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On Periodic Solutions of a Delay Integral Equation Modelling Epidemics

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Summary

A delay-integral equation, proposed by Cooke and Kaplan in [1] as a model of epidemics, is studied. The focus of this work is on the qualitative behavior of solutions as a certain parameter is allowed to vary. It is shown that if a certain threshold is not exceeded then solutions tend to zero exponentially while if this threshold is exceeded, periodic solutions exist. Many features of the numerical studies in [1] are explained.

1. Introduction

In a recent paper [1], K. L. Cooke and J. L. Kaplan formulate a model to explain the observed periodic outbreaks of certain infectious diseases. This was accomplished by allowing for a periodic contact rate between infectious and susceptibles in the population. The equation obtained from their model is

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds. \tag{1.1}$$

In (1.1), x represents the proportion of infectious individuals in the population, τ represents the length of time an individual is infective, and f is a nonnegative function which is ω-periodic in s.

Cooke and Kaplan obtained sufficient conditions for (1.1) to have nonzero ω-periodic solutions and sufficient conditions for all solutions to approach zero as $t \to \infty$. If f is fixed these conditions involve the parameter τ. In the particular case where

$$f(t, x) = a(t)\, x(1-x) \tag{1.2}$$

with $a(t) = 1 + \frac{1}{2}\sin 2\pi t$, $\omega = 1$, their results indicate that 1-periodic solutions of (1.1) exist for $\tau > 2$ and that solutions tend to zero as $t \to \infty$ when $\tau < \frac{2}{3}$. On the other hand, numerical experiments reported in [1] indicate that for $\tau \geq 1$ solutions tend to zero for large t and for $\tau > 1$, 1-periodic solutions are approached as t becomes large. Furthermore, for $\tau > 1$ the periodic solution increased with τ, that is, the solution corresponding to a smaller value of τ was smaller than a solution for a larger value of τ.

Recent work of R. Nussbaum [10] has shown that nontrivial ω-periodic solutions of (1.1) bifurcate from the trivial solution $(f(t,0)\equiv 0)$ precisely when τ exceeds a certain threshold value τ_0. Regarding the right hand side of (1.1) as defining a mapping from the Banach space of ω-periodic, continuous functions on $\mathbb{R}$ with the supremum norm into itself, τ_0 is obtained as the unique value of τ for which the spectral radius of the linearization of this mapping is equal to 1. Estimates for τ_0 were obtained which give $\tau_0=1$ for the special form of f mentioned above. Further, it was shown that ω-periodic solutions exist for (1.1) whenever $\tau>\tau_0$. The work of Nussbaum was carried out for a more general equation than (1.1) with the use of a global bifurcation theorem.

Certain features of the numerical results in [1] remain without a mathematical explanation. It is the purpose of this note to provide such justification. In particular we show for a class of f's which include (1.2), where $a(t)$ need not be $(1+\frac{1}{2}\sin 2\pi t)$, that for each $\tau\in(\tau_0,\tau^0]$ where $\tau^0>\tau_0$ depends on f and is computable, $(1.1)_\tau$ has exactly one ω-periodic solution, x_τ, which is positive on $\mathbb{R}$. Moreover, the map $\tau\mapsto x_\tau$ for $\tau\in[\tau_0,\tau^0]$ is continuous $(\tau_0\mapsto 0)$ and increasing, that is, if $\tau_0\le\tau_1<\tau_2\le\tau^0$ then $x_{\tau_1}(t)<x_{\tau_2}(t)$ for all $t\in\mathbb{R}$. On the other hand, if $\tau<\tau_0$ we show that all solutions approach zero exponentially fast. For the sake of completeness we include proofs of the existence of a critical value τ_0 of the parameter τ from which nonzero ω-periodic solutions bifurcate. Assuming slightly more about the function f than in [10] we are able to simplify the argument concerning bifurcation at τ_0 and show that the bifurcating branch is a simple curve $\tau\mapsto x_\tau$ parametrized by $\tau\in[\tau_0,\tau^0]$.

Related models of infectious diseases may be found in [2, 4, 5, 6, 9].

The organization of the paper is as follows: In section 2 we show how (1.1) is obtained from the model of Cooke and Kaplan and list our main assumptions concerning the function f. In section 3 the results are stated and biological implications are discussed. The proofs of results stated in section 3 are given in section 4.

2. Construction of the Model

It is known that some infectious diseases have a periodically varying incidence in human populations (see [9, 11, 13]). Records kept during the years 1929—1970 in New York City show that chickenpox and mumps have annual peaks in incidence while measles incidence peaks biennially. Periodic variation in incidence has also been observed for the common cold, influenza, pneumonia, streptococcal sore throat, and meningitis ([11]). A reasonable explanation for the observed periodicity in incidence is periodicity in the contact rates for these diseases. Roughly speaking, the contact rate is a measure of the amount of effective interaction between infected individuals and susceptible persons per unit time. That the contact rate varies periodically is not surprising if one considers seasonal variations in the weather and the fact that children are in school certain months of the year.

We consider an isolated population satisfying the following assumptions:

(1) The population has constant size N.

(2) The population consists of those currently infectious and those susceptible; there being no overlap in the two groups.

(3) The disease is not lethal and, on recovery, the individual is again susceptible.

(4) The disease is contracted upon exposure, i.e., there is no period of latency.

(5) Once an individual becomes infective he remains infective for precisely τ units of time.

(6) The population is homogeneous and uniformly mixing. The contact rate, defined as the average number of effective contacts with other individuals per infective per unit time, is a given periodic function of time. By an effective contact, we mean an encounter which would result in the infection of the other individual, were that individual susceptible.

In addition to the assumptions above, we must assume that the population size, N, is so large that both the proportion of individuals currently infected, $I(t)$, and the proportion of individuals currently susceptible, $S(t)$, behave like continuous variables.

Let $a(t)$ denote the contact rate. The product $a(t)S(t)$ gives the number of contacts which result in infection of the other individual per infective per unit time. Multiplying this term by $NI(t)$ one obtains $N a(t) I(t) S(t)$, the number of contacts resulting in infection at time t per unit time. This is just the number of new infectives introduced into the population per unit time at time t. Recognizing that $I(t)+S(t)=1$ by assumption (2), we may write

$$N a(t) I(t) \left(1-I(t)\right)$$

for the number of new infectives introduced into the population per unit time at time t. Since the duration of infectiousness is precisely τ,

$$N a(t-\tau) I(t-\tau) \left(1-I(t-\tau)\right)$$

is the number of individuals leaving the infected class per unit time at time t. Hence we obtain

$$NI'(t)=N a(t) I(t) \left(1-I(t)\right)-N a(t-\tau) I(t-\tau) \left(1-I(t-\tau)\right)$$

or

$$I'(t)=a(t) I(t) \left(1-I(t)\right)-a(t-\tau) I(t-\tau) \left(1-I(t-\tau)\right).$$

This last equation can be integrated, yielding

$$I(t)=\int_{t-\tau}^{t} a(s) I(s) \left(1-I(s)\right) ds+c. \tag{2.1}$$

In (2.1), the constant c should be taken to be zero since if there were no infectives in the past then no new infectives should appear.

The equation (1.1) is obtained by an obvious change in notation in (2.1) and by replacing $a(s)\, x(s) \left(1-x(s)\right)$ by $f(s, x(s))$ for greater generality.

At this point it is convenient to list the assumptions to be made concerning the function f. Let $f: R \times R^+ \to \mathbb{R}^+$ be a continuous function for which the following assumptions hold:

(H 1) $f(t, 0) \equiv 0$ for all $t \in \mathbb{R}$.

(H 2) There exists a least positive number ω such that $f(t+\omega, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^+$.

(H 3) $f_x(t, x)$, the partial derivative of f with respect to x, exists and is continuous for all $t \in \mathbb{R}$ and $0 \leq x \leq x_0$ where x_0 is some positive number.

(H 4) The set $\{t \in \mathbb{R} : f_x(t, 0) = 0\}$ has Lebesgue measure zero.

(H 5) $\limsup_{x \to \infty} \frac{f(t, x)}{x} = 0$ uniformly in $t \in [0, \omega]$.

For obtaining our bifurcation result we will later employ the following two additional assumptions:

(H 6) $f(t, x) < f_x(t, 0)\, x$ for all $(t, x) \in \mathbb{R} \times (0, \infty)$ except where $f_x(t, 0) = 0$.

(H 7) $f(t, x) = a(t)\, g(x)$ where we assume the existence of a positive number γ such that g is nondecreasing on $[0, \gamma]$ and, if $0 < x \leq \gamma$ and $0 < \lambda < 1$, then $g(\lambda x) > \lambda g(x)$.

Although (H 6) is not used in [10], it simplifies the proof and gives more qualitative information about the nature of the bifurcation. We use (H 7) to show that the branch of nontrivial ω-periodic solutions bifurcating from the trivial solution at τ_0 is a simple curve continuous in the parameter τ.

3. Statement of Results

Theorem 3.1: (Nussbaum): If f satisfies (H 1)—(H 5) then there exists a uniquely determined value $\tau_0 > 0$ of the parameter τ such that for $\tau > \tau_0$, $(1.1)_\tau$ has at least one positive, ω-periodic solution.

The statement of Theorem 3.1 requires some clarification. The number τ_0 is uniquely determined by the derivative $f_x(t, 0)$ as implied in the following paragraph. (This point is clarified in the proof of Theorem 3.1.) For a given f satisfying (H 1)—(H 5) it is possible that there is a number $\tau^1 < \tau_0$ such that for all $\tau > \tau^1$, $(1.1)_\tau$ has at least one positive, ω-periodic solution. This cannot occur, however, if (H 6) holds as indicated in Theorem 3.2.

If K is the cone of nonnegative functions in the Banach space, E, of ω-periodic continuous functions on $\mathbb{R}$ where if $x \in E$, $\| x \| = \sup_{0 \leq t \leq \omega} |x(t)|$, and if A_τ is the mapping from K into K defined by the right hand side of $(1.1)_\tau$, then A_τ has a Frechet derivative T_τ at 0 in the direction of the cone K. The value τ_0 of the parameter τ will be shown to be the unique value τ for which $\rho(T_\tau) = 1$, where $\rho(T_\tau)$ is the spectral radius of the linear operator T_τ. Simple estimates giving upper and lower bounds on the function $\rho(T_\tau)$ will be obtained in the course of the proof of Theorem 3.1. In certain situations these estimates will allow a precise determination of τ_0. It will be seen that if for some positive integer n, $\int_0^{n\omega} f_x(s,0)\,ds=1$, then $\tau_0=n\,\omega$. For a detailed discussion of the smoothness properties and computability of the function $\rho(T_\tau)$ the interested reader is referred to [10].

The biological implication of Theorem 3.1 is obvious: periodic incidence occurs if the duration of the disease exceeds a certain threshold value τ_0 which depends on the "birth" rate for infectives f.

Theorem 3.1 is contained in Theorem 3 in [10] and the remarks following its proof. Included in [10], Theorem 3, is the fact that bifurcation from the trivial solution occurs at τ_0. By assuming (H 6) we are able to show slightly more using relatively simple arguments.

Theorem 3.2: Assume (H 1)—(H 6) are satisfied. Then for $\tau\le\tau_0$ there are no nontrivial, ω-periodic solutions of (1.1). In fact, for $\tau<\tau_0$ all bounded solutions of (1.1) on $[0,\infty)$ approach zero exponentially fast as t becomes large. If $\tau>\tau_0$, then the following holds: for every $\varepsilon>0$, there exists $\delta>0$ such that if τ satisfies $\tau_0<\tau<\tau_0+\delta$ and if x is a nontrivial (nonnegative) ω-periodic solution of $(1.1)_\tau$, then $\sup\limits_{t\in\mathbb{R}} x(t)<\varepsilon$.

By Theorem 3.1, nontrivial ω-periodic solutions of (1.1) exist for $\tau>\tau_0$ but uniqueness of solutions has not yet been established. Theorem 3.2 implies that bifurcation does occur from the trivial solution at τ_0 and that the bifurcation is in some sense continuous. Also a very sharp difference in the qualitative behavior of solutions of (1.1) occurs at τ_0. For the case where f is bounded and thus solutions of (1.1) are bounded, Theorem 3.2 says that if $x(t)$ satisfies (1.1) for $t\ge 0$, when $\tau<\tau_0$, then there exist positive numbers M and η such that $x(t)\le M\,e^{-\eta t}$ for $t\ge 0$.

From the biological viewpoint, we may draw the following conclusions from Theorem 3.2: first, if the duration of the disease is less than the threshold value τ_0 then incidence dies out exponentially and the disease becomes extinct. On the other hand, if the duration of the disease exceeds the threshold value then periodic incidence is possible and if τ exceeds τ_0 but is very close to τ_0 then the peak incidence will be small.

Our next result requires the stronger assumption (H 7).

Theorem 3.3: Assume (H 1)—(H 7) hold. Then there exists $\tau^0>\tau_0$ such that for each $\tau\in(\tau_0,\tau^0]$, $(1.1)_\tau$ has exactly one nontrivial ω-periodic solution (this solution is positive). If x_τ denotes the unique solution, then the map $\tau\mapsto x_\tau$ is continuous for $\tau\in[\tau_0,\tau^0]$. Moreover, if $\tau_0\le\tau_1<\tau_2\le\tau^0$ then $x_{\tau_1}(t)<x_{\tau_2}(t)$ for all $t\in\mathbb{R}$.

In the course of the proof it will be shown that τ^0 is chosen so that if $\tau\in(\tau_0,\tau^0]$ and x is an ω-periodic solution of $(1.1)_\tau$, then $\sup x(t)\le\gamma$, γ as in (H 7). Theorem 3.3 shows that the bifurcating branch of ω-periodic solutions is a simple curve parameterized by τ.

Rather than comment on the biological implications of Theorem 3.3 at this point, we prefer to summarize what has been learned about equation (1.1) by applying the results obtained so far to the particular case used in the numerical results of Cooke and Kaplan. In particular, assume $\omega=1$, $f(t,x)=a(t)g(x)$ where

$$g(x)=\begin{cases} x(1-x) & 0\leq x\leq 1 \\ 0 & x>1 \end{cases}$$

and $\int_0^1 a(t)\,dt=1$. In order that $(H\,4)$ is satisfied we need to assume that $a(t)$ does not vanish on any set of positive measure. Notice that $(H\,1)$—$(H\,7)$ hold for this choice of f.

Theorem 3.4: Assume the particular form of f described above. Then $\tau_0=1$, $\tau^0=2$ and if $\tau<1$ and $x(t)$ is a solution of (1.1) on $t\geq 0$, then $x(t)\to 0$ exponentially as $t\to\infty$. For $\tau>1$ there exists at least one positive 1-periodic solution of (1.1) and any such solution satisfies $\sup x(t)\leq 1$. If $\tau\in(1,2]$ then there exists a unique nontrivial 1-periodic solution x_τ of $(1.1)_\tau$ and $0<x_\tau(t)\leq\frac{1}{2}\equiv x_{\tau=2}(t)$. The map $\tau\mapsto x_\tau$ on $[1,2]$ is continuous and increasing, that is, $1\leq\tau_1<\tau_2\leq 2$ implies $x_{\tau_1}(t)<x_{\tau_2}(t)$ for all $t\in\mathbb{R}$.

Although plausible, we have been unable to extend the domain of uniqueness (and thus continuity) beyond $\tau^0=2$. The special form of f assumed in Theorem 3.4 is precisely the one derived in section 2 from the model of Cooke and Kaplan. Notice that $\omega=1$ corresponds to seasonal variation in the contact rate. Theorem 3.4 implies that no periodic incidence will be observed unless the duration of the disease exceeds 1 year. This rather implausible result occurs only because we assume $\int_0^1 a(t)\,dt=1$. A smaller value of τ_0 would result if $\int_{t-\gamma}^{t} a(s)\,ds\geq 1$ for all t where $\gamma<1$ (see the estimates for $\rho\,(T_r)$ in the proof of Theorem 3.1). The assumption $\int_0^1 a(t)\,dt=1$ is made only to simplify computation of τ_0 and τ^0 and to compare with the numerical results in [1]. Theorem 3.4 does allow the interesting interpretation that if the duration of the disease is decreased by some means while f remains unchanged, then the periodic outbreaks are less severe. In fact, if the duration τ could be decreased so as not to exceed τ_0, then the disease could be eradicated entirely.

Theorem 3.4 serves to explain many of the features of the numerical results of Kaplan and Cooke in [1]. It is not difficult to show using the special form of f in Theorem 3.4 that if $\tau=\tau_0$ then all solutions approach zero. The stability properties of the ω-periodic solutions when $\tau>\tau_0$ remains an open problem although the numerical results indicate that these solutions possess some stability.

4. Proofs of the Theorems

Our proof of Theorem 3.1 is based on the following fixed point theorem of J. A. Gatica and the author [3]:

Fixed Point Theorem: Let $A: K\to K$ be a completely continuous operator on the cone K. Suppose $A\,0=0$, A is Frechet differentiable in the direction of the cone at $x=0$, and

(a) $A'(0)$ has an eigenvector $k \in K$ corresponding to an eigenvalue $\lambda > 1$, and 1 is not an eigenvalue corresponding to an eigenvector in K;

(b) There exists a positive number R such that if $x \in K$, $\| x \| = R$, and $Ax = \mu x$ then $\mu \leq 1$.

Then A has a nonzero fixed point $x \in K$ with $\| x \| \leq R$.

This theorem may also be found in [10]. We will use extensively the following theorem of M. G. Krein and M. A. Rutman [8, Theorem 6.2].

Theorem A: Let A be a completely continuous linear operator satisfying the following two conditions:

(α) $A(K) \subset K$ (K is a cone),

(β) there exists an element $u \in K$, $\| u \| = 1$, a scalar $c > 0$, and a natural number p such that $A^p u \geq c u$.

Then A has nonzero eigenvalues; among those of maximal modulus there is a positive one not less than $c^{1/p}$, to which corresponds a characteristic vector $v \in K$ of the operator A:

$$A v = \rho v \quad (\rho \geq c^{1/p}, v \in K, v \neq 0).$$

This theorem may also be found in Krasnoselskii's book [7, p. 67]. It should be noted that the requirement $\| u \| = 1$ in the above theorem is not essential.

Let E be the Banach space of ω-periodic continuous functions on $\mathbb{R}$ with supremum norm and let K be the cone in E of nonnegative functions. Define the map A_τ on K as follows:

$$(A_\tau x)(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \quad t \in \mathbb{R}, \quad x \in K.$$

It is not hard to show that A_τ maps K into itself, $A_\tau 0 = 0$, and A_τ is completely continuous for each $\tau > 0$. Moreover, A_τ is Frechet differentiable at 0 in the direction of the cone K (see [7]) with Frechet derivative T_τ defined by

$$(T_\tau h)(t) = \int_{t-\tau}^{t} f_x(s, 0)\, h(s)\, ds, \quad t \in \mathbb{R}, \quad h \in E.$$

Clearly, finding nontrivial ω-periodic solutions of (1.1) is equivalent to finding nonzero fixed points of A_τ.

The fixed point theorem will be applied to the operator A_τ after we have obtained the necessary information concerning the derivative T_τ contained in the next several lemmas.

Lemma 1: If (H 1)—(H 4) are satisfied then T_τ has an eigenvector $x_\tau \in K$ corresponding to a positive eigenvalue λ_τ.

Proof: It is well-known [7, p. 102] that the linear operator T_τ is compact. Letting $u(t) \equiv 1$ and $\alpha = \inf_{0 \leq t \leq \omega} \int_{t-\tau}^{t} f_x(s, 0)\, ds$ ($\alpha > 0$ by (H 2)–(H 4)) we obtain: $T_\tau u \geq \alpha u$, where the partial ordering is that induced by the cone K ($x, y \in E$ then $x \leq y$ if

$y-x\in K$). The lemma now follows from Theorem A. This theorem also gives the estimate $\lambda_\tau \geq \alpha$.

Lemma 1 will be strengthened below (see Lemma 3). It will be shown that there is precisely one eigenvalue (λ_τ) of T_τ having an associated eigenvector in K and this eigenvalue equals the spectral radius of T_τ. Moreover, though not needed for our analysis, x_τ spans the generalized eigenspace, $\{x\in E: (\lambda_\tau I - T_\tau)^n x=0$ for some positive integer $n\}$. These assertions follow from the fact that T_τ is strongly positive.

Definition ([7], p. 60): Let the linear operator B map a cone, K, with nonempty interior into itself. Then B is called *strongly positive* if for all nonzero $x\in K$, there exists $n=n(x)$, a positive integer, such that $B^n x\in \operatorname{int} K$.

Notice that for our choice of K, $\operatorname{int} K=\{x\in K: x(t)>0 \text{ for all } t\in\mathbb{R}\}\neq\emptyset$.

Lemma 2: Let $\tau>0$ and $(H\,1)$—$(H\,4)$ hold. Then T_τ is strongly positive.

Proof: It is trivial to verify that if n satisfies $n\tau>\omega$ then $T_\tau^n(K-\{0\})\subseteq \operatorname{int} K$.

Lemma 3: Let $\tau>0$ and $(H\,1)$—$(H\,4)$ hold. Then T_τ has exactly one positive eigenvalue corresponding to an eigenvector in K. Moreover, if λ_τ is this unique positive eigenvalue then $\lambda_\tau=\rho(T_\tau)\equiv r(\tau)$, where $\rho(T_\tau)$ is the spectral radius of T_τ.

Proof: The lemma is a direct consequence of Theorem 6.3 of Krein and Rutman [8]. It is here that we use the strong positiveness of T_τ.

Lemma 4: If $(H\,1)$—$(H\,4)$ are satisfied then $r(\tau)$ is a strictly increasing, continuous function on $(0,\infty)$ satisfying the estimate:

$$\min_{t\in[0,\omega]} \int_{t-\tau}^t f_x(x,0)\,ds \leq r(\tau) \leq \max_{t\in[0,\omega]} \int_{t-\tau}^t f_x(s,0)\,ds.$$

In particular, there exists a unique value τ_0 of the parameter τ such that $r(\tau_0)=1$ and if $\tau>\tau_0$, then $r(\tau)>1$ and for $\tau<\tau_0$, $r(\tau)<1$.

Proof of Lemma 4: We first remark that if $x\in K-\{0\}$ were an eigenvector of T_τ corresponding to the eigenvalue $r(\tau)$, then $x\in\operatorname{int} K$. This follows since there exists a positive integer n such that $(r(\tau))^n x=T_\tau^n x\in\operatorname{int} K$ and that

$$r(\tau)\geq \min_{0\leq t\leq\omega} \int_{t-\tau}^t f_x(s,0)\,ds>0.$$

If $0<\tau_1<\tau_2$ and $x_1\in\operatorname{int} K$ is an eigenvector of T_{τ_1} corresponding to $r(\tau_1)$ then

$$\begin{aligned} T_{\tau_2} x_1(t) &= \int_{t-\tau_2}^t f_x(s,0)\,x_1(s)\,ds > \int_{t-\tau_1}^t f_x(s,0)\,x_1(s)\,ds \\ &= r(\tau_1)\,x_1(t). \end{aligned}$$

Hence, there exists $\eta>0$ such that $T_{\tau_2}x_1\geq(1+\eta)\,r(\tau_1)\,x_1$ where the partial ordering is that induced by the cone. Appealing again to theorem A, this last inequality implies that $r(\tau_2)\geq(1+\eta)\,r(\tau_1)>r(\tau_1)$.

The estimate $r(\tau)\geq \min\limits_{0\leq t\leq\omega} \int_{t-\tau}^t f_x(s,0)\,ds$ has already been verified. It is well known that the spectral radius is smaller than the norm of the operator. Hence

$$r(\tau)\leq\|T_\tau\| = \max_{0\leq t\leq\omega} \int_{t-\tau}^t f_x(s,0)\,ds.$$

Since r is strictly increasing, continuity will follow if $r(\tau-0)=r(\tau)=r(\tau+0)$ where $r(\tau-0)$ and $r(\tau+0)$ are, respectively, the left and right hand limits of r at τ which are known to exist. Let $\tau_n \nearrow \tau$ and let $x \in K-\{0\}$ be an eigenvector for T_τ corresponding to the eigenvalue $r(\tau)$: $T_\tau x=r(\tau)x$. Then

$$T_{\tau_n} x(t)=\int_{t-\tau_n}^{t} f_x(x,0)\,x(s)\,ds=\int_{t-\tau}^{t} f_x(s,0)\,x(s)\,ds-\int_{t-\tau}^{t-\tau_n} f_x(s,0)\,x(s)\,ds$$

$$=r(\tau)\,x(t)-\int_{t-\tau}^{t-\tau_n} f_x(s,0)\,x(s)\,ds \geq r(\tau)\,x(t)-M\,\|x\|\,(\tau-\tau_n)$$

where $M=\sup f_x(s,0)$. Given $\varepsilon>0$, we have

$$T_{\tau_n} x(t) \geq \big(r(\tau)-\varepsilon\big)\,x(t)+[\varepsilon\,x(t)-M\,\|x\|\,(\tau-\tau_n)].$$

The quantity in brackets is positive for large enough n since $x(t)$ is bounded below. Thus for large enough n, $T_{\tau_n} x \geq \big(r(\tau)-\varepsilon\big)\,x$. Appealing to Theorem A we have $r(\tau_n) \geq r(\tau)-\varepsilon$ for large enough n. This, together with $r(\tau_n) \leq r(\tau)$ shows $r(\tau-0)=r(\tau)$. Now it is well known that if the spectral radius of the linear operator B is less than some positive number m, then the spectral radius of all linear operators sufficiently close to B in norm will also be less than m [12, p. 239]. If $\tau_n \searrow \tau$ then $r(\tau_n) \searrow r$ where $r \geq r(\tau)$ $\big(\text{since } r(\tau) \leq r(\tau_n)\big)$. If $r(\tau)<r$ then since $\tau \mapsto T_\tau$ is easily shown to be continuous, it must be true that $r(\tau')<r$ for all τ' sufficiently near τ. In particular, $r(\tau_n)<r$ for large enough n. Clearly this is impossible hence $r=r(\tau)$ and $r(\tau+0)=r(\tau)$.

The last statement of the lemma is now obvious provided it is seen that $r(0+)=0$ and $r(\infty)=+\infty$.

Notice that if $\int_0^{n\omega} f_x(s,0)\,ds=1$ for some positive integer n then $\tau_0=n\,\omega$. This follows immediately from the estimate in Lemma 4.

Proof of Theorem 3.1: From the previous lemmas it is clear that (a) of the Fixed Point Theorem is satisfied precisely when $\tau>\tau_0$. We show that $(H\,5)$ implies that (b) also holds. Choose $L>0$ so that $\dfrac{f(t,x)}{x}<\dfrac{1}{2\tau}$ for $x \geq L$ and let $M=\sup\limits_{\substack{0 \leq t \leq \omega \\ 0 \leq x \leq L}} f(t,x)$. Then $f(t,x) \leq \dfrac{x}{2\tau}+M$. Let $R=2M\tau$. If $x \in K$ and $\|x\|=R$ then

$$(A_\tau x)(t)=\int_{t-\tau}^{t} f\big(s,x(s)\big)\,ds \leq \frac{\|x\|\,\tau}{2\tau}+M\tau$$
$$=2M\tau=R.$$

Hence $\|x\|=R$ implies $\|A_\tau x\| \leq \|x\|$. If in addition $A_\tau x=\mu x$ then $\mu\,\|x\| \leq \|x\|$ so $\mu \leq 1$.

The fixed point theorem may now be applied to obtain a nontrivial solution $x_\tau \in K$ of $(1.1)_\tau$ when $\tau>\tau_0$. To see that $x_\tau(t)>0$ for all t, is suffices to recognize that

$$x'(t)=f\big(t,x(t)\big)-f\big(t-\tau,x(t-\tau)\big) \leq f\big(t,x(t)\big).$$

Thus, if x_τ were ever to vanish on $\mathbb{R}$, standard differential inequality arguments would imply that $x(t) \equiv 0$ on $\mathbb{R}$, in contradiction to x_τ being a nontrivial solution of $(1.1)_\tau$.

Proof of Theorem 3.2: If $\tau=\tau_0$ and $x\in K-\{0\}$ such that $A_{\tau_0}x=x$ then by $(H\,6)$ and the fact that $x(t)>0$ we have

$$T_{\tau_0}x(t)>A_{\tau_0}x(t)=x(t) \quad \text{for} \quad t\in\mathbb{R}.$$

Hence, there must exist a $\eta>0$ such that $T_{\tau_0}x\geq(1+\eta)x$. But this implies (Theorem A) that $r(\tau_0)\geq 1+\eta>1$ which is impossible since $r(\tau_0)=1$. Thus no nontrivial, ω-periodic solution of (1.1) exists for $\tau=\tau_0$.

Now suppose $\tau<\tau_0$ and for simplicity in notation let $a(t)=f_x(t,0)$. Since $\tau<\tau_0$, $\rho(T_\tau)<1$ so there exists N, a positive integer, such that $\|T_\tau^N\|<1$, where the norm is the operator norm. It is easy to show that

$$\|T_\tau^n\|=\sup_{0\leq t\leq\omega}\int_{t-\tau}^{t}a(s_1)\int_{s_1-\tau}^{s_1}a(s_2)\ldots\int_{s_{n-1}-\tau}^{s_{n-1}}a(s_n)\,ds_n\,ds_{n-1}\ldots ds_1.$$

If $x(t)$ is a solution of $(1.1)_\tau$ for $t>0$ satisfying $x(t)=\varphi(t)$, $t\in[-\tau,0]$ and if $x(t)\leq M$ for $t\geq-\tau$ then $(H\,6)$ implies

$$x(t)\leq M\int_{t-\tau}^{t}a(s)\,ds$$

for $t\geq 0$. If $t\geq\tau$, then

$$x(t)\leq\int_{t-\tau}^{t}a(s)\,x(s)\,ds\leq M\int_{t-\tau}^{t}a(s_1)\int_{s_1-\tau}^{s_1}a(s_2)\,ds_2\,ds_1.$$

It is clear how to show that if $t\geq(N-1)\tau$, then

$$x(t)\leq M\int_{t-\tau}^{t}a(s_1)\int_{s_1-\tau}^{s_1}a(s_2)\ldots\int_{s_{N-1}-\tau}^{s_{N-1}}a(s_N)\,ds_N\ldots ds_1\leq M\,e^{-\gamma}$$

for some $\gamma>0$. An induction argument shows that if $t\geq(kN-1)\tau$, then $x(t)\leq M\,e^{-k\gamma}$ $(k=0,1,2,\ldots)$. It is now trivial to see that $x(t)\leq\dfrac{M}{e^{-\gamma}}\cdot e^{-\gamma t/N\tau}$ for $t\geq 0$.

Now let $b>\tau_0$ and define $F_b=\{x\in K-\{0\}: x \text{ satisfies } (1.1)_\tau \text{ for } \tau_0<\tau\leq b\}$. A straightforward argument shows that $(H\,5)$ implies the existence of an $L(b)>0$ such that $F_b\subseteq\{x\in K:\|x\|\leq L(b)\}$. A simple argument involving the Ascoli-Arzela Theorem shows that F_b is precompact. If the last statement of Theorem 3.2 were false then for some $\varepsilon_0>0$ we can find sequences $\{\tau_n\}$ and $\{x_n\}\subseteq K-\{0\}$ with $\tau_n\searrow\tau_0$ and x_n satisfying $(1.1)_{\tau_n}$ and $\|x_n\|\geq\varepsilon_0$. Since $\{x_n\}$ is precompact we may as well assume $x_n\to x$ where $x\in K-\{0\}$ with $\|x\|\geq\varepsilon_0$. We show that x must satisfy $(1.1)_{\tau_0}$ which contradicts our earlier argument that $(1.1)_{\tau_0}$ has no nontrivial, ω-periodic solutions. Now

$$\|x-A_{\tau_0}x\|\leq\|x-x_n\|+\|A_{\tau_n}x_n-A_{\tau_0}x_n\|+\|A_{\tau_0}x_n-A_{\tau_0}x\|.$$

The first and last terms on the right hand side of the above inequality go to zero as $n\to\infty$ and it is easily seen that $\|A_{\tau_n}x_n-A_{\tau_0}x_n\|\leq L\,|\tau_n-\tau_0|$ where $L=\sup_{\substack{0\leq t\leq\omega\\0\leq z\leq M}}f(t,z)$, $M=\sup_n\|x_n\|$. Thus $A_{\tau_0}x=x$, a contradiction. This completes the proof of Theorem 3.2.

Proof of Theorem 3.3: Choose $\tau^0>\tau_0$ such that if $\tau_0<\tau\leq\tau^0$ and $x\in K-\{0\}$ satisfies $(1.1)_\tau$ then $\|x\|\leq\gamma$. Suppose for some $\tau\in(\tau_0,\tau^0]$ there exists $x,y\in K-\{0\}$ with $x\neq y$ such that both x and y satisfy $(1.1)_\tau$. We may assume $y-x\notin K$. Since

K is closed, there exists $\lambda>0$ such that $0<\lambda<1$ and $y-\lambda x \notin K$. Since $y \in \operatorname{int} K$ there exists $v>0$ such that $y-v x \in K$. It is easy to see that $v<\lambda$. We may choose $\mu>0$ such that $\mu x \leq y$ and such that if $l>0$ and $l x \leq y$ then $l \leq \mu$ (see [7], lemma 1.2). Clearly $0<v \leq \mu<\lambda<1$. By the monotonicity of g we have $A_\tau(\mu x) \leq A_\tau y=y$. Applying the concavity assumption on g and the fact that $x \in \operatorname{int} K$ we obtain

$$A_\tau(\mu x)(t)>\mu A_\tau x(t).$$

Hence there must exist $\eta>0$ such that

$$A_\tau(\mu x) \geq(1+\eta) \mu A_\tau x=(1+\eta) \mu x.$$

Putting the inequalities together we have $(1+\eta) \mu x \leq y$. The maximality of μ then implies that $(1+\eta) \mu \leq \mu$. This contradiction proves the uniqueness assertion of Theorem 4.

Define the map from $[\tau_0, \tau^0] \rightarrow K-\{0\}$ by $\tau_0 \mapsto 0$, $\tau \mapsto x_\tau$ where x_τ is the unique solution in $K-\{0\}$ of $(1.1)_\tau$. That this map is continuous at τ_0 has already been shown. Suppose $\tau_n \rightarrow \tau$ where $\{\tau_n\} \subseteq(\tau_0, \tau^0]$, $\tau \in(\tau_0, \tau^0]$. Let $\underline{x_n}=x_{\tau_n}$. If $x_n \not\rightarrow x_\tau$ then, since cluster points of $\{x_n\}$ are solutions of $(1.1)_\tau$ and $\{x_n\}$ is compact, we may as well assume that $x_n \rightarrow 0$ since some subsequence does. We write

$$x_n=A_{\tau_n} x_n=T_\tau x_n+(T_{\tau_n}-T_\tau) x_n+(A_{\tau_n} x_n-T_{\tau_n} x_n),$$

or

$$x_n=T_\tau x_n+(T_{\tau_n}-T_\tau) x_n+R_{\tau_n}(x_n).$$

Dividing both sides of this last equality by $\| x_n \|$ and letting $\delta_n=\frac{x_n}{\| x_n \|}$ we get $\delta_n=T_\tau \delta_n+(T_{\tau_n}-T_\tau) \delta_n+\frac{R_{\tau_n}(x_n)}{\| x_n \|}$. It is not hard to see that $R_\tau(x)=A_\tau x-T_\tau x$ satisfies $R_\tau(x)=0(\| x \|)$ uniformly in τ on compact τ-sets. As a consequence of this, $\frac{R_{\tau_n}(x_n)}{\| x_n \|} \rightarrow 0$ as $n \rightarrow \infty$. Also, $\|(T_{\tau_n}-T_\tau) \delta_n \| \leq \| T_{\tau_n}-T_\tau \| \rightarrow 0$ as $n \rightarrow \infty$. Since T_τ is compact, we may as well assume that $T_\tau \delta_n \rightarrow w \in K$. It follows that $\delta_n \rightarrow w$ and $w \in K-\{0\}$ since $\| \delta_n \|=1$. Thus $T_\tau w=w$. But this implies that 1 is an eigenvalue corresponding to an eigenvector in K of the operator T_τ. This contradicts Lemma 4 and the fact that $\tau>\tau_0$ so $r(\tau)>1$. The contradiction establishes the continuity assertion of Theorem 3.3.

Now let $\tau_0<\tau_1<\tau_2 \leq \tau^0$ and $x_1, x_2 \in K-\{0\}$ be the unique solutions of $(1.1)_{\tau_1}$ and $(1.1)_{\tau_2}$ respectively. We first show that $x_1 \leq x_2$. If not, then $x_2-x_1 \notin K$. Arguing exactly as before, we obtain μ, $0<\mu<1$, such that $\mu x_1 \leq x_2$ and μ is maximal with this property. Now $A_{\tau_1}(\mu x_1) \leq A_{\tau_1} x_2 \leq A_{\tau_2} x_2=x_2$ and there exists $\eta>0$, exactly as before, such that $A_{\tau_1}(\mu x_1) \geq(1+\eta) \mu A_{\tau_1} x_1=(1+\eta) \mu x_1$. Again we obtain $(1+\eta) \mu x_1 \leq x_2$, a contradiction to the maximality of μ. Hence $x_1 \leq x_2$. But

$$x_1(t)=A_{\tau_1} x_1(t) \leq A_{\tau_1} x_2(t)<A_{\tau_2} x_2(t)=x_2(t).$$

Proof of Theorem 3.4: That $\tau_0=1$ is an immediate consequence of the remark following Lemma 4. Using the fact that any solution satisfies a particular differential equation, it is easily seen that $\| x \| \leq 1$ for all solutions x of $(1.1)_\tau$, $\tau > 1$. Also, if $\tau < 2$ and x satisfies $(1.1)_\tau$ then

$$x(t) = \int_{t-\tau}^{t} a(s)\, x(s)\left(1 - x(s)\right) ds \leq \tfrac{1}{4} \int_{t-\tau}^{t} a(s)\, ds \leq \tfrac{1}{4} \int_{0}^{2} a(s)\, ds$$

so $x(t) \leq \frac{1}{2}$. Clearly $(H\,7)$ is satisfied for $\gamma = \frac{1}{2}$ and τ^0 can be taken to be 2. Notice that $x_{\tau=2} = \frac{1}{2}$. The theorem now follows from the four previous theorems.

Note added in proof: Uniqueness of solutions of (1.1), where f is given by (1.2), for all $\tau > \tau_0$ has recently been established by R. Nussbaum.

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