omega, X)$ at the point $X = X(\tau, \Gamma)$.) Thus for each different value of $X(\tau;\Gamma)$ we obtain a problem with constant coefficients. This *canonical problem** is, however, much simpler than the original problem; and it can be solved exactly. (This is done in chapter 4.) Then the asymptotic expansion of the solution of the canonical problem is found to be identical to the result obtained above (and specialized to constant coefficients), except that $\tilde{z}(\tau;\Gamma)$ is now given explicitly. If we assume that $\tilde{z}$ depends only on the local value of the coefficients, the $\tilde{z}$ obtained for the canonical problem can be used to complete the construction of the asymptotic solution of the original problem. The details of the indirect method are best understood by examining it in specific applications such as those of chapter 5 and [*1, 2, 4, 5, 7, 10, 11, 12*].

* The term "canonical problem", and the idea of the "indirect method" were introduced by J. B. KELLER in studying asymptotic solutions of the reduced wave equation and the phenomenon of diffraction [*6, 7, 12*].

4. Exact Solution of the System of Equations with Constant Coefficients

4.1. The fourier integral representation

In chapter 3 we presented a method for obtaining asymptotic solutions of the system of equations (2.1.26, 27). With constant coefficients that system becomes

$$\boldsymbol{v}_t+\sum_{\nu=1}^{n} A^{\nu}\boldsymbol{u}_{x_\nu}=\boldsymbol{f}(t,X), \tag{1}$$

$$\boldsymbol{v}(t,X)=\int F(\tau)\,\boldsymbol{u}(t-\tau,X)\,d\tau. \tag{2}$$

Here $\boldsymbol{u}$ and $\boldsymbol{v}$ are column vectors of dimension m, $X=(x_1,\ldots,x_n)$, and

$$F(t)=\frac{\lambda}{2\pi}\int e^{-i\lambda\omega t}\tilde{\mathscr{E}}(\omega)\,d\omega. \tag{3}$$

The constant $m\times m$ matrices A^{ν} are hermitian, and the $m\times m$ matrix $\tilde{\mathscr{E}}(\omega)$ satisfies conditions 1 – 5 of appendix H. As explained in section 2.1, we consider the following *problem* for (1, 2): The source function $\boldsymbol{f}$ is identically zero for $t<0$. It is required to find a function $\boldsymbol{u}(t, X)$ which satisfies (1, 2) and the "initial condition"

$$\boldsymbol{u}(t,X)\equiv 0 \quad \text{for} \quad t<0. \tag{4}$$

It can be shown that the leading term of the asymptotic expansion of the solution $\boldsymbol{u}$ of this problem is determined by the first two terms of the expansion

$$\tilde{\mathscr{E}}(\omega;\lambda)=\mathscr{E}(\omega)-(i\lambda)^{-1}\mathscr{D}(\omega)+O(\lambda^{-2}) \tag{5}$$

of $\tilde{\mathscr{E}}$. The "dissipation term" $\mathscr{D}(\omega)$ introduces considerable difficulties into the exact solution of the problem. Since the modifications of the asymptotic solution produced by this term can be obtained by the method of chapter 3, we shall take $\mathscr{D}\equiv 0$ in this chapter. Therefore we may replace $\tilde{\mathscr{E}}$ by $\mathscr{E}$ in (3)

In order to solve (1, 2, 4) we set $\boldsymbol{K}=(k_1,\ldots,k_n)$ and introduce the fourier transform

$$\hat{\boldsymbol{u}}(\omega,\boldsymbol{K})=\int \exp\{-i\lambda(k_\nu x_\nu-\omega t)\}\,\boldsymbol{u}(t,X)\,dt\,dX. \tag{6}$$

Then

$$\boldsymbol{u}(t,X)=\left(\frac{\lambda}{2\pi}\right)^{n+1}\int \exp\{i\lambda(k_\nu x_\nu-\omega t)\}\,\hat{\boldsymbol{u}}(\omega,\boldsymbol{K})\,d\omega\,d\boldsymbol{K}. \tag{7}$$

Similarly we introduce the fourier transforms $\hat{\boldsymbol{v}}(\omega, \boldsymbol{K})$ and $\hat{f}(\omega, \boldsymbol{K})$. From (3) we see that $\tilde{\mathscr{E}}(\omega)=\mathscr{E}(\omega)$ is the one-dimensional fourier transform of $F(t)$. Because of the convolution integral relation (2), it is easy to show that

$$\hat{\boldsymbol{v}}(\omega,\boldsymbol{K})=\mathscr{E}(\omega)\,\hat{\boldsymbol{u}}(\omega,\boldsymbol{K}). \tag{8}$$

From (1), we find that

$$-i\lambda\omega\,\hat{\boldsymbol{v}}+i\lambda k_\nu A^{\nu}\hat{\boldsymbol{u}}=\hat{\boldsymbol{f}}, \tag{9}$$

hence from (8),

$$G\,\hat{\boldsymbol{u}}=(i\lambda)^{-1}\hat{\boldsymbol{f}}, \tag{10}$$

where

$$G = k_\nu A^\nu - \omega \mathscr{E}(\omega) = G(\omega, \boldsymbol{K}). \tag{11}$$

Thus, from (7) we obtain the representation

$$\boldsymbol{u}(t, \boldsymbol{X}) = (i\lambda)^{-1} \left(\frac{\lambda}{2\pi}\right)^{n+1} \int \exp\{i\lambda(k_\nu x_\nu - \omega t)\}\, G^{-1}(\omega, \boldsymbol{K})\, \hat{\boldsymbol{f}}(\omega, \boldsymbol{K})\, d\omega\, d\boldsymbol{K}. \tag{12}$$

We assume that $\boldsymbol{f}(t, \boldsymbol{X})$ is sufficiently well-behaved as $|\boldsymbol{X}| \to \infty$ to justify the interchanges of integrations we have performed. For the behavior of $\boldsymbol{f}$ as $t \to \infty$ we require only that each component f_ν of the column-vector $\boldsymbol{f}$ is such that for all $\boldsymbol{X}$

$$\int_0^\infty |f_\nu(t, \boldsymbol{X})|\, e^{-\lambda \alpha_0 t} dt < \infty \tag{13}$$

for some $\alpha_0 \geqq 0$. Then $\hat{\boldsymbol{f}}(\omega, \boldsymbol{K})$ is an analytic function of ω in $\operatorname{Im}\omega > \alpha_0$.

In (12) it is understood that the variables $k_1, \ldots, k_n$ are integrated over the real axis from $-\infty$ to $+\infty$. The variable ω is integrated over a countour C parallel to the real axis, and lying in the region $\operatorname{Im}\omega > \alpha_0$. From condition 4 we see that $\mathscr{E}$ and hence G are analytic in that region, and it follows from condition 5 that $G^{-1}(\omega)$ is also analytic there. If we now insert (12) in (1, 2) we obtain

$$\boldsymbol{v}_t + A^\nu \boldsymbol{u}_{x_\nu} = \left(\frac{\lambda}{2\pi}\right)^{n+1} \int d\boldsymbol{K} \int_C d\omega \exp\{i\lambda(k_\nu x_\nu - \omega t)\}\, \hat{\boldsymbol{f}}(\omega, \boldsymbol{K}) = \boldsymbol{f}(t, \boldsymbol{X}), \tag{14}$$

by standard theorems on fourier integrals. Thus equations (1) and (2) are satisfied.

Since $\boldsymbol{f} \equiv 0$ for $t < 0$,

$$\hat{\boldsymbol{f}}(\omega, \boldsymbol{K}) = \int d\xi \int_0^\infty d\tau \exp\{-i\lambda(k_\nu \xi_\nu - \omega\tau)\}\, \boldsymbol{f}(\tau, \xi)\, d\tau\, d\xi; \quad \operatorname{Im}\omega > \alpha_0. \tag{15}$$

We now insert (15) in (12). This yields

$$\boldsymbol{u}(t, \boldsymbol{X}) = \left(\frac{\lambda}{2\pi}\right)^n \int d\boldsymbol{K}\, d\xi \exp\{i\lambda[k_\nu(x_\nu - \xi_\nu)]\} \int_0^\infty d\tau\, \mathscr{T}(t-\tau, \boldsymbol{K})\, \boldsymbol{f}(\tau, \xi), \tag{16}$$

where

$$\mathscr{T}(\sigma, \boldsymbol{K}) = \frac{1}{2\pi i} \int_C \exp\{-i\lambda\omega\sigma\}\, G^{-1}(\omega, \boldsymbol{K})\, d\omega. \tag{17}$$

For $\sigma < 0$ we may close the contour C by an infinite semi-circle in the upper half of the ω-plane*. Since G^{-1} is analytic in $\operatorname{Im}\omega > \alpha_0$, we conclude that

$$\mathscr{T} = 0 \quad \text{for} \quad \sigma < 0. \tag{18}$$

From (16) and (18) we see that $\boldsymbol{u} \equiv 0$ for $t < 0$, *i.e.*, $\boldsymbol{u}$ is the required solution of the problem (1, 2, 4). We also see that for $t > 0$

$$\boldsymbol{u}(t, \boldsymbol{X}) = \left(\frac{\lambda}{2\pi}\right)^n \int d\boldsymbol{K}\, d\xi \exp\{i\lambda[k_\nu(x_\nu - \xi_\nu)]\} \int_0^t d\tau\, \mathscr{T}(t-\tau, \boldsymbol{K})\, \boldsymbol{f}(\tau, \xi). \tag{19}$$

* From condition 4 we see that, uniformly with respect to $\arg\omega$ in $0 \leqq \arg\omega \leqq \pi$, $\det G = O(\omega^m)$, as $|\omega| \to \infty$, while the cofactors of G are of order ω^{m-1}. It follows that $G^{-1}(\omega) \to 0$ as $|\omega| \to \infty$, and the integral over the infinite semi-circle vanishes, by Jordan's Lemma, [15].

In section 4.3 we shall evaluate the integral (17) asymptotically by obtaining residue contributions from the singularities of $G^{-1}(\omega)$. It is clear that, for $\mathrm{Im}\,\omega > 2\beta_0$ ($\beta_0 < 0$), these singularities occur either at the zeros of $\det G(\omega)$ or at the poles of $\mathscr{E}(\omega)$. We shall demonstrate that the poles of $\mathscr{E}(\omega)$, which (by condition 4) are all real, do not produce singularities of $G^{-1}(\omega)$: From condition 1 we note that $\mathscr{E}(\omega)$ and hence $G(\omega)$ is hermitian for real ω. Therefore there exists a unitary matrix $U(\omega)$ such that $G = U\Gamma U^*$ and $\Gamma(\omega) = (\gamma_j \delta_{ij})$ is a diagonal matrix. The reciprocals $1/\gamma_j(\omega)$ are bounded in a neighborhood of each pole. Since the elements of a unitary matrix are bounded in absolute value by unity, $G^{-1} = U\Gamma^{-1}U^*$ has singularities, in $\mathrm{Im}\,\omega > 2\beta_0$ only at the (real) zeros of $\det G(\omega)$.

4.2. The eigenvalue problem for G

In order to obtain the residue evaluation of (17), we shall introduce the representation of G^{-1} in terms of the eigenvectors and eigenvalues of G.

For real ω and $\boldsymbol{K}$, $G = k_\nu A^\nu - \omega\,\mathscr{E}(\omega)$ is hermitian, therefore posesses a complete set of mutually orthogonal eigenvectors $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^m$ such that

$$G\,\boldsymbol{r}^j = \gamma^j \boldsymbol{r}^j; \qquad (\boldsymbol{r}^i, \boldsymbol{r}^j) = 0; \qquad i, j = 1, \ldots, m. \tag{1}$$

Let $\gamma(\omega) = \gamma(\omega, \boldsymbol{K}) = \gamma^1 = \cdots = \gamma^q$ be an eigenvalue of multiplicity q. By condition 3 of appendix H, q is independent of ω. By differentiating (1) with respect to ω we find, for $j = 1, \ldots, q$, that

$$-A^0 \boldsymbol{r}^j + G\,\boldsymbol{r}^j_\omega = \gamma_\omega \boldsymbol{r}^j + \gamma\,\boldsymbol{r}^j_\omega, \tag{2}$$

where

$$A^0 = -G_\omega = (\omega\,\mathscr{E})_\omega. \tag{3}$$

Then, since $(\boldsymbol{r}^i, G\,\boldsymbol{r}^j_\omega) = (G\,\boldsymbol{r}^i, \boldsymbol{r}^j_\omega) = \gamma(\boldsymbol{r}^i, \boldsymbol{r}^j_\omega)$, we see from (2) that

$$(\boldsymbol{r}^i, A^0\,\boldsymbol{r}^j) = -\gamma_\omega(\boldsymbol{r}^i, \boldsymbol{r}^j); \qquad i, j = 1, \ldots, q. \tag{4}$$

Since, by condition 2, A^0 is positive-definite, we may normalize the eigenvectors by the condition $(\boldsymbol{r}^j, A^0\,\boldsymbol{r}^j) = 1$. Then (4) yields

$$-\gamma_\omega(\boldsymbol{r}^i, \boldsymbol{r}^j) = (\boldsymbol{r}^i, A^0\,\boldsymbol{r}^j) = \delta_{ij}; \qquad i, j = 1, \ldots, q, \tag{5}$$

and

$$\gamma_\omega = -[(\boldsymbol{r}^j, \boldsymbol{r}^j)]^{-1}; \qquad j = 1, \ldots, q. \tag{6}$$

Of course, if $\omega = h(\boldsymbol{K})$ is a zero of $\gamma(\omega)$, then, at $\omega = h$,

$$G\,\boldsymbol{r}^j = 0; \qquad j = 1, \ldots, q. \tag{7}$$

4.3. Residue evaluation of the integral representation

For an arbitrary column vector $\boldsymbol{f}$, and for real ω, we may introduce the expansion of $\boldsymbol{f}$ in terms of the complete set of eigenvectors of the hermitian matrix $G(\omega)$:

$$\boldsymbol{f} = \sum_{j=1}^{m} b_j(\omega)\,\boldsymbol{r}^j(\omega). \tag{1}$$

Then

$$G^{-1}(\omega) f = \sum_{j=1}^{m} [b_j(\omega)/\gamma^j(\omega)]\, \boldsymbol{r}^j(\omega). \tag{2}$$

At least one eigenvalue γ^j vanishes at each real root $\omega = h$ of $\det G = 0$. We order the eigenvalues so that $\gamma^1 = \cdots = \gamma^q = \gamma$ vanishes at $\omega = h$. From (4.2.6) we see that $\gamma_\omega(h) \neq 0$; *i.e.*, $\gamma(\omega)$ has a simple zero at $\omega = h$.

For $\sigma > 0$ we shift the contour C in (4.1.17) to the line $\operatorname{Im} \omega = \beta_0 < 0$ below the real axis. This yields residue contributions from the singularities of $G^{-1}(\omega)$, which, as we have seen at the end of section 4.1, occur only at the (real) zeros of $\det G(\omega)$. The integral over the new contour is a remainder of order $e^{\lambda \sigma \beta_0}$, which, since $\beta_0 < 0$, we neglect* for $\lambda \to \infty$. Since the contours which yield the residue terms encircle the zeros in the clock-wise direction, we see from (4.1.17) and (2) that the residue contribution from the point $\omega = h$ to $\mathscr{T}(\sigma, \boldsymbol{K}) f$ is given by

$$-\exp\{-i\lambda h \sigma\} \sum_{j=1}^{q} [b_j(h)/\gamma_\omega(h)]\, \boldsymbol{r}^j(h). \tag{3}$$

But, from (1) and (4.2.1, 6),

$$(\boldsymbol{r}^j, f) = b_j(\boldsymbol{r}^j, \boldsymbol{r}^j) = -b_j/\gamma_\omega; \tag{4}$$

hence, summing over the real roots $\omega = h(\boldsymbol{K})$ of $\det G$, we find that

$$\mathscr{T}(\sigma, \boldsymbol{K}) f \sim \sum_{\omega = h} \exp\{-i\lambda \omega \sigma\} \sum_{j=1}^{q} [\boldsymbol{r}^j(\omega), f]\, \boldsymbol{r}^j(\omega). \tag{5}$$

We now insert (5) in (4.1.19). The result is, for $t > 0$,

$$\begin{aligned} \boldsymbol{u}(t, X) \sim \left(\frac{\lambda}{2\pi}\right)^n \int d\boldsymbol{K}\, d\xi \int_0^t d\tau \times \\ \times \sum_{\omega = h(\boldsymbol{K})} \exp\{i\lambda[k_\nu(x_\nu - \xi_\nu) - \omega(t-\tau)]\} \sum_{j=1}^{q} [\boldsymbol{r}^j, f(\tau, \xi)]\, \boldsymbol{r}^j. \end{aligned} \tag{6}$$

Here, for each $\boldsymbol{K}$, we sum over the real roots $\omega = h(\boldsymbol{K})$ of $\det G(\omega, \boldsymbol{K}) = 0$, and then over $j = 1, \ldots, q$; where q is the nullity of $G(h, \boldsymbol{K})$ and $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^q$ are the null eigenvectors of $G(h, \boldsymbol{K})$. From section (4.2) we see that they satisfy the equations

$$G(h, \boldsymbol{K})\, \boldsymbol{r}^j = 0; \qquad [\boldsymbol{r}^i, A^0(h)\, \boldsymbol{r}^j] = \delta_{ij}; \qquad i, j = 1, \ldots, q. \tag{7}$$

The eigenvectors are uniquely determined by (7), up to a unitary transformation (of the form $\hat{\boldsymbol{r}}^i = \sum_{j=1}^{q} \alpha_{ij}\, \boldsymbol{r}^j$, where (α_{ij}) is a unitary matrix) which leaves the value of (6) unchanged. Since equations (7) are identical to (3.2.3, 4), we may, for the case of constant coefficients (and $\mathscr{D} = 0$), identify the eigenvectors of chapter 3 and this chapter. For that reason we have used the same notation in both chapters.

* From (4.1.19) we note that $\sigma = t - \tau$ lies in the interval $0 \leqq \sigma \leqq t$. Therefore, in a neighborhood of $\sigma = 0$, the neglect of the remainder would be questionable. However, we shall be interested only in points (t, X) outside of the "source region", *i.e.* the support of the function f. Because of the finite time of propagation, $\boldsymbol{u}$ at such points cannot depend on $f(\tau, \xi)$ for values of τ in a neighborhood of $\tau = t$. Hence we may delete a neighborhood of $\sigma = 0$, and the neglect of the remainder is justified.

Our asyptotic solution (6) of the problem (4.1.1, 2, 4) is the main result of this chapter. For various choices of the source function $f(t, X)$ (which may depend on λ), the integrals with respect to K, ξ, and τ in (6) may be evaluated asyptotically by the method of stationary phase, leading to formulas for u which are free of integrals. Two representative examples are discussed in chapter 5. Others are treated in [2, 4, 5].

5. Radiation from Sources

In this chapter we consider the problem

$$v_t + \sum_{\nu=1}^{n} A^\nu u_{x_\nu} = f(t, X), \quad v(t, X) = \int F(\tau, X)\, u(t-\tau, X)\, d\tau; \tag{1}$$

$$u(t, X) \equiv 0 \quad \text{for} \quad t<0. \tag{2}$$

Here $f(t, X) \equiv 0$ for $t<0$ and

$$F(t, X) = \frac{\lambda}{2\pi} \int e^{-i\lambda\omega t}\, \tilde{\mathscr{E}}(\omega, X)\, d\omega. \tag{3}$$

The matrix $\tilde{\mathscr{E}}$ and the hermitian matrices A^ν are smooth functions of X in all of X-space, and $\tilde{\mathscr{E}}$ satisfies conditions 1–6. The source function f may depend also on λ. It is possible to obtain the leading term of the asymptotic expansion for $\lambda \to \infty$ of the solution $u(t, X)$ of (1, 2) for a variety of types of source functions f. In order to illustrate our theory, and in particular to demonstrate the "indirect method" outlined in section 3.11, we shall consider two representative examples in this chapter. Others are discussed in [1, 2, 4, 5, 10].

5.1. Rapidly varying source

The source function f will be called "rapidly varying" if it is of the form

$$f(t, X) = \lambda^{n+1}\, g[\lambda(t-\tau), \lambda(X-X_0)] \tag{4}$$

where τ is a positive constant and X_0 is a constant vector. An important example of (4) is obtained by setting

$$g(t, X) = \delta(t)\, \delta(x_1) \ldots \delta(x_n)\, g_0 = \delta(t)\, \delta(X)\, g_0, \tag{5}$$

where g_0 is any constant vector. Then, since $\lambda\, \delta(\lambda x) = \delta(x)$, (4) becomes

$$f(t, X) = \delta(t-\tau)\, \delta(X-X_0)\, g_0. \tag{6}$$

In general we shall assume that the function $g(t, X)$ has "compact support", *i.e.*, vanishes outside of a bounded region in t, X-space*. Then as $\lambda \to \infty$ the support of $f(t, X)$ shrinks to the manifold $\mathscr{M}$ consisting of the single point $(t, X) = (\tau, X_0)$. We now proceed as outlined in section 3.3: Since $\mathscr{M}$ consists of a single point, s has some constant value s_0 on $\mathscr{M}$. This constant value may be absorbed into the undetermined function $\tilde{z}$ therefore it is convenient to take $s_0 = 0$. Since the dimension r of $\mathscr{M}$ is zero, there are no conditions (3.3.7) and we may choose

* This assumption can be weakened. It is sufficient to assume that g decays sufficiently rapidly at infinity for the integral (31) to exist.

as our parameters

$$\gamma_j = k_{0j}; \qquad j=1,\dots,n. \tag{7}$$

This yields the initial conditions

$$\boldsymbol{X}=\boldsymbol{X}_0(\boldsymbol{\Gamma})\equiv\boldsymbol{X}_0, \quad t=\tau(\boldsymbol{\Gamma})\equiv\tau, \quad \boldsymbol{K}=\boldsymbol{K}_0(\boldsymbol{\Gamma})\equiv\boldsymbol{\Gamma}, \quad s=s_0(\boldsymbol{\Gamma})\equiv 0. \tag{8}$$

The initial value

$$\omega=\omega_0(\boldsymbol{\Gamma}) \tag{9}$$

is determined by the dispersion relation (3.2.1) or (3.2.2) in the form

$$\det G=\det\left[\gamma_\nu A^\nu(X_0)-\omega_0\,\mathscr{E}(\omega_0, X_0)\right]=0 \tag{10}$$

or

$$\omega_0(\boldsymbol{\Gamma})=h[\boldsymbol{\Gamma}, \boldsymbol{X}_0]. \tag{11}$$

(11) is, in general, a multiple-valued function of the vector $\boldsymbol{\Gamma}$ which may be thought of as a point in n-dimensional space, E^n. We may make ω_0 a single-valued function of $\boldsymbol{\Gamma}$, by letting $\boldsymbol{\Gamma}$ be a point in a parameter space $\mathscr{P}$ consisting of several duplicates of E^n, one for each value of (11)*. By this simple device we conform to the presentation of chapter 3, in which $\boldsymbol{X}_0$, τ, $\boldsymbol{K}_0$, s_0, and ω_0 were assumed to be single-valued functions of $\boldsymbol{\Gamma}$.

From (3.3.14) s is given by

$$s(t;\boldsymbol{\Gamma})=\int_\tau^t \ell(t';\boldsymbol{\Gamma})\,dt' \tag{12}$$

where

$$\ell(t;\boldsymbol{\Gamma})=k_\nu(t;\boldsymbol{\Gamma})\,g_\nu(t;\boldsymbol{\Gamma})-\omega_0(\boldsymbol{\Gamma}) \tag{13}$$

and

$$g_\nu(t;\boldsymbol{\Gamma})=h_{k_\nu}[\boldsymbol{K}(t;\boldsymbol{\Gamma}), \boldsymbol{X}(t;\boldsymbol{\Gamma})]; \qquad \nu=1,\dots,n. \tag{14}$$

Here $\boldsymbol{X}(t;\boldsymbol{\Gamma})$, $\boldsymbol{K}(t;\boldsymbol{\Gamma})$ is the solution of the ray equations (3.3.1) with initial conditions (8). From these equations we find, as in section 3.4, that

$$x_\nu(t;\boldsymbol{\Gamma})=x_{0\nu}+(t-\tau)\,h_{k_\nu}(\boldsymbol{\Gamma}, X_0)+O\left[(t-\tau)^2\right], \tag{15}$$

$$\frac{\partial x_\nu}{\partial \gamma_\mu}=(t-\tau)\,h_{k_\nu k_\mu}(\boldsymbol{\Gamma}, X_0)+O\left[(t-\tau)^2\right], \tag{16}$$

and

$$j(t;\boldsymbol{\Gamma})=(t-\tau)^n\det\left[h_{k_\nu k_\mu}(\boldsymbol{\Gamma}, X_0)\right]+O\left[(t-\tau)^{n+1}\right]. \tag{17}$$

It follows that $\nu=n$ and

$$\tilde{j}(\tau;\boldsymbol{\Gamma})=\det\left[h_{k_\nu k_\mu}(\boldsymbol{\Gamma}, X_0)\right]. \tag{18}$$

We recall that, in chapter 3, we were able to solve the transport equations if the nullity q of G is 1 or 2. If $q=1$, $z(t)=z(t;\boldsymbol{\Gamma})$ is given by (3.10.7). If $q=2$ then $z(t)$ is given by (3.10.3, 4, 6) provided conditions 6, 7, 8, 9 are satisfied. In either case it is necessary to determine $\tilde{z}(\tau;\boldsymbol{\Gamma})$ which appears in these formulas for z. Let us suppose that we have determined z. Then with s given by (12), the

* To be more precise each space E^n is restricted to those values of $\boldsymbol{\Gamma}$ for which the corresponding value of (11) is real.

asymptotic solution of (1, 2, 4) is given parametrically by

$$u \sim e^{i\lambda s(t;\Gamma)} z(t;\Gamma); \tag{19}$$

$$X = X(t;\Gamma). \tag{20}$$

(As in section (3.11), for each point (t, X), (19) is to be summed over all values of Γ which satify (20), *i.e.* over all rays through (t, X).)

To determine $\tilde{z}$ we use the "indirect method" outlined in section 3.11. We first consider the "canonical problem" with constant coefficients, $A^\nu = A^\nu(X_0)$ and $\tilde{\mathscr{E}}(\omega) = \tilde{\mathscr{E}}(\omega, X_0)$. For this problem, we find (as in section 3.6) that the solution of the ray equations is given by

$$k_\nu(t;\Gamma) = \gamma_\nu; \quad x_\nu(t;\Gamma) = x_{0\nu} + (t-\tau)\, g_\nu; \quad g_\nu = h_{k_\nu}[\Gamma; X_0], \tag{21}$$

and (13) becomes

$$\ell(t;\Gamma) = \gamma_\nu\, g_\nu - \omega, \tag{22}$$

with ω given by (11). Now (12) takes the form

$$s(t;\Gamma) = (t-\tau)\,\ell = (t-\tau)[\gamma_\nu\, g_\nu - \omega]. \tag{23}$$

As pointed out at the end of section 3.10, z is now given by (3.10.9), *i.e.*

$$z(t) = \left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}} \exp\{-(t-\tau)\,\eta\}\, \tilde{z}(\tau). \tag{24}$$

We see from (21) that (15–18) now hold exactly with no remainder terms, hence

$$\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}} = (t-\tau)^{-\frac{1}{2}n}. \tag{25}$$

Thus (19) and (20) become

$$u \sim \exp\{i\lambda[(t-\tau)(\gamma_\nu\, g_\nu - \omega)] - (t-\tau)\,\eta]\}\, (t-\tau)^{-\frac{1}{2}n}\, \tilde{z}; \tag{26}$$

$$x_\nu = x_{0\nu} + (t-\tau)\, g_\nu. \tag{27}$$

The next step in the indirect method is to compare (26, 27) with the asymptotic expansion of the exact solution of the same problem*. To do this, we first insert (4) in (4.3.6). Then

$$\begin{aligned} u \sim \lambda^{n+1}\left(\frac{\lambda}{2\pi}\right)^n \int dY \int d\Gamma \int_0^t d\sigma \sum_{\omega = h} \exp\{i\lambda[\gamma_\nu(x_\nu - y_\nu) - \omega(t-\sigma)]\} \times \\ \times \sum_j \{r^j(\Gamma), g[\lambda(\sigma-\tau), \lambda(Y - X_0)]\}\, r^j(\Gamma); \qquad t > 0. \end{aligned} \tag{28}$$

Here we sum over every real zero $\omega = h(\Gamma; X_0)$ of $\det[\gamma_\nu\, A^\nu(X_0) - \omega\, \mathscr{E}(\omega, X_0)] = 0$, and then over $j = 1, 2$ if $q = 2$. If $q = 1$ we may omit the index j. We now introduce the transformation

$$t' = \lambda(\sigma - \tau), \quad X' = \lambda(Y - X_0), \quad dt' = \lambda\, d\sigma, \quad dX' = \lambda^n\, dY, \tag{29}$$

* As in chapter 4 we now take $\mathscr{D} \equiv 0$. Then, to make the comparison, we shall take $\eta \equiv 0$ in (26).

in (28). The result is

$$u \sim \left(\frac{\lambda}{2\pi}\right)^n \int d\Gamma \sum_{\omega=h} \exp\{i\lambda[\gamma_\nu(x_\nu - x_{\nu 0}) - h(t-\tau)]\} \sum_j [r^j(\Gamma), a(h,\Gamma)]\, r^j(\Gamma) \tag{30}$$

where

$$a(h,\Gamma) = \int dX' \int_{-\lambda\tau}^{\lambda(t-\tau)} dt' \exp\{i[ht' - \gamma_\nu x'_\nu]\}\, g(t', X') \sim$$

$$\sim \begin{cases} 0 & \text{for } t<\tau \\ \int dX' \int_{-\infty}^{\infty} dt' \exp\{i[ht' - \gamma_\nu x'_\nu]\}\, g(t', X') & \text{for } \tau<t \end{cases}. \tag{31}$$

We note that (the asymptotic form of) a is simply the fourier transform of g. It is independent of λ. For the special case (6) we see from (5) and (31) that $a(h, \Gamma) \sim g_0$.

We now apply the method of stationary phase* to obtain the asymptotic expansion of the integral (30): We introduce the function

$$\varphi(\Gamma) = \gamma_\nu(x_\nu - x_{\nu 0}) - h(t-\tau). \tag{32}$$

Then

$$\varphi_{\gamma_\nu} = x_\nu - x_{\nu 0} - (t-\tau)\, g_\nu; \qquad g_\nu = h_{k_\nu}(\Gamma; X_0). \tag{33}$$

Stationary points are determined by the condition that (33) vanish for $\nu = 1, \ldots, n$. At the stationary points,

$$x_\nu = x_{0\,\nu} + (t-\tau)\, g_\nu, \tag{34}$$

$$\varphi = (t-\tau)[\gamma_\nu g_\nu - \omega_0], \qquad \omega = h(\Gamma; X_0), \tag{35}$$

and

$$\varphi_{\gamma_\nu \gamma_\mu} = -(t-\tau)\, h_{k_\nu k_\mu}(\Gamma; X_0). \tag{36}$$

Therefore, for $\tau < t$ the stationary phase formula yields

$$u \sim \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}n} (t-\tau)^{\frac{1}{2}n} \sum_{\omega=h} [|\det(h_{k_\nu k_\mu})|]^{-\frac{1}{2}} \times$$
$$\times \exp\{i\lambda[(t-\tau)(\gamma_\nu g_\nu - \omega_0)] - i\pi/4\, \mathrm{sig}(h_{k_\nu k_\mu})\} \sum_j [r^j, a(h,\Gamma)]\, r^j. \tag{37}$$

Here $\mathrm{sig}(h_{k_\nu k_\mu})$ denotes the signature of the matrix $(h_{k_\nu k_\mu})$. At each point (t, X) we must sum over all Γ satisfying (34). Alternatively, if we adopt the extended definition of the parameter space introduced earlier in this section, we may omit the first summation sign in (37). It is then understood that (37) is to be summed over all Γ in $\mathscr{P}$ that satisfy (34), as well as over the index j.

We now compare (34, 37) with (26, 27) for $\eta = 0$. We find that they are identical provided

$$\tilde{z}(\tau; \Gamma) = \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}n} [|\det(h_{k_\nu k_\mu})|]^{-\frac{1}{2}} e^{-i\pi/4\, sig\,(h_{k_\nu k_\mu})} \sum_j [r^j, a(h,\Gamma)]\, r^j. \tag{38}$$

* A simple discussion of this method, in n dimensions, is given in [10].

Here

$$h=h(\Gamma, X_0) \quad \text{and} \quad h_{k_\nu k_\mu}=\frac{\partial^2}{\partial k_\nu \partial k_\mu} h(K, X_0)|_{K=\Gamma},$$

where $\omega=h(K, X_0)$ is the (multiple-valued) solution of $\det[k_\nu A^\nu(X_0)-\omega \mathscr{E}(\omega, X_0)]=0$. Furthermore, we see from (4.3.7) that the vectors $r^j=r^j(\Gamma)$ satisfy the equations

$$G r^j=0; \quad j=1,\ldots,q; \quad G=\gamma_\nu A^\nu(X_0)-h\mathscr{E}(h, X_0); \tag{39}$$

and

$$(r^i, A^0(h) r^j)=\delta_{ij}; \quad i,j=1,\ldots,q. \tag{40}$$

These equations determine the r^j except for a unitary transformation which leaves (38) invariant.

With $\tilde{z}$ given by (38) we may now, in principle, complete the solution of (1, 2, 4). Of course this requires the solution of the ray equations, which is not in general possible by analytic means. Under special assumptions about the functions $(A^\nu(X)$ and $\mathscr{E}(\omega, X)$ this is sometimes possible* and our method leads to explicit formulas for the asymptotic solutions. We shall not carry out the details of this procedure here for any problem with non-constant coefficients.

5.2. Oscillatory source

In this section we shall, for convenience, restrict our attention to three-dimensional space ($n=3$) and we shall assume that the dispersion relation is *isotropic*, i.e.

$$\omega=h(K, X)=h(k, X); \quad k=|K|. \tag{1}$$

We further assume that the multiple-valued function (1) has (for each X) a single-valued inverse

$$k=m(\omega)=m(\omega; X); \quad m'(\omega)\neq 0. \tag{2}$$

The function (2) is defined for ω in some subset $\Omega(X)$ of the real axis. For each X, $\Omega(X)$ consists of all (real) values which (1) assumes when K takes on all real values (or k takes on all non-negative real values). We shall see in chapter 7 that the above assumptions are suitable for electromagnetic waves in isotropic media.

We now consider source functions of the form

$$f(t, X)=\lambda^3 g[\lambda(X-X_0), t]\, e^{-i\lambda q_0(t)}; \quad g\equiv 0 \quad \text{for} \quad t<0. \tag{3}$$

If we take

$$g(X, t)=\delta(X) g_0(t), \tag{4}$$

then (3) becomes

$$f(t, X)=\delta(X-X_0) g_0(t) e^{-i\lambda q_0(t)}. \tag{5}$$

(5) represents an *oscillatory point source.* If g_0 is constant and $q_0(t)=\psi t$, the source is *time-harmonic* with *frequency* $\psi\lambda$. If g_0 is not constant but $q_0(t)=\psi t$, we may call it an *amplitude-modulated time-harmonic* source of frequency $\psi\lambda$.

* See, e.g., [1].

We assume that, in general, $g(X, t)$ has compact support* for each t. Then as $\lambda\to\infty$ the support of (3) shrinks to the one-dimensional manifold $\mathcal{M}$, defined parametrically (with parameter τ) by

$$X=X_0, \quad t=\tau; \quad \tau\geqq 0. \tag{6}$$

From (3) it is reasonable to assume that on $\mathcal{M}$, s is given by

$$s[\tau, X_0]=-q_0(\tau). \tag{7}$$

We shall verify this assumption shortly by comparison with the solution of the canonical problem. Differention of (7) yields

$$\omega=\dot{q}_0(\tau). \tag{8}$$

Proceeding as in section 3.3 we choose, as independent parameters, $\Gamma=(\tau, \alpha, \gamma)$, where α and γ are polar angles defined by

$$[\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma]=A_0=K_0/k_0; \qquad k_0=|K_0|. \tag{9}$$

The initial values of X, t, s, ω, K are then given as functions of Γ by (6), (7), (8), and

$$K_0=k_0 A_0; \quad k_0=m(\omega; X_0)=m[\dot{q}_0(\tau); X_0]. \tag{10}$$

Since $m(\omega)$ is single-valued, all the initial values are given as single-valued functions of Γ, as required in section 3.3, and the parameter space $\mathcal{P}$ consists of all values of Γ for which $\dot{q}_0(\tau)$ is in $\Omega(X_0)$, $\tau\geqq 0$, and the polar angles are in the intervals $0\leqq\alpha\leqq\pi$, $0\leqq\gamma\leqq 2\pi$.

We now denote by $X=X(t;\Gamma)$, $K=K(t;\Gamma)$ the solution of the ray equations (3.3.1). (For each value of $\Gamma=(\tau, \alpha, \gamma)$ we use, in the ray equations, that value of the multiple-valued function (1) for which $h[m(\dot{q}_0, X_0); X_0]=\dot{q}_0(\tau)$.) We note that, since ω is constant, (8) holds along every ray, and we set

$$A(t;\Gamma)=(a_1, a_2, a_3)=k^{-1}K(t;\Gamma)=\{m[\omega; X(t;\Gamma)]\}^{-1}K(t;\Gamma), \tag{11}$$

$$\begin{gathered} g_\nu=h_{k_\nu}=h'(k)\,a_\nu=[m'(\omega)]^{-1}a_\nu; \\ G=(g_1, g_2, g_3)=\{m'[\omega; X(t;\Gamma)]\}^{-1}A(t;\Gamma), \end{gathered} \tag{12}$$

and

$$\ell(t;\Gamma)=K\cdot G-\omega=\frac{m[\omega; X(t;\Gamma)]}{m'[\omega; X(t;\Gamma)]}-\omega. \tag{13}$$

Then

$$s(t;\Gamma)=-q_0(\tau)+\int_\tau^t \ell(t';\Gamma)\,dt'. \tag{14}$$

As in section 3.4 we see that the ray equations may be used to obtain the expansion

$$\begin{aligned} X(t;\Gamma)&=(x_1, x_2, x_3) \\ &=X_0+\frac{t-\tau}{m'[\dot{q}_0(\tau); X_0]}[\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma]+O[(t-\tau)^2]. \end{aligned} \tag{15}$$

* It is sufficient to assume that $g\to 0$, as $|X|\to\infty$, sufficiently rapidly so that the integral (30) converges.

By differentiating (15) we find that

$$\frac{\partial \boldsymbol{X}}{\partial \tau}=\left(\frac{\partial x_1}{\partial \tau}, \frac{\partial x_2}{\partial \tau}, \frac{\partial x_3}{\partial \tau}\right) = -\frac{[1+(t-\tau)\, m''\, \ddot{q}_0/m']}{m'}\,[\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma]+O(t-\tau), \tag{16}$$

$$\frac{\partial \boldsymbol{X}}{\partial \alpha}=\left(\frac{\partial x_1}{\partial \alpha}, \frac{\partial x_2}{\partial \alpha}, \frac{\partial x_3}{\partial \alpha}\right)=\frac{t-\tau}{m'}\,[-\sin\alpha, \cos\alpha\cos\gamma, \cos\alpha\sin\gamma]+O[(t-\tau)^2], \tag{17}$$

and

$$\frac{\partial \boldsymbol{X}}{\partial \gamma}=\left(\frac{\partial x_1}{\partial \gamma}, \frac{\partial x_2}{\partial \gamma}, \frac{\partial x_3}{\partial \gamma}\right)=\frac{t-\tau}{m'}\,[0, -\sin\alpha\sin\gamma, \sin\alpha\cos\gamma]+O[(t-\tau)^2]. \tag{18}$$

From these equations it is easy to compute the jacobian,

$$j(t;\boldsymbol{\Gamma})=\det\begin{pmatrix} \frac{\partial x_1}{\partial \tau} & \frac{\partial x_2}{\partial \tau} & \frac{\partial x_3}{\partial \tau} \\ \frac{\partial x_1}{\partial \alpha} & \frac{\partial x_2}{\partial \alpha} & \frac{\partial x_3}{\partial \alpha} \\ \frac{\partial x_1}{\partial \gamma} & \frac{\partial x_2}{\partial \gamma} & \frac{\partial x_3}{\partial \gamma} \end{pmatrix}. \tag{19}$$

The result is*

$$j(t;\boldsymbol{\Gamma})=\frac{-\sin\alpha}{(m')^3}\,(t-\tau)^2\,[1+(t-\tau)\, m''\, \ddot{q}_0/m']+O[(t-\tau)^3] \tag{20}$$

or

$$j(t;\boldsymbol{\Gamma})=\frac{-\sin\alpha}{(m')^3}\,(t-\tau)^2+O[(t-\tau)^3]. \tag{21}$$

It follows that $\nu=2$ and

$$\tilde{j}(\tau;\boldsymbol{\Gamma})=\frac{-\sin\alpha}{(m')^3}=\frac{-\sin\alpha}{\{m'[\dot{q}_0(\tau);\boldsymbol{X}_0]\}^3}. \tag{22}$$

Just as in section 5.1 we can now, in principle, complete the asymptotic solution of our problem if $q=1$, or if $q=2$ and conditions 6, 7, 8, 9 are satisfied; except that $\tilde{z}=\tilde{z}(\tau;\boldsymbol{\Gamma})$ is undetermined.

In order to determine $\tilde{z}$, we again consider the canonical problem. We set $A^\nu=A^\nu(\boldsymbol{X}_0)$ and $\tilde{\mathscr{E}}(\omega)=\tilde{\mathscr{E}}(\omega, \boldsymbol{X}_0)$. Then $m(\omega)=m(\omega;\boldsymbol{X}_0)$, and on each ray,

$$\omega=\dot{q}_0(\tau), \quad k=m(\omega)=m[\dot{q}_0(\tau)], \quad \boldsymbol{K}=k\boldsymbol{A}, \qquad \boldsymbol{A}=\boldsymbol{A}_0=[\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma]. \tag{23}$$

Furthermore

$$\boldsymbol{G}=\boldsymbol{A}/m'(\omega), \quad s=-q_0(\tau)+(t-\tau)\ell; \quad \ell=[m(\omega)/m'(\omega)]-\omega, \tag{24}$$

* In (16) and (20) the term $(t-\tau)\, m''\, \ddot{q}_0/m'$ could be omitted since it is of the same order as the remainder. We keep it to facilitate our later discussion of the problem with constant coefficients.

and the rays are given by

$$X=X_0+(t-\tau)A/m'(\omega). \tag{25}$$

We now find that (16–20) are valid for all $t>\tau$ with the remainder terms equal to zero. Hence

$$\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}=(t-\tau)^{-1}[1+(t-\tau)m''(\dot{q}_0)\ddot{q}_0/m'(\dot{q}_0)]^{-\frac{1}{2}}. \tag{26}$$

Now (3.10.9) yields

$$z(t)=(t-\tau)^{-1}[1+(t-\tau)m''(\dot{q}_0)\ddot{q}_0/m'(\dot{q}_0)]^{-\frac{1}{2}}\exp\{-(t-\tau)\eta\}\,\tilde{z}(\tau) \tag{27}$$

and from (3.11.1, 2) we have

$$\begin{aligned} u\sim(t-\tau)^{-1}&[1+(t-\tau)m''(\dot{q}_0)\ddot{q}_0/m'(\dot{q}_0)]^{-\frac{1}{2}}\times\\ &\times\exp\left\{i\lambda\left[(t-\tau)\left(\frac{m(\dot{q}_0)}{m'(\dot{q}_0)}-\dot{q}_0\right)-q_0(\tau)\right]-(t-\tau)\eta\right\}\tilde{z}(\tau); \qquad (28)\\ &X=X_0+(t-\tau)A/m'(\dot{q}_0). \end{aligned}$$

We must now compare this result with the asymptotic expansion of the exact solution of the canonical problem. Therefore we insert (3) in (4.3.6). We then introduce the transformation $X'=\lambda(\xi-X_0)$, $dX'=\lambda^3 d\xi$, of the integration variables. Thus we obtain

$$\begin{aligned} u(t,X)\sim\left(\frac{\lambda}{2\pi}\right)^3\int dK\int_0^t d\tau\sum_{\omega=h}&\exp\{i\lambda[k_\nu(x_\nu-x_{\nu 0})-\omega(t-\tau)-q_0(\tau)]\}\times\\ &\times\sum_j[r^j(K),a(K,\tau)]\,r^j(K), \end{aligned} \tag{29}$$

where

$$a(K,\tau)=\int e^{-ik_\nu x'_\nu}g(X',\tau)\,dX'. \tag{30}$$

In (29) we sum over all values of (1). If $q=1$ we omit the index j. If $q=2$ we sum over $j=1,2$. In the special case (4) we find that $a(K,\tau)=g_0(\tau)$. We now apply the method of stationary phase [*10*] to the 4-fold integral (29): We introduce the function

$$\varphi=k_\nu(x_\nu-x_{\nu 0})-h(t-\tau)-q_0(\tau). \tag{31}$$

Stationary points are determined by the conditions

$$\begin{aligned} &\varphi_{k_\nu}=(x_\nu-x_{\nu 0})-(t-\tau)\,g_\nu=0; \qquad \nu=1,2,3\\ &(g_\nu=h_{k_\nu}=h'(k)\,a_\nu;\ a_\nu=k_\nu/k); \end{aligned} \tag{32}$$

and

$$\varphi_\tau=h-\dot{q}_0(\tau)=0. \tag{33}$$

At the stationary points,

$$\varphi=(t-\tau)[k_\nu g_\nu-h]-q_0(\tau)=(t-\tau)[k\,h'(k)-h]-q_0(\tau); \tag{34}$$

and the second derivatives of φ are given by

$$\varphi_{k_\nu\tau}=h_{k_\nu}=h'(k)\,a_\nu; \qquad \varphi_{\tau\tau}=-\ddot{q}_0(\tau); \tag{35}$$

and

$$\varphi_{k_\nu k_\mu}=-(t-\tau)\,h_{k_\nu k_\mu}=-(t-\tau)\left[h''(k)\,a_\nu a_\mu+\frac{h'(k)}{k}(\delta_{\nu\mu}-a_\nu a_\mu)\right]. \tag{36}$$

We now introduce the matrix

$$\Phi=\begin{pmatrix}\varphi_{k_1k_1} & \varphi_{k_1k_2} & \varphi_{k_1k_3} & \varphi_{k_1\tau}\\ \varphi_{k_2k_1} & \varphi_{k_2k_2} & \varphi_{k_2k_3} & \varphi_{k_2\tau}\\ \varphi_{k_3k_1} & \varphi_{k_3k_2} & \varphi_{k_3k_3} & \varphi_{k_3\tau}\\ \varphi_{\tau k_1} & \varphi_{\tau k_2} & \varphi_{\tau k_3} & \varphi_{\tau\tau}\end{pmatrix}. \tag{37}$$

By choosing a co-ordinate system in which $A=(a_1, a_2, a_3)=(1, 0, 0)$ (35) and (36) simplify greatly, and it becomes easy to evaluate the determinant of (37). We find that

$$\det\Phi=-\frac{(t-\tau)^2(h')^4}{k^2}\left[1-\frac{(t-\tau)h''\ddot{q}_0}{(h')^2}\right]. \tag{38}$$

In order to determine the signature of (Φ) we may find its eigenvalues. They are the solutions, x, of the equation $\det(\Phi-xI)=0$. This determinant is also easily evaluated, and the equation for x factors into the two equations

$$\left[(t-\tau)\frac{h'}{k}+x\right]^2=0, \tag{39}$$

and

$$[(t-\tau)h''+x][\ddot{q}_0+x]-(h')^2=0. \tag{40}$$

The two roots $x=-(t-\tau)h'/k$ of (39) vanish at $t=\tau$, and for $t>\tau$ $\operatorname{sgn}x=-\operatorname{sgn}h'=-\operatorname{sgn}m'[\dot{q}_0(\tau)]$. The two roots of (40) do not vanish at $t=\tau$ and are given by

$$2x=-\ddot{q}_0\pm\sqrt{(\ddot{q}_0)^2+4(h')^2}. \tag{41}$$

It is clear that one of these roots is positive and the other is negative. Hence in a neighborhood of $t=\tau(t>\tau)$; i.e., up to the next caustic point, $\operatorname{sig}\Phi=-2\operatorname{sgn}h'=-2\operatorname{sgn}m'(\dot{q}_0)$.

The stationary phase formula now yields for (29)

$$u\sim\frac{\lambda}{2\pi}\sum\left\{(t-\tau)^2\frac{(h')^4}{k^2}\left[1-\frac{(t-\tau)h''\ddot{q}_0}{(h')^2}\right]\right\}^{-\frac{1}{2}}\times$$
$$\times\exp\{i\lambda[(t-\tau)(kh'-h)-q_0]-i\pi/2\operatorname{sgn}h'\}[r^j(K), a(K,\tau)]\,r^j(K), \tag{42}$$
$$h=\dot{q}_0(\tau),\quad x_\nu=x_{\nu0}+(t-\tau)h'(k)a_\nu;\quad K=kA=k(a_1,a_2,a_3).$$

Since

$$h'(k)=\frac{1}{m'(\omega)}\quad\text{and}\quad h''(k)=-\frac{m''(\omega)}{(m')^3},$$

this becomes

$$u\sim\frac{\lambda}{2\pi}\sum(t-\tau)^{-1}m(\dot{q}_0)[m'(\dot{q}_0)]^2\left[1+(t-\tau)\frac{m''\ddot{q}_0}{m'}\right]^{-\frac{1}{2}}\times \tag{43}$$
$$\times\exp\left\{i\lambda\left[(t-\tau)\left(\frac{m}{m'}-\dot{q}_0\right)-q_0\right]-i\pi/2\operatorname{sgn}m'\right\}[r^j(K), a(K,\tau)]\,r^j(K);$$
$$K=m(\dot{q}_0)A;\quad X=X_0+(t-\tau)A/m'(\dot{q}_0).$$

In (42) $u(t, X)$ is represented parametrically with parameters τ, and $K=(k_1, k_2, k_3)$. These parameters are not independent, because they are related by the equation $h=\dot{q}_0(\tau)$. In (42) the indicated sum is to be taken over the values of (1) as well as over the index j (if $q=2$). In (43), $u(t, X)$ is represented parametrically by 3

independent parameters. These parameters are τ and two angles which determine the unit vector $\boldsymbol{A}$. They may be taken as in (23). In (43) the indicated sum is to be taken only over the index j (if $q=2$).

We may now compare (28) (with $\eta=0$) and (43). We see that they agree exactly if we set

$$\tilde{z}(\tau;\Gamma)=\frac{\lambda}{2\pi}m(\dot{q}_0)[m'(\dot{q}_0)]^2 e^{-\frac{1}{2}i\pi\,\mathrm{sgn}\,m'(\dot{q}_0)}\sum_j[\boldsymbol{r}^j(\boldsymbol{K}),\boldsymbol{a}(\boldsymbol{K},\tau)]\,\boldsymbol{r}^j(\boldsymbol{K}). \tag{44}$$

Here $\boldsymbol{K}=m[\dot{q}_0(\tau)]\boldsymbol{A}$, $\boldsymbol{A}=[\cos\alpha, \sin\alpha\cos\gamma, \sin\alpha\sin\gamma]$, and $\Gamma=(\tau,\alpha,\gamma)$. As in section 5.1, (44) may now be used to complete the asymptotic solution of the problem (5.1.1, 2), (5.2.3) for the case of non-constant coefficients.

It is interesting to note that, for the case of constant coefficients the jacobian has not only the zero of order 2 at $t=\tau$ but also a zero of order 1 at the caustic point $t=t_1>\tau$, determined by

$$1+(t_1-\tau)\,m''\,\ddot{q}_0/m'=1-\frac{(t_1-\tau)\,h''\,\ddot{q}_0}{(h')^2}=0. \tag{45}$$

Such a point exists if and only if $h''\,\ddot{q}_0>0$. Then for $t>t_1$ the signature of Φ changes. To find sig Φ we write (40) in the form

$$x^2+Bx+C=0; \quad B=(t-\tau)\,h''+\ddot{q}_0, \quad C=(t-\tau)\,h''\,\ddot{q}_0-(h')^2. \tag{46}$$

Then

$$2x=-B\pm\sqrt{B^2-4C}, \tag{47}$$

and $\mathrm{sgn}\,C=\mathrm{sgn}\,(t-t_1)$. Thus for $t<t_1$ the two solutions (47) have opposite signs, while for $t>t_1$ they both have the sign of $-B$. However (46) shows that $\mathrm{sgn}\,B=\mathrm{sgn}\,h''=\mathrm{sgn}\,\ddot{q}_0$, since $h''\,\ddot{q}_0>0$. It follows from the stationary phase formula that, for $t>t_1$, (43) must be multiplied by a factor

$$e^{i\frac{\pi}{4}[2\,\mathrm{sgn}\,(-B)]}=e^{-i\frac{\pi}{2}\mathrm{sgn}\,\ddot{q}_0}. \tag{48}$$

Our result (48) can also be obtained by applying the "phase-shift rule" of appendix F.

6. Dispersive and Non-dispersive Systems

6.1. Non-dispersive systems

We shall say that our general system of equations is *non-dispersive* if $\mathscr{E}=A(\boldsymbol{X})$ is independent of ω. If, furthermore, $\tilde{\mathscr{E}}=\mathscr{E}-(i\lambda)^{-1}\mathscr{D}$, where

$$\mathscr{E}(\omega,\boldsymbol{X})=A(\boldsymbol{X}), \quad \text{and} \quad \mathscr{D}(\omega,\boldsymbol{X})=C(\boldsymbol{X})/\omega, \tag{1}$$

then, from section 2.3, we see that the general system of equations (2.1.26, 27) becomes the symmetric-hyperbolic system

$$A\,\boldsymbol{u}_t+\sum_{\nu=1}^{n}A^\nu\,\boldsymbol{u}_{x_\nu}+C\,\boldsymbol{u}=\boldsymbol{f}. \tag{2}$$

We also see that $A^0=(\omega\,\mathscr{E})_\omega=A$ and the dispersion relation becomes

$$\det G=\det(k_\nu A^\nu-\omega A)=0. \tag{3}$$

By condition 2, A must be positive-definite.

It is now easy to verify that the dispersion equation (*i.e.* the partial differential equation (3) for s, with $\omega=-s_t$ and $k_\nu=s_{x_\nu}$) is the *characteristic equation**, the level surfaces of $s(t, X)$ are *characteristic hypersurfaces**, and the rays are the *bicharacteristics** of (2).

If we express the dispersion relation (3) in the equivalent form

$$\omega=h(\boldsymbol{K}; \boldsymbol{X}), \tag{4}$$

we see from (3) that the (multiple-valued) function h is homogeneous of degree one in $\boldsymbol{K}$; *i.e.* for all real ε

$$h(\varepsilon \boldsymbol{K}; \boldsymbol{X})=\varepsilon\, h(\boldsymbol{K}; \boldsymbol{X}). \tag{5}$$

From the Euler equation for homogeneous functions it follows that

$$\ell=k_\nu h_{k_\nu}-h=\boldsymbol{K}\cdot\boldsymbol{G}-\omega=0, \tag{6}$$

hence from (3.3.3) we see that s is constant on rays. This verifies the fact that the rays are bicharacteristics, *i.e.*, they generate the characteristic hypersurfaces. In section 3.3 we pointed out that the term "ray" is sometimes used for the projections $X=X(t; \Gamma)$ of the space-time curves $(t, X)=[t, X(t; \Gamma)]$ into X-space. For non-dispersive systems we shall refer to the space-time curves as bicharacteristics and to their projections as rays. By differentiating (6) with respect to k_μ we find that

$$\sum_{\nu=1}^{n} k_\nu h_{k_\nu k_\mu}=0, \tag{7}$$

and from this equation it follows that the matrix $(h_{k_\nu k_\mu})$ is singular, *i.e.*

$$\det(h_{k_\nu k_\mu})=0. \tag{8}$$

Since a non-dispersive system is a special case of our general system, the results of this paper apply with very few exceptions, and we may construct asymptotic solutions of (2) according to the general theory of the preceeding chapters. Of course, the parameter λ no longer appears in the system of equations (2), but does appear in the data of the problem (*i.e.* the function $\boldsymbol{f}$, initial data, boundary data, *etc.*). The asymptotic theory of (2) with $\boldsymbol{f}=0$, and with *oscillatory initial data* of the form

$$u(0, X)=e^{i\lambda s_0(X)}\,\boldsymbol{z}^0(X), \tag{9}$$

was first discussed by LAX [9].

It is interesting to note that although the non-dispersive system is certainly simpler than the general system, singularities appear in the special case which do not occur in general. Thus, for example the discussion of section 5.1 is not valid for non-dispersive systems, for then we see from (8) and (5.1.38) that $\tilde{z}$ is infinite. On the other hand, the discussion of section 5.2 is valid for non-dispersive (isotropic) systems, and the results in this case simplify considerably by virtue of the fact that $h''(k)=0$.

For non-dispersive systems the factor $j(t_0)/j(t)$ which appears in all of our formulas for z has a special geometrical interpretation which is most easily

* See [3].

examined in the case of 3 space dimensions ($n=3$): Since s is constant on bicharacteristics,

$$s[t, \boldsymbol{X}(t; \boldsymbol{\Gamma})] = s_0(\boldsymbol{\Gamma}). \tag{10}$$

Let us choose our parameters $\boldsymbol{\Gamma}=(\gamma_1, \gamma_2, \gamma_3)$ so that

$$s_0(\boldsymbol{\Gamma})=\gamma_1. \tag{11}$$

Then for each fixed value of γ_1, $(t, \boldsymbol{X})=[t, \boldsymbol{X}(t; \gamma_1, \gamma_2, \gamma_3)]$ is the parametric equation of a characteristic hypersurface with parameters (t, γ_2, γ_3) and for each fixed value of (t, γ_1), $\boldsymbol{X}=\boldsymbol{X}(t; \gamma_1, \gamma_2, \gamma_3)$ is the parametric equation of a surface*, with parameters (γ_2, γ_3). The vectors

$$\boldsymbol{X}_2=\frac{\partial \boldsymbol{X}}{\partial \gamma_2}, \quad \boldsymbol{X}_3=\frac{\partial \boldsymbol{X}}{\partial \gamma_3} \tag{12}$$

are tangent to the phase-front. Now differentation of (10, 11) yields

$$\boldsymbol{K} \cdot \frac{\partial \boldsymbol{X}}{\partial \gamma_1}=\boldsymbol{K} \cdot \boldsymbol{X}_1=1, \quad \boldsymbol{K} \cdot \boldsymbol{X}_2=\boldsymbol{K} \cdot \boldsymbol{X}_3=0, \tag{13}$$

hence $\boldsymbol{K}$ is orthogonal to $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$; *i.e.*

$$\boldsymbol{X}_2 \times \boldsymbol{X}_3=b\,\boldsymbol{K}, \tag{14}$$

where b is a scalar.

For fixed γ_1, let γ_2 and γ_3 vary in infinitesimal intervals of length $d\gamma_2$ and $d\gamma_3$. Then, for fixed t, the point $\boldsymbol{X}(t; \gamma_1, \gamma_2, \gamma_3)$ varies in an infinitesimal region on the phase-front of area $d\sigma$, and the rays vary in an infinitesimal "tube" of rays of cross-sectional area da. Since the vector $\boldsymbol{G}$ is parallel to the rays, it is easy to see that

$$g\,da=\boldsymbol{X}_2 \times \boldsymbol{X}_3 \cdot \boldsymbol{G}\,d\gamma_2\,d\gamma_3=b\,\boldsymbol{K} \cdot \boldsymbol{G}\,d\gamma_2\,d\gamma_3 \tag{15}$$

and

$$k\,d\sigma=\boldsymbol{X}_2 \times \boldsymbol{X}_3 \cdot \boldsymbol{K}\,d\gamma_2\,d\gamma_2=b\,k^2\,d\gamma_2\,d\gamma_3. \tag{16}$$

Here $g=|\boldsymbol{G}|$ and $k=|\boldsymbol{K}|$. Now, by definition,

$$j(t)=j(t; \boldsymbol{\Gamma})=\boldsymbol{X}_1 \cdot \boldsymbol{X}_2 \times \boldsymbol{X}_3=b\,\boldsymbol{K} \cdot \boldsymbol{X}_1=b. \tag{17}$$

Hence (6), (15), (16) and (17) yield

$$j(t)=\frac{g}{\omega}\frac{da}{d\gamma_2\,d\gamma_3}=\frac{1}{k}\frac{d\sigma}{d\gamma_2\,d\gamma_3}. \tag{18}$$

Since ω, $d\gamma_2$, and $d\gamma_3$ are independent of t we find that

$$\frac{j(t_0)}{j(t)}=\frac{g(t_0)}{g(t)}\frac{da(t_0)}{da(t)}=\frac{k(t)}{k(t_0)}\frac{d\sigma(t_0)}{d\sigma(t)}. \tag{19}$$

Here $j(t)=j(t; \boldsymbol{\Gamma})$, $k(t)=k(t; \boldsymbol{\Gamma})$ and $g(t)=g(t; \boldsymbol{\Gamma})$. $d\sigma(t)$ is the area of the portion of the phase-front intersected by the tube of cross-sectional area $da(t)$. It is important to note that, since γ_1 was held fixed, the tube of rays is formed from *rays associated with a single characteristic hypersurface*, i.e., projections of bicharacteristics which lie on a single characteristic hypersurface. Since $\boldsymbol{G}$ is parallel to the rays and $\boldsymbol{K}$ is normal to the phase-front we see that if ϑ is the

* This surface is called a "phase-front". See section 6.3.

angle between these two vectors, then

$$\frac{da}{d\sigma}=\cos\vartheta=\frac{\boldsymbol{K}\cdot\boldsymbol{G}}{k\,g}=\frac{\omega}{k\,g} \tag{20}$$

in agreement with (18).

For the non-dispersive system, either of the expressions (19) may be substituted for $j(t_0)/j(t)$ wherever it appears in our earlier formulas.

6.2. *Progressing wave solutions of the non-dispersive system*

We have seen in section 3.3 that the function

$$\boldsymbol{u}=e^{i\lambda s(t;\Gamma)}\,\boldsymbol{z}(t;\Gamma);\qquad \boldsymbol{X}=\boldsymbol{X}(t;\Gamma), \tag{1}$$

is the leading term of the asymptotic expansion of a solution of the (homogeneous) general system of equations. In the non-dispersive case that system of equations becomes (6.1.2) (with $\boldsymbol{f}\equiv 0$). Since, in that case, the equations are independent of λ, we may multiply (1) by an arbitrary function $\hat{e}(\lambda)$ and then integrate with respect to λ. Thus we obtain the formal solution

$$\boldsymbol{u}=\boldsymbol{e}[s(t;\Gamma)]\,\boldsymbol{z}(t;\Gamma);\qquad \boldsymbol{X}=\boldsymbol{X}(t;\Gamma), \tag{2}$$

which involves the arbitrary "wave-form" function,

$$e(s)=\int\hat{e}(\lambda)\,e^{i\lambda s}\,d\lambda. \tag{3}$$

(According to our usual convention, at each point $(t, \boldsymbol{X})$, (2) is to be summed over all values of Γ for which $\boldsymbol{X}(t;\Gamma)=\boldsymbol{X}$, i.e. over all bicharacteristics which pass through the point.)

Equation (2) is the leading term of a formal series solution of the non-dispersive system. It is easy to see, from (3.1.5), that the successive terms of the series involve successive integrals of the function $e(s)$. Such solutions are called *progressing waves*. Applications and interpretations of the formal series are discussed in [*11*, *13*]. Here we mention only that in using progressing waves to obtain solutions of specific problems for the non-dispersive system of equations, the data of the problem yield not only the initial values of s and $\boldsymbol{z}$, as we have seen earlier, but also the wave-form function $e(s)$.

6.3. *Group velocity, phase speed, and wave-speed for dispersive and non-dispersive systems*

In this section we return to the consideration of dispersive system, but we shall make some comparisons with the non-dispersive case. Many dispersive systems satisfy condition 10 of appendix H. The system of electromagnetic field equations, with a dielectric permeability function of the kind discussed in appendix D is of this type. For a system which satisfies condition 10, it is convenient to refer to the system with $\mathscr{E}$ replaced by its limiting value A and with $D\equiv 0$ as the *corresponding non-dispersive system.* (The latter system is given by (6.1.2) with $C=0$.)

From the ray equations (3.3.1) we see that the space point $\boldsymbol{X}=\boldsymbol{X}(t;\Gamma)$ which remains on a given ray as t increases moves with a velocity $\dot{\boldsymbol{X}}=\boldsymbol{G}=(g_1,\ldots,g_n)$. The vector $\boldsymbol{G}$ is often called the *group velocity* (or *ray velocitiy*) vector, and its magnitude g is called the *group speed* (or *ray speed*).

From (3.1.6) we see that the vector $\boldsymbol{K}=(k_1, \ldots, k_n)$ is identical to the vector $\nabla s=(s_{x_1}, \ldots, s_{x_n})$. Hence for each fixed t the unit vector

$$\boldsymbol{A}=(a_1, \ldots, a_n), \quad a_\nu=k_\nu/k, \quad k^2=\boldsymbol{K}\cdot\boldsymbol{K}; \tag{1}$$

is normal to the level surfaces of the phase function $s(t, \boldsymbol{X})$. These level surfaces, which are defined by the equation

$$s(t, \boldsymbol{X})=\text{constant}, \quad t=\text{constant} \tag{2}$$

are called *phase-fronts*. As t increases the phase-fronts move through space. Let $\boldsymbol{X}=\boldsymbol{X}(t)=(x_1, \ldots, x_n)$ be a point constrained to remain on a phase-front (2). By differentiating (2) with respect to t we find that

$$-\omega+k\boldsymbol{A}\cdot\dot{\boldsymbol{X}}=-\omega+k\,a_\nu\dot{x}_\nu=-\omega+k_\nu\dot{x}_\nu=0. \tag{3}$$

The quantity

$$|p|=\left|\frac{\omega}{k}\right|=(\operatorname{sgn}\omega)\frac{\omega}{k}=(\operatorname{sgn}\omega)\,\boldsymbol{A}\cdot\dot{\boldsymbol{X}}=\boldsymbol{N}\cdot\dot{\boldsymbol{X}} \tag{4}$$

which is the component of the velocity $\dot{\boldsymbol{X}}$ of the constrained point, in the direction of the unit normal vector $\boldsymbol{N}=(\operatorname{sgn}\omega)\boldsymbol{A}$ to the phase-front, is called the *phase speed.* From the dispersion relation (3.2.1) we see that p is a solution of the equation

$$\det(a_\nu A^\nu-p\mathscr{E})=0. \tag{5}$$

There is one phase speed corresponding to each value of h, *i.e.*, corresponding to each value of the multiple-valued function s. For the corresponding non-dispersive system the phase speeds are the *normal speeds** $|v|$ which satisfy

$$\det(a_\nu A^\nu-v\,A)=0, \tag{6}$$

and from (6.1.6) we see that

$$\boldsymbol{G}\cdot\boldsymbol{N}=\boldsymbol{G}\cdot\boldsymbol{K}\,\frac{(\operatorname{sgn}\omega)}{k}=\frac{|\omega|}{k}=|p|=|v|. \tag{7}$$

This equation, which asserts that the component of the group velocity in the direction of the normal to the phase-front is equal to the phase-speed, holds *only in the non-dispersive case.* For the general case (subject to condition 11) it is replaced by an inequality which we now derive:

Since $a_\nu A^\nu$ is hermitian and A is positive definite, there exist vectors $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^m$ such that

$$(a_\nu A^\nu-v_j A)\boldsymbol{s}^j=0; \quad (\boldsymbol{s}^i, A\,\boldsymbol{s}^j)=\delta_{ij}; \quad i, j=1, \ldots, m. \tag{8}$$

$v_1, \ldots, v_n$ are the roots of (6). An arbitrary vector $\boldsymbol{r}$ may be expressed in the form

$$\boldsymbol{r}=\sum_{j=1}^{m}\rho_j\boldsymbol{s}^j. \tag{9}$$

It follows easily from (8) and (9) that

$$(\boldsymbol{r}, a_\nu A^\nu\boldsymbol{r})=\sum_{j=1}^{m}v_j\rho_j^2, \quad (\boldsymbol{r}, A\,\boldsymbol{r})=\sum_{j=1}^{m}\rho_j^2. \tag{10}$$

Hence

$$|(\boldsymbol{r}, a_\nu A^\nu\boldsymbol{r})|\leqq v_{\max}(\boldsymbol{r}, A\,\boldsymbol{r}); \quad v_{\max}=\max_j|v_j|. \tag{11}$$

* See [3].

Now let $\boldsymbol{r}$ be one of the null eigenvectors of the matrix G (section 3.2). From the basic identity (3.2.8) we see that

$$\boldsymbol{A} \cdot \boldsymbol{G} = a_\nu g_\nu = (\boldsymbol{r}, a_\nu A^\nu \boldsymbol{r}), \tag{12}$$

while condition 10 and the basic identity (with $\nu = 0$) imply that

$$(\boldsymbol{r}, A\boldsymbol{r}) \leqq (\boldsymbol{r}, A^0 \boldsymbol{r}) = 1. \tag{13}$$

(11), (12) and (13) now yield the inequality

$$|\boldsymbol{G} \cdot \boldsymbol{N}| \leqq v_{\max}, \tag{14}$$

which is the required generalization of (7). It is well known that $v_{\max}$ is the normal speed of a *wave front*, *i.e.*, a moving surface which represents the first arrival of non-zero values of a solution $\boldsymbol{u}(t, \boldsymbol{X})$ of (6.1.2). For this reason we refer to $v_{\max}$ as the *wave speed.* Thus (14) implies that the component of the group velocity in the direction of the normal to the wave-front does not exceed the wave speed of the corresponding non-dispersive system. [For the electromagnetic field equations, the wave speed is c.] For isotropic media, $\boldsymbol{G}$, $\boldsymbol{K}$, and $\boldsymbol{N}$ are parallel, and (14) implies that the group speed g does not exceed the wave speed. However, for anisotropic media, (7) and (14) impose no upper bound on the magnitude of the group speed. Since we have seen, in section 3.7, that energy is transported with the group velocity, we conclude that *the theory of relativity imposes no restriction on the speed of energy transport in anisotropic media.*

7. Electromagnetic Waves in Isotropic Media

In chapter 2 we motivated the introduction of the general system of equations by discussing the electromagnetic field equations for dispersive media. We return now to those equations in order to see how our general results apply to them. We consider here only the case of isotropic media. This means that the dielectric and magnetic permeability $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ are scalar functions of ω and $\boldsymbol{X}$. Then from (2.1.24) and condition 1 of appendix H we see that $\tilde{\varepsilon}_\nu = \varepsilon_\nu - (i\lambda)^{-1} \delta_\nu + O(\lambda^{-2})$ and

$$\mathscr{E} = \begin{bmatrix} \varepsilon_1 I_3 & 0 \\ 0 & \varepsilon_2 I_3 \end{bmatrix}, \quad \mathscr{D} = \begin{bmatrix} \delta_1 I_3 & 0 \\ 0 & \delta_2 I_3 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We also impose conditions 2–5 which are listed in appendix H. They imply obvious contitions for ε_1 and ε_2. Specific examples satisfying these conditions are discussed in appendix D.

7.1. The dispersion equation and the null eigenvectors

The matrix $G = k_\nu A^\nu - \omega \mathscr{E}$ may be obtained from the above expression for $\mathscr{E}$ and from (2.1.21). Thus

$$G = \begin{bmatrix} -\omega \varepsilon_1 I_3 & -c(\boldsymbol{K}) \\ c(\boldsymbol{K}) & -\omega \varepsilon_2 I_3 \end{bmatrix}. \tag{1}$$

Let us denote the null eigenvectors of G by $\boldsymbol{r} = (\boldsymbol{R}_1, \boldsymbol{R}_2)$ where $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are 3-vectors. Then the equation $G\boldsymbol{r} = 0$ becomes

$$\omega \varepsilon_1 \boldsymbol{R}_1 + c\boldsymbol{K} \times \boldsymbol{R}_2 = 0, \quad c\boldsymbol{K} \times \boldsymbol{R}_1 - \omega \varepsilon_2 \boldsymbol{R}_2 = 0. \tag{2}$$

We first note that (2) is satisfied with $\omega=0$ and $\boldsymbol{K}\times\boldsymbol{R}_1=\boldsymbol{K}\times\boldsymbol{R}_2=0$, i.e., $\omega\equiv 0$ is a root of the dispersion relation $\det G=0$. However for this root the group velocity $\boldsymbol{G}=(g_1, g_2, g_3)$ is zero because $g_\nu=\partial\omega/\partial k_\nu=0$; hence the rays never leave the "source region" (*i.e.*, the projection of the manifold $\mathscr{M}$ into $\boldsymbol{X}$-space). Thus the zero root does not contribute to the asymptotic solution outside of the source region and is therefore uninteresting.

If $\varepsilon_1(\omega)$ vanishes for some real value $\omega=\omega_0$, then (2) is satisfied with $\omega\equiv\omega_0$, $\boldsymbol{R}_2\equiv 0$, and $\boldsymbol{R}_1$ parallel to $\boldsymbol{K}$. For such solutions the group velocity is also zero (because ω is independent of $\boldsymbol{K}$), and the corresponding asymptotic solution is again uninteresting. We shall not consider either of the solutions $\omega\equiv 0$ or $\omega\equiv\omega_0$ further.

If $\omega\,\varepsilon_\nu\neq 0$, (2) implies that

$$\boldsymbol{R}_1\cdot\boldsymbol{K}=0 \tag{3}$$

and

$$\boldsymbol{R}_2=\frac{c}{\omega\,\varepsilon_2}\,\boldsymbol{K}\times\boldsymbol{R}_1\,. \tag{4}$$

The last equation determines $\boldsymbol{R}_2$ in terms of $\boldsymbol{R}_1$ and from (3) we see that there are two linearly independent eigenvectors $\boldsymbol{r}^1=(\boldsymbol{R}_1^1, \boldsymbol{R}_2^1)$, $\boldsymbol{r}^2=(\boldsymbol{R}_1^2, \boldsymbol{R}_2^2)$; *i.e.*, $q=2$. If we insert (4) in (2), expand the vector triple product, and use (3), we obtain $(\omega^2\varepsilon_1\varepsilon_2-c^2k^2)\,\boldsymbol{R}_1=0$. This implies that the dispersion relation is given by

$$\omega^2\,\varepsilon_1(\omega, \boldsymbol{X})\,\varepsilon_2(\omega, \boldsymbol{X})=c^2\,k^2. \tag{5}$$

The matrix A^0 is given by

$$A^0=(\omega\mathscr{E})_\omega=\begin{bmatrix}(\omega\varepsilon_1)_\omega I_3 & 0\\ 0 & (\omega\varepsilon_2)_\omega I_3\end{bmatrix}, \tag{6}$$

hence the orthonormality condition (3.2.4) becomes

$$(\omega\varepsilon_1)_\omega\,\boldsymbol{R}_1^i\cdot\boldsymbol{R}_1^j+(\omega\varepsilon_2)_\omega\,\boldsymbol{R}_2^i\cdot\boldsymbol{R}_2^j=\delta_{ij}\,. \tag{7}$$

But (4) implies that

$$\boldsymbol{R}_2^i\cdot\boldsymbol{R}_2^j=\frac{c^2\,k^2}{(\omega\,\varepsilon_2)^2}\,\boldsymbol{R}_1^i\cdot\boldsymbol{R}_1^j=\frac{\varepsilon_1}{\varepsilon_2}\,\boldsymbol{R}_1^i\cdot\boldsymbol{R}_1^j\,. \tag{8}$$

Hence

$$\boldsymbol{R}_1^1\cdot\boldsymbol{R}_1^2=\boldsymbol{R}_2^1\cdot\boldsymbol{R}_2^2=0\,, \tag{9}$$

and

$$\varepsilon_1(\boldsymbol{R}_1^j)^2=\varepsilon_2(\boldsymbol{R}_2^j)^2=\zeta \tag{10}$$

where

$$\zeta=\left[\frac{(\omega\varepsilon_1)_\omega}{\varepsilon_1}+\frac{(\omega\varepsilon_2)_\omega}{\varepsilon_2}\right]^{-1}=\frac{\omega\,\varepsilon_1\,\varepsilon_2}{(\omega\,\varepsilon_2)(\omega\,\varepsilon_1)_\omega+(\omega\,\varepsilon_1)(\omega\,\varepsilon_2)_\omega}\,. \tag{11}$$

We also note that

$$(\boldsymbol{r}^i, \omega\,\mathscr{D}\,\boldsymbol{r}^j)=\omega[\delta_1\,\boldsymbol{R}_1^i\cdot\boldsymbol{R}_1^j+\delta_2\,\boldsymbol{R}_2^i\cdot\boldsymbol{R}_2^j]=\eta\,\delta_{ij}\,. \tag{12}$$

Thus condition 7 is satisfied with

$$\eta=\omega[\delta_1(\boldsymbol{R}_1^j)^2+\delta_2(\boldsymbol{R}_2^j)^2]=\omega\,\zeta\left[\frac{\delta_1}{\varepsilon_1}+\frac{\delta_2}{\varepsilon_2}\right]. \tag{13}$$

Conditions 6 and 9 are satisfied provided ε_ν and δ_ν are real; and condition 8 is satisfied with $\beta=0$ because the matrices A^ν are constant.

From the dispersion relation (5) it is clear that ω is independent of $\boldsymbol{A}=\boldsymbol{K}/k$. The function $\omega=h(k, \boldsymbol{X})$ is defined implicitly by (5) and is in general multiple-valued. The inverse function is however single-valued and is given explicitly by

$$k=m(\omega; \boldsymbol{X})=\frac{|\omega|}{c}\sqrt{\varepsilon_1 \varepsilon_2}. \tag{14}$$

It is defined for all real ω such that $\varepsilon_1(\omega)\,\varepsilon_2(\omega)\geqq 0$. From (11) and (14) it is easy to show that

$$\zeta=\frac{m}{2\omega m'} \qquad (m'=m_\omega). \tag{15}$$

Since $g_\nu=\dfrac{\partial h}{\partial k_\nu}=h'(k)\,a_\nu$, $a_\nu=k_\nu/k$, we see that

$$\boldsymbol{G}=(g_1, g_2, g_3)=h'(k)\,\boldsymbol{A}=\boldsymbol{A}/m'(\omega), \tag{16}$$

and

$$g^2=\boldsymbol{G}\cdot\boldsymbol{G}=\frac{1}{(m')^2}=\frac{4\omega^2\zeta^2}{m^2}=\frac{4c^2\zeta^2}{\varepsilon_1\varepsilon_2}. \tag{17}$$

Thus

$$\boldsymbol{G}=g\boldsymbol{T}, \qquad \boldsymbol{T}=[\operatorname{sgn} m'(\omega)]\,\boldsymbol{A}, \tag{18}$$

where g is the positive root of (17).

7.2. The polarization vector

In the preceding section we have seen that conditions 6, 7, 8 and 9 are satisfied with $\beta=0$ and η given by (7.1.13). Furthermore we found that $q=2$. Therefore the formulas (3.10.3, 4, 6) for $z(t)$ are applicable. However, rather than use these formulas, we prefer to return to section 3.9 and derive equivalent formulas for z, in which the physical interpretation is emphasized.

The unit vector $\boldsymbol{T}$ is a function of $(t, \boldsymbol{X})$. Let $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ be mutually orthogonal unit vector functions of $(t, \boldsymbol{X})$ which satisfy*

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}, \quad \boldsymbol{N}=\boldsymbol{B}\times\boldsymbol{T}, \quad \boldsymbol{T}=\boldsymbol{N}\times\boldsymbol{B}. \tag{1}$$

From (7.1.3) we see that we may choose any two directions for $\boldsymbol{R}_1^1$ and $\boldsymbol{R}_1^2$ which are orthogonal to $\boldsymbol{K}=\pm k\,\boldsymbol{T}$. Therefore we set

$$\boldsymbol{R}_1^1=\sqrt{\frac{\zeta}{\varepsilon_1}}\,\boldsymbol{B}, \quad \boldsymbol{R}_1^2=\sqrt{\frac{\zeta}{\varepsilon_1}}\,\boldsymbol{N}. \tag{2}$$

Then it is easily seen that

$$\boldsymbol{R}_2^1=-\sqrt{\frac{\zeta}{\varepsilon_2}}\,\boldsymbol{N}, \quad \boldsymbol{R}_2^2=\sqrt{\frac{\zeta}{\varepsilon_2}}\,\boldsymbol{B}, \tag{3}$$

and (7.1.3, 4, 7) are satisfied. From (3.5.1) we note that

$$z=(\boldsymbol{Z}_1, \boldsymbol{Z}_2)=\sigma_1\boldsymbol{r}^1+\sigma_2\boldsymbol{r}^2. \tag{4}$$

Therefore

$$\boldsymbol{Z}_1=\sqrt{\frac{\zeta}{\varepsilon_1}}\,(\sigma_1\boldsymbol{B}+\sigma_2\boldsymbol{N}), \quad \boldsymbol{Z}_2=\sqrt{\frac{\zeta}{\varepsilon_2}}\,(-\sigma_1\boldsymbol{N}+\sigma_2\boldsymbol{B}). \tag{5}$$

* From (7.1.18) we see that the unit vector $\boldsymbol{T}$ is tangent to (the space projection of) a ray. Later we shall take $\boldsymbol{N}$ and $\boldsymbol{B}$ to be unit normal and binormal vectors to the ray.

We now introduce the (complex) *polarization vector*,

$$\boldsymbol{P}=(\beta_1 \boldsymbol{B}+\beta_2 \boldsymbol{N})=\boldsymbol{P}'+i\boldsymbol{P}'', \tag{6}$$

where $\boldsymbol{P}'$ and $\boldsymbol{P}''$ are vectors with real components and β_1, β_2 are defined by (3.9.6). Then (3.9.12) implies that

$$(\boldsymbol{P}')^2+(\boldsymbol{P}'')^2=\boldsymbol{P}^*\cdot\boldsymbol{P}=|\beta_1|^2+|\beta_2|^2=1, \tag{7}$$

and from (1) and (5) we see that

$$\boldsymbol{Z}_1=\sqrt{\frac{w\zeta}{\varepsilon_1}}\,\boldsymbol{P}, \qquad \boldsymbol{Z}_2=\sqrt{\frac{w\zeta}{\varepsilon_2}}\,\boldsymbol{A}\times\boldsymbol{P}. \tag{8}$$

By taking the real and imaginary parts of (6) we find that

$$\boldsymbol{P}'=\beta_1'\boldsymbol{B}+\beta_2'\boldsymbol{N}, \qquad \boldsymbol{P}''=\beta_1''\boldsymbol{B}+\beta_2''\boldsymbol{N} \tag{9}$$

where

$$\beta_1=\beta_1'+i\beta_1'', \qquad \beta_2=\beta_2'+i\beta_2''. \tag{10}$$

It is clear that β_1', β_2' and β_1'', β_2'' satisfy (3.9.10), therefore

$$|\boldsymbol{P}'|=\sqrt{(\beta_1')^2+(\beta_2')^2}=p', \qquad |\boldsymbol{P}''|=\sqrt{(\beta_1'')^2+(\beta_2'')^2}=p'', \tag{11}$$

where p' and p'' are independent of t. Thus

$$\boldsymbol{P}'=p'[\boldsymbol{N}\cos\alpha'+\boldsymbol{B}\sin\alpha'], \qquad \boldsymbol{P}''=p''[\boldsymbol{N}\cos\alpha''+\boldsymbol{B}\sin\alpha''], \tag{12}$$

where $p'\sin\alpha'=\beta_1'$, $p'\cos\alpha'=\beta_2'$, etc., and from (3.9.10) (with t_0 replaced by τ) we find that

$$\alpha'(\tau)-\alpha'(t)=\alpha''(\tau)-\alpha''(t)=\delta=\int_\tau^t \tau_2(t')\,dt'. \tag{13}$$

The real and imaginary parts $\boldsymbol{P}'$ and $\boldsymbol{P}''$ of the polarization vector are illustrated in Fig. 1.

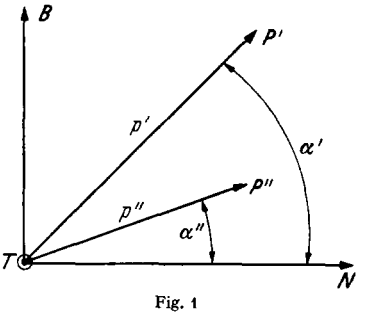

Fig. 1

The lengths p', p'' of the vectors and the angle $\alpha'(t)-\alpha''(t)$ between them are fixed, but as t increases the pair of vectors rotates around the ray direction $\boldsymbol{T}$, according to (13). [We shall shortly investigate the geometrical significance of the quantity τ_2 that appears in (13).]

From (3.10.1) we recall that the energy density w which appears in (8) is given by

$$w(t)=\tilde{w}(\tau)\,\frac{\tilde{j}(\tau)}{j(t)}\exp\left\{-2\int_\tau^t \eta(t')\,dt'\right\}, \tag{14}$$

and if we let $t\to\tau$ in (3.7.5), (4), and (8) we find that

$$\tilde{w}(\tau)=(\tilde{\boldsymbol{z}},A^0\tilde{\boldsymbol{z}})=[(\omega\varepsilon_1)_\omega\tilde{\boldsymbol{Z}}_1^*\cdot\tilde{\boldsymbol{Z}}_1+(\omega\varepsilon_2)_\omega\tilde{\boldsymbol{Z}}_2^*\cdot\tilde{\boldsymbol{Z}}_2], \tag{15}$$

$$\tilde{\boldsymbol{z}}=[\tilde{\boldsymbol{Z}}_1,\tilde{\boldsymbol{Z}}_2], \tag{16}$$

and

$$\boldsymbol{P}'(\tau)+i\boldsymbol{P}''(\tau)=\boldsymbol{P}(\tau)=\sqrt{\frac{\varepsilon_1}{\tilde{w}\zeta}}\,\tilde{\boldsymbol{Z}}_1. \tag{17}$$

Our results so far enable us to determine $z(t)=(Z_1, Z_2)$ along a ray in terms of the value of $\tilde{z}(\tau)$. (The determination of $\tilde{z}$ is discussed in section 3.11 and is illustrated in chapter 5.) Let us summarize these results: From (15) we may determine $\tilde{w}$. Then (17) and (12) (at $t=\tau$) may be used to determine the constants p', p'' and $\alpha'(\tau)$, $\alpha''(\tau)$. Once τ_2 is computed (13) yields the values of $\alpha'(t)$ and $\alpha''(t)$, (12) yields the values of $\boldsymbol{P}(t)=\boldsymbol{P}'(t)+i\boldsymbol{P}''(t)$, and (8) and (14) provide formulas for $\boldsymbol{Z}_1(t)$ and $\boldsymbol{Z}_2(t)$.

It remains to compute the value of τ_2 which, from (13), determines the rate of rotation of the real and imaginary parts of the polarization vector $\boldsymbol{P}$: From (7.1.6, 9) we find that

$$(\boldsymbol{r}^1, A_t^0 \boldsymbol{r}^2)=0 \tag{18}$$

therefore (3.9.4) yields

$$\tau_2=(\boldsymbol{r}^1, A^0 \boldsymbol{r}_t^2)+\sum_{\nu=1}^{3}(\boldsymbol{r}^1, A^\nu \boldsymbol{r}_{x_\nu}^2). \tag{19}$$

But from (2.1.21), (2), (3), and (7.1.17)

$$\begin{aligned}\sum_{\nu=1}^{3}(\boldsymbol{r}^1, A^\nu \boldsymbol{r}_{x_\nu}^2)=c[\boldsymbol{R}_2^1\cdot\nabla\times\boldsymbol{R}_1^2-\boldsymbol{R}_1^1\cdot\nabla\times\boldsymbol{R}_2^2]&=-\sqrt{\frac{c^2\zeta^2}{\varepsilon_1\varepsilon_2}}[\boldsymbol{N}\cdot\nabla\times\boldsymbol{N}+\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}]\\&=-\tfrac{1}{2}g[\boldsymbol{N}\cdot\nabla\times\boldsymbol{N}+\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}].\end{aligned} \tag{20}$$

By using (E.13) of appendix E and (7.1.18), we now find that

$$\sum_{\nu=1}^{3}(\boldsymbol{r}^1, A^\nu \boldsymbol{r}_{x_\nu}^2)=-g\boldsymbol{N}\cdot(\boldsymbol{T}\cdot\nabla)\boldsymbol{B}=-\boldsymbol{N}\cdot(\boldsymbol{G}\cdot\nabla)\boldsymbol{B}. \tag{21}$$

From (7.1.6), (2), (3) we see that the first term of (19) is

$$(\boldsymbol{r}^1, A^0 \boldsymbol{r}_t^2)=(\omega\varepsilon_1)_\omega \boldsymbol{R}_1^1\cdot(\boldsymbol{R}_1^2)_t+(\omega\varepsilon_2)_\omega \boldsymbol{R}_2^1\cdot(\boldsymbol{R}_2^2)_t=\alpha_1\boldsymbol{B}\cdot\boldsymbol{N}_t-\alpha_2\boldsymbol{N}\cdot\boldsymbol{B}_t, \tag{22}$$

where $\alpha_\nu=(\zeta/\varepsilon_\nu)(\omega\varepsilon_\nu)_\omega$. From (7.1.11) we see that $\alpha_1+\alpha_2=1$, therefore

$$(\boldsymbol{r}^1, A^0 \boldsymbol{r}_t^2)=\alpha_1(\boldsymbol{B}\cdot\boldsymbol{N}_t+\boldsymbol{N}\cdot\boldsymbol{B}_t)-\boldsymbol{N}\cdot\boldsymbol{B}_t=\alpha_1(\boldsymbol{B}\cdot\boldsymbol{N})_t-\boldsymbol{N}\cdot\boldsymbol{B}_t=-\boldsymbol{N}\cdot\boldsymbol{B}_t. \tag{23}$$

By inserting (23) and (21) in (19) we obtain

$$\tau_2=-\boldsymbol{N}\cdot\left[\frac{\partial}{\partial t}+\boldsymbol{G}\cdot\nabla\right]\boldsymbol{B}=-\boldsymbol{N}\cdot\frac{d\boldsymbol{B}}{dt}. \tag{24}$$

In (24) d/dt denotes differentiation, with respect to time, along a ray. We now choose $\boldsymbol{N}$ and $\boldsymbol{B}$ to be unit normal and binormal vectors to the projection of that ray, which is a space-curve. Since $\boldsymbol{T}$ is a unit tangent vector to the projection, $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ satisfy the Frenet equations

$$\frac{d\boldsymbol{T}}{d\sigma}=\kappa\boldsymbol{N}, \quad \frac{d\boldsymbol{N}}{d\sigma}=-\kappa\boldsymbol{T}+\tau_0\boldsymbol{B}, \quad \frac{d\boldsymbol{B}}{d\sigma}=-\tau_0\boldsymbol{N}. \tag{25}$$

Here κ is the curvature and τ_0 is the torsion of the projected ray; σ denotes an arclength parameter. But (7.1.18) and the ray equations imply that

$$\frac{d\boldsymbol{X}}{dt}=\boldsymbol{G}=g\boldsymbol{T}=g\frac{d\boldsymbol{X}}{d\sigma}, \tag{26}$$

hence $d\sigma = g\,dt$. Thus (24) and (25) yield

$$\tau_2 = -g\,\boldsymbol{N}\cdot\frac{d\boldsymbol{B}}{d\sigma} = g\,\tau_0\,, \tag{27}$$

and from (13),

$$\delta = \int_{\tau}^{t} g\,\tau_0\,dt' = \int_{\sigma(\tau)}^{\sigma(t)} \tau_0\,d\sigma\,. \tag{28}$$

Thus we see that the rate of rotation of the vectors $\boldsymbol{P}'$ and $\boldsymbol{P}''$ is determined by the torsion τ_0 of the projected ray. If this space curve is a straight line (as it is for a homogeneous medium) τ_0 is zero and the vectors $\boldsymbol{P}'$ and $\boldsymbol{P}''$ are constant on the ray. If the projected curve is a plane curve, τ_0 is again zero and the vectors $\boldsymbol{P}'$ and $\boldsymbol{P}''$ remain fixed in the $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$-frame.

Appendices

A. *A theorem for homogeneous media*

Theorem. *If A^ν $(\nu = 1, \ldots, n)$ and $\mathscr{E}$ are independent of $\boldsymbol{X}$ and condition 6 holds, then*

$$\tau_{\ell m} = \left[\tfrac{1}{2}\nabla\cdot\boldsymbol{G} + \eta_\ell\right]\delta_{\ell m}\,.$$

Proof. Our assumptions imply that

$$G = \sum_{\nu=1}^{n} k_\nu A^\nu - \omega\,\mathscr{E}(\omega)$$

is independent of $\boldsymbol{X}$. Therefore the roots $\omega = h(\boldsymbol{K})$ of $\det G = 0$ are independent of $\boldsymbol{X}$, and it follows from the ray equations (3.3.1) that k_ν, ω, and G are constant on each ray. Thus the null eigenvectors $\boldsymbol{r}^m$ of G may be so chosen that they are constant on each ray, and

$$\sum_{\nu=0}^{n} g_\nu \boldsymbol{r}^m_{x_\nu} = \frac{d}{dt}\,\boldsymbol{r}^m = 0; \qquad m = 1, \ldots, q\,. \tag{1}$$

Since $G_\omega = -A^0$ and $g_j = \dfrac{\partial\omega}{\partial k_j} = \dfrac{\partial h}{\partial k_j}$, $G_{k_j} = A^j - A^0 g_j$. Hence

$$(\boldsymbol{r}^\ell, A^j\,\boldsymbol{r}^m_{x_j}) = (\boldsymbol{r}^\ell, G_{k_j}\,\boldsymbol{r}^m_{x_j}) + (\boldsymbol{r}^\ell, g_j A^0\,\boldsymbol{r}^m_{x_j}); \qquad \ell, m = 1, \ldots, q; \quad j = 1, \ldots, n\,. \tag{2}$$

From (3.5.6, 7) and (2)

$$\begin{aligned}\tau_{\ell m} - \tfrac{1}{2}(\boldsymbol{r}^\ell, A^0_t\,\boldsymbol{r}^m) - \eta_\ell\,\delta_{\ell m} &= (\boldsymbol{r}^\ell, A^0\,\boldsymbol{r}^m_{x_0}) + \sum_{j=1}^{n}(\boldsymbol{r}^\ell, A^j\,\boldsymbol{r}^m_{x_j}) \\ &= \sum_{j=1}^{n}(\boldsymbol{r}^\ell, G_{k_j}\,\boldsymbol{r}^m_{x_j}) + \sum_{\nu=0}^{n}(\boldsymbol{r}^\ell, g_\nu A^0\,\boldsymbol{r}^m_{x_\nu})\,.\end{aligned} \tag{3}$$

But from (1) we see that the last term in (3) vanishes. Furthermore, since $G_{k_j}\boldsymbol{r}^\ell + G\,\boldsymbol{r}^\ell_{k_j} = (G\,\boldsymbol{r}^\ell)_{k_j} = 0$,

$$\sum_{j=1}^{n}(\boldsymbol{r}^\ell, G_{k_j}\,\boldsymbol{r}^m_{x_j}) = -\sum_{j=1}^{n}(\boldsymbol{r}^\ell_{k_j}, G\,\boldsymbol{r}^m_{x_j})\,. \tag{4}$$

Hence (3) and (4) yield

$$\tau^*_{m\ell} = \tfrac{1}{2}(\boldsymbol{r}^\ell, A^0_t\,\boldsymbol{r}^m) + \eta_\ell\,\delta_{\ell m} - \sum_{j=1}^{n}(G\,\boldsymbol{r}^\ell_{x_j}, \boldsymbol{r}^m_{k_j})\,. \tag{5}$$

Since $k_j = s_{x_j}$ is constant on a ray

$$s_{x_j x_0} + \sum_{\nu=1}^{n} g_\nu s_{x_j x_\nu} = \sum_{\nu=0}^{n} g_\nu \frac{\partial k_j}{\partial x_\nu} = \frac{dk_j}{dt} = 0, \tag{6}$$

therefore

$$G_{x_j} = \sum_{\nu=1}^{n} G_{k_\nu} \frac{\partial k_\nu}{\partial x_j} = \sum_{\nu=1}^{n} (A^\nu - A^0 g_\nu) s_{x_\nu x_j} = \sum_{\nu=0}^{n} A^\nu s_{x_\nu x_j}. \tag{7}$$

Furthermore

$$\boldsymbol{r}^m_{x_\nu} = \sum_{j=1}^{n} \boldsymbol{r}^m_{k_j} \frac{\partial k_j}{\partial x_\nu} = \sum_{j=1}^{n} s_{x_j x_\nu} \boldsymbol{r}^m_{k_j}. \tag{8}$$

Now (3.5.6), (8) and (7) yield

$$\begin{aligned} \tau_{\ell m} - \tfrac{1}{2}(\boldsymbol{r}^\ell, A^0_t \boldsymbol{r}^m) - \eta_\ell \delta_{\ell m} &= \sum_{\nu=0}^{n} (\boldsymbol{r}^\ell, A^\nu \boldsymbol{r}^m_{x_\nu}) \\ &= \sum_{\nu=0}^{n} \sum_{j=1}^{n} (\boldsymbol{r}^\ell, A^\nu s_{x_j x_\nu} \boldsymbol{r}^m_{k_j}) \\ &= \sum_{j=1}^{n} (\boldsymbol{r}^\ell, G_{x_j} \boldsymbol{r}^m_{k_j}) = -\sum_{j=1}^{n} (\boldsymbol{r}^\ell_{x_j}, G \boldsymbol{r}^m_{k_j}) \\ &= -\sum_{j=1}^{n} (G \boldsymbol{r}^\ell_{x_j}, \boldsymbol{r}^m_{k_j}). \end{aligned} \tag{9}$$

(The last equation in (9) follows from the fact that G is hermitian.)

We now compare (5) and (9) and find that

$$\tau_{\ell m} = \tau^*_{m\ell}. \tag{10}$$

Then (10), (3.7.3), condition 6 and (3.2.11) yield the required result,

$$\tau_{\ell m} = \tfrac{1}{2}[\tau_{\ell m} + \tau^*_{m\ell}] = \tfrac{1}{2}(\nabla \cdot \boldsymbol{G}) \delta_{\ell m} + (\boldsymbol{r}^\ell, \omega \mathscr{D} \boldsymbol{r}^m) = [\tfrac{1}{2} \nabla \cdot \boldsymbol{G} + \eta_\ell] \delta_{\ell m}. \tag{11}$$

B. *Generalized energy density and Poynting vector*

If $\boldsymbol{u}$ is any real solution of the general system of equations (2.1.26, 7) we define the *generalized Poynting vector* $\boldsymbol{S} = (S_1, \ldots, S_n)$ corresponding to $\boldsymbol{u}$ by

$$S_\nu = 2(\boldsymbol{u}, A^\nu \boldsymbol{u}); \quad \nu = 1, \ldots, n. \tag{1}$$

For real solutions $\boldsymbol{u} = (\boldsymbol{E}, \boldsymbol{H})$ of the electromagnetic field equations we see from (2.1.21) that if $\boldsymbol{K}$ is an arbitrary vector,

$$\boldsymbol{K} \cdot \boldsymbol{S} = \sum_{\nu=1}^{3} k_\nu S_\nu = 2c[\boldsymbol{H} \cdot \boldsymbol{K} \times \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{K} \times \boldsymbol{H}] = 4c \boldsymbol{K} \cdot \boldsymbol{E} \times \boldsymbol{H} \tag{2}$$

hence

$$S = 16\pi S_E, \quad \text{where} \quad \boldsymbol{S}_E = \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{H}. \tag{3}$$

But $\boldsymbol{S}_E$ is the usual "Poynting vector" of electromagnetic theory. For this reason we have called $\boldsymbol{S}$ the "generalized Poynting vector".

For complex solutions we define $\boldsymbol{S}$ in terms of the real part of $\boldsymbol{u}$. Thus

$$\begin{aligned} S_\nu = 2(\operatorname{Re} \boldsymbol{u}, A^\nu \operatorname{Re} \boldsymbol{u}) &= \tfrac{1}{2}[(\boldsymbol{u} + \boldsymbol{u}^*), A^\nu (\boldsymbol{u} + \boldsymbol{u}^*)] \\ &= \tfrac{1}{2}[(\boldsymbol{u}, A^\nu \boldsymbol{u}) + (\boldsymbol{u}^*, A^\nu, \boldsymbol{u}^*) + (\boldsymbol{u}^*, A^\nu \boldsymbol{u}) + (\boldsymbol{u}, A^\nu \boldsymbol{u}^*)]. \end{aligned} \tag{4}$$

Throughout this paper we have studied complex asymptotic solutions of the form $e^{i\lambda s}z$. For such solutions (4) becomes

$$S_\nu=\tfrac{1}{2}[(z, A^\nu z)+(z^*, A^\nu z^*)+e^{2i\lambda s}(z^*, A^\nu z)+e^{-2i\lambda s}(z, A^\nu z^*)]. \tag{5}$$

We note that the last two terms in (5) contain oscillatory factors, hence if we average (5) over a short time interval*, the average of the last two terms will be asymptotically zero for $\lambda\to\infty$. Thus the average value of S_ν is given by

$$\langle S_\nu\rangle=\tfrac{1}{2}[(z, A^\nu z)+(z^*, A^\nu z^*)]. \tag{6}$$

If we assume that A^ν is real for $\nu=1, \ldots, n$, then $(z^*, A^\nu z^*)=(z, A^\nu z)^*=(A^\nu z, z)=(z, A^\nu z)$, and (6) becomes

$$\langle S_\nu\rangle=(z, A^\nu z). \tag{7}$$

The vector $\langle \boldsymbol{S}\rangle=(\langle S_1\rangle, \ldots, \langle S_n\rangle)$ is the *average Poynting vector* of the given asymptotic solution.

We now introduce (3.5.1) in (7), and we use the basic identity (3.2.8) and the definition (3.7.5) of w. We find that

$$\langle S_\nu\rangle=g_\nu\sigma_t^*\sigma_t=w\,g_\nu, \tag{8}$$

hence

$$\langle \boldsymbol{S}\rangle=w\boldsymbol{G}. \tag{9}$$

We also find, from (3.7.7) that

$$w_t+\nabla\cdot\langle \boldsymbol{S}\rangle=w_t+(w\,g_\nu)_{x_\nu}=\frac{dw}{dt}+(g_\nu)_{x_\nu}w=-2(\operatorname{Re}\eta+\beta)\,w. \tag{10}$$

Thus if our system is conservative and the matrices $A^1, \ldots, A^n$ are constant, then $\eta=\beta=0$ and

$$w_t+\nabla\cdot\langle \boldsymbol{S}\rangle=0. \tag{11}$$

This equation is the well known energy-conservation equation. It explains why we have called w the "energy density". To be more precise we should call it the *average energy density*. From (9) we note that the Poynting vector $\langle \boldsymbol{S}\rangle$ has the direction of the (projected) ray and the magnitude $w\,g$, where g is the group speed. Thus it is particularly clear in the asymptotic theory that the Poynting vector measures the energy flux. From (3) we see that the average energy density of the electromagnetic field is given by

$$w_E=w/16\pi. \tag{12}$$

It is interesting to note that equations 1, 4, 5, 6, 7, 8 remain valid with $\nu=0$, hence $w=\langle S_0\rangle$ where $S_0=2(\boldsymbol{u}, A_0\boldsymbol{u})$. Now the matrix $A_0(\omega)$ and the above quantity S_0 are defined only for *asymptotic* solutions, because then $\omega=-s_t$ is defined. For *exact* solutions S_0 is undefined unless the system is non-dispersive, for then $A_0=\mathscr{E}$ is independent of ω. For the non-dispersive electromagnetic field, it is easy to show that $S_0=16\pi\,S_{0E}$, where S_{0E} is the usual electromagnetic energy density,

$$S_{0E}=\frac{1}{8\pi}[\varepsilon E^2+\mu H^2]. \tag{13}$$

* A space-average or space-time average will yield the same result.

For dispersive systems it does not appear to be possible to give a satisfactory definition of the energy density in terms of exact solutions, but only in the framework of the asymptotic theory. A similar remark applies to the group velocity vector, $\boldsymbol{G}$.*

As we have seen in section (3.11) an asymptotic solution is often given by a sum of terms of the form

$$\boldsymbol{u}(t, X)=\sum_j \boldsymbol{u}_{(j)}; \qquad \boldsymbol{u}_{(j)}=e^{i\lambda s_{(j)}}\, z_{(j)}. \tag{14}$$

(Each ray that passes through the point (t, X) contributes one term to this sum.) For such solutions, (4) becomes

$$\begin{aligned} S_\nu=\tfrac{1}{2}\sum_j\sum_m \{&e^{i\lambda[-s_{(j)}+s_{(m)}]}(z_{(j)}, A^\nu z_{(m)})+e^{i\lambda[s_{(j)}-s_{(m)}]}(z^*_{(j)}, A^\nu z^*_{(m)})+ \\ &+e^{i\lambda[s_{(j)}+s_{(m)}]}(z^*_{(j)}, A^\nu z_{(m)})+e^{i\lambda[-s_{(j)}-s_{(m)}]}(z_{(j)}, A^\nu z^*_{(m)})\}. \end{aligned} \tag{15}$$

We assume that $|s_j|\neq|s_m|$ for $j\neq m$. Then by averaging (15) we find that

$$\langle S_\nu\rangle=\tfrac{1}{2}\sum_j\{(z_{(j)}, A^\nu z_{(j)})+(z^*_{(j)}, A^\nu z^*_{(j)})\}=\sum_j (z_{(j)}, A^\nu z_{(j)}). \tag{16}$$

Thus

$$\langle \boldsymbol{S}\rangle=\sum_j \langle \boldsymbol{S}_{(j)}\rangle \tag{17}$$

where $\langle \boldsymbol{S}_{(j)}\rangle$ is the average Poynting vector for the asymptotic solution $\boldsymbol{u}_{(j)}=e^{i\lambda s_{(j)}}z_{(j)}$. If $w_{(j)}$ is the energy density of the same solution, then (11) implies that

$$w_{(j)\,t}+\nabla\cdot\langle \boldsymbol{S}_{(j)}\rangle=0. \tag{18}$$

Thus we see from (17) and (18) that

$$w_t+\nabla\cdot\langle \boldsymbol{S}\rangle=0 \tag{19}$$

where

$$w=\sum_j w_{(j)}. \tag{20}$$

From (19) and (20) we conclude that the energy density of an asymptotic solution consisting of a sum of terms (14) is the sum of the energy densities associated with each term.

C. *Solution of the equation for the energy density*

The energy density function w satisfies the equation (3.7.7)

$$\frac{dw}{dt}+[\nabla\cdot\boldsymbol{G}+2(\operatorname{Re}\eta+\beta)]\,w=0. \tag{1}$$

In order to solve this equation, we set

$$w=\gamma\alpha; \qquad \gamma=\exp\left\{-2\int_{t_0}^{t}[\operatorname{Re}\eta(t')+\beta(t')]\,dt'\right\}; \tag{2}$$

then it is easily seen that α satisfies the simpler equation,

$$\frac{d\alpha}{dt}+(\nabla\cdot\boldsymbol{G})\,\alpha=0. \tag{3}$$

* "Approximate" definitions, under special conditions, are sometimes given. See, *e.g.* [8].

We now set $t=x_0$, $g_0=1$ and introduce the $(n+1)$-vectors

$$\hat{\boldsymbol{G}}=(g_0, g_1, \dots, g_n)=(g_0, \boldsymbol{G}),$$
$$\hat{\nabla}=\left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_\nu}\right)=\left(\frac{\partial}{\partial x_0}, \nabla\right). \tag{4}$$

Then from (3)

$$\hat{\nabla}\cdot(\alpha\hat{\boldsymbol{G}})=\hat{\nabla}\alpha\cdot\hat{\boldsymbol{G}}+\alpha(\hat{\nabla}\cdot\hat{\boldsymbol{G}})=\frac{d\alpha}{dt}+\alpha(\nabla\cdot\boldsymbol{G})=0. \tag{5}$$

Let $\mathscr{G}$ be an arbitrary point set in the space $\mathscr{P}$ of parameters $\boldsymbol{\Gamma}$ which label the rays $(t, \boldsymbol{X})=[t, \boldsymbol{X}(t; \boldsymbol{\Gamma})]$. Let $\mathscr{T}$ be the set of points in space-time filled by the "tube" of ray segments for which $\boldsymbol{\Gamma}$ is in $\mathscr{G}$ and $t_1\leqq t\leqq t_2$. The region $\mathscr{T}$ is illustrated in Fig. 2.

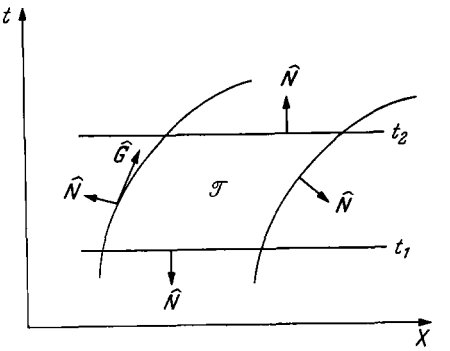

Fig. 2

We now apply Gauss' theorem to the region $\mathscr{T}$. From (5) it follows that

$$0=\int_{\mathscr{T}}\hat{\nabla}\cdot(\alpha\hat{\boldsymbol{G}})\,dx_0\,dx_1\dots dx_n$$
$$=\int_{\mathscr{S}}\alpha\hat{\boldsymbol{G}}\cdot\hat{\boldsymbol{N}}\,dS, \tag{6}$$

where $\hat{\boldsymbol{N}}$ is the unit outward normal vector to the surface $\mathscr{S}$ of $\mathscr{T}$. On that portion of $\mathscr{S}$ generated by rays $\hat{\boldsymbol{G}}\cdot\hat{\boldsymbol{N}}=0$. On the hyperplanes $t=t_1$ and $t=t_2$ $\hat{\boldsymbol{G}}\cdot\hat{\boldsymbol{N}}=\pm g_0=\pm 1$. Hence (6) becomes

$$\{\int_{t_2}-\int_{t_1}\}\alpha\,dx_1\dots dx_n=0. \tag{7}$$

But $dx_1\dots dx_n=j(t;\boldsymbol{\Gamma})\,d\gamma_1\dots d\gamma_n=j(t)\,d\boldsymbol{\Gamma}$, therefore

$$\int_{\mathscr{G}}[\alpha(t_2)\,j(t_2)-\alpha(t_1)\,j(t_1)]\,d\boldsymbol{\Gamma}=0, \tag{8}$$

and since $\mathscr{G}$, t_1 and t_2 were arbitrary we conclude that $\alpha(t)\,j(t)$ is constant on a ray, or

$$\alpha(t)=\alpha(t_0)\frac{j(t_0)}{j(t)}. \tag{9}$$

Now (2) implies that $w(t_0)=\alpha(t_0)$, therefore from (2) and (9) we find that the solution of (1) is given by

$$w(t)=w(t_0)\frac{j(t_0)}{j(t)}\exp\left\{-2\int_{t_0}^{t}[\operatorname{Re}\eta(t')+\beta(t')]\,dt'\right\}. \tag{10}$$

In order to solve equation (3.6.2) we set $w=\sigma^2$ and $\beta=0$. From (1) we find that

$$\frac{d\sigma}{dt}+\left[\frac{1}{2}\nabla\cdot\boldsymbol{G}+\eta\right]\sigma=0, \tag{11}$$

while (10) yields

$$\sigma(t)=\sigma(t_0)\left[\frac{j(t_0)}{j(t)}\right]^{\frac{1}{2}}\exp\left\{-\int_{t_0}^{t}\eta(t')\,dt'\right\}. \tag{12}$$

Since (3.6.2) and (11) are the same, (12) is the required solution.

D. *The dielectric permeability function*

Theoretical derivations [*14*] of the dielectric permeability for isotropic media with a single "resonance frequency" lead to formulas of the form

$$\hat{\varepsilon}_1(\hat{\omega}, X) - 1 = \frac{\varphi^2}{\rho^2 - \hat{\omega}^2 - i\nu\hat{\omega}}. \tag{1}$$

Here, as in chapter 2 and 7, $\hat{\omega}$ denotes the frequency and $\hat{\varepsilon}_1$ denotes the dielectric permeability. φ^2 is proportinal to the number density of oscillators of resonance frequency ρ, and ν is proportional to the damping force. φ, ρ, and ν may be functions of X, and $\nu \geqq 0$. For media with n resonance frequencies, the right side of (1) is replaced by a sum of n such terms. For an isotropic plasma, (1) is valid with $\rho = 0$. Then φ is called the "plasma frequency" and ν the "collision frequency".

If φ is independent of X, we set $\lambda = \varphi$. Otherwise we take λ to be some average value of φ (or an average value of ρ) and we set $r = \rho/\lambda$, $p = \varphi/\lambda$. Then

$$\rho = \lambda r(X), \quad \varphi = \lambda p(X). \tag{2}$$

As in section 2.1 we set

$$\omega = \hat{\omega}/\lambda; \quad \tilde{\varepsilon}_1(\omega, X) = \hat{\varepsilon}_1(\hat{\omega}, X). \tag{3}$$

Then

$$\tilde{\varepsilon}_1 = 1 + \frac{p^2}{r^2 - \omega^2 - i\omega\nu/\lambda}. \tag{4}$$

We now expand (4) for large λ and find that

$$\tilde{\varepsilon}_1 = \varepsilon_1 - (i\lambda)^{-1}\delta_1 + O(\lambda^{-2}) \tag{5}$$

where

$$\varepsilon_1 = 1 - \frac{p^2}{\omega^2 - r^2}, \quad \delta_1 = \frac{\omega\nu p^2}{(\omega^2 - r^2)^2}. \tag{6}$$

If we take the magnetic permeability of the medium to be $\hat{\varepsilon}_2 \equiv 1$, then from (2.1.24)

$$\tilde{\mathscr{E}} = \begin{bmatrix} \tilde{\varepsilon}_1 I_3 & 0 \\ 0 & I_3 \end{bmatrix} = \mathscr{E} - (i\lambda)^{-1}\mathscr{D} + O(\lambda^{-2}), \quad \mathscr{E} = \begin{bmatrix} \varepsilon_1 I_3 & 0 \\ 0 & I_3 \end{bmatrix}, \quad \mathscr{D} = \begin{bmatrix} \delta_1 I_3 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Thus we see that condition 1 of appendix G is satisfied. From (6) we find that

$$\frac{\partial}{\partial\omega}(\omega\varepsilon_1) = 1 + \frac{p^2(\omega^2 + r^2)}{(\omega^2 - r^2)^2}, \tag{8}$$

hence

$$A^0 = \frac{\partial}{\partial\omega}(\omega\mathscr{E}) \geqq I_6,$$

where I_6 is the 6×6 identity matrix. Since

$$\lim_{\omega\to\infty} \tilde{\mathscr{E}} = I_6$$

we see that condition 10 is satisfied with $A = I_6$. Condition 2 then follows from condition 10. Condition 4 is easily verified from (7) and (6).

In order to examine condition 5 we may compute the roots of $\det G=0$, with $\varepsilon_2=1$ and ε_1 given by (6). Proceeding as in section (7.1) we find, in addition to the roots

$$\omega=0, \quad \omega=\pm\sqrt{p^2+r^2}; \tag{9}$$

which are independent of $\boldsymbol{K}$ (and therefore uninteresting); the roots $\omega=\pm h_1(k)$, $\omega=\pm h_2(k)$, where

$$2(h_1)^2=(c^2k^2+r^2+p^2)\pm\sqrt{(c^2k^2+r^2+p^2)^2-4c^2k^2r^2}. \tag{10}$$

Since the roots are all real, condition 5 is satisfied.

Condition 3 may be verified by explicit calculation of the eigenvalues of the matrix G. There are two eigenvalues of multiplicity one and two of multiplicity two. They are distinct except at $\omega=0$*.

E. *Some vector identities*

We begin with the standard vector identity

$$\nabla\times(\boldsymbol{A}_2\times\boldsymbol{A}_3)=\boldsymbol{A}_2(\nabla\cdot\boldsymbol{A}_3)-\boldsymbol{A}_3(\nabla\cdot\boldsymbol{A}_2)+(\boldsymbol{A}_3\cdot\nabla)\boldsymbol{A}_2-(\boldsymbol{A}_2\cdot\nabla)\boldsymbol{A}_3. \tag{1}$$

If $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$ are orthonormal then (1) yields

$$\boldsymbol{A}_1\cdot\nabla\times(\boldsymbol{A}_2\times\boldsymbol{A}_3)=\boldsymbol{A}_1\cdot[(\boldsymbol{A}_3\cdot\nabla)\boldsymbol{A}_2-(\boldsymbol{A}_2\cdot\nabla)\boldsymbol{A}_3]. \tag{2}$$

We now apply (2) to the vectors $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ which satisfy (7.2.1). This yields

$$\boldsymbol{N}\cdot\nabla\times\boldsymbol{N}=\boldsymbol{N}\cdot\nabla\times(\boldsymbol{B}\times\boldsymbol{T})=\boldsymbol{N}\cdot[(\boldsymbol{T}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{T}], \tag{3}$$

$$\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}=\boldsymbol{B}\cdot\nabla\times(\boldsymbol{T}\times\boldsymbol{N})=\boldsymbol{B}\cdot[(\boldsymbol{N}\cdot\nabla)\boldsymbol{T}-(\boldsymbol{T}\cdot\nabla)\boldsymbol{N}], \tag{4}$$

and

$$\boldsymbol{T}\cdot\nabla\times\boldsymbol{T}=\boldsymbol{T}\cdot\nabla\times(\boldsymbol{N}\times\boldsymbol{B})=\boldsymbol{T}\cdot[(\boldsymbol{B}\cdot\nabla)\boldsymbol{N}-(\boldsymbol{N}\cdot\nabla)\boldsymbol{B}]. \tag{5}$$

But

$$\boldsymbol{N}\cdot(\boldsymbol{T}\cdot\nabla)\boldsymbol{B}+\boldsymbol{B}\cdot(\boldsymbol{T}\cdot\nabla)\boldsymbol{N}=(\boldsymbol{T}\cdot\nabla)(\boldsymbol{B}\cdot\boldsymbol{N})=0, \tag{6}$$

$$\boldsymbol{T}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{N}+\boldsymbol{N}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{T}=(\boldsymbol{B}\cdot\nabla)(\boldsymbol{N}\cdot\boldsymbol{T})=0, \tag{7}$$

and

$$\boldsymbol{B}\cdot(\boldsymbol{N}\cdot\nabla)\boldsymbol{T}+\boldsymbol{T}\cdot(\boldsymbol{N}\cdot\nabla)\boldsymbol{B}=(\boldsymbol{N}\cdot\nabla)(\boldsymbol{B}\cdot\boldsymbol{T})=0. \tag{8}$$

If we add (3) and (4) and subtract (5) we find that

$$\boldsymbol{N}\cdot\nabla\times\boldsymbol{N}+\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}-\boldsymbol{T}\cdot\nabla\times\boldsymbol{T}=2\boldsymbol{N}\cdot(\boldsymbol{T}\cdot\nabla)\boldsymbol{B}. \tag{9}$$

Now, by definition, $\boldsymbol{K}=k\boldsymbol{A}$, hence

$$\nabla\times\boldsymbol{K}=\nabla k\times\boldsymbol{A}+k\nabla\times\boldsymbol{A} \tag{10}$$

and

$$\boldsymbol{A}\cdot\nabla\times\boldsymbol{K}=k\boldsymbol{A}\cdot\nabla\times\boldsymbol{A}. \tag{11}$$

Therefore, since $\boldsymbol{K}=\nabla s$,

$$\boldsymbol{A}\cdot\nabla\times\boldsymbol{A}=\frac{1}{k}\boldsymbol{A}\cdot\nabla\times\boldsymbol{K}=\frac{1}{k}\boldsymbol{A}\cdot\nabla\times\nabla s=0. \tag{12}$$

* The confluence of eigenvalues at special points is a phenomenon that requires further study.

But $\boldsymbol{T}=\pm\boldsymbol{A}$; therefore $\boldsymbol{T}\cdot\nabla\times\boldsymbol{T}=0$ and (9) becomes

$$\boldsymbol{N}\cdot\nabla\times\boldsymbol{N}+\boldsymbol{B}\cdot\nabla\times\boldsymbol{B}=2\boldsymbol{N}\cdot(\boldsymbol{T}\cdot\nabla)\boldsymbol{B}. \tag{13}$$

F. *The phase-shift rule*

In chapter 5, we have evaluated the integral (4.3.6) asymptotically, by the method of stationary phase, for two choices of the source function $f=f(t, X; \lambda)$. If we perform the integrations first with respect to ξ and τ, we are led to a sum of integrals of the form

$$u(t,X)\sim\int g(K)\,e^{i\lambda\varphi(K)}\,dK, \tag{1}$$

where

$$\varphi(K)=k_\nu x_\nu-h(K)t-f(K). \tag{2}$$

In fact, whenever the source function is such that the integrals with respect to ξ and τ can be performed (exactly or asymptotically) we are led to integrals of the form (1, 2)*. Furthermore, integral representations of the solution of initial-value problems for linear hyperbolic equations also lead to integrals of the form (1, 2) when all integrations except those with respect to K are performed*.

We now evaluate (1) by the method of stationary phase [*10*]. At a stationary point,

$$\varphi_\nu=\frac{\partial\varphi}{\partial k_\nu}=x_\nu-\frac{\partial h}{\partial k_\nu}t-\frac{\partial f}{\partial k_\nu}=0, \tag{3}$$

hence

$$x_\nu=\xi_\nu+g_\nu t;\qquad \xi_\nu=\frac{\partial f}{\partial k_\nu},\qquad g_\nu=\frac{\partial h}{\partial k_\nu};\qquad \nu=1,\ldots,n. \tag{4}$$

Furthermore, at the stationary point,

$$\varphi_{\nu\mu}=\frac{\partial^2\varphi}{\partial k_\nu\partial k_\mu}=-\frac{\partial^2 h}{\partial k_\nu\partial k_\mu}t-\frac{\partial^2 f}{\partial k_\nu\partial k_\mu}=-\frac{\partial x_\nu}{\partial k_\mu};\qquad \nu,\mu=1,\ldots,n, \tag{5}$$

and

$$\varphi=(k_\nu g_\nu-h)t+s_0(K) \tag{6}$$

where

$$s_0=k_\nu\xi_\nu-f=k_\nu\frac{\partial f}{\partial k_\nu}-f. \tag{7}$$

The stationary phase formula then yields

$$u\sim\left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}n}|\det(\varphi_{\nu\mu})|^{-\frac{1}{2}}g(k)\,e^{i\lambda\{(k_\nu g_\nu-h)t+s_0\}-i\frac{\pi}{4}\operatorname{sig}J(t)},\qquad x_\nu=\xi_\nu+g_\nu t; \tag{8}$$

where sig $J(t)$ denotes the signature of the matrix

$$J(t)=(-\varphi_{\nu\mu})=\left(\frac{\partial x_\nu}{\partial k_\mu}\right).$$

We now suppose that K is a function of a parameter $\Gamma=(\gamma_1,\ldots,\gamma_n)$ i.e.

$$k_\nu=k_\nu(\Gamma). \tag{9}$$

* See, *e.g.* [*4, 5, 10*].

Differentiating this equation with respect to k_μ yields

$$\delta_{\nu\mu}=\frac{\partial k_\nu}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial k_\mu}. \tag{10}$$

But

$$\frac{\partial x_\nu}{\partial k_\mu}=\frac{\partial x_\nu}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial k_\mu}. \tag{11}$$

Hence

$$J(t)=\left(\frac{\partial x_\nu}{\partial k_\mu}\right)=\left(\frac{\partial x_\nu}{\partial \gamma_j}\right)\left(\frac{\partial k_\nu}{\partial \gamma_j}\right)^{-1}. \tag{12}$$

If we introduce the new parameter Γ, we may rewrite (8) in the form

$$\boldsymbol{u}\sim e^{i\lambda s(t;\Gamma)}\boldsymbol{z}(t;\Gamma);\qquad x_\nu=\xi_\nu+g_\nu t, \tag{13}$$

where

$$s=(k_\nu g_\nu-h)t+s_0, \tag{14}$$

and

$$z(t;\Gamma)=z(t)=\frac{1}{|j(t)|^{\frac{1}{2}}}e^{-i\frac{\pi}{4}\operatorname{sig}J(t)}\boldsymbol{a}(\Gamma). \tag{15}$$

Here $\boldsymbol{a}(\Gamma)$ is independent of t, and

$$j(t)=j(t;\Gamma)=\det\left(\frac{\partial x_\nu}{\partial \gamma_\mu}\right)=\det(\varphi_{\nu\mu})\det\left(\frac{\partial k_\nu}{\partial \gamma_\mu}\right). \tag{16}$$

We now compare (13) with (3.11.1) and (15) with (3.10.9) (with $\eta=0$). For $\tau<t<\tau'$ (where τ' is the first caustic point past $t=\tau$ on the ray), $\tilde{j}(\tau)/j(t)$ is positive, and (3.10.9) and (15) agree if we set

$$\tilde{z}(\tau)=e^{-i\frac{\pi}{4}\operatorname{sig}J(t)}\boldsymbol{a}(\Gamma)/|\tilde{j}(\tau)|^{\frac{1}{2}};\qquad \tau<t<\tau'. \tag{17}$$

We note that $J(t)$ is constant between caustic points, but may change discontinuously at such points. For t larger than τ' (but smaller than the next caustic point τ''), (17) and (15) yield

$$z(t)=\left|\frac{\tilde{j}(\tau)}{j(t)}\right|^{\frac{1}{2}}e^{-i\frac{\pi}{4}[\operatorname{sig}J(\tau'+0)-\operatorname{sig}J(\tau'-0)]}\tilde{z};\qquad \tau'<t<\tau''. \tag{18}$$

By comparing (18) with (3.10.9) we arive at the following rule* for the interpretation of the ambigious factor $[\tilde{j}(\tau)/j(t)]^{\frac{1}{2}}$.

*Phase-shift rule**:

$$\left[\frac{\tilde{j}(\tau)}{j(t)}\right]^{\frac{1}{2}}=\left|\frac{\tilde{j}(\tau)}{j(t)}\right|^{\frac{1}{2}}e^{-ip\frac{\pi}{2}}.$$

* Note that the phase-shift rule has been derived here by examining the asymptotic solution of problems with constant coefficients. Since problems with variable coefficients may be approximated, in the neighborhood of a caustic point, by problems with constant coefficients, we *assume* that the rule remains valid for the general problem. (This, then, is another application of the "indirect method" outlined in section 3.11.) In general, the functions $x_\nu=x_\nu(t;\Gamma)$ and $k_\nu=k_\nu(t;\Gamma)$ which appear in (20), are the solutions of the ray equations of section 3.3.

For $\tau<t<\tau'$, $p=0$. p is constant between caustic points. At a caustic point $t=\tau'$, p changes discontinuously by the amount

$$[p]=p(\tau'+0)-p(\tau'-0)=\tfrac{1}{2}[\operatorname{sig}J(\tau'+0)-\operatorname{sig}J(\tau'-0)], \tag{19}$$

where

$$J(t)=\left(\frac{\partial x_\nu}{\partial k_j}\right)=\left(\frac{\partial x_\nu}{\partial \gamma_j}\right)\left(\frac{\partial k_\nu}{\partial \gamma_j}\right)^{-1}. \tag{20}$$

(Since $[\operatorname{sig}J(\tau'+0)-\operatorname{sig}J(\tau'-0)]$ is an even number, p is always an integer.)

G. *Notation*

In the system of equations studied in this paper, the independent variables are t and $\boldsymbol{X}$. $\boldsymbol{X}$ is a *vector* with n components. The dependent variables $\boldsymbol{u}$ and $\boldsymbol{v}$ are *column vectors* with m components. In general, $\boldsymbol{A}$ denotes an n-vector with components $A_1, \ldots, A_n$ or $a_1, \ldots, a_n$. If $n=3$, $\boldsymbol{A}$ is an ordinary 3-vector. $\boldsymbol{a}$ denotes a column vector with m components. To save space we often write the components horizontally rather then vertically. Thus $\boldsymbol{a}=(a_1, \ldots, a_m)$. A denotes an $m\times m$ matrix or a scalar. Thus $A\boldsymbol{a}$ is a column vector. a denotes a scalar or (rarely) an $n\times n$ matrix.

The *inner product* of two column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar defined by

$$(\boldsymbol{a}, \boldsymbol{b})=\sum_{j=1}^{m} a_j^* b_j.$$

The star denotes the complex conjugate (or the hermitian conjugate for matrices). The following properties of the inner product are used. They follow easily from the definition

$$(\boldsymbol{a}, \boldsymbol{b})^*=(\boldsymbol{b}, \boldsymbol{a}); \qquad (c\boldsymbol{a}, \boldsymbol{b})=c^*(\boldsymbol{a}, \boldsymbol{b}); \quad (\boldsymbol{a}, c\boldsymbol{b})=c(\boldsymbol{a}, \boldsymbol{b});$$
$$(\boldsymbol{a}, A\boldsymbol{b})=(A^*\boldsymbol{a}, \boldsymbol{b}); \quad \text{if } A \text{ is hermitian } (\boldsymbol{a}, A\boldsymbol{b})=(A\boldsymbol{a}, \boldsymbol{b}).$$

The *summation convention* with respect to repeated indices is used. Usually the indices run from 1 to n. If not, a parenthetical note is inserted. Thus

$$a_\nu b_\nu \quad \text{denotes} \quad \sum_{\nu=1}^{n} a_\nu b_\nu,$$
$$a_j b_j, \ \left(\sum_{j=1}^{q}\right) \quad \text{denotes} \quad \sum_{j=1}^{q} a_j b_j.$$

Some column vectors $\boldsymbol{a}$ with 6 components are defined by an ordered pair of two 3-vectors. Thus $\boldsymbol{a}=(\boldsymbol{A}, \boldsymbol{B})=(A_1, A_2, A_3, B_1, B_2, B_3)$. The symbol $(\boldsymbol{A}, \boldsymbol{B})$ is not an inner product. Only column vectors $\boldsymbol{a}$, $\boldsymbol{b}$ appear in inner products.

H. *Summary of conditions*

The following *basic conditions* are imposed throughout this paper:

1. $\tilde{\mathscr{E}}(\omega, \boldsymbol{X}; \lambda)=\mathscr{E}(\omega, \boldsymbol{X})-(i\lambda)^{-1}\mathscr{D}(\omega, \boldsymbol{X})+O(\lambda^{-2})$. For real ω, $\mathscr{E}$ is hermitian.
2. For real ω, $A^0(\omega, \boldsymbol{X})=\dfrac{\partial}{\partial\omega}(\omega\mathscr{E})$ is positive-definite.

3. For real ω and for $\boldsymbol{K}=(k_1, \ldots, k_n)$ with real components, the multiplicity of each eigenvalue of the matrix $G=k_\nu A^\nu(\boldsymbol{X})-\omega\mathscr{E}(\omega, \boldsymbol{X})$ is independent of ω and $\boldsymbol{K}$.

4. For each $\boldsymbol{X}$, $\mathscr{E}(\omega, \boldsymbol{X})$ is a meromorphic* function of ω in a region $\operatorname{Im}\omega>2\beta_0$ $(\beta_0<0)$ which includes the upper half-plane and the real axis, and all the poles lie on the real axis. As $|\omega|\to\infty$, $\mathscr{E}(\omega, \boldsymbol{X})\to A(\boldsymbol{X})$ uniformly with respect to $\arg\omega$ in $0\leqq\arg\omega\leqq\pi$; where $A(\boldsymbol{X})$ is positive-definite.

5. For $\boldsymbol{K}=(k_1, \ldots, k_n)$ with real components the solutions $\omega=h(\boldsymbol{K}, \boldsymbol{X})$ of the determinantal equation $\det G=\det[k_\nu A^\nu(\boldsymbol{X})-\omega\mathscr{E}(\omega, \boldsymbol{X})]=0$ are real or lie in the region $\operatorname{Im}\omega<\beta_0(\beta_0<0)$.

Conditions 1, 2, and 3 are used repeatedly in the expansion procedure of chapter 3. Conditions 4 and 5 are used in obtaining the fourier integral representation of chapter 4 and its residue evaluation.

The following *special conditions* are required for certain purposes ($\boldsymbol{r}^1, \ldots, \boldsymbol{r}^q$ are null eigenvectors of G):

6. For real ω, $\mathscr{D}$ is hermitian.

7. $[\boldsymbol{r}^i, \omega\mathscr{D}(\omega)\,\boldsymbol{r}^j]=\eta(\omega)\,\delta_{ij}$; $\quad i, j=1, \ldots, q.$

8. $-\sum_{\nu=1}^{n}(\boldsymbol{r}^\ell, A^\nu_{x_\nu}\boldsymbol{r}^m)=2\beta\,\delta_{\ell m}$; $\quad \ell, m=1, \ldots, q.$

9. The hermitian matrices $A^1(\boldsymbol{X}), \ldots, A^n(\boldsymbol{X})$ and $\mathscr{E}(\omega, \boldsymbol{X})$ are real (hence symmetric) for real ω.

10. For real ω, $A^0(\omega, \boldsymbol{X})\geqq A(\boldsymbol{X})$; *i.e.*, the matrix A^0-A is non-negative.

Conditions 6, 7, 8, and 9 are required for the solution of the transport equations in the various cases discussed in sections 3.6–3.9.

Condition 10 is used in section 6.3 to derive the inequality (6.3.14).

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* A function is *meromorphic* in a region if it is analytic in the region except for isolated poles.

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