Existence in the Large of Periodic Solutions of Hyperbolic Partial Differential Equations

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The problem of existence of solutions $\varphi(x, y)$ periodic in x and in y of period T for a hyperbolic partial differential system of the form

$$u_{xy}=f(x, y, u, u_x, u_y), \tag{1}$$

where $u=(u_1, \dots, u_n)$, and $f=(f_1, \dots, f_n)$ is periodic in x and y of period T, presents a number of difficulties when no damping of any sort is assumed. In this paper we analyze this difficult problem in the line of our previous work on ordinary and partial differential equations. We conclude with criteria of existence for solutions to the problem above. These criteria can then be used for the analogous problem for the equation

$$u_{xx}-u_{yy}=g(x, y, u, u_x, u_y). \tag{2}$$

1. The Modified Problem

1. Modified Problem. We shall first associate with (1) the following analogous weaker problem or modified problem:

Given two periodic functions $u_0(x)$, $v_0(y)$ of class C^1 in $(-\infty, +\infty)$ and of period T,

$$u_0(x+T)=u_0(x), \quad v_0(x+T)=v_0(x),$$

determine a function $\Phi(x, y)$ continuous with its partial derivatives Φ_x, Φ_y, Φ_{xy}, two functions $m(y)$ and $n(x)$ both continuous, and a constant μ, such that

$$\begin{gathered}\Phi(x+T, y)=\Phi(x, y)=\Phi(x, y+T), \quad m(x+T)=m(x), \quad n(y+T)=n(y), \\ \int_0^T m(y)\,dy=0, \quad \int_0^T n(x)\,dx=0,\end{gathered} \tag{3}$$

and

$$\Phi_{xy}=f(x, y, \Phi, \Phi_x, \Phi_y)-m(y)-n(x)-\mu. \tag{4}$$

For this modified problem we shall prove theorems of existence, uniqueness, and continuous dependence on the boundary values and parameters. In (4) we assume

$$f(x+T, y, z, p, q)=f(x, y, z, p, q)=f(x, y+T, z, p, q).$$

Then the function Φ is a periodic solution of the original problem (1) if and only if we can determine $u_0(x)$, $v_0(y)$ in such a way that

$$\mu=0, \quad m(y)\equiv 0, \quad n(x)\equiv 0.$$

Criteria for this occurrence are given in Sections 12—19.

2. Theorem I (existence theorem for the modified problem). *If* $T>0$, *and* $N, N_1, N_2, L, b_1, b_2, M_1, M_2, M_3 \geqq 0$ *are constants, if* A *and* R *are the sets*

$$A=[0\leqq x\leqq T,\ 0\leqq y\leqq T],$$

$$R=[0\leqq x\leqq T,\ 0\leqq y\leqq T,\ |z|\leqq M_1,\ |p|\leqq M_2,\ |q|\leqq M_3],$$

if

$$M_1\geqq N+(N_1+N_2)T/2+3LT^2, \quad M_2\geqq N_1+3LT, \quad M_3\geqq N_2+3LT, \tag{5}$$

if

$$u_0(x), \quad 0\leqq x\leqq T, \quad v_0(y), \quad 0\leqq y\leqq T,$$

are vector functions which are continuous with $u_0'(x)$, $v_0'(y)$, *if*

$$f(x, y, z, p, q), \quad (x, y, z, p, q)\in R,$$

is continuous in R, *and*

$$u_0(T)=u_0(0), \quad v_0(T)=v_0(0)=0, \quad u_0'(T)=u_0'(0), \quad v_0'(T)=v_0'(0), \tag{6}$$

$$|u_0(0)|\leqq N,\ |u_0(x_1)-u_0(x_2)|\leqq N_1|x_1-x_2|,\ |v_0(y_1)-v_0(y_2)|\leqq N_2|y_1-y_2|, \tag{7}$$

$$f(T, y, z, p, q)=f(0, y, z, p, q), \quad f(x, T, z, p, q)=f(x, 0, z, p, q), \tag{8}$$

$$\begin{gathered}|f(x, y, z, p, q)|\leqq L,\\ |f(x, y, z, p_1, q_1)-f(x, y, z, p_2, q_2)|\leqq b_1|p_1-p_2|+b_2|q_1-q_2|;\end{gathered} \tag{9}$$

then for

$$2Tb_1<1, \quad 2Tb_2<1, \tag{10}$$

there exist a vector function $\varphi(x, y)$, $(x, y)\in A$, *continuous in* A *together with* $\varphi_x, \varphi_y, \varphi_{xy}$, *continuous vector functions* $m(y)$, $0\leqq y\leqq T$, $n(x)$, $0\leqq x\leqq T$, *and a constant* μ, *such that*

$$\varphi(x, 0)=\varphi(x, T)=u_0(x), \quad \varphi_y(x, 0)=\varphi_y(x, T), \tag{11}$$

$$\varphi(0, y)=\varphi(T, y)=u_0(0)+v_0(y), \quad \varphi_x(0, y)=\varphi_x(T, y), \tag{12}$$

$$m(0)=m(T), \quad n(0)=n(T), \tag{13}$$

$$\varphi_{xy}(x, y)=f\big(x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)\big)-m(y)-n(x)-\mu, \tag{14}$$

$$\int_0^T m(\eta)\,d\eta=0, \quad \int_0^T n(\xi)\,d\xi=0, \tag{15}$$

$$\mu=T^{-2}\int_0^T\int_0^T f\big(\xi, \eta, \varphi(\xi, \eta), \varphi_x(\xi, \eta), \varphi_y(\xi, \eta)\big)\,d\xi\,d\eta, \tag{16}$$

$$m(y)=T^{-1}\int_0^T f\big(\xi, y, \varphi(\xi, y), \varphi_x(\xi, y), \varphi_y(\xi, y)\big)\,d\xi-\mu, \tag{17}$$

$$n(x)=T^{-1}\int_0^T f\big(x, \eta, \varphi(x, \eta), \varphi_x(x, \eta), \varphi_y(x, \eta)\big)\,d\eta-\mu, \tag{18}$$

for all $0 \leqq x \leqq T$, $0 \leqq y \leqq T$. *Thus, by extending all functions* $\varphi(x, y)$, $m(y)$, $n(x)$, $f(x, y, z, p, q)$ *for all* $-\infty < x, y < +\infty$, $|z| \leqq M_1$, $|p| \leqq M_2$, $|q| \leqq M_3$, *by means of the periodicity of period* T *in* x *and* y, *equation* (14) *is satisfied in the whole* xy*-plane.*

3. Proof of Theorem 1. First let us note that relations (11), (12), (14), (15), imply (16), (17), (18). Indeed, by integration of (14) on A, we deduce (16). Then, by integration of (14) again on $0 \leqq x \leqq T$, or on $0 \leqq y \leqq T$, we deduce (17) and (18), respectively. Note that (8), (11), (12), (17), (18) imply (13), and that (16), (17), (18) imply (15).

Let us first prove that every vector function $\varphi(x, y)$, $(x, y) \in A$, satisfying

$$\begin{gathered} \varphi(x,0) = \varphi(x,T) = u_0(x), \qquad \varphi(0,y) = \varphi(T,y) = u_0(0) + v_0(y), \\ |\varphi(x_1,y_1) - \varphi(x_1,y_2) - \varphi(x_2,y_1) + \varphi(x_2,y_2)| \leqq 6L|x_1 - x_2|\,|y_1 - y_2|, \end{gathered} \tag{19}$$

also satisfies the relations

$$\begin{gathered} |\varphi(x,y)| \leqq M_1, \qquad |\varphi(x_1,y) - \varphi(x_2,y)| \leqq M_2\,|x_1 - x_2|, \\ |\varphi(x,y_1) - \varphi(x,y_2) \leqq M_3\,|y_1 - y_2|. \end{gathered} \tag{20}$$

Indeed, we have, for $0 \leqq x \leqq T$, $0 \leqq y \leqq T$,

$$|\varphi(x_1,y) - \varphi(x_2,y) - \varphi(x_1,0) + \varphi(x_2,0)| \leqq 6L|x_1 - x_2|\,y,$$

where

$$|\varphi(x_1,0) - \varphi(x_2,0)| = |u_0(x_1) - u_0(x_2)| \leqq N_1\,|x_1 - x_2|,$$

and hence

$$|\varphi(x_1,y) - \varphi(x_2,y)| \leqq (N_1 + 6Ly)\,|x_1 - x_2|.$$

Analogously, we have

$$|\varphi(x_1,y) - \varphi(x_2,y)| \leqq (N_1 + 6L(T-y))\,|x_1 - x_2|.$$

Since either $0 \leqq y \leqq T/2$ or $0 \leqq T - y \leqq T/2$, we have

$$|\varphi(x_1,y) - \varphi(x_2,y)| \leqq (N_1 + 3LT)\,|x_1 - x_2| \leqq M_2\,|x_1 - x_2|.$$

Analogously, we prove that

$$|\varphi(x,y_1) - \varphi(x,y_2)| \leqq (N_2 + 3LT)\,|y_1 - y_2| \leqq M_3\,|y_1 - y_2|.$$

Finally, for $0 \leqq x, y \leqq T/2$,

$$\begin{aligned} |\varphi(x,y)| &\leqq |\varphi(0,0)| + |\varphi(0,y) - \varphi(0,0)| + |\varphi(x,y) - \varphi(0,y)| \\ &\leqq N + N_2\,y + (N_1 + 6Ly)\,x \leqq N + (N_1 + N_2)\,T/2 + 3LT^2 \leqq M_1. \end{aligned}$$

Analogous reasoning holds for (x, y) in the remaining quadrants of A. Thus $|\varphi(x, y)| \leqq M_1$, $(x, y) \in A$. We have proved that relations (19) imply (20). Also, the vector functions $\varphi(x, y)$ satisfying (20) are all Lipschitzian in A, and hence have partial derivatives φ_x, φ_y a.e. in A satisfying $|\varphi_x| \leqq M_2$, $|\varphi_y| \leqq M_3$ a.e. in A.

The vector function $f(x, y, z, p, q)$ is continuous in R. Hence, there are continuous monotone functions $\omega_1(\alpha)$, $\omega_2(\beta)$, $\omega_3(\gamma)$ in $[0, +\infty)$ such that $\omega_1(0)=\omega_2(0)=\omega_3(0)=0$, and

$$\begin{aligned} &|f(x_1, y, z, p, q)-f(x_2, y, z, p, q)|\leqq\omega_1(|x_1-x_2|),\\ &|f(x, y_1, z, p, q)-f(x, y_2, z, p, q)|\leqq\omega_2(|y_1-y_2|),\\ &|f(x, y, z_1, p, q)-f(x, y, z_2, p, q)|\leqq\omega_3(|z_1-z_2|), \end{aligned} \tag{21}$$

for all

$$0\leqq x, x_1, x_2, y, y_1, y_2\leqq T, \quad |z|, |z_1|, |z_2|\leqq M_1, \quad |p|\leqq M_2, \quad |q|\leqq M_3.$$

The vector functions $u_0'(x)$, $v_0'(y)$ are continuous in $[0, T]$. Hence, there are continuous monotone functions $\omega_4(\alpha)$, $\omega_5(\beta)$, $0\leqq\alpha, \beta<+\infty$, with $\omega_4(0)=\omega_5(0)=0$, such that

$$|u_0'(x_1)-u_0'(x_2)|\leqq\omega_4(|x_1-x_2|), \quad |v_0'(y_1)-v_0'(y_2)|\leqq\omega_5(|y_1-y_2|). \tag{22}$$

Let

$$\eta_1(\beta)=(1-2Tb_2)^{-1}[\omega_5(\beta)+2T\omega_2(\beta)+2T\omega_3(M_3\beta)+12LTb_1\beta], \tag{23}$$

$$\eta_2(\alpha)=(1-2Tb_1)^{-1}[\omega_4(\alpha)+2T\omega_1(\alpha)+2T\omega_3(M_2\alpha)+12LTb_2\alpha]. \tag{24}$$

Both $\eta_1(\beta)$ and $\eta_2(\alpha)$, $0\leqq\alpha, \beta<+\infty$, are continuous monotone functions with $\eta_1(0)=\eta_2(0)=0$.

Let E be the linear space of all vector functions $\varphi(x, y)$, $(x, y)\in A$, continuous in A together with their partial derivatives φ_x, φ_y with norm $\|\varphi\|=\max|\varphi|+\max|\varphi_x|+\max|\varphi_y|$, where max is taken in A.

Let K be the subset of E made up of all vector functions $\varphi(x, y)\in E$ satisfying relations (19) and in addition

$$\begin{aligned} &\varphi_x(0, y)=\varphi_x(T, y), \quad \varphi_y(x, 0)=\varphi_y(x, T),\\ &|\varphi_x(x_1, y)-\varphi_x(x_2, y)|\leqq\eta_2(|x_1-x_2|), \quad |\varphi_x(x, y_1)-\varphi_x(x, y_2)|\leqq 6L|y_1-y_2|,\\ &|\varphi_y(x_1, y)-\varphi_y(x_2, y)|\leqq 6L|y_1-y_2|, \quad |\varphi_y(x, y_1)-\varphi_y(x, y_2)|\leqq\eta_1(|y_1-y_2|). \end{aligned} \tag{25}$$

Then the vector functions $\varphi\in K$ satisfy relations (20), and then $|\varphi|\leqq M_1$, $|\varphi_x|\leqq M_2$, $|\varphi_y|\leqq M_3$ everywhere in A. As a consequence the vector function

$$F(x, y)=f(x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)), (x, y)\in A, \tag{26}$$

is defined everywhere in A and is continuous in A.

For $\varphi\in K$ the vector functions $m(y)$ and $n(x)$ defined by (17) and (18) are continuous in $[0, T]$. With μ defined by (16), the vector function

$$\psi(x, y)=u_0(x)+v_0(y)+\int_0^x\int_0^y[F(\xi, \eta)-m(\eta)-n(\xi)-\mu]\,d\xi\,d\eta \tag{27}$$

is continuous in A together with its partial derivatives

$$\psi_x(x, y)=u_0'(x)+\int_0^y[F(x, \eta)-m(\eta)-n(x)-\mu]\,d\eta, \tag{28}$$

$$\psi_y(x, y)=v_0'(y)+\int_0^x[F(\xi, y)-m(y)-n(\xi)-\mu]\,d\xi. \tag{29}$$

Thus, relations (16), (17), (18), (26), and (27) define a map $\mathcal{T}:\psi=\mathcal{T}\varphi$, or $\mathcal{T}:K\to E$. Let us prove that actually $\mathcal{T}:K\to K$.

Since $|f|\leqq L$, by (16), (17), (18) we deduce

$$|\mu|\leqq L, \quad |m(y)|, |n(x)|\leqq 2L.$$

On the other hand, by (6), (8), (19), (25), (26), (27), (28), (29), we have with the usual conventions

$$\begin{gathered} F(0,y)=F(T,y), \quad F(x,0)=F(T,0), \quad m(0)=m(T), \quad n(0)=n(T), \\ \int_0^T m(\eta)\,d\eta=0, \quad \int_0^T n(\xi)\,d\xi=0. \end{gathered} \tag{30}$$

$$\begin{gathered} \psi(x,0)=\psi(x,T)=u_0(x), \qquad \psi_y(x,0)=\psi_y(x,T), \\ \psi(0,y)=\psi(T,y)=u_0(0)+v_0(y), \; \psi_x(0,y)=\psi_x(T,y), \end{gathered} \tag{31}$$

$$\begin{aligned} &|\psi(x_1,y_1)-\psi(x_1,y_2)-\psi(x_2,y_1)+\psi(x_2,y_2)| \\ &\quad=\left|\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(\xi,\eta)-m(\eta)-n(\xi)-\mu]\,d\xi\,d\eta\right|\leqq 6L|x_1-x_2|\,|y_1-y_2|. \end{aligned} \tag{32}$$

In other words $\psi=\mathcal{T}\varphi$ for $\varphi\in K$ satisfies relations (19) and, hence, relations (20) as proved above. Also, we have

$$|\psi_x(x,y_1)-\psi_x(x,y_2)|=\left|\int_{y_1}^{y_2}[F(x,\eta)-m(\eta)-n(x)-\mu]\,d\eta\right|,$$

$$|\psi_y(x_1,y)-\psi_y(x_2,y)|=\left|\int_{x_1}^{x_2}[F(\xi,y)-m(y)-n(\xi)-\mu]\,d\xi\right|,$$

and hence

$$|\psi_x(x,y_1)-\psi_x(x,y_2)|\leqq 6L|y_1-y_2|, \quad |\psi_y(x_1,y)-\psi_y(x_2,y)|\leqq 6L|x_1-x_2|. \tag{33}$$

Further, from (17) we have

$$\begin{aligned} |m(y_1)-m(y_2)|&=\left|T^{-1}\int_0^T[f(\xi,y_1,\varphi(\xi,y_1),\varphi_x(\xi,y_1),\varphi_y(\xi,y_1))\right. \\ &\qquad\left.-f(\xi,y_2,\varphi(\xi,y_2),\varphi_x(\xi,y_2),\varphi_y(\xi,y_2))]\,d\xi\right| \\ &\leqq T^{-1}\int_0^T[\omega_2(|y_1-y_2|)+\omega_3(|\varphi(\xi,y_1)-\varphi(\xi,y_2)|)+ \\ &\qquad+b_1|\varphi_x(\xi,y_1)-\varphi_x(\xi,y_2)|+b_2|\varphi_y(\xi,y_1)-\varphi_y(\xi,y_2)|]\,d\xi \\ &\leqq \omega_2(|y_1-y_2|)+\omega_3(M_3|y_1-y_2|)+6Lb_1|y_1-y_2|+ \\ &\qquad+b_2\,\eta_1(|y_1-y_2|), \end{aligned}$$

and analogously

$$|n(x_1)-n(x_2)|\leqq\omega_1(|x_1-x_2|)+\omega_3(M_2|x_1-x_2|)+b_1\,\eta_2(|x_1-x_2|)+6Lb_2|x_1-x_2|.$$

We have now from (29) and (23)

$$\begin{aligned}|\psi_y(x,y_1)-\psi_y(x,y_2)|&=|v_0'(y_1)-v_0'(y_2)+\\&\quad+\int_0^x[f(\xi,y_1,\varphi(\xi,y_1),\varphi_x(\xi,y_1),\varphi_y(\xi,y_1))-m(y_1)-\\&\quad-f(\xi,y_2,\varphi(\xi,y_2),\varphi_x(\xi,y_2),\varphi_y(\xi,y_2))+m(y_2)]\,d\xi|\\&\leqq\omega_5(|y_1-y_2|)+\int_0^T[\omega_2(|y_1-y_2|)+\\&\quad+\omega_3(|\varphi(\xi,y_1)-\varphi(\xi,y_2)|)+b_1|\varphi_x(\xi,y_1)-\varphi_x(\xi,y_2)|+\\&\quad+b_2|\varphi_y(\xi,y_1)-\varphi_y(\xi,y_2)|+|m(y_1)-m(y_2)|]\,d\xi\\&\leqq\omega_5(|y_1-y_2|)+2T\omega_2(|y_1-y_2|)+2T\omega_3(M_3|y_1-y_2|)+\\&\quad+12LTb_1|y_1-y_2|+2Tb_2\,\eta_1(|y_1-y_2|)\\&=(1-2Tb_2)\,\eta_1(|y_1-y_2|)+2Tb_2\,\eta_1(|y_1-y_2|)\\&=\eta_1(|y_1-y_2|).\end{aligned}\tag{34}$$

Analogously, we have

$$|\psi_x(x_1,y)-\psi_x(x_2,y)|\leqq\eta_2(|x_1-x_2|).\tag{35}$$

Relations (31), (32), (33), (34) show that $\psi=\mathscr{T}\varphi$ for $\varphi\in K$ satisfies all relations (19) and (25). Thus $\psi\in K$, and $\mathscr{T}:K\to K$.

The transformation $\mathscr{T}:K\to K$, $K\subset E$, is continuous in K with respect to the norm $\|\varphi\|$ of E. Indeed, for two vector functions $\varphi_j\in K, j=1,2$, we have $\psi_j=\mathscr{T}\varphi_j$, $F_j=F_{\varphi_j}$, $m_j=m_{\varphi_j}(y)$, $n_j=n_{\varphi_j}(x)$, $\mu_j=\mu_{\varphi_j}$, $j=1,2$, and from (16)

$$\begin{aligned}|\mu_1-\mu_2|&=\Big|T^{-2}\int_0^T\int_0^T[f(\xi,\eta,\varphi_1(\xi,\eta),\varphi_{1x}(\xi,\eta),\varphi_{1y}(\xi,\eta))-\\&\quad-f(\xi,\eta,\varphi_2(\xi,\eta),\varphi_{2x}(\xi,\eta),\varphi_{2y}(\xi,\eta))]\,d\xi\,d\eta\Big|\\&\leqq[\omega_3(\|\varphi_1-\varphi_2\|)+b_1\|\varphi_1-\varphi_2\|+b_2\|\varphi_1-\varphi_2\|].\end{aligned}$$

Then from (17) we have

$$\begin{aligned}|m_1(y)-m_2(y)|&=\Big|T^{-1}\int_0^T[f(x,y,\varphi_1(x,y),\varphi_{1x}(x,y),\varphi_{1y}(x,y))-\\&\quad-f(x,y,\varphi_2(x,y),\varphi_{2x}(x,y),\varphi_{2y}(x,y))]\,dx-\mu_1+\mu_2\Big|\\&\leqq2[\omega_3(\|\varphi_1-\varphi_2\|)+b_1\|\varphi_1-\varphi_2\|+b_2\|\varphi_1-\varphi_2\|],\end{aligned}$$

and analogously from (18)

$$|n_1(x)-n_2(x)|\leqq2[\omega_3(\|\varphi_1-\varphi_2\|)+b_1\|\varphi_1-\varphi_2\|+b_2\|\varphi_1-\varphi_2\|].$$

From (27) we deduce

$$|\psi_1(x,y)-\psi_2(x,y)|$$
$$=\left|\int_0^x\int_0^y [F_1(\xi,\eta)-m_1(\eta)-n_1(\xi)-\mu_1-F_2(\xi,\eta)+m_2(\eta)+n_2(\xi)+\mu_2]\,d\xi\,d\eta\right|$$
$$\leqq 6T^2[\omega_3(\|\varphi_1-\varphi_2\|)+(b_1+b_2)\|\varphi_1-\varphi_2\|].$$

Analogously, from (28) and (29),

$$|\psi_{1x}(x,y)-\psi_{2x}(x,y)|\leqq 6T[\omega_3(\|\varphi_1-\varphi_2\|)+(b_1+b_2)\|\varphi_1-\varphi_2\|],$$
$$|\psi_{1y}(x,y)-\psi_{2y}(x,y)|\leqq 6T[\omega_3(\|\varphi_1-\varphi_2\|)+(b_1+b_2)\|\varphi_1-\varphi_2\|].$$

Thus $\|\psi_1-\psi_2\|\to 0$ as $\|\varphi_1-\varphi_2\|\to 0$ uniformly in K. Finally, the set K is obviously convex, closed and compact with respect to the norm $\|\varphi\|$ of E. By SCHAUDER's fixed point theorem we conclude that there is at least an element $\varphi(x,y)\in K$ such that $\varphi=\mathscr{T}\varphi$, or

$$\varphi(x,y)=u_0(x)+v_0(y)+$$
$$+\int_0^x\int_0^y [f(\xi,\eta,\varphi(\xi,\eta),\varphi_x(\xi,\eta),\varphi_y(\xi,\eta))-m(\eta)-n(\xi)-\mu]\,d\xi\,d\eta,$$

$$m(y)=T^{-1}\int_0^T f(\xi,y,\varphi(\xi,y),\varphi_x(\xi,y),\varphi_y(\xi,y))\,d\xi-\mu,$$

$$n(x)=T^{-1}\int_0^T f(x,\eta,\varphi(x,\eta),\varphi_x(x,\eta),\varphi_y(x,\eta))\,d\eta-\mu,$$

$$\mu=T^{-2}\int_0^T\int_0^T f(\xi,\eta,\varphi(\xi,\eta),\varphi_x(\xi,\eta),\varphi_y(\xi,\eta))\,d\xi\,d\eta,$$

for all $0\leqq x, y\leqq T$. Obviously φ_x, φ_y, φ_{xy} exist everywhere in A, are continuous in A, and, everywhere in A, we have

$$\varphi_{xy}=f(x,y,\varphi,\varphi_x,\varphi_y)-m(y)-n(x)-\mu.$$

Theorem I is thereby proved.

4. Remark 1. If f is Lipschitzian with respect to all variables x, y, z, p, q in R, and if $u_0'(x)$, $v_0'(y)$ also are Lipschitzian, then $m(y)$, $n(x)$, as well as φ, φ_x, φ_y, φ_{xy} are Lipschitzian. Indeed, if $\omega_1(\alpha)=k_1\alpha$, $\omega_2(\beta)=k_2\beta$, $\omega_3(\gamma)=b_0\gamma$, $\omega_4(\alpha)=k_4\alpha$, $\omega_5(\beta)=k_5\beta$, then

$$\eta_1(\beta)=(1-2Tb_2)^{-1}(k_5+2Tk_2+2Tb_0M_3+12LTb_1)\beta=k_6\beta,$$
$$\eta_2(\alpha)=(1-2Tb_1)^{-1}(k_4+2Tk_1+2Tb_0M_2+12LTb_2)\alpha=k_7\alpha,$$

and then

$$|m(y_1)-m(y_2)|\leqq(k_2+b_0M_3+6Lb_1+b_2k_6)|y_1-y_2|=k_8|y_1-y_2|,$$
$$|n(x_1)-n(x_2)|\leqq(k_1+b_0M_2+b_1k_7+6Lb_2)|x_1-x_2|=k_9|x_1-x_2|.$$

Formulas (33), (34), (35) show that φ_x, φ_y are also uniformly Lipschitzian and so is $\varphi_{xy}=f-m-n-\mu$.

5. Remark 2. The conditions of Theorem I do not assure uniqueness, as the following example shows. Take $T=1$, $u_0(x)=0$, $v_0(y)=0$, $f=|z|^{\frac{1}{2}}\sin 2\pi x \sin 2\pi y$, for $0\leqq x, y\leqq 1$, and all y, z, p, q. Then the equation

$$u_{xy}=|u|^{\frac{1}{2}}\sin 2\pi x \sin 2\pi y, \tag{36}$$

besides the trivial solution $\varphi_1(x,y)=0$, has also the solution $\varphi_2(x,y)=(16\pi^4)^{-1}\times \sin^4\pi x \sin^4 \pi y$, $0\leqq x, y\leqq 1$, and both satisfy the boundary conditions. Here we have $m_1(y)=m_2(y)=0$, $n_1(x)=n_2(x)=0$. Note that we may take $N=N_1=N_2=0$, $L=1$, $M_1=1$, $M_2=M_3=2$, $b_1=b_2=0$. All conditions of Theorem I are satisfied.

6. The Lipschitzian Case. We shall assume now that $\omega_3(\gamma)=b_0|\gamma|$, so that f is now Lipschitzian in z, p, q with constants b_0, b_1, b_2. In this situation, for given boundary values $u_0(x)$, $v_0(x)$ and different functions $\varphi_1, \varphi_2\in K$ we have

$$|\mu_1-\mu_2|\leqq(b_0+b_1+b_2)\,\|\varphi_1-\varphi_2\|,$$

$$|m_1(y)-m_2(y)|,\quad |n_1(x)-n_2(x)|\leqq 2(b_0+b_1+b_2)\,\|\varphi_1-\varphi_2\|,$$

$$|\psi_1(x,y)-\psi_2(x,y)|\leqq 6T^2(b_0+b_1+b_2)\,\|\varphi_1-\varphi_2\|,$$

$$|\psi_{1x}(x,y)-\psi_{2x}(x,y)|,\quad |\psi_{1y}(x,y)-\psi_{2y}(x,y)|\leqq 6T(b_0+b_1+b_2)\,\|\varphi_1-\varphi_2\|.$$

Thus

$$\|\psi_1-\psi_2\|=\|\mathscr{T}\varphi_1-\mathscr{T}\varphi_2\|\leqq 6T(T+2)(b_0+b_1+b_2)\,\|\varphi_1-\varphi_2\|.$$

If we assume now that

$$6T(T+2)(b_0+b_1+b_2)<1, \tag{37}$$

then $\mathscr{T}:K\to K$ is a contraction into. This remark yields

7. Theorem II (uniqueness). *Under the same hypotheses of Theorem I, if* $\omega_3(\gamma)=b_0|\gamma|$, *and* (37) *holds, then* $\mathscr{T}:K\to K$ *is a contraction, and problem* (11)–(18) *has one and only one solution.*

The boundary values are represented by the pair of functions $w=(u_0(x), v_0(y))$ of class C^1 and satisfying (6) and (7). Therefore, they form a subset $\mathscr{B}$ of the linear space of all w of class C^1 satisfying (6) only, and we take in this linear space the norm

$$\|w\|=\max|u_0(x)|+\max|u_0'(x)|+\max|v_0(y)|+\max|v_0'(y)|. \tag{38}$$

The solution of the problem (11)–(18) is actually the system $W=[\varphi(x,y), m(y), n(x), \mu]$. These quadruples also can be thought of as imbedded in a linear space on which we take the norm

$$\|W\|=\max|\varphi|+\max|\varphi_x|+\max|\varphi_y|+\max|m|+\max|n|+|\mu|. \tag{39}$$

We shall prove that the solution, or system W, is a continuous function $\mathscr{F}$ of the boundary values, or system w, and we write

$$W=\mathscr{F}w,\qquad w\in\mathscr{B}.$$

We shall need the numbers

$$\Delta=(1-6Tb_1)(1-6Tb_2)-36T^2b_1b_2,\quad k=6T^2b_0+72T^3\Delta^{-1}b_0(b_1+b_2). \tag{40}$$

8. Theorem III (continuous dependence upon the boundary values). Under the *conditions of Theorem II, if in addition* $\Delta>0$, *and* $0<k<1$, *then the solution* $W=(\varphi, m, n, \mu)$ *of problem* (11)−(18) *is a continuous function* $\mathscr{F}$ *of the boundary values* $w=(u_0, v_0)\in\mathscr{B}$ *in the topologies determined by the norms* (38) *and* (39).

9. Proof of Theorem III. Let $w_1=[u_{01}(x), v_{01}(y)]$, $w_2=[u_{02}(x), v_{02}(y)]$ be a pair of boundary values as in Theorems I and II, and let $W_1=[\varphi_1, m_1, n_1, \mu_1]$, $W_2=[\varphi_2, m_2, n_2, \mu_2]$ be the corresponding solutions. Let

$$\varepsilon=\|w_1-w_2\|=\max|u_{01}(x)-u_{02}(x)|+\max|u'_{01}(x)-u'_{02}(x)|+$$
$$+\max|v_{01}(y)-v_{02}(y)|+\max|v'_{01}(y)-v'_{02}(y)|,$$

$$\alpha=\max|\varphi_1(x,y)-\varphi_2(x,y)|, \qquad \beta=\max|\varphi_{1x}(x,y)-\varphi_{2x}(x,y)|,$$

$$\gamma=\max|\varphi_{1y}(x,y)-\varphi_{2y}(x,y)|, \qquad \delta=\max|m_1(y)-m_2(y)|,$$

$$\delta'=\max|n_1(x)-n_2(x)|, \qquad \delta''=|\mu_1-\mu_2|.$$

We shall denote by F_1 and F_2 the functions F relative to φ_1 and φ_2. Then we have

$$\delta''=|\mu_1-\mu_2|=\left|T^{-2}\int_0^T\int_0^T[F_1(x,y)-F_2(x,y)]\,dx\,dy\right|\leqq b_0\alpha+b_1\beta+b_2\gamma,$$

$$|m_1(y)-m_2(y)|=\left|T^{-1}\int_0^T[F_1(x,y)-F_2(x,y)]\,dx-\mu_1+\mu_2\right|\leqq b_0\alpha+b_1\beta+b_2\gamma+\delta'',$$

$$|n_1(x)-n_2(x)|=\left|T^{-1}\int_0^T[F_1(x,y)-F_2(x,y)]\,dy-\mu_1+\mu_2\right|\leqq b_0\alpha+b_1\beta+b_2\gamma+\delta''.$$

Hence

$$\delta''\leqq b_0\alpha+b_1\beta+b_2\gamma, \qquad \delta,\delta'\leqq b_0\alpha+b_1\beta+b_2\gamma+\delta''. \tag{41}$$

Analogously,

$$|\varphi_1(x,y)-\varphi_2(x,y)|=\Big|u_{01}(x)+v_{01}(y)-u_{02}(x)-v_{02}(y)+$$
$$+\int_0^x\int_0^y[F_1(x,y)-m_1(y)-n_1(x)-\mu_1-F_2(x,y)+m_2(y)+n_2(x)+\mu_2]\,dx\,dy\Big|$$
$$\leqq\varepsilon+T^2(b_0\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta''),$$

$$|\varphi_{1x}(x,y)-\varphi_{2x}(x,y)|\leqq\varepsilon+T(b_0\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta''),$$

$$|\varphi_{1y}(x,y)-\varphi_{2y}(x,y)|\leqq\varepsilon+T(b_0\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta''),$$

and hence

$$\begin{aligned}&\alpha\leqq\varepsilon+T^2(b_0\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta''),\\&\beta,\gamma\leqq\varepsilon+T(b_0\alpha+b_1\beta+b_2\gamma+\delta+\delta'+\delta'').\end{aligned}\tag{42}$$

Relations (41), (42) yield

$$\begin{aligned}&\delta''\leqq b_0\alpha+b_1\beta+b_2\gamma, &&\delta,\delta'\leqq 2(b_0\alpha+b_1\beta+b_2\gamma),\\&\alpha\leqq\varepsilon+6T^2(b_0\alpha+b_1\beta+b_2\gamma), &&\beta,\gamma\leqq\varepsilon+6T(b_0\alpha+b_1\beta+b_2\gamma).\end{aligned}\tag{43}$$

The last relation can be written in the form

$$\beta=6Tb_0\xi_1\alpha+6Tb_1\xi_1\beta+6Tb_2\xi_1\gamma+\varepsilon\xi_1,$$
$$\gamma=6Tb_0\xi_2\alpha+6Tb_1\xi_2\beta+6Tb_2\xi_2\gamma+\varepsilon\xi_2,$$

where $0\leqq\xi_1,\ \xi_2\leqq 1$ are convenient numbers, and then

$$(1-6Tb_1\xi_1)\beta-6Tb_2\xi_1\gamma=6Tb_0\xi_1\alpha+\varepsilon\xi_1,$$
$$-6Tb_1\xi_2\beta+(1-6Tb_2\xi_2)\gamma=6Tb_0\xi_2\alpha+\varepsilon\xi_2.$$

If Δ' is the determinant of this system, we have $\Delta'\geqq\Delta>0$, $0<6Tb_j<1$, $j=1, 2$, and

$$\beta=\Delta'^{-1}\{(1-6Tb_2\xi_2)(6Tb_0\xi_1\alpha+\varepsilon\xi_1)+(6Tb_2\xi_1)(6Tb_0\xi_2\alpha+\varepsilon\xi_2)\}$$
$$\leqq 2\Delta^{-1}(\varepsilon+6Tb_0\alpha).$$

Analogously, we have

$$\gamma\leqq 2\Delta^{-1}(\varepsilon+6Tb_0\alpha).$$

Finally, by (43)

$$\alpha\leqq\varepsilon+72T^3\Delta^{-1}b_0(b_1+b_2)\alpha+6T^2b_0\alpha+12T^2\Delta^{-1}(b_1+b_2)\varepsilon.$$

Since the number k defined in (40) lies in the interval $0<k<1$, we have

$$\alpha\leqq(1-k)^{-1}[1+12\Delta^{-1}T^2(b_1+b_2)]\varepsilon.$$

This proves that $\alpha, \beta, \gamma, \delta, \delta', \delta''\to 0$ as $\varepsilon\to 0$ uniformly in K. Theorem III is thereby proved.

10. A Method of Successive Approximations. Under the hypotheses of Theorem III, the usual method of successive approximations defined by $\varphi_{k+1}=\mathcal{T}\varphi_k$, $k=0, 1 \ldots$, converges toward the solution φ of problem (11)–(18), where φ_0 is an arbitrary element of K. It may be convenient to use as first approximation

$$\varphi_0(x\, y)=u_0(x)+v_0(y).$$

Then

$$F_0(x, y)=f\big(x, y, u_0(x)+v_0(y),\ \ u_0'(x), v_0'(y)\big),\quad \mu_0=T^{-2}\int_0^T\int_0^T F_0(\xi,\eta)\,d\xi\,d\eta,$$
$$m_0(y)=T^{-1}\int_0^T F_0(\xi, y)\,d\xi,\qquad u_0(x)=T^{-1}\int_0^T F_0(x,\eta)\,d\eta, \tag{44}$$
$$\varphi_1(x, y)=u_0(x)+v_0(y)+\int_0^x\int_0^y[F_0(\xi,\eta)-m_0(\eta)-n_0(\xi)-\mu_0]\,d\xi\,d\eta,$$

and successively,

$$F_k(x, y)=f\big(x, y, \varphi_k(x, y), \varphi_{kx}(x, y)+\varphi_{ky}(x, y)\big),$$
$$\mu_k=T^{-2}\int_0^T\int_0^T F_k(\xi,\eta)\,d\xi\,d\eta,\qquad m_k(y)=T^{-1}\int_0^T F_k(\xi, y)\,d\xi,$$
$$n_k(x)=T^{-1}\int_0^T F_k(x,\eta)\,d\eta,\qquad \varphi_{k+1}(x, y)=u_0(x)+v_0(y)+ \tag{45}$$
$$+\int_0^T\int_0^T[F_k(\xi,\eta)-m_k(\eta)-n_k(\xi)-\mu_k]\,d\xi\,d\eta,\qquad k=1, 2, \ldots.$$

Then we have $\varphi_k \to \varphi$, $\varphi_{kx} \to \varphi_x$, $\varphi_{ky} \to \varphi_y$, $m_k \to m$, $n_k \to n$, $\mu_k \to \mu$ uniformly for $0 \leqq x \leqq T$, $0 \leqq y \leqq T$, and consequently we have also $\varphi_{kxy} \to \varphi_{xy}$ as $k \to \infty$ uniformly.

11. Smoothness of the Solution. Two theorems can now be stated concerning the smoothness of the solution (φ, m, n, μ) of the problem (11)–(18).

(α) Under the conditions of Theorems I, II, III, if $f(x, y, z, p, q)$ is of class C^1 in R and $u_0(x)$, $v_0(y)$ of class C^2, then $m(y)$, $n(x)$ are of class C^1 and $\varphi(x, y)$ of class C^2.

This statement was essentially proved in Section 12 of [*2a*]. A more precise statement is as follows:

(β) Under the conditions of Theorems I, II, III, if $f(x, y, z, p, q)$ is of class C^1 with Lipschitzian first order partial derivatives, if $u_0(x)$, $v_0(y)$ are of class C^2 with Lipschitzian second order derivatives, then $m(y)$, $n(x)$ are of class C^1 with Lipschitzian first derivatives and $\varphi(x, y)$ of class C^2 also with Lipschitzian second order partial derivatives.

The proof is the same as for (α). An analogous statement holds:

(γ) Under the conditions of Theorem I, II, III, if $f(x, y, z, p, q)$ is of class C^{1+r} in R with Lipschitzian partial derivatives of the order $1+r$, if $u_0(x)$, $v_0(y)$ are of class C^{2+r} with Lipschitzian derivatives of the order $2+r$, then $m(y)$, $n(x)$ are of class C^{1+r} with Lipschitzian derivatives of order $1+r$, and $\varphi(x, y)$ is of class C^{2+r} with Lipschitzian partial derivatives of the order $2+r$.

2. Criteria for the Existence of Periodic Solutions in the Large of the Original Problem

12. A Differential Equation Containing a Small Parameter. Let us consider the differential equation

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y), \\ &f = \varepsilon[\psi(x, y) + C u + \psi_1(y) u_x + \psi_2(x) u_y] + \varepsilon^2 g(x, y, u, u_x, u_y), \end{aligned} \tag{46}$$

where ε is a small parameter, and ψ, ψ_1, ψ_2, g are periodic functions of period $T = 2\pi/\omega$ in x and y. The Fourier series of ψ, ψ_1, ψ_2 will be denoted by

$$\psi(x, y) \sim \sum_{m,n=0}^{\infty} (a_{mn} \cos m\omega x \cos n\omega y + b_{mn} \cos m\omega x \sin n\omega y +$$

$$+ c_{mn} \sin m\omega x \cos n\omega y + d_{mn} \sin m\omega x \sin n\omega y),$$

$$\psi_1(y) \sim e_0 + \sum_1^{\infty} (e_n \cos n\omega y + f_n \sin n\omega y),$$

$$\psi_2(x) \sim g_0 + \sum_1^{\infty} (g_n \cos n\omega x + h_n \sin n\omega x).$$

If $u_0(x)$, $v_0(y)$ denote arbitrary boundary values with $v_0(0) = 0$, and u_0, v_0 both of class C^1 and u_0', v_0' Lipschitzian with constants k_1, k_2 respectively, then it is convenient for us to denote their Fourier series as follows:

$$u_0(x) \sim \alpha_0 + \sum_1^{\infty} (\alpha_n \cos n\omega x - \alpha_n + \beta_n \sin n\omega x),$$
$$v_0(y) \sim \sum_1^{\infty} (\gamma_n \cos n\omega y - \gamma_n + \delta_n \sin n\omega y), \tag{47}$$

where both series $\sum \alpha_m$, $\sum \gamma_n$ are absolutely convergent. With this notation we have

$$u_0(0) = \alpha_0, \quad v_0(0) = 0, \quad T^{-1} \int_0^T u_0(x)\,dx = \alpha_0 - \sum_1^{\infty} \alpha_n,$$
$$T^{-1} \int_0^T v_0(y)\,dy = -\sum_1^{\infty} \gamma_n.$$

If we apply *formally* the method of successive approximations of Section 10 to equation (46) with initial values $u_0(x)$, $v_0(y)$, we obtain at the first approximation and preserving only the terms in ε, namely a quadruple $[\varphi_0, \varepsilon m_0, \varepsilon n_0, \varepsilon \mu_0]$, with $\varphi_0, m_0, n_0, \mu_0$ given by

$$\varphi_0(x, y) = u_0(x) + v_0(y) + \varepsilon \int_0^x \int_0^y [F_0(\xi, \eta) - m_0(\xi) - n_0(\eta) - \mu_0]\,d\xi\,d\eta,$$
$$m_0(y) = T^{-1} \int_0^T F_0(\xi, y)\,d\xi - \mu_0, \quad n_0(x) = T^{-1} \int_0^T F_0(x, \eta)\,d\eta - \mu_0, \tag{48}$$
$$\mu_0 = T^{-2} \int_0^T \int_0^T F_0(\xi, \eta)\,d\xi\,d\eta,$$
$$F_0(x, y) = \psi(x, y) + C\,u_0(x) + C\,v_0(y) + \psi_1(y)\,u_0'(x) + \psi_2(x)\,v_0'(y).$$

If we write

$$m_0(y) \sim \sum_1^{\infty} (B_n \cos n\omega y + C_n \sin n\omega y), \qquad n_0(x) \sim \sum_1^{\infty} (D_n \cos n\omega x + E_n \sin n\omega x),$$
$$\mu_0 = A_0,$$

we obtain

$$\mu_0 = A_0 = a_{0\,0} + C\left(\alpha_0 - \sum_1^{\infty} \alpha_s - \sum_1^{\infty} \gamma_s\right),$$
$$m_0(y) = a_{0\,0} + \kappa_1(y) + C\left(\alpha_0 - \sum_1^{\infty} \alpha_s\right) + C\,v_0(y) + g_0\,v_0'(y) - \mu_0, \tag{49}$$
$$n_0(x) = a_{0\,0} + \kappa_2(x) + C\,u_0(x) + e_0\,u_0'(x) + C\left(-\sum_1^{\infty} \gamma_s\right) - \mu_0,$$

$$\kappa_1(y) = T^{-1} \int_0^T \psi(\xi, y)\,d\xi - a_{0\,0} \sim \sum_1^{\infty} (a_{0\,n} \cos n\omega y + b_{0\,n} \sin n\omega y),$$
$$\kappa_2(x) = T^{-1} \int_0^T \psi(x, \eta)\,d\eta - a_{0\,0} \sim \sum_1^{\infty} (a_{n_0} \cos n\omega x + c_{n_0} \sin n\omega x). \tag{50}$$

In terms of the Fourier constants relations (49) become

$$\begin{aligned} &A_0=a_{00}+C\left(\alpha_0-\sum_1^\infty\alpha_s-\sum_1^\infty\gamma_s\right), && B_n=C\gamma_n+n\omega g_0\delta_n+a_{0n},\\ &C_n=C\delta_n-n\omega g_0\gamma_n+b_{0n}, && D_n=C\alpha_n+n\omega e_0\beta_n+a_{n0},\\ &E_n=C\beta_n-n\omega e_0\alpha_n+c_{n0}, && n=1,2,\dots. \end{aligned}\tag{51}$$

We shall denote by $u(x)$, $v(y)$ the excesses of the functions $u_0(x)$, $v_0(y)$ over their mean values:

$$\begin{aligned} u(x)&=u_0(x)-\alpha_0+\sum_1^\infty\alpha_s\sim\sum_1^\infty(\alpha_s\cos s\omega x+\beta_s\sin s\omega x),\\ v(y)&=v_0(y)+\sum_1^\infty\gamma_s\sim\sum_1^\infty(\gamma_s\cos s\omega y+\delta_s\sin s\omega y). \end{aligned}\tag{52}$$

If we require $\mu_0=0$, $m_0(y)\equiv 0$, $n_0(x)\equiv 0$, then relations (49) reduce to

$$C\,v(y)+g_0\,v'(y)=-\kappa_1(y),\qquad C\,u(x)+e_0\,u'(x)=-\kappa_2(x).$$

For $e_0=0$, we have

$$u(x)=-C^{-1}\kappa_2(x);\tag{53}$$

for $e_0\neq 0$, we have

$$\begin{aligned} u(x)&=\exp(-e_0^{-1}Cx)\left[K+e_0^{-1}\int_0^x\exp(e_0^{-1}C\xi)\,\kappa_2(\xi)\,d\xi\right],\\ K&=-e_0^{-2}C\big(1-\exp(-e_0^{-1}CT)\big)^{-1}\int_0^T\exp(e_0^{-1}Cx)\,dx\times\\ &\quad\times\int_0^x\exp(e_0^{-1}C\xi)\,\kappa_2(\xi)\,d\xi, \end{aligned}\tag{54}$$

where the constant K is determined in such a way that

$$\int_0^T u(x)\,dx=0.$$

Analogous relations hold for $v(y)$. This determines all the coefficients α_n, β_n, γ_n, δ_n, $n=1,2,\dots$. Actually, relations (53), (54) are equivalent to those we obtain from (51) by taking $B_n=C_n=D_n=E_n=0$ and solving with respect to α_n, β_n, γ_n, δ_n:

$$\begin{aligned} \alpha_n&=(C^2+n^2\omega^2e_0^2)^{-1}(-C\,a_{n0}+n\omega e_0\,c_{n0}),\\ \beta_n&=(C^2+n^2\omega^2e_0^2)^{-1}(-n\omega e_0\,a_{n0}-C\,c_{n0}),\\ \gamma_n&=(C^2+n^2\omega^2g_0^2)^{-1}(-C\,a_{0n}+n\omega g_0\,b_{0n}),\\ \delta_n&=(C^2+n^2\omega^2g_0^2)^{-1}(-n\omega g_0\,a_{0n}-C\,b_{0n}),\\ n&=1,2,\dots. \end{aligned}$$

The coefficients α_n, β_n, γ_n, δ_n, $n=1,2,\dots$, being so determined, then equation $\mu_0=A_0=0$ yields

$$\alpha_0=-C^{-1}a_{00}+\sum_1^\infty\alpha_s+\sum_1^\infty\gamma_s,$$

provided the series $\sum \alpha_s$, $\sum \gamma_s$ converge. This will be the case under the hypotheses of Criterion I below. We shall denote the corresponding functions $u_0(x)$, $v_0(y)$ so determined by $U(x)$, $V(y)$, or

$$\begin{aligned} U(x) &= u(x)+\alpha_0-\sum_1^\infty \alpha_s \sim \alpha_0+\sum_1^\infty (\alpha_s \cos s\omega x-\alpha_s+\beta_s \sin s\omega x), \\ V(y) &= v(y)-\sum_1^\infty \gamma_s \sim \sum_1^\infty (\gamma_s \cos s\omega y-\gamma_s+\delta_s \sin s\omega y). \end{aligned} \tag{55}$$

Under the conditions of Criterion I we shall require that these functions be interior points of the set defined by relations (7) of Theorem I.

13. Criterion I. If the function f given in (46) for all $|\varepsilon|<\varepsilon_0$ satisfies all conditions of Theorems I, II, III with given constants T, N, N_1, N_2, L, M_1, M_2, M_3, b_0, b_1, b_2, and in addition if f is Lipschitzian with respect to x and y in R, if $C \neq 0$, and the functions $\kappa_1(x)$, $\kappa_2(y)$, $U(x)$, $V(y)$ defined in (50) and (55) are of class C^1 with Lipschitzian first derivatives, and

$$|U(0)| \leqq N_0 < N, \quad |U(x_1)-U(x_2)| \leqq N_{10}|x_1-x_2|, \quad N_{10}<N_1,$$
$$|V(y_1)-V(y_2) \leqq N_{20}|y_1-y_2|, \quad N_{20}<N_2,$$

then there is some $\bar{\varepsilon}_0$, $0<\bar{\varepsilon}_0 \leqq \varepsilon_0$, such that equation (46) for all $|\varepsilon| \leqq \bar{\varepsilon}_0$ possesses at least one periodic solution $\varphi(x, y)$ of period T in x and y, which is Lipschitzian in E_2 together with φ_x, φ_y, φ_{xy}:

$$\varphi_{xy}=f(x, y, \varphi, \varphi_x, \varphi_y), \quad \varphi(x+T, y)=\varphi(x, y)=\varphi(x, y+T).$$

Moreover, the periodic functions $u_0(x)=\varphi(x, 0)=\varphi(x, T)$, $v_0(y)=\varphi(0, y)-\varphi(0, 0)$ $=\varphi(T, y)-\varphi(T, 0)$, satisfy relations (7) of Theorem I.

14. Proof of Criterion I. Let us denote by k_{40}, k_{50} the Lipschitzian constants of $U(x)$ and $V(y)$ respectively, and let k_4, k_5 be arbitrary numbers $k_4>k_{40}$, $k_5>k_{50}$. Let us denote as usual by k_1, k_2 the Lipschitzian constants of f with respect to x and y respectively in R.

Let S be the set of all pairs $w=[u_0(x), v_0(y)]$ of functions $u_0(x)$, $v_0(y)$ periodic of period T, of class C^1, with derivatives $u_0'(x)$, $v_0'(y)$ Lipschitzian of constants k_4, k_5, and satisfying relations (7) of Theorem I, that is

$$|u_0(0)| \leqq N, \quad |u_0(x_1)-u_0(x_2)| \leqq N_1|x_1-x_2|,$$
$$v_0(0)=0, \quad |v_0(y_1)-v_0(y_2)| \leqq N_2|y_1-y_2|.$$

Then $w_0=[U, V] \in S$. We shall consider S imbedded in the Banach space of all pairs of periodic fuctions of class C^1 with norm

$$\|w\|=\max|u(x)|+\max|u'(x)|+\max|v(y)|+\max|v'(y)|. \tag{56}$$

For every $w=[u_0(x), v_0(y)] \in S$ we shall determine the solution $W=[\varphi, \varepsilon m, \varepsilon n, \varepsilon\mu]$ of the modified problem relative to (46). Since this solution can be determined by the method of successive approximations of Section 10, we see that W can be written in the form

$$\kappa(x, y, \varepsilon) = \varphi_0(x, y) + \tilde{\varphi}(x, y, \varepsilon), \qquad m(y, \varepsilon) = m_0(y) + \tilde{m}(y, \varepsilon),$$

$$n(x, \varepsilon) = n_0(x) + \tilde{n}(x, \varepsilon), \qquad \mu(\varepsilon) = \mu_0 + \tilde{\mu}(\varepsilon),$$

where $\tilde{\varphi}, \tilde{m}, \tilde{n}, \tilde{\mu} = O(1)$ uniformly as $\varepsilon \to 0$ and $\varphi_0, m_0, n_0, \mu_0$ are given by formulas (48).

By use of the functions $u(x)$, $v(y)$ as in Section 8, equations $\mu = 0$, $m(y) \equiv 0$, $n(x) \equiv 0$ reduce to

$$\begin{aligned} &C v(y) + g_0 v'(y) = -\kappa_1(y) - \tilde{m}(y, \varepsilon),\\ &C u(x) + e_0 u'(x) = -\kappa_2(x) - \tilde{n}(x, \varepsilon), \\ &a_{00} + C\left(\alpha_0 - \sum_1^\infty \alpha_s - \sum_1^\infty \gamma_s\right) - \tilde{\mu}(\varepsilon) = 0. \end{aligned} \tag{57}$$

For $e_0 = 0$ we have

$$u(x) = -C^{-1}(\kappa_2(x) + \tilde{n}(x, \varepsilon)_n; \tag{58}$$

for $e_0 \neq 0$ we have

$$\begin{aligned} &u(x) = \exp(-e_0^{-1} C x)\left[K + e_0^{-1} \int_0^x \exp(e_0^{-1} C \xi)(\kappa_2(\xi) + \tilde{n}(\xi, \varepsilon))\, d\xi\right], \\ &K = -e_0^{-2} C(1 - \exp(-e_0^{-1} C T))^{-1} \int_0^T \exp(e_0^{-1} C x)\, dx \int_0^x \exp(e_0^{-1} C \xi) \times \\ &\quad \times (\kappa_2(\xi) + \tilde{n}(\xi, \varepsilon))\, d\xi, \end{aligned} \tag{59}$$

and hence

$$\begin{aligned} &u'(x) = -C^{-1}(d/dx)(\kappa_2(x) + \tilde{n}(x, \varepsilon)) \quad \text{if} \quad e_0 = 0, \\ &u'(x) = -e_0^{-1}\,(C u(x) + \kappa_2(x) + \tilde{n}(x, \varepsilon)) \quad \text{if} \quad e_0 \neq 0. \end{aligned} \tag{60}$$

Analogous formulas hold for $v(y)$.

This determines $u(x)$, $v(y)$ and hence all coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n$, $n = 1, 2, \ldots$. By Remark 1 we know that m, n are Lipschitzian functions, and so are $\tilde{m}, \tilde{n}$ as well as κ_1, κ_2. Thus $u(x)$, $u(y)$ are periodic functions of mean value zero, of class C^1, with Lipschitzian first derivatives. Thus, the series $\sum \alpha_n$, $\sum \gamma_n$ are absolutely convergent, and (47), (57) yield

$$\begin{aligned} \alpha_0 &= -C^{-1}\left(a_{00} - \tilde{\mu}(\varepsilon)\right) + \sum_1^\infty \alpha_s + \sum_1^\infty \gamma_s, \\ &= -C^{-1}(a_{00} - \tilde{\mu}(\varepsilon)) - u(0) - v(0)), \end{aligned} \tag{61}$$

$$u_0(x) = u(x) + \alpha_0 - \sum_1^\infty \alpha_s, \qquad v_0(y) = v(y) - \sum_1^\infty \gamma_s. \tag{62}$$

Note that these functions, when we take $\tilde{m} = \tilde{n} = 0$, reduce to $U(x)$ and $V(y)$ respectively, and thus the convergence of the series $\sum \alpha_n$, $\sum \gamma_n$ of Section 12 is proved above.

Actually, for every $w = [u(x), v(y)] \in S$, we can first determine m, n, μ as in Theorems I, II, III, using the method of successive approximations of Section 10; then we determine $\tilde{m}, \tilde{n}$, and finally the second members of formulae (58), (59), (60), (62), and analogous ones determine new functions, say $\bar{w}=[\bar{u}_0(x), \bar{v}_0(y)]$. Thus, we have a map $\mathscr{F}$,

$$\bar{w}=\mathscr{F} w, \quad w \in S,$$

whose fixed elements $w=\mathscr{F} w$, if any, have the property that $\mu=0$, $m(y)\equiv 0$, $n(x)\equiv 0$.

We have already chosen the uniform topology of class C^1 on w and $\bar{w}$ by means of (56). Let us choose the uniform topology of class C^0 on m, n as in Theorem III, as well as on $\tilde{m}, \tilde{n}$. We know already from Theorem III that m, n are continuous functions of w, and so are $\tilde{m}, \tilde{n}$. The second members of (58), (59), (60), (62) define continuous functions of $\tilde{m}, \tilde{n}$. Thus $\mathscr{F}$ is a continuous function of w for $w\in S$ in the topology defined by (56).

By Theorems I, II, III we know that $m(y, \varepsilon)$, $n(x, \varepsilon)$ are Lipschitzian functions. The same property holds for $\tilde{m}(y, \varepsilon)$, $\tilde{n}(x, \varepsilon)$, but these functions – as well as their Lipschitzian constants – have a uniform bound of the form $M\varepsilon$ for some $M>0$ and all $|\varepsilon|<\varepsilon_0$. Then, by choosing convenient constants k, k_1, we have

$$|\tilde{m}(y,\varepsilon)|, |\tilde{n}(x,\varepsilon)| \leqq k\varepsilon, \quad |\tilde{\mu}| \leqq k\varepsilon, \quad |\tilde{m}(y_1,\varepsilon)-\tilde{m}(y_2,\varepsilon)| \leqq k\varepsilon |y_1-y_2|,$$
$$|\tilde{n}(x_1,\varepsilon)-\tilde{n}(x_2,\varepsilon)| \leqq k\varepsilon|x_1-x_2|, \quad |\bar{u}_0-U| \leqq k_1\varepsilon, \quad |\bar{v}_0-V| \leqq k_1\varepsilon,$$
$$|\bar{u}_0'-U'| \leqq k_1\varepsilon, \quad |\bar{v}_0'-V'| \leqq k_1\varepsilon, \quad |\bar{u}_0(x_1)-U(x_1)-\bar{u}_0(x_2)+U(x_2)| \leqq k_1\varepsilon,$$
$$|\bar{v}_0(y_1)-V(y_1)-\bar{v}(y_2)+V(y_2)| \leqq k_1\varepsilon, \quad |\bar{u}_0'(x_1)-U'(x_1)-\bar{u}_0'(x_2)+U'(x_2)| \leqq k_1\varepsilon,$$
$$|\bar{v}_0'(y_1)-V'(y_1)-\bar{v}'(y_2)+V'(y_2)| \leqq k_1\varepsilon.$$

If k_4, k_5 are the Lipschitzian constants of U', V', and

$$\bar{\varepsilon}_0=\min\left[\varepsilon_0,\, k_1^{-1}(N-N_0),\, k_1^{-1}(N_1-N_{10}),\, k_1^{-1}(N_2-N_{20})\right],$$

then for $|\varepsilon|\leqq\bar{\varepsilon}_0$ we have

$$|\bar{u}_0(0)| \leqq |U(0)|+k_1\varepsilon \leqq N_0+k_1\varepsilon \leqq N,$$
$$|\bar{u}_0(x_1)-\bar{u}_0(x_2)| \leqq (N_{10}+k_1\varepsilon)|x_1-x_2| \leqq N_1|x_1-x_2|,$$
$$|\bar{u}_0'(x_1)-\bar{u}_0'(x_2)| \leqq (k_4+k_1\varepsilon)|x_1-x_2|,$$
$$|\bar{v}_0(y_1)-\bar{v}_0(y_2)| \leqq (N_{20}+k_1\varepsilon)|y_1-y_2| \leqq N_2|y_1-y_2|,$$
$$|\bar{v}_0'(y_1)-\bar{v}_0'(y_2)| \leqq (k_5+k_1\varepsilon)|y_1-y_2|.$$

This shows that, for $|\varepsilon|\leqq\bar{\varepsilon}_0$, $\mathscr{F}$ maps S into itself, $\mathscr{F}: S\to S$, and S is a convex closed compact subset of a Banach space. By SCHAUDER's fixed point theorem $\mathscr{F}$ possesses at least one fixed element $w=\mathscr{F} w\in S$, $w=[u_0(x), v_0(y)]$, with $u_0(x)$, $v_0(y)$ satisfying relations (7) of Theorem I. Criterion I is thereby proved.

15. Example. The equation

$$u_{xy}=\varepsilon(1-u)+\varepsilon^2 g(x, y, u, u_x, u_y),$$

where g is periodic of period 2π in x and y, has a periodic solution $\varphi(x, y)$ of the same period,

$$\varphi(x, y)=1+O(\varepsilon).$$

The analogous equation

$$u_{xy}=\varepsilon(\sin x-\cos x-\sin y+\cos y+u+u_x+u_y)+\varepsilon^2 g(x,y,u,u_x,u_y)$$

with g as above has a periodic solution

$$\varphi=\cos x-\cos y+O(\varepsilon).$$

16. Another Equation Containing a Small Parameter. Let us consider the differential equation

$$\begin{aligned}&u_{xy}=f(x,y,u,u_x,u_y),\\&f=\psi(x,y)+Cu+\psi_1(y)u_x+\psi_2(x)u_y+\varepsilon g(x,y,u,u_x,u_y),\end{aligned}\tag{63}$$

where ε is a small parameter, and ψ,ψ_1,ψ_2,g are as in Section 12. We assume here that, for $\varepsilon=0$, equation (63) possesses a known periodic solution of period T in x and y, of the form

$$\varphi_0(x,y)=u_0(x)+v_0(y),$$

where $u_0(x)$, $v_0(y)$ have Fourier series (47), and hence

$$\psi(x,y)=-Cu_0(x)-Cv_0(y)-\psi_1(y)u_0'(x)-\psi_2(x)v_0'(y).$$

Under the hypotheses below, we shall prove that for $|\varepsilon|\neq 0$ sufficiently small, (63) possesses a solution $\varphi(x,y)=\varphi_0(x,y)+O(\varepsilon)$ which is periodic of period T in x and y.

17. Criterion II. If the function f defined in (63) for all $|\varepsilon|\leqq\varepsilon_0$ satisfies all conditions of Theorems I, II, III with given constants T, N, N_1, N_2, L, M_1, M_2, M_3, b_0, b_1, b_2 and in addition f is Lipschitzian with respect to x and y in R, if $C\neq 0$, if (63) possesses for $\varepsilon=0$ a solution $\varphi_0(x,y)=U(x)+V(y)$ with U, V periodic of period T, if the functions $U(x)$, $V(y)$, $\kappa_1(x)$, $\kappa_2(y)$ are of class C^1 with Lipschitzian first derivative, and

$$|U(0)|\leqq N_0<N,\quad |U(x_1)-U(x_2)|\leqq N_{10}|x_1-x_2|,\quad N_{10}<N_1,$$
$$|V(y_1)-V(y_2)|\leqq N_{20}|y_1-y_2|,\quad N_{20}<N_2,$$

then there is some $\bar\varepsilon_0$, $0<\bar\varepsilon_0\leqq\varepsilon_0$, such that equation (63) for all $|\varepsilon|\leqq\bar\varepsilon_0$, possesses at least one periodic solution $\varphi(x,y)$ of period T in x and y, which is Lipschitzian in E_2 together with φ_x, φ_y, φ_{xy}:

$$\varphi_{xy}=f(x,y,\varphi,\varphi_x,\varphi_y),\qquad \varphi(x+T,y)=\varphi(x,y)=\varphi(x,y+T).$$

Moreover, the periodic functions $u_0(x)=\varphi(x,0)=\varphi(x,T)$, $v_0(y)=\varphi(0,y)-\varphi(0,0)$ $=\varphi(T,y)-\varphi(T,0)$ satisfy relations (7) of Theorem I.

18. Proof of Criterion II. As in Section 12 let us apply formally the method of successive approximations of Section 10 to equation (63) with arbitrary initial values $u_0(x)$, $v_0(y)$. Then the quadruple $[\varphi,m,n,\mu]$, the solution of the modified problem for equation (63), is given by

$$\varphi(x,y,\varepsilon)=\varphi_0(x,y)+\tilde\varphi(x,y,\varepsilon),\qquad m(y,\varepsilon)=m_0(y)+\tilde m(y,\varepsilon),$$
$$n(x,\varepsilon)=n_0(x)+\tilde n(x,\varepsilon),\qquad \mu(\varepsilon)=\mu_0+\tilde\mu(\varepsilon),$$

where $\tilde{\varphi}, \tilde{m}, \tilde{n}, \tilde{\mu}=0(1)$ uniformly as $\varepsilon \to 0$, and $\varphi_0, m_0, n_0, \mu_0$ are given by formulas (48). In addition, we know that for $u_0(x)\equiv U(x)$, $v_0(y)\equiv V(y)$ we have $\mu_0=0$, $m_0(y)\equiv 0$, $n_0(x)\equiv 0$. We can now repeat with obvious variants the argument of the proof of Criterion I.

19. Examples. The equation

$$u_{xy}=-1+u+\psi_1(y)\,u_x+\psi_2(x)\,u_y+\varepsilon\, g(x, y, u, u_x, u_y)$$

for $\varepsilon=0$ has the obvious solution $u=1$. Since $C\neq 0$, the same equation has a periodic solution φ of period T in x and y for all $|\varepsilon|$ sufficiently small.

Analogously, the equation

$$u_{xy}=-\cos x+\sin x+\cos y-\sin y+u+u_x+u_y+\varepsilon\, g(x, y, u, u_x, u_y)$$

has, for $\varepsilon=0$, the obvious solution $u=\cos x-\cos y$. Since $C\neq 0$, the same equation has a periodic solution of period 2π in x and y for every $|\varepsilon|\neq 0$ sufficiently small.

20. Application to the Wave Equation. Let us consider the differential equation

$$u_{tt}-u_{\xi\xi}=f(t, \xi, u, u_t, u_\xi), \tag{64}$$

where f is periodic in t and ξ of period T. Then the transformation

$$t=x+y, \quad \xi=x-y, \quad x=2^{-1}(t+\xi), \quad y=2^{-1}(t-\xi), \tag{65}$$

changes (64) into

$$u_{xy}=F(x, y, u, u_x, u_y),$$

where

$$F=f(x+y, x-y, u, 2^{-1}u_x+2^{-1}u_y, 2^{-1}u_x-2^{-1}u_y), \tag{66}$$

and F is periodic of period T in x and y. Theorems I, II, III and the criteria should now be applied to (66). Other transformations beside (65) can be used.

As an example, let us consider the equation

$$u_{tt}-u_{\xi\xi}=\varepsilon[\lambda(t, \xi)+C_0\, u+\lambda_1(t, \xi)\, u_t+\lambda_2(t, \xi)\, u_\xi]+\varepsilon^2 g(t, \xi, u, u_t, u_\xi), \tag{67}$$

$$\begin{aligned}
\lambda(t, \xi)&=A_0+B_1\cos 2t+C_1\sin 2t+B_2\cos 2\xi+C_2\sin 2\xi+\\
&\quad+D_1\cos(t+\xi)+E_1\sin(t+\xi)+D_2\cos(t-\xi)+E_2\sin(t-\xi),\\
\lambda_1(t, \xi)&=A+B\cos(t+\xi)+C\sin(t+\xi)+D\cos(t-\xi)+E\sin(t-\xi),\\
\lambda_2(t, \xi)&=A'-B\cos(t+\xi)-C\sin(t+\xi)+D\cos(t-\xi)+E\sin(t-\xi),
\end{aligned} \tag{68}$$

where $A_0, B, \ldots, E$ are constants, $C_0\neq 0$, and g is of period π in t and ξ. By the transformation

$$t=2^{-1}(x+y), \quad \xi=2^{-1}(x-y), \quad x=t+\xi, \quad y=t-\xi, \tag{69}$$

equation (67) is changed into

$$\begin{aligned}
u_{xy}&=\varepsilon[\psi(x, y)+4^{-1}C_0\, u+\psi_1(y)\, u_x+\psi_2(x)\, u_y]+\\
&\quad+\varepsilon^2 g(2^{-1}x+2^{-1}y, 2^{-1}x-2^{-1}y, u, u_x+u_y, u_x-u_y),
\end{aligned} \tag{70}$$

where the second member has period 2π in x and y, and

$$\psi(x, y)=a_{00}+a_{10}\cos x+c_{01}\sin x+a_{01}\cos y+b_{01}\sin y+a_{11}\cos x\cos y+ \\ +b_{11}\cos x\sin y+c_{11}\sin x\cos y+d_{11}\sin x\sin y,$$

$$\psi_1(y)=e_0+e_1\cos y+f_1\sin y,\quad \psi_2(x)=g_0+g_1\cos x+h_1\sin y,$$

$$4a_{00}=A_0,\quad 4a_{10}=D_1,\quad 4c_{01}=E_1,\quad 4a_{01}=D_2,\quad 4b_{01}=E_2, \tag{71}$$

$$4a_{11}=B_1+B_2,\quad 4b_{11}=C_1-C_2,\quad 4c_{11}=C_1+C_2,\quad 4d_{11}=-B_1+B_2,$$

$$4e_0=A+A',$$

$$2e_1=D,\quad 2f_1=E,\quad 4g_0=A-A',\quad 2g_1=B,\quad 2h_1=C.$$

By Criterion I we conclude that if $C_0\neq 0$ and $|\varepsilon|$ sufficiently small, then equation (70) has at least a periodic solution $\varphi(x, y, \varepsilon)$ of period 2π in x and y, and then equation (67) has at least a solution

$$u(t, \xi, \varepsilon)=\varphi(t+\xi, t-\xi, \varepsilon),$$

also of period 2π in t and ξ.

21. Another Example. Let us consider the differential equation

$$u_{tt}-u_{\xi\xi}=\lambda(t, \xi)+C_0 u+\lambda_1(t, \xi)u_t+\lambda_2(t, \xi)u_\xi+\varepsilon g(t, \xi, u, u_t, u_\xi), \tag{72}$$

where λ, λ_1, λ_2 are given by (68) and again $C_0\neq 0$. By the same transformation (69) equation (72) is changed into

$$u_{xy}=\psi(x, y)+4^{-1}C_0 u+\psi_1(y)u_x+\psi_2(x)u_y+ \\ +\varepsilon g(2^{-1}x+2^{-1}y, 2^{-1}x-2^{-1}y, u, u_x+u_y, u_x-u_y), \tag{73}$$

where ψ, ψ_1, ψ_2 are given by formulas (71). It is immediately seen that (73) for $\varepsilon=0$ has a solution of the form

$$u(x, y)=U(x)+V(y),\quad U(x)=\alpha_0+\alpha_1\cos x+\beta_1\sin x,$$

$$V(y)=\gamma_1\cos y+\delta_1\sin y$$

if and only if

$$\begin{aligned} B_1&=\Delta_1 C_0(ED_1+DE_1)+\Delta_1(A+A')(DD_1-EE_1)+ \\ &\quad+\Delta_2 C_0(CD_2+BE_2)+\Delta_2(A-A')(BD_2-CE_2), \\ B_2&=\Delta_1 C_0(-ED_1+DE_1)+\Delta_1(A+A')(DD_1-EE_1)+ \\ &\quad+\Delta_2 C_0(-CD_2+BE_2)+\Delta_2(A-A')(BD_2+CE_2), \\ C_1&=\Delta_1 C_0(-DD_1+EE_1)+\Delta_1(A+A')(ED_1+DE_1)+ \\ &\quad+\Delta_2 C_0(-BD_2+CE_2)+\Delta_2(A-A')(CD_2+BE_2), \\ C_2&=\Delta_1 C_0(-DD_1-EE_1)+\Delta_1(A+A')(-ED_1+DE_1)+ \\ &\quad+\Delta_2 C_0(BD_2+CE_2)+\Delta_2(A-A')(CD_2-BE_2), \\ \Delta_1&=\left(C_0^2+(A+A')^2\right)^{-1},\quad \Delta_2=\left(C_0^2+(A-A')^2\right)^{-1}. \end{aligned} \tag{74}$$

In this situation then

$$\alpha_1=\Delta_1\big(-C_0 D_1+(A+A')E_1\big),\quad \beta_1=\Delta_1\big(-C_0 E_1-(A+A')D_1\big),$$
$$\gamma_1=\Delta_2\big(-C_0 D_2+(A-A')E_2\big),\quad \delta_1=\Delta_2\big(-C_0 E_2-(A-A')D_2\big),\quad \alpha_0=-C_0^{-1}A_0.$$

By Criterion II we conclude that if $C_0 \neq 0$, $|\varepsilon|$ sufficiently small, and relations (74) hold, then equation (73) has a periodic solution $\varphi(x, y)$ of period 2π in x and y, and (72) has a solution

$$u(t,\xi,\varepsilon)=\varphi(t+\xi,t-\xi,\varepsilon)$$

also of period 2π in t and ξ.

For instance, for the equation

$$\begin{aligned} u_{tt}-u_{\xi\xi}=D_1\cos(t+\xi)+E_1\,sin(t+\xi)+D_2\cos(t-\xi)+\\ +E_2\sin(t-\xi)+u+u_t+\varepsilon\, g(t,\xi,u,u_t,u_\xi),\end{aligned} \tag{75}$$

where D_1, E_1, D_2, E_2 are arbitrary constants and g periodic of period π in t and ξ, we see that relations (74) are all satisfied with

$$B_1=C_1=B_2=C_2=0,\quad B=C=D=E=0,\quad A_0=0,\quad C_0=1,\quad A=1,$$
$$A'=0,\quad \Delta_1=\Delta_2=2^{-1}.$$

The corresponding equation (73) is

$$4u_{xy}=D_1\cos x+E_1\sin x+D_2\cos y+E_2\sin y+u+u_x+u_y+\varepsilon\, g.$$

For $\varepsilon=0$ this equation has the periodic solution $\varphi_0(x, y)$ of period 2π in x, y given by

$$-2\varphi_0(x,y)=(D_1-E_1)\cos x+(D_1+E_1)\sin x+(D_2-E_2)\cos y+(D_2+E_2)\sin y,$$

and hence (75) for $\varepsilon=0$ has the periodic solution

$$\begin{aligned} u_0(t,\xi)=-2^{-1}(D_1-E_1)\cos(t+\xi)-2^{-1}(D_1+E_1)\sin(t+\xi)-\\ -2^{-1}(D_2-E_2)\cos(t-\xi)-2^{-1}(D_2+E_2)\sin(t-\xi).\end{aligned}$$

Thus, for all $|\varepsilon|$ sufficiently small equation (75) has a periodic solution of period 2π in t and ξ of the form $u(t, \xi)=u_0(t, \xi)+O(\varepsilon)$.

22. Remark. In the autonomous case, that is, when f does not depend on x and y, then $\psi=a_{00}$, $\psi_1=e_0$, $\psi_2=g_0$, $g=g(u, u_x, u_y)$. It is easy to verify that the periodic solution φ of equation (46), whose existence is proved by Criterion I, is a constant.

Concerning Criterion II, let us note first that equation (63) in the autonomous case reduces to

$$u_{xy}=a_{00}+C\,u+e_0\,u_x+g_0\,u_y+\varepsilon\, g(u,u_x,u_y). \tag{76}$$

Under the hypothesis of Criterion II we have $C\neq 0$ and

$$|C(z_1-z_2)+\varepsilon\, g(z_1,p,q)-\varepsilon\, g(z_2,p,q)|\leqq b_0|z_1-z_2| \tag{77}$$

for all z_1, z_2, p, q, ε with $|z_1|$, $|z_2|\leqq M_1$, $|p|\leqq M_2$, $|q|\leqq M_3$, $|\varepsilon|\leqq\varepsilon_0$. We may well assume $b_0>0$. From (77) we deduce first, taking $\varepsilon=0$, $|C|<b_0$, and then, taking $\varepsilon=\varepsilon_0$, $|g(z_1,p,q)-g(z_2,p,q)|\leqq(b_0+|C|)\,\varepsilon_0^{-1}\,|z_1-z_2|\leqq 2b_0\,\varepsilon_0^{-1}\,|z_1-z_2|$,

and hence g is uniformly Lipschitzian with constant $2b_0\,\varepsilon_0^{-1}$. Then, for $|\varepsilon|\leqq\varepsilon_1=\min[\varepsilon_0,\varepsilon_0\,|C|/4b_0]$, we see that $\varepsilon g(z,p,q)$ is uniformly Lipschitzian in z with constant $|C|/2$. Then for $|\varepsilon|\leqq\varepsilon_1$ and $C\neq 0$, the expression $a_{00}+Cz+\varepsilon g(z,0,0)$ is monotone, namely strictly increasing or strictly decreasing as Cz. Note that, for $\gamma_0=C^{-1}\,|a_{00}|<M_1$, there is a unique $\gamma_0=\gamma(0)=C^{-1}\,a_{00}$ such that $\gamma=\gamma_0$, $\varepsilon=0$ satisfy the equation

$$a_{00}+C\,\gamma+\varepsilon\,g(\gamma,0,0)=0, \tag{78}$$

and thus there is some ε_2, $0<\varepsilon_2\leqq\varepsilon_1$, and a constant $\gamma=\gamma(\varepsilon)$ such that (78) is satisfied for $|\varepsilon|\leqq\varepsilon_2$. Then equation (76) is satisfied for $\varepsilon=0$ by the constant function $\varphi(x,y)=U(x)+V(y)=\gamma_0$, with $U=\gamma_0$, $V=0$ and equation (76) is satisfied for $|\varepsilon|\leqq\varepsilon_2$ by the constant function $\varphi(x,y)=\gamma(\varepsilon)$.

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