An Approximation Theorem for Functionals, with Applications in Continuum Mechanics

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Contents

1. Introduction

We start with a brief discussion of the physical motivation behind the mathematical considerations to be presented in Part I of this paper.

Often in theoretical physics one is concerned with a functional relationship,

$$\boldsymbol{\sigma} = \mathop{\mathfrak{F}}_{s=0}^{\infty} (\boldsymbol{g}(s)), \tag{1.1}$$

which states that the present value $\boldsymbol{\sigma}$ of a physical quantity is determined by the values $\boldsymbol{g}(s)$ of a second quantity at all times s in the past. In (1.1), $\boldsymbol{g}(s)$ is the value of the second quantity s time-units ago. If the functional $\mathfrak{F}$ is given, the relation (1.1) may be used to predict the present value $\boldsymbol{\sigma}$ of the first quantity from a knowledge of the "history" $\boldsymbol{g}$ of the second quantity. The relation (1.1) may be interpreted as expressing the causal nature of a class of physical processes. For definiteness we assume that the values of $\boldsymbol{\sigma}$ and $\boldsymbol{g}(s)$ lie in appropriate normed vector spaces, not necessarily finite-dimensional.

In many physical situations, the value $\boldsymbol{\sigma}$ in (1.1) will be, in some sense, more sensitive to the values of $\boldsymbol{g}$ for small s than for large s. Intuitively speaking, the "memory" of the system described by (1.1) will "fade away" in time. In order to make this idea precise, we shall introduce a *norm* $\|\boldsymbol{g}\|$ in the function space of the histories $\boldsymbol{g}$ for which (1.1) is meaningful. We first choose a number p,

$1 \leqq p \leqq \infty$, which will be kept fixed, and then define the norm of $\boldsymbol{g}$ to be

$$\|\boldsymbol{g}\| = \begin{cases} \sqrt[p]{\int\limits_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p\, ds} & \text{if} \quad 1 \leqq p < \infty \\ \sup\limits_{s \geqq 0} |\boldsymbol{g}(s)|\, h(s) & \text{if} \quad p = \infty, \end{cases} \tag{1.2}$$

where $|\boldsymbol{g}(s)|$ is the magnitude of $\boldsymbol{g}(s)$ and where $h(s)$ is a real-valued function which approaches zero rapidly as $s \to \infty$. Thus, in computing the norm $\|\boldsymbol{g}\|$ we assign a greater influence to the values of $\boldsymbol{g}$ for small s (recent past) than for very large s (distant past). We call the function h in (1.2) an *influence function.* The physical idea that the memory of the system is fading corresponds to the mathematical assumption that the functional $\mathfrak{F}$ in (1.1) is *continuous* with respect to convergence in the norm (1.2) of the function space of histories. The influence function h characterizes the rapidity with which the memory is fading.

It is always possible to reduce the relation (1.1) to one which is *normalized* in the sense that the possible histories $\boldsymbol{g}$ have the value $\boldsymbol{g}(0) = \boldsymbol{0}$ at the present instant and that the value of $\mathfrak{F}$ for the zero history $\boldsymbol{g}(s) \equiv \boldsymbol{0}$ is zero. Now, physical experience indicates that phenomena which one would expect to be described exactly by a normalized functional relation (1.1) often follow a simpler relation of the form

$$\boldsymbol{\sigma} = \mathfrak{l}(\overset{(1)}{\boldsymbol{g}}), \qquad \overset{(1)}{\boldsymbol{g}} = \frac{d}{ds}\boldsymbol{g}(s)\Big|_{s=0}, \tag{1.3}$$

where $\mathfrak{l}$ is a linear transformation. In particular, it appears that (1.3) accounts well for the observed phenomena in the case of slow processes. This observation leads to the conjecture that, in some mathematically precise sense, the relation (1.3) approximates the normalized relation (1.1) for slow processes. It is the purpose of the present paper to prove an approximation theorem of this kind. Theorem 2 of §5 asserts not only that (1.3) is the first-order approximation to the normalized relation (1.1) for slow processes but shows also the form of the approximations of higher order. The theorem is based on the assumption that the functional $\mathfrak{F}$ of (1.1) is not only continuous but also Fréchet differentiable at the zero history $\boldsymbol{g}(0) \equiv \boldsymbol{0}$ in the function space with norm (1.2). A normalized functional $\mathfrak{F}$ satisfying this assumption will be called a *memory functional.* The continuity of $\mathfrak{F}$ for histories $\boldsymbol{g}$ different from the zero history is not needed to prove the approximation theorem.

In Part II of the paper we apply the approximation theorem to constitutive equations of continuum mechanics. Our main interest there is in the logical status of the theory of Newtonian fluids within the framework of a recently proposed general theory of fluids with memory effects*.

The theory of compressible Newtonian fluids is based on the following constitutive equation for the stress tensor $\boldsymbol{S}$:

$$\boldsymbol{S} = -p\,\boldsymbol{I} + 2\eta\,\boldsymbol{D} + \lambda(\operatorname{tr}\boldsymbol{D})\,\boldsymbol{I}; \tag{1.4}$$

here the rate of deformation tensor $\boldsymbol{D}$ is the symmetric part of the velocity gradient tensor; $p = p(\varrho)$ is the hydrostatic pressure the fluid would be supporting

* Cf. [1], [2] and [3].

if it were at rest at its present density ϱ; η and λ are functions of ϱ alone and are called coefficients of viscosity.

Experience shows that for some substances, such as water, the theory of Newtonian fluids accounts remarkably well for experimental measurements over a very wide range of conditions. Other substances, such as molten plastics, definitely do not obey (1.4) exactly, but yet their behavior appears to approximate that of Newtonian fluids in the limit of slow motions.

In §6 we review the mathematical definition of a *simple fluid*. As we have frequently remarked in the past, we believe this definition is capable of covering the behavior of nearly all real fluids including such substances as molten plastics which exhibit "hereditary", "non-Newtonian" and "viscoelastic" effects. Here we add to the definition of a simple fluid a new requirement: we require that the functionals occurring in the definition be memory functionals in the sense of the definition used in this paper. As we have indicated, this requirement is related to the physical assumption that simple fluids have a fading memory. If this assumption were not satisfied, the term "fluid" would hardly be appropriate.

In §7 we conclude that the theory of Newtonian fluids is indeed the complete first-order approximation to the theory of simple fluids for slow flows. We also indicate what an experimenter should expect to find as second-order corrections to the constitutive equation of a Newtonian fluid. We point out that several special flow problems for incompressible second-order fluids lead to third-order linear partial differential equations.

In §8 we apply our approximation theorem to the theory of the general simple materials defined in [*1*], Part III.

I. The Approximation Theorem

2. *Influence Functions and Histories*

An *influence function* h of order $r>0$ is a real-valued function of a real variable with the following properties:

(i) $h(s)$ is defined and continuous for $0 \leqq s < \infty$.

(ii) $h(s)$ is positive, $h(s)>0$.

(iii) For each $\sigma>0$, there is a constant M_σ, independent of α, such that

$$\sup_{s\geqq\sigma} \frac{h(s/\alpha)}{\alpha^r h(s)} \leqq M_\sigma \quad \text{for} \quad 0<\alpha\leqq 1. \tag{2.1}$$

The last condition (iii) means that $h(s)$ must decay to zero at a fast enough rate as $s\to\infty$. In fact, we have

$$\sup_{s>0} s^r h(s) = N < \infty, \tag{2.2}$$

which follows from (2.1) by taking $\sigma=1$ and observing that

$$\left(\frac{1}{\alpha}\right)^r h\left(\frac{1}{\alpha}\right) \leqq h(1) \sup_{s\geqq 1} \frac{h(s/\alpha)}{\alpha^r h(s)} \leqq h(1)\, M_1.$$

Let a real Banach space $\mathscr{S}$ with norm $|\;|$ be given. We then define a ***history*** $\boldsymbol{g}$ to be a measurable function defined for $0\leqq s<\infty$ with values $\boldsymbol{g}(s)$ in $\mathscr{S}$. Two such functions will be regarded as the same if they differ only on a set of measure zero.

For a given influence function h and a given number p, $1 \leqq p \leqq \infty$, we define the $\mathscr{L}_{h,p}$-*norm*, $\|\boldsymbol{g}\|_{h,p}$, or simply $\|\boldsymbol{g}\|$, of a history $\boldsymbol{g}$ by

$$\|\boldsymbol{g}\|_{h,p} = \sqrt[p]{\int_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p\, ds} \quad \text{if} \quad 1 \leqq p < \infty, \tag{2.3}$$

$$\|\boldsymbol{g}\|_{h,\infty} = \sup_{s \geqq 0} |\boldsymbol{g}(s)|\, h(s) \qquad \text{if} \quad p = \infty. \tag{2.4}$$

In (2.4) and subsequently *sup* $f(s)$ stands for the essential supremum of $f(s)$, *i.e.* for the greatest lower bound of the suprema of all functions which differ from $f(s)$ only on a set of measure zero.

The set of all histories with finite $\mathscr{L}_{h,p}$-norm forms a Banach space, which we denote by $\mathscr{L}_{h,p}$.

We remark that (2.2) is a necessary but not a sufficient condition for the decay relation (2.1). However, if a function $h(s)$ satisfies (i), (ii), and the limit relation

$$\lim_{s \to \infty} s^r h(s) = 0 \quad \text{monotonically for large } s, \tag{2.5}$$

then it also satisfies the decay relation (2.1) and hence is an influence function of order r. To prove this fact we consider the expression

$$\frac{h(s/\alpha)}{\alpha^r h(s)} = \frac{(s/\alpha)^r h(s/\alpha)}{s^r h(s)}, \qquad 0 < \alpha \leqq 1. \tag{2.6}$$

Since $s^r h(s)$ is monotonically decreasing for s larger than some value s_1, it follows that (2.6) is not greater than 1 for all $s > s_1$. For $0 < \sigma \leqq s \leqq s_1$ the denominator $s^r h(s)$ of (2.6) has a positive minimum because h satisfies the conditions (i) and (ii). The numerator of (2.6) is bounded by the maximum of $s^r h(s)$, which exists and is finite because of (2.5). It follows that (2.6) is bounded, for $\sigma \leqq s < \infty$, by a constant M_σ independent of α, which is the content of (2.1).

The function

$$h(s) = \frac{1}{(s+1)^r}$$

is an influence function of order r. An exponential,

$$h(s) = e^{-\beta s}, \qquad \beta > 0,$$

is an influence function of any order.

3. Retardation

The *retardation* Γ_α with *retardation factor* α, $0 < \alpha \leqq 1$, is the linear transformation $\boldsymbol{g} \to \boldsymbol{g}_\alpha$ defined, for all histories $\boldsymbol{g}$, by

$$(\Gamma_\alpha \boldsymbol{g})(s) = \boldsymbol{g}_\alpha(s) = \boldsymbol{g}(\alpha s). \tag{3.1}$$

We show that Γ_α maps the space $\mathscr{L}_{h,p}$ into itself. For $p \neq \infty$, we have*

$$\begin{aligned} \|\boldsymbol{g}_\alpha\|^p &= \int_0^\infty (|\boldsymbol{g}(\alpha s)|\, h(s))^p\, ds \\ &= \frac{1}{\alpha} \int_0^\infty (|\boldsymbol{g}(s)|\, h(s))^p \left(\frac{h(s/\alpha)}{h(s)}\right)^p ds \leqq \|\boldsymbol{g}\|^p \frac{1}{\alpha} \left(\sup_{s>0} \frac{h(s/\alpha)}{h(s)}\right)^p. \end{aligned} \tag{3.2}$$

* Since no confusion can arise, we omit the indices h, p on the norm.

Since h is continuous and positive,

$$\sup_{\sigma \geqq s \geqq 0} \frac{h(s/\alpha)}{h(s)} < \infty$$

for any $\sigma > 0$. The decay condition (2.1) implies that

$$\sup_{s \geqq \sigma} \frac{h(s/\alpha)}{h(s)} \leqq \alpha^r M_\sigma < \infty .$$

Hence,

$$\sup_{s \geqq 0} \frac{h(s/\alpha)}{h(s)} = K_\alpha < \infty . \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$\|\boldsymbol{g}_\alpha\| \leqq \alpha^{-\frac{1}{p}} K_\alpha \|\boldsymbol{g}\| . \tag{3.4}$$

It is easily verified that this inequality remains valid for $p = \infty$ when we put $\sqrt[\infty]{\alpha} = 1$. It follows from (3.4) that when $\boldsymbol{g}$ has a finite norm $\boldsymbol{g}_\alpha$ has a finite norm and is, therefore, in $\mathscr{L}_{h,p}$. Intuitively, retardation replaces a given history by one which is essentially the same, but slower.

If, possibly after a suitable alteration of $\boldsymbol{g}(s)$ on a set of measure zero, the limits

$$\overset{(0)}{\boldsymbol{g}} = \lim_{s \to 0} \boldsymbol{g}(s) , \qquad \overset{(k)}{\boldsymbol{g}} = \lim_{s \to 0} \frac{k!}{s^k} \left(\boldsymbol{g}(s) - \sum_{j=0}^{k-1} \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}} \right) \tag{3.5}$$

exist for $k = 0, 1, \ldots, n$, then we shall say that the history $\boldsymbol{g}$ has n *generalized derivatives* at $s = 0$. Of course, the existence of ordinary derivatives implies the existence of the corresponding generalized derivatives. Here we shall use the term "derivative" always in the generalized sense of (3.5). Generalized derivatives at $s = 0$ of any order may exist even though $\boldsymbol{g}(s)$ is not continuous near $s = 0$.

The *Taylor transformation* Π_n is the linear transformation $\boldsymbol{g} \to \Pi_n \boldsymbol{g}$ defined, for all histories $\boldsymbol{g}$ which are n times differentiable at $s = 0$, by

$$(\Pi_n \boldsymbol{g})(s) = \sum_{j=0}^{n} \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}} , \tag{3.6}$$

where the $\overset{(j)}{\boldsymbol{g}}$ are the derivatives (3.5).

This Taylor transformation replaces $\boldsymbol{g}$ by its n^{th} "Taylor approximation" $\Pi_n \boldsymbol{g}$. The history $\Pi_n \boldsymbol{g}$ is a polynomial of degree $\leqq n$.

We note that retardation Γ_α preserves the differentiability of a history and that Π_n and Γ_α commute; *i.e.*, for all histories $\boldsymbol{g}$ having n derivatives at $s = 0$, we have

$$\Pi_n \Gamma_\alpha \boldsymbol{g} = \Gamma_\alpha \Pi_n \boldsymbol{g} . \tag{3.7}$$

The set of all histories which have n derivatives at $s = 0$ and which also belong to the Banach space $\mathscr{L}_{h,p}$ will be denoted by $\mathscr{D}_n$. This set $\mathscr{D}_n$ is a linear subspace of $\mathscr{L}_{h,p}$, but it is not closed in $\mathscr{L}_{h,p}$ and hence not a Banach space.

The following theorem is an analogue of the classical Taylor approximation theorem.

Theorem 1. *Assume that n, p, and the order r of the influence function h satisfy the inequality*

$$n < r - \frac{1}{p} \qquad \left(\frac{1}{p} = 0 \quad \textit{if} \quad p = \infty \right) . \tag{3.8}$$

Then the Taylor transformation Π_n maps the subspace $\mathscr{D}_n$ of $\mathscr{L}_{h,p}$ into itself, and, for all $\boldsymbol{g}$ in $\mathscr{D}_n$,

$$\lim_{\alpha\to 0}\frac{1}{\alpha^n}\|\Gamma_\alpha(\boldsymbol{g}-\Pi_n\boldsymbol{g})\|_{h,p}=0. \tag{3.9}$$

We can also write (3.9), using (3.7), in the form

$$\Gamma_\alpha\boldsymbol{g}-\Pi_n\Gamma_\alpha\boldsymbol{g}=\boldsymbol{g}_\alpha-\Pi_n\boldsymbol{g}_\alpha=o(\alpha^n), \tag{3.10}$$

where the order symbol o must be understood in terms of the norm of the function space $\mathscr{L}_{h,p}$. Roughly speaking, the theorem states that in the space $\mathscr{D}_n\in\mathscr{L}_{h,p}$ a slow history is close to its Taylor approximation and that the distance between them is $o(\alpha^n)$.

Proof of Theorem 1. We consider only the case when p is finite. The case when $p=\infty$ can be treated analogously.

First, we show that every polynomial of degree $\leqq n$ has a finite norm. For this purpose it is sufficient to prove that

$$\int_0^\infty (s^k h(s))^p\,ds=L_k<\infty \quad\text{for}\quad k\leqq n. \tag{3.11}$$

It follows from (2.2) that

$$\int_1^\infty (s^k h(s))^p\,ds=\int_1^\infty (s^r h(s))^p s^{p(k-r)}\,ds\leqq N^p\int_1^\infty s^{-p(r-k)}\,ds. \tag{3.12}$$

The inequalities (3.8) and $k\leqq n$ imply that $p(r-k)>1$ and hence that the integral

$$\int_1^\infty s^{-p(r-k)}\,ds=\frac{1}{p(r-k)-1}$$

is finite. Since $s^k h(s)$ is continuous, it then follows from (3.12) that the integral (3.11) is finite.

We have shown that any polynomial of degree $\leqq n$ belongs to $\mathscr{L}_{h,p}$ and hence to $\mathscr{D}_n$, because it has n derivatives at $s=0$. Since a Taylor approximation $\Pi_n\boldsymbol{g}$ is a polynomial of degree $\leqq n$, it follows that Π_n maps $\mathscr{D}_n$ into itself.

The definitions (3.5) and (3.6) imply that the history

$$\boldsymbol{f}=\boldsymbol{g}-\Pi_n\boldsymbol{g}\in\mathscr{D}_n \tag{3.13}$$

satisfies the limit relation

$$\lim_{s\to 0}\frac{|\boldsymbol{f}(s)|}{s^n}=0. \tag{3.14}$$

The definitions (2.3), (3.1), and (3.13) show that the assertion (3.9) of the theorem is equivalent to

$$\lim_{\alpha\to 0}\frac{1}{\alpha^{np}}\int_0^\infty (|\boldsymbol{f}(\alpha s)|\,h(s))^p\,ds=0. \tag{3.15}$$

We observe that

$$\frac{1}{\alpha^{np}}\int_0^\infty (|\boldsymbol{f}(\alpha s)|\,h(s))^p\,ds=\frac{1}{\alpha^{np+1}}\int_0^\infty \left(|\boldsymbol{f}(s)|\,h\left(\frac{s}{\alpha}\right)\right)^p ds \tag{3.16}$$

and investigate the latter expression. Let $\varepsilon>0$ be given. It follows from (3.14) that we can find a $\sigma(\varepsilon)>0$ such that

$$\frac{|f(s)|}{s^n}\leqq\varepsilon \quad \text{for} \quad 0<s\leqq\sigma(\varepsilon).$$

Hence

$$\begin{aligned}\frac{1}{\alpha^{np+1}}\int_0^{\sigma(\varepsilon)}\left(|f(s)|\,h\left(\frac{s}{\alpha}\right)\right)^p ds &\leqq \frac{1}{\alpha^{np+1}}\,\varepsilon^p\int_0^{\sigma(\varepsilon)}\left(s^n h\left(\frac{s}{\alpha}\right)\right)^p ds\\ &\leqq \varepsilon^p\int_0^{\infty}\left(\frac{s^n}{\alpha^n}h\left(\frac{s}{\alpha}\right)\right)^p d\left(\frac{s}{\alpha}\right)=\varepsilon^p L_n,\end{aligned} \tag{3.17}$$

where L_n is given by (3.11). On the other hand, we have

$$\frac{1}{\alpha^{np+1}}\int_{\sigma(\varepsilon)}^{\infty}\left(|f(s)|\,h\left(\frac{s}{\alpha}\right)\right)^p ds=\alpha^w\int_{\sigma(\varepsilon)}^{\infty}\left(|f(s)|\,h(s)\right)^p\left(\frac{h(s/\alpha)}{\alpha^r h(s)}\right)^p ds, \tag{3.18}$$

where $w=p(r-n)-1$. Applying the decay condition (2.1) to (3.18), we find that

$$\frac{1}{\alpha^{np+1}}\int_{\sigma(\varepsilon)}^{\infty}\left(|f(s)|\,h\left(\frac{s}{\alpha}\right)\right)^p ds\leqq\alpha^w M^p_{\sigma(\varepsilon)}\,\|f\|^p. \tag{3.19}$$

On combining (3.17) and (3.19), we see that

$$\frac{1}{\alpha^{np+1}}\int_0^{\infty}\left(|f(s)|\,h\left(\frac{s}{\alpha}\right)\right)^p ds\leqq\varepsilon^p L_n+\alpha^w M^p_{\sigma(\varepsilon)}\,\|f\|^p \tag{3.20}$$

holds for all $\varepsilon>0$ and all $0<\alpha\leqq 1$. The assumed inequality (3.8) insures that $w=p(r-n)-1$ is positive. Therefore, by choosing first ε and then α sufficiently small, we see that the right side of (3.20) can be made as small as desired. It then follows from (3.16) that the limit relation (3.15) holds, which completes the proof of the theorem.

In the special case $n=0$, $r>\frac{1}{p}$, $\overset{(0)}{g}=\mathbf{0}$ Theorem 1 states the following. If $g\in\mathscr{L}_{h,p}$ is continuous at $s=0$ with value $\mathbf{0}$, then

$$\lim_{\alpha\to 0}\|\Gamma_\alpha g\|=0. \tag{3.21}$$

If $1\leqq n<r-\frac{1}{p}$ and $\overset{(0)}{g}=\mathbf{0}$, instead of (3.21), we have the stronger result

$$g_\alpha=\Gamma_\alpha g=O(\alpha). \tag{3.22}$$

4. *Memory Functionals*

We first recall some definitions* from the theory of functions defined on a real vector space $\mathscr{H}$ with norm $\|\;\|$ and having values in another real vector space $\mathscr{T}$ with norm $|\;|$.

(1) Suppose $g_1,\ldots,g_k$ are variables in $\mathscr{H}$. Then a function $\mathfrak{P}(g_1,\ldots,g_k)$, defined for all values of the variables g_i in $\mathscr{H}$ and having values in $\mathscr{T}$, is called

* These definitions are analogous to those given in Chap. XXVI of [4].

a *bounded k-linear form* if it is linear in each variable $\boldsymbol{g}_i$ separately and if there is a constant M, independent of $\boldsymbol{g}_i$, such that

$$|\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)| \leqq M \|\boldsymbol{g}_1\| \cdots \|\boldsymbol{g}_k\|. \tag{4.1}$$

The form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is said to be *symmetric* if any permutation of the variables leaves the value unchanged.

(2) A function $\mathfrak{P}(\boldsymbol{g})$, defined for all $\boldsymbol{g} \in \mathscr{H}$ and having values in $\mathscr{T}$, is called a *bounded homogeneous polynomial of degree k* if there is a bounded symmetric k-linear form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ such that

$$\mathfrak{P}(\boldsymbol{g}) = \mathfrak{P}(\boldsymbol{g}, \ldots, \boldsymbol{g}). \tag{4.2}$$

The symmetric n-linear form $\mathfrak{P}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is uniquely determined by the homogeneous polynomial $\mathfrak{P}(\boldsymbol{g})$ and is called the *polar form* of the polynomial.

(3) A function $\mathfrak{F}$, defined on a neighborhood of $\mathbf{0} \in \mathscr{H}$ and having values in $\mathscr{T}$ is said to be *n times Fréchet-differentiable* at $\mathbf{0} \in \mathscr{H}$ if there are bounded homogeneous polynomials $\delta^k \mathfrak{F}(\boldsymbol{g})$ of degree $k = 0, 1, \ldots, n$ such that

$$\mathfrak{F}(\boldsymbol{g}) = \sum_{k=0}^{n} \frac{1}{k!}\, \delta^k \mathfrak{F}(\boldsymbol{g}) + \|\boldsymbol{g}\|^n \mathfrak{R}(\boldsymbol{g}), \tag{4.3}$$

where

$$\lim_{\|\boldsymbol{g}\| \to 0} |\mathfrak{R}(\boldsymbol{g})| = 0. \tag{4.4}$$

The polynomial $\delta^k \mathfrak{F}(\boldsymbol{g})$ or its polar form $\delta^k \mathfrak{F}(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ is called the k^{th} *Fréchet-differential* or the k^{th} *variation* of $\mathfrak{F}$ at $\mathbf{0} \in \mathscr{H}$. The differentials may be obtained recursively by

$$\delta^0 \mathfrak{F}(\boldsymbol{g}) = \mathfrak{F}(\mathbf{0}), \quad \delta^k \mathfrak{F}(\boldsymbol{g}) = k! \lim_{\lambda \to 0} \frac{1}{\lambda^k} \left[\mathfrak{F}(\lambda \boldsymbol{g}) - \sum_{j=0}^{k-1} \frac{\lambda^j}{j!}\, \delta^j \mathfrak{F}(\boldsymbol{g})\right]. \tag{4.5}$$

We here consider the case in which $\mathscr{H}$ is the function space of all histories $\boldsymbol{g}$ with the following properties:

(α) $\boldsymbol{g}$ has a finite $\mathscr{L}_{h,p}$-norm,

(β) $\boldsymbol{g}$ has n generalized derivatives at $s = 0$,

(γ) $\boldsymbol{g}$ has a zero limit at $s = 0$:

$$\lim_{s \to 0} \boldsymbol{g}(s) = \overset{(0)}{\boldsymbol{g}} = \mathbf{0}, \tag{4.6}$$

(δ) n, p, and the order r of the influence function h obey the inequality (3.8).

The conditions (α) and (β) state that $\mathscr{H} \subset \mathscr{D}_n$. Condition ($\delta$) insures that the conclusion (3.9) of Theorem 1 is valid for all $\boldsymbol{g} \in \mathscr{H}$.

A function $\mathfrak{F}$ defined on a neighborhood in $\mathscr{H}$ of the zero function $\mathbf{0} \in \mathscr{H} \subset \mathscr{L}_{h,p}$ and having values in a real Banach space $\mathscr{T}$ will be called a *memory functional of type* (h, n) if it is n times Fréchet-differentiable at $\mathbf{0} \in \mathscr{H}$ and if it is normalized by

$$\delta^0 \mathfrak{F}(\boldsymbol{g}) = \mathfrak{F}(\mathbf{0}) = \mathbf{0}. \tag{4.7}$$

In some applications it may be more natural to assume that $\mathfrak{F}$ is defined and Fréchet-differentiable on a neighborhood of zero in the entire space $\mathscr{L}_{h,p}$. However, the approximation theorem of the following section applies only to histories which belong to the subspace $\mathscr{H}$ of $\mathscr{L}_{h,p}$.

5. The Approximation Theorem

Theorem 2. *Let $\mathfrak{F}$ be a memory functional of type (h, n); the histories $\boldsymbol{g}$ in the domain space $\mathscr{H}$ of $\mathfrak{F}$ have values in the space $\mathscr{S}$, and $\mathfrak{F}$ itself has values in $\mathscr{T}$. Then, for each k-tuple of indices $(j_1, j_2, \ldots, j_k)$ such that*

$$1 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_k \leqq n, \quad j_1 + j_2 + \cdots + j_n \leqq n, \tag{5.1}$$

there exists a bounded k-linear form $\mathfrak{l}_{j_1 \ldots j_k}$ with variables in $\mathscr{S}$ and values in $\mathscr{T}$ such that, for all $\boldsymbol{g} \in \mathscr{H}$,

$$\mathfrak{F}(\boldsymbol{g}_\alpha) = \sum_{(j_1, \ldots, j_k)} \mathfrak{l}_{j_1 \ldots j_k}(\overset{(j_1)}{\boldsymbol{g}}_\alpha, \ldots, \overset{(j_k)}{\boldsymbol{g}}_\alpha) + o(\alpha^n), \tag{5.2}$$

where

$$\boldsymbol{g} = \Gamma_\alpha \boldsymbol{g}, \quad \text{i.e.,} \quad \boldsymbol{g}_\alpha(s) = \boldsymbol{g}(\alpha s), \tag{5.3}$$

and where

$$\overset{(j)}{\boldsymbol{g}}_\alpha = \alpha^j \overset{(j)}{\boldsymbol{g}} \tag{5.4}$$

is the generalized j^{th} derivative at $s=0$ of $\boldsymbol{g}_\alpha$, defined according to (3.5). The sum in (5.2) is to be extended over all sets $(j_1, \ldots, j_k)$ of indices satisfying (5.1), and the order symbol $o(\alpha^n)$ has the usual sense. The multilinear forms $\mathfrak{l}_{j_1 \ldots j_k}$ are uniquely determined by $\mathfrak{F}$.

This theorem permits the asymptotic approximation of a memory *functional*, for "slow" histories, by a polynomial *function* of the derivatives at $s=0$ of the argument function of the functional. It is worth noting that the approximating expression

$$\mathfrak{F}_n(\boldsymbol{g}) = \sum_{(j_1, \ldots, j_k)} \mathfrak{l}_{j_1 \ldots j_k}(\overset{(j_1)}{\boldsymbol{g}}, \ldots, \overset{(j_k)}{\boldsymbol{g}}), \tag{5.6}$$

regarded as a functional of $\boldsymbol{g}$, is not a memory functional in the sense of the previous section because it is not even continuous at the zero function $\boldsymbol{g} = \boldsymbol{0} \in \mathscr{H}$.

For $n=1$ and $n=2$, (5.6) reduces to

$$\mathfrak{F}_1(\boldsymbol{g}) = \mathfrak{l}_1(\overset{(1)}{\boldsymbol{g}}), \quad \mathfrak{F}_2(\boldsymbol{g}) = \mathfrak{l}_1(\overset{(1)}{\boldsymbol{g}}) + \mathfrak{l}_{11}(\overset{(1)}{\boldsymbol{g}}, \overset{(1)}{\boldsymbol{g}}) + \mathfrak{l}_2(\overset{(2)}{\boldsymbol{g}}) \tag{5.7}$$

respectively, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are linear and $\mathfrak{l}_{11}$ is bilinear.

Proof of Theorem 2. For $n=0$, the theorem is a trivial consequence of (3.21). For $n \geqq 1$, the proof is based on a combination of the Fréchet-differentiability assumption (4.3) and equation (3.10) of Theorem 1 which, in the notation of (5.3), may be written as

$$\boldsymbol{g}_\alpha = \Pi_n \boldsymbol{g}_\alpha + o(\alpha^n). \tag{5.8}$$

Since $\overset{(0)}{\boldsymbol{g}} = \boldsymbol{0}$ by (4.6), the definition (3.6) gives

$$(\Pi_n \boldsymbol{g}_\alpha)(s) = \sum_{j=1}^{n} \alpha^j \frac{s^j}{j!} \overset{(j)}{\boldsymbol{g}}. \tag{5.9}$$

The result (3.22) applies to both $\boldsymbol{g} \in \mathscr{H}$ and $\Pi_n \boldsymbol{g} \in \mathscr{H}$:

$$\Pi_n \boldsymbol{g}_\alpha = O(\alpha), \tag{5.10}$$

$$\boldsymbol{g}_\alpha = O(\alpha). \tag{5.11}$$

Combining (5.11) with (4.3), (4.4), and (4.7), we obtain

$$\mathfrak{F}(g_\alpha) = \sum_{k=1}^{n} \frac{1}{k!} \delta^k \mathfrak{F}(g_\alpha) + o(\alpha^n). \tag{5.12}$$

Consider now a bounded homogeneous polynomial $\mathfrak{P}(g)$ of degree k, $1 \leqq k \leqq n$, in the sense of (2) of §4. The differentials $\delta^k \mathfrak{F}$, $k > 1$ are such polynomials. The polar form of $\mathfrak{P}$ will also be denoted by $\mathfrak{P}$. Using the multilinearity of this polar form $\mathfrak{P}$, we obtain by (4.2), (5.8), and (5.10)

$$\begin{aligned}\mathfrak{P}(g_\alpha) &= \mathfrak{P}\big(\Pi_n g_\alpha + o(\alpha^n), \ldots, \Pi_n g_\alpha + o(\alpha^n)\big)\\ &= \mathfrak{P}(\Pi_n g_\alpha, \ldots, \Pi_n g_\alpha) + \sum \mathfrak{P}\big(O(\alpha), \ldots, o(\alpha^n), \ldots, O(\alpha)\big),\end{aligned}$$

where each term of the sum contains at least one variable $o(\alpha^n)$. It follows from the boundedness (4.1) of $\mathfrak{P}$ that the terms in the sum are all $o(\alpha^{n+k-1})$. Since $k \geqq 1$, we have

$$\mathfrak{P}(g_\alpha) = \mathfrak{P}(\Pi_n g_\alpha) + o(\alpha^n). \tag{5.13}$$

We now investigate $\mathfrak{P}(\Pi_n g_\alpha)$. Using (5.9) and the multilinearity of the polar form $\mathfrak{P}$, we obtain

$$\mathfrak{P}(\Pi_n g_\alpha) = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\alpha^{j_1+\cdots+j_k}}{j_1! \cdots j_k!} \mathfrak{P}(s^{j_1} \overset{(j_1)}{g}, \ldots, s^{j_k} \overset{(j_k)}{g}). \tag{5.14}$$

Due to the symmetry of $\mathfrak{P}$, all terms of (5.14) which differ only in the order of the indices $j_1, \ldots, j_k$ are the same. Collecting these terms and separating all terms of order higher than n in α, we get

$$\mathfrak{P}(\Pi_n g_\alpha) = \sum_{\substack{(j_1, \ldots, j_k)\\ k \text{ fixed}}} \alpha^{j_1+\cdots+j_k} m_{j_1 \ldots j_k} \mathfrak{P}(s^{j_1} \overset{(j_1)}{g}, \ldots, s^{j_k} \overset{(j_k)}{g}) + O(\alpha^{n+1}) \tag{5.15}$$

where the sum is to be extended over all k-tuples $(j_1, \ldots, j_k)$ satisfying (5.1) and where the $m_{j_1 \ldots j_k}$ are positive rational numbers.

Now, the function

$$\mathfrak{p}_{j_1 \ldots j_k}(a_1, \ldots, a_k) = \mathfrak{P}(s^{j_1} a_1, \ldots, s^{j_k} a_k), \tag{5.16}$$

with variables $a_i \in \mathscr{S}$ and values in $\mathscr{T}$ is clearly multilinear, because $\mathfrak{P}$ is. Also, $\mathfrak{p}_{j_1 \ldots j_k}$ is bounded. Indeed, application of (4.1) to (5.16) yields

$$|\mathfrak{p}_{j_1 \ldots j_k}(a_1, \ldots, a_k)| \leqq M \|s^{j_1} a_1\| \ldots \|s^{j_k} a_k\|,$$

and (3.11) shows

$$\|s^{j_i} a_i\|_h = |a_i| \sqrt[p]{L_{j_i}},$$

hence

$$|\mathfrak{p}_{j_1 \ldots j_k}(a_1, \ldots, a_k)| \leqq \bar{M} |a_1| \ldots |a_k|. \tag{5.17}$$

Substituting (5.16) into (5.15) and using (5.4) yields

$$\mathfrak{P}(\Pi_n g_\alpha) = \sum_{\substack{(j_1, \ldots, j_k)\\ k \text{ fixed}}} \mathfrak{l}_{j_1 \ldots j_k}(\overset{(j_1)}{g}_\alpha, \ldots, \overset{(j_k)}{g}_\alpha) + O(\alpha^{n+1}) \tag{5.18}$$

where the $\mathfrak{l}_{j_1 \ldots j_k}$ are bounded k-linear forms.

Applying the results (5.13) and (5.18) to $\mathfrak{P}=\frac{1}{k!}\delta^k\mathfrak{F}$ in (5.12), we obtain formula (5.2).

The uniqueness of the $\mathfrak{l}_{j_1\ldots j_k}$ follows easily from the observation that they are linearly independent and of order $O(\alpha^n)$ or lower.

We remark that the multilinear forms $\mathfrak{l}_{j_1\ldots j_k}$ of (5.2) are not necessarily symmetric.

II. Applications

6. *The Concept of a Simple Fluid*

The notion of a simple fluid has been given a definition within the framework of a general theory of the mechanical behavior of materials*. This definition is based on the following two physical assumptions**:

(a) The present stress depends on the past history of the first spatial gradient of the displacement function.

(b) A fluid has no preferred configurations.

Using the principle of material objectivity ([*1*], §11), it was shown in reference [*1*] that the constitutive equation of a simple fluid can be written in the form (22.12) of [*1*]. This functional relation, in a slightly different notation, reads

$$\boldsymbol{S}(t)=\underset{s=0}{\overset{\infty}{\mathfrak{H}}}\left(\boldsymbol{C}_t(t-s);\varrho(t)\right). \tag{6.1}$$

Here $\boldsymbol{S}(t)$ is the stress and $\varrho(t)$ the density at time t. $\boldsymbol{C}_t(\tau)$ is called the right Cauchy-Green tensor at time τ relative to the configuration at time t. This tensor is defined by

$$\boldsymbol{C}_t(\tau)=\boldsymbol{F}_t^T(\tau)\,\boldsymbol{F}_t(\tau) \tag{6.2}$$

where

$$\boldsymbol{F}_t(\tau)=\nabla_{\boldsymbol{x}}\boldsymbol{\chi}_t(\boldsymbol{x},\tau) \tag{6.3}$$

is the gradient the displacement function $\boldsymbol{\xi}=\boldsymbol{\chi}_t(\boldsymbol{x},\tau)$ which gives the position at time τ of the material point having the position $\boldsymbol{x}$ at time t. The stress tensor $\boldsymbol{S}(t)$ is symmetric. The Cauchy-Green tensor $\boldsymbol{C}_t(\tau)$ is positive definite and symmetric; for $\tau=t$ it reduces to the unit tensor $\boldsymbol{I}$:

$$\boldsymbol{C}_t(t)=\boldsymbol{I}. \tag{6.4}$$

The functional $\mathfrak{H}$ in (6.1) is isotropic; *i.e.*, $\mathfrak{H}$ obeys

$$\boldsymbol{Q}\underset{s=0}{\overset{\infty}{\mathfrak{H}}}\left(\boldsymbol{C}(s);\varrho\right)\boldsymbol{Q}^T=\underset{s=0}{\overset{\infty}{\mathfrak{H}}}\left(\boldsymbol{Q}\,\boldsymbol{C}(s)\,\boldsymbol{Q}^T;\varrho\right) \tag{6.5}$$

identically in the history $\boldsymbol{C}(s)=\boldsymbol{C}_t(t-s)$ and the orthogonal tensor $\boldsymbol{Q}$; here $\boldsymbol{Q}^T$ is the transpose of $\boldsymbol{Q}$. It follows from (6.5) that the value of $\mathfrak{H}$ for the "rest

* [*1*], § 21.

** In [*3*] we give a survey of the theory of simple fluids with emphasis on physical applications. In that paper we anticipate some of the results rigorously derived here. Although the language and the definitions of the present paper are slightly different from those used in §§ 6 and 7 of [*3*], the arguments presented here prove also the theorems stated there.

history" $\boldsymbol{C}(s) \equiv \boldsymbol{I}$ is a scalar multiple $-p(\varrho)$ of the unit tensor $\boldsymbol{I}$. Defining

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{G}(s), \varrho\right) = p(\varrho)\,\boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{H}}}\left(\boldsymbol{I} + \boldsymbol{G}(s), \varrho\right), \tag{6.6}$$

we may rewrite (6.1) in the form

$$\boldsymbol{S}(t) = -p\left(\varrho(t)\right)\boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}, \varrho(t)\right). \tag{6.7}$$

The functional $\underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{G}(s); \varrho\right)$ is defined for functions $\boldsymbol{G}(s)$ with the property

$$\boldsymbol{G}(0) = \boldsymbol{0}, \tag{6.8}$$

and it has the value $\boldsymbol{0}$ for the zero function $\boldsymbol{G}(s) \equiv \boldsymbol{0}$; *i.e.*,

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}}(\boldsymbol{0}, \varrho) = \boldsymbol{0}. \tag{6.9}$$

It is also isotropic in the sense of (6.5).

We now assume that for each simple fluid defined by a constitutive equation (6.7) there exists an influence function h of an order r such that the functional $\mathfrak{F}$ of (6.7) is a memory functional of type (h, n) in the sense of §4. The domain of $\mathfrak{F}$ is a class $\mathscr{H}$ of histories $\boldsymbol{G}$ whose values $\boldsymbol{G}(s)$ are in the space $\mathscr{S}$ of all symmetric tensors. For the norm $|\ |$ in $\mathscr{S}$ we use

$$|\boldsymbol{A}| = \sqrt{\operatorname{tr} \boldsymbol{A}^2} \quad \text{for} \quad \boldsymbol{A} \in \mathscr{S}. \tag{6.10}$$

The range space $\mathscr{T}$ of $\mathfrak{F}$ is the same as the range space of the histories $\boldsymbol{G}$, *i.e.* the space $\mathscr{T} = \mathscr{S}$ of all symmetric tensors with norm (6.10). Equation (6.9) insures that $\mathfrak{F}$ has the normalization (4.7) required for a memory functional. The density ϱ enters into (6.7) only as a real parameter. The assumption that $\mathfrak{F}$ is a memory functional implies that its domain of definition contains a neighborhood of zero in a function space $\mathscr{H}$ which is defined by the conditions $(\alpha)-(\delta)$ of §4. The condition (γ) expresses the assumption that all histories $\boldsymbol{G} \in \mathscr{H}$ correspond to motions which are continuous at the present instant $s=0$ $\left(cf.\ (6.8)\right)$.

If the simple fluid under consideration is *incompressible*, we must make some alterations in our starting assumptions. For every possible motion in such a fluid the density ϱ is constant and the tensor $\boldsymbol{C}_t(\tau)$ is unimodular. In addition, the stress is determined by the history of the motion only up to a hydrostatic pressure p. Consequently, the equation (6.7) must be replaced by

$$\boldsymbol{S}(t) = -p\,\boldsymbol{I} + \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}\right) \tag{6.11}$$

in which the indeterminate pressure p and the functional $\mathfrak{F}$ may be normalized by

$$0 = \operatorname{tr} \boldsymbol{S}(t) + 3p = \operatorname{tr} \underset{s=0}{\overset{\infty}{\mathfrak{F}}}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}\right). \tag{6.12}$$

If we were to limit the domain of the functional $\mathfrak{F}$ in (6.11) to kinematically possible histories, this domain would not contain a neighborhood of the zero function in an appropriate function space, $\mathscr{H}$; therefore, $\mathfrak{F}$ could not then be a memory functional. We assume, however, that $\mathfrak{F}$ becomes a memory functional when its domain is extended by putting

$$\mathop{\mathfrak{F}}_{s=0}^{\infty}\left(\boldsymbol{C}(s)-\boldsymbol{I}\right)=\mathop{\mathfrak{F}}_{s=0}^{\infty}\left((\det \boldsymbol{C}(s))^{-1}\boldsymbol{C}(s)-\boldsymbol{I}\right) \tag{6.13}$$

when $\boldsymbol{C}(s)$ is not unimodular.

Aside from the added properties (6.12) and (6.13), the functional $\mathfrak{F}$ occurring in (6.11) is assumed to be of the same type as that in (6.7) with respect to both isotropy and memory.

7. *Approximations of Order n for Simple Fluids*

We now apply Theorem 2, §5, to the memory functional occurring in the constitutive equations for simple fluids (6.7) or (6.11).

Suppose a motion with Cauchy-Green tensor $\boldsymbol{C}_t(\tau)$ is given. The corresponding history $\boldsymbol{G}$ is defined by

$$\boldsymbol{G}(s)=\boldsymbol{C}_t(t-s)-\boldsymbol{I}. \tag{7.1}$$

If $\boldsymbol{C}_t(\tau)$ is n times differentiable with respect to τ at $\tau=t$, the k^{th} Rivlin-Ericksen tensor $\boldsymbol{A}_k$, $k=1, 2, \ldots, n$ is defined as follows:

$$\boldsymbol{A}_k=\frac{d^k}{d\tau^k}\boldsymbol{C}_t(\tau)\Big|_{\tau=t}=(-1)^k\overset{(k)}{\boldsymbol{G}}, \tag{7.2}$$

where $\overset{(k)}{\boldsymbol{G}}$ is the k^{th} derivative of $\boldsymbol{G}(s)$ at $s=0$, as in §2.

We now consider histories $\boldsymbol{G}_\alpha$ obtained from $\boldsymbol{G}\in\mathscr{H}$ by retardation as in (5.3). The corresponding Rivlin-Ericksen tensors are

$$\boldsymbol{A}_k^\alpha=\alpha^k\boldsymbol{A}_k=(-1)^k\overset{(k)}{\boldsymbol{G}}_\alpha. \tag{7.3}$$

They differ only by the inessential factor $(-1)^k$ from the tensors $\overset{(k)}{\boldsymbol{G}}_\alpha$ to be substituted for $\overset{(k)}{\boldsymbol{g}}_\alpha$ in the approximation formula (5.2). This formula, applied to the constitutive equations (6.7) or (6.11), yields the following expression for the stress tensor $\boldsymbol{S}_\alpha$ corresponding to the retarded history $\boldsymbol{G}_\alpha$:

$$\boldsymbol{S}_\alpha=-p\,\boldsymbol{I}+\sum_{(j_1\ldots j_k)}\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}^\alpha,\ldots,\boldsymbol{A}_{j_k}^\alpha)+o(\alpha^n), \tag{7.4}$$

where the summation is to be extended over all sets of indices $(j_1,\ldots,j_k)$ obeying (5.1). The terms $\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}^\alpha,\ldots,\boldsymbol{A}_{j_k}^\alpha)$ are linear in each of the variables. For compressible fluids, it is understood that p and $\mathfrak{m}_{j_1\ldots j_k}$ depend on the density ϱ.

The equation (7.4) remains valid even when the derivatives shown in (7.2) exist only in the generalized sense of (3.5).

The multilinear forms $\mathfrak{m}_{j_1\ldots j_k}$ in (7.4) are isotropic functions, which means that they obey the identities

$$\boldsymbol{Q}\,\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{U}_1,\ldots,\boldsymbol{U}_k)\,\boldsymbol{Q}^T=\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{Q}\,\boldsymbol{U}_1\,\boldsymbol{Q}^T,\ldots,\boldsymbol{Q}\,\boldsymbol{U}_k\,\boldsymbol{Q}^T) \tag{7.5}$$

for all orthogonal $\boldsymbol{Q}$ and all symmetric tensors $\boldsymbol{U}_1,\ldots,\boldsymbol{U}_k$. This proposition follows easily from the fact that the memory functional $\mathfrak{F}$ occurring in (6.7) or (6.11) is isotropic in the sense of (6.5), from the observation that the conjugation $\boldsymbol{G}(s)\to\boldsymbol{Q}\,\boldsymbol{G}(s)\,\boldsymbol{Q}^T$ leaves the norm $\|\boldsymbol{G}\|$ unchanged, and from the uniqueness of the multilinear forms $\mathfrak{m}_{j_1\ldots j_k}$ asserted in Theorem 2, §5.

It follows from known theorems on isotropic functions that each form $\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_k})$, because it is isotropic and multilinear, may be expressed as a sum

$$\mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_k}) = \sum_\iota \boldsymbol{\zeta}_\iota \tag{7.6}$$

of products of the form

$$\boldsymbol{\zeta}_\iota = \mu_\iota \varphi_1 \varphi_2 \cdots \varphi_{\mathfrak{m}_\iota}(\boldsymbol{A}_{l_1}\boldsymbol{A}_{l_2}\cdots\boldsymbol{A}_{l_{q_i}} + \boldsymbol{A}_{l_{q_i}}\cdots\boldsymbol{A}_{l_2}\boldsymbol{A}_{l_1}) \tag{7.7}$$

where the φ_j's are traces of products of some of the tensors $\boldsymbol{A}_{j_r}$ and are such that each $\boldsymbol{A}_{j_r}$, $r = 1, \ldots, k$, occurs precisely once in each term $\boldsymbol{\zeta}_\iota$. In the case of an incompressible simple fluid the coefficients μ_ι are constants, whereas for simple fluids in general the μ_i are functions of the density. Thus, for simple fluids, a finite number of scalar material functions $p(\varrho)$, $\mu_\iota(\varrho)$ suffices to determine the stress $\boldsymbol{S}_\alpha$ to within terms of order n in α.

The case $n=1$ of (7.4) is of particular interest. With use of (7.6), (7.7), we obtain

$$\boldsymbol{S}_\alpha = -p\boldsymbol{I} + \eta \boldsymbol{A}_1^\alpha + \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{A}_1^\alpha)\boldsymbol{I} + o(\alpha). \tag{7.8}$$

Now, the first Rivlin-Ericksen tensor $\boldsymbol{A}_1 = 2\boldsymbol{D}$ differs from the rate of deformation tensor $\boldsymbol{D}$ only by the factor 2*. It follows that (7.8) is, to within terms of order $o(\alpha)$, simply the constitutive equation (1.4) of a Newtonian fluid.

When the fluid is incompressible, the case $n=2$ of (7.4) takes a remarkably simple form. The observation that $\operatorname{tr}\boldsymbol{A}_1 = 0$ for isochoric motions and use of (7.6), (7.7) yield

$$\boldsymbol{S}_\alpha = -\tilde{p}\boldsymbol{I} + \eta\boldsymbol{A}_1^\alpha + \beta(\boldsymbol{A}_1^\alpha)^2 + \gamma\boldsymbol{A}_2^\alpha + o(\alpha^2), \tag{7.9}$$

where η, β, and γ are material constants and where $\tilde{p}$ is an indeterminate pressure. This pressure $\tilde{p}$ differs, in general, from the mean pressure p defined by the normalization (6.12), because it is obtained from p by absorbing all scalar multiples of $\boldsymbol{I}$ arising from $\mathfrak{m}_{11}$ and $\mathfrak{m}_2$ through use of (7.6) and (7.7).

Motivated by (7.9), we can define an *incompressible second-order fluid* by the constitutive equation

$$\boldsymbol{S} = -\hat{p}\boldsymbol{I} + \eta\boldsymbol{A}_1 + \beta\boldsymbol{A}_1^2 + \gamma\boldsymbol{A}_2. \tag{7.10}$$

Incompressible Newtonian fluids correspond to the special case $\beta=\gamma=0$ of (7.10).

In some dynamical situations equation (7.10) leads to a linear partial differential equation for the velocity, just as in the Newtonian case. For example, consider a rectilinear shearing flow which, in Cartesian coordinates x, y, z, has a velocity field with components

$$\{v^i\} = \{0, v(x,t), 0\}. \tag{7.11}$$

If the body forces are conservative, substitution of (7.10) and (7.11) into CAUCHY's dynamical equations leads to the following third-order partial differential equation for $v(x,t)$:

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^3 v}{\partial x^2 \partial t}, \quad a = \frac{\eta}{\varrho}, \quad b = \frac{\gamma}{\varrho}. \tag{7.12}$$

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* *Cf.* [1], (9.7).

A non-steady flow of the Couette type, in cylindrical coordinates r, ϑ, z, has a velocity field with contravariant components

$$\{v^i\} = \{0, \omega(r, t), 0\}. \tag{7.13}$$

For such a flow, instead of (7.12), we get

$$r^3 \frac{\partial \omega}{\partial t} = \frac{\partial}{\partial r}\left[r^3\left(a\frac{\partial \omega}{\partial r} + b\frac{\partial^2 \omega}{\partial r\, \partial t}\right)\right]. \tag{7.14}$$

In the Newtonian case, since $b=0$, equations (7.12) and (7.14) reduce to diffusion equations. As for the diffusion equations, many physically interesting solutions of (7.12) and (7.14) may be obtained by separation of variables*. It would be desirable to develop a mathematical theory of third-order partial differential equations of the type (7.12), (7.14).

8. *Simple Materials in General*

The considerations of the previous two sections may easily be extended to the general simple materials defined in Reference [*1*], Part III.

The form of the constitutive equation of an *isotropic simple material* may be obtained from (6.1) by replacing the scalar density $\varrho(t)$ by the left Cauchy-Green tensor $\boldsymbol{B}(t)$, taken relative to an undistorted reference state (*cf.* [*1*] (22.10)). A consideration analogous to the one which led to (6.7) shows that the constitutive equation of an isotropic simple material may be written in the form

$$\boldsymbol{S}(t) = \mathfrak{h}\left(\boldsymbol{B}(t)\right) + \mathop{\mathfrak{F}}_{s=0}^{\infty}\left(\boldsymbol{C}_t(t-s) - \boldsymbol{I}; \boldsymbol{B}(t)\right). \tag{8.1}$$

Here, the functional $\mathfrak{F}$ depends on a tensor parameter $\boldsymbol{B}$, instead of on a scalar parameter ϱ as in (6.7). We assume again that there is an influence function h of order n such that the functional $\mathfrak{F}$ in (8.1) is a memory functional of type (h, n). In place of (7.4), we then obtain the following approximation formula for the stress $\boldsymbol{S}_\alpha$ produced by a slow motion of an isotropic simple material:

$$\boldsymbol{S}_\alpha = \mathfrak{h}(\boldsymbol{B}) + \sum_{(j_1,\ldots,j_k)} \mathfrak{m}_{j_1\ldots j_k}(\boldsymbol{A}^\alpha_{j_1}, \ldots, \boldsymbol{A}^\alpha_{j_k}; \boldsymbol{B}) + o(\alpha^n), \tag{8.2}$$

where $\mathfrak{m}_{j_1\ldots j_k}$ is linear in each of its first k tensor variables but not necessarily in the last tensor variable $\boldsymbol{B}$.

The formulae corresponding to (8.1) and (8.2) in the case of *anisotropic simple materials* are obtained from (8.1) and (8.2) simply by replacing all tensors $\boldsymbol{T}$ occurring in these formulae by their conjugates $\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}$ with the rotation tensor $\boldsymbol{R}=\boldsymbol{R}(t)$ of the displacement from the reference state (*cf.* [*1*], (22.8)).

In the case of isotropic materials, the function $\mathfrak{h}$ and the functional $\mathfrak{F}$ in (8.1) are isotropic in the sense that they obey the identities

$$\boldsymbol{Q}\,\mathfrak{h}(\boldsymbol{B})\,\boldsymbol{Q}^T = \mathfrak{h}(\boldsymbol{Q}\,\boldsymbol{B}\,\boldsymbol{Q}^T), \tag{8.3}$$

$$\mathop{\mathfrak{F}}_{s=0}^{\infty}\left(\boldsymbol{Q}\,\boldsymbol{G}(s)\,\boldsymbol{Q}^T, \boldsymbol{Q}\,\boldsymbol{B}\,\boldsymbol{Q}^T\right) = \boldsymbol{Q}\mathop{\mathfrak{F}}_{s=0}^{\infty}\left(\boldsymbol{G}(s), \boldsymbol{B}\right)\boldsymbol{Q}^T \tag{8.4}$$

* In particular, the sinusoidal vibration problems discussed for Newtonian fluids in §§ 345—346 of LAMB's treatise [*5*] are easily solved for second-order fluids. Also, special solutions of (7.14) corresponding to sinusoidal vibrations of a fluid between coaxial cylinders can readily be found.

for all orthogonal tensors $\boldsymbol{Q}$. As in §7, it follows that the $\mathfrak{m}_{j_1 \ldots j_k}$ in (8.2) are isotropic functions of all their variables; *i.e.*, they obey the identities

$$\boldsymbol{Q}\,\mathfrak{m}_{j_1 \ldots j_k}(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k; \boldsymbol{B})\,\boldsymbol{Q}^T = \mathfrak{m}_{j_1 \ldots j_k}(\boldsymbol{Q}\,\boldsymbol{U}_1\,\boldsymbol{Q}^T, \ldots, \boldsymbol{Q}\,\boldsymbol{U}_k\,\boldsymbol{Q}^T; \boldsymbol{Q}\,\boldsymbol{B}\,\boldsymbol{Q}^T) \quad (8.5)$$

for all orthogonal $\boldsymbol{Q}$ and all symmetric tensors $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k$, $\boldsymbol{B}$ for which $\mathfrak{m}_{j_1 \ldots j_k}$ is defined. The methods developed by SPENCER & RIVLIN ([6], [7] and [8]) may be used to derive explicit representations for the $\mathfrak{m}_{j_1 \ldots j_k}$ of a type analogous to but more complicated than (7.6), (7.7). Using such a representation, one can show that, in the case $n=1$, the approximation formula (8.2) reduces to

$$\boldsymbol{S}_\alpha = \mathfrak{h}(\boldsymbol{B}) + \boldsymbol{A}_1^\alpha\,\mathfrak{t}_1(\boldsymbol{B}) + \mathfrak{t}_1(\boldsymbol{B})\,\boldsymbol{A}_1^\alpha + \operatorname{tr}\left(\boldsymbol{A}_1^\alpha\,\mathfrak{t}_2(\boldsymbol{B})\right)\mathfrak{t}_3(\boldsymbol{B}), + o(\alpha), \quad (8.6)$$

where $\mathfrak{h}$, $\mathfrak{t}_1$, $\mathfrak{t}_2$ and $\mathfrak{t}_3$ are isotropic functions of the one variable $\boldsymbol{B}$ and hence have representations of the form

$$\mathfrak{h}(\boldsymbol{B}) = \beta_0\,\boldsymbol{I} + \beta_1\,\boldsymbol{B} + \beta_2\,\boldsymbol{B}^2, \quad (8.7)$$

in which β_0, β_1 and β_2 are scalar functions of the three principal invariants of $\boldsymbol{B}$.

The first term $\mathfrak{h}(\boldsymbol{B})$ in the expression (8.2) for the stress $\boldsymbol{S}_\alpha$ corresponds to purely elastic response. The sum in (8.2) may be interpreted as representing the internal friction for slow motions.

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