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Oblique Dipole Layer Potentials Applied to Electrocardiology

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Abstract. We study the properties of the potential field generated by an oblique dipole layer. This field arises, for instance, in describing the potential elicited by a depolarization wavefront spreading in the myocardium when a dependence of the potential on the cardiac fiber orientation is introduced. The representation of cardiac bioelectric sources by means of an oblique dipole layer leads to a mathematical structure which generalizes the classical solid angle theory used in electrocardiology, which has been challenged by recent experimental evidence, and links models previously proposed with a view to adequately reproduce the potential observed in experiments. We investigate also the relationship between our model and an intracellular current model and we derive potential jump formulae for some models which account for the anisotropic structure of the myocardium. The potential generated by an oblique dipole layer is considered both for unbounded and bounded domains. In the latter case an integral boundary equation is derived and we study its solvability. A numerical procedure for solving this integral equation by means of the finite element method with collocation is outlined.

Key words: Potential theory – Integral equations – Electrocardiology

1. Introduction

Experimental findings by Corbin and Scher [9], Baruffi et al. [4], Spach et al. [27] have shown that the classical uniform double layer model of the bioelectric cardiac sources does not always predict correctly the potential field generated by an excitation wavefront spreading in the heart tissue.

Corbin and Scher [9] first proposed an "axial" model in which the dipoles on the wavefront are locally parallel to the direction of the myocardium fibers.

In our paper [7] a generalization of the "axial" model has been proposed which takes into account also a transverse dipole distribution on the wavefront i.e. with dipole axis perpendicular to the fiber direction.

In [7] it was shown that this model, representing the depolarization wavefront as an oblique dipole layer, reproduced adequately the pattern of the potential field

elicited by paced dog hearts in a volume conductor; we refer again to [7] for the comparison between experimental data and simulated results.

In this paper we study the properties of the potential field generated by an oblique dipole layer in an unbounded isotropic medium; subsequently we investigate the potential problem in a bounded and insulated medium deriving an integral boundary equation, which is proved solvable, and we develop a numerical finite element procedure for its approximation.

A feature of the potential distribution due to an oblique dipole layer, defined by a superposition of an axial and transverse dipole layer, is its equivalence to a field obtained by a linear combination of a normal and an axial dipole layer. Hence the model can be viewed as an extension of the classical uniform and normal dipole layer model obtained by adding to it an axial perturbation which makes the potential dependent on the shape of the front and on the orientation of the fibers.

With the aim of interpreting the dipole moment of the oblique dipole layer in terms of electrophysiological quantities and to establish a more satisfactory foundation of the oblique dipole layer theory, we show that this mathematical structure can be obtained as a limit case of an intracellular current model, used for simulation purposes by Spach et al. [27], when the intracellular action potential approaches a step function.

Subsequently we introduce two different "local" oblique dipole layer models taking into account the anisotropic structure of the heart tissue. We study their relationship and we show that for one of them the predicted potential jump across the wavefront coincides with the jump formula proposed by Roberts et al. [25].

The dependence of the potential field on the fiber orientation, introduced by the oblique dipole layer structure of the depolarization wavefront, raises the difficult problem of actually defining the geometry of the cardiac fibers. We show that it is possible to recover the geometry of the fiber orientation from the knowledge of the motion of the depolarization wavefront, when a suitable relationship between the conduction velocity and the anisotropic tissue conductivity is assumed.

2. Preliminaries and Notations

We recall some matrix calculus notations. Let $\mathbf{x}, \mathbf{y}, \ldots$, denote points or vectors in the euclidean space $\mathbb{R}^3$.

The scalar product of $\mathbf{x}$, $\mathbf{y}$ is denoted, as usual, by $\mathbf{x} \cdot \mathbf{y}$. The gradient vector operator ∇ has components $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$; furthermore $\mathbf{x}$, $\mathbf{y}, \ldots$ and $\mathbf{x}^T, \mathbf{y}^T, \ldots$, denote respectively column and row vectors; if $A, B, \ldots$, are matrices then, $A^T, B^T, \ldots$ are the transposed matrices. Therefore:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}, \qquad \mathbf{a} \cdot \nabla = \mathbf{a}^T\nabla,$$

$$\mathbf{a} \cdot (C\mathbf{b}) = \mathbf{a}^T C\mathbf{b}, \qquad \mathbf{a} \cdot (C\nabla) = \mathbf{a}^T C\nabla.$$

We shall use the dot $\cdot$ or the matrix notation according to convenience. We shall also use the following property (given in $\mathbb{R}^n$).

If $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ are square matrices, then

$$AB^T = \sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_i^T.$$

When using different systems of orthogonal coordinates, we shall distinguish between the global or standard system to be used everywhere and the local system to be used in the neighbourhood of a point. The corresponding axes or equivalently the unit vectors parallel to them define respectively the standard and the local basis. These bases are assumed to be both left or right-handed so that they can be made to coincide by means of a rigid displacement and a rotation. The local bases will be used to describe anisotropic properties of a medium and will be determined by the principal axes (p.a.). We recall briefly the definition of p.a. (e.g. [17]). Let $\gamma > 0$ be the coefficient related to a property of the medium (e.g. the conductivity) which depends generally both on the point and on the direction. In the frame of mathematical physics it is shown that (e.g. [17]) at any point there are three mutually orthogonal planes of symmetry for γ and the intersections of these planes are the principal axes. If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the unit vectors parallel to these axes, let $\gamma_1, \gamma_2, \gamma_3$ be the corresponding non-negative values of γ. If γ is the same for all directions perpendicular to $\mathbf{a}_3$, then $\gamma_1 = \gamma_2$ and $\mathbf{a}_1, \mathbf{a}_2$ are defined up to a rotation around $\mathbf{a}_3$; in this case, we set

$$\gamma_1 = \gamma_2 = \gamma_t, \qquad \gamma_3 = \gamma_l, \qquad \mathbf{a}_3 = \mathbf{a}_l.$$

If γ is the same for all directions, then $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (mutually orthogonal) can be chosen arbitrarily and $\gamma_1 = \gamma_2 = \gamma_3$; the medium is isotropic at that point.

For any direction characterized by the unit vector $\mathbf{e}$ with component e'_1, e'_2, e'_3 in the local basis, the value of γ is given by

$$\gamma = \gamma_1 e'^2_1 + \gamma_2 e'^2_2 + \gamma_3 e'^2_3.$$

Since

$$e'_i = \mathbf{e} \cdot \mathbf{a}_i \qquad \text{and} \qquad e'^2_i = \mathbf{e}^T \mathbf{a}_i \mathbf{a}_i^T \mathbf{e},$$

we have also

$$\gamma = \mathbf{e}^T(\gamma_1 \mathbf{a}_1 \mathbf{a}_1^T + \gamma_2 \mathbf{a}_2 \mathbf{a}_2^T + \gamma_3 \mathbf{a}_3 \mathbf{a}_3^T)\mathbf{e} = \mathbf{e}^T C \mathbf{e}$$

with

$$C = \gamma_1 \mathbf{a}_1 \mathbf{a}_1^T + \gamma_2 \mathbf{a}_2 \mathbf{a}_2^T + \gamma_3 \mathbf{a}_3 \mathbf{a}_3^T.$$

Setting

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3), \qquad C' = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3), \qquad \gamma_i \geqslant 0, \qquad i = 1, 2, 3,$$

we have also

$$C = AC'A^T, \tag{1}$$

where C, C' are both symmetric and positive semidefinite (positive definite if $\gamma_i > 0$, $i = 1, 2, 3$) and they characterize the cartesian tensor of the property measured by γ in the standard and in the local basis. A, A^T with $A^T A = AA^T = I$ (I identity matrix) are the rotation matrices from the local to the standard basis and vice versa. Of course, we could have started from the symmetric positive semidefinite matrix C defining γ for a direction $\mathbf{e}$ as

$$\gamma = \mathbf{e}^T C \mathbf{e}.$$

From well known properties of symmetric matrices, there exists an orthogonal matrix A of eigenvectors and a diagonal matrix C' of eigenvalues

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3), \qquad C' = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), \qquad \gamma_i \geqslant 0, \qquad i = 1, 2, 3$$

such that

$$CA = AC', \qquad \text{i.e.} \qquad C = AC'A^T = \gamma_1 \mathbf{a}_1 \mathbf{a}_1^T + \gamma_2 \mathbf{a}_2 \mathbf{a}_2^T + \gamma_3 \mathbf{a}_3 \mathbf{a}_3^T.$$

We remark also that, given the unit vector $\mathbf{n}$, it is

$$\mathbf{n} = (\mathbf{n} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n} \cdot \mathbf{a}_2)\mathbf{a}_2 + (\mathbf{n} \cdot \mathbf{a}_3)\mathbf{a}_3.$$

If we consider the vector

$$\mathbf{a} = (\mathbf{n} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n} \cdot \mathbf{a}_2)\mathbf{a}_2 = \mathbf{n} - (\mathbf{n} \cdot \mathbf{a}_3)\mathbf{a}_3$$

it follows that $\mathbf{a}$ is coplanar with $\mathbf{a}_1, \mathbf{a}_2$ and with $\mathbf{n}, \mathbf{a}_3$; moreover $\mathbf{a}$ is perpendicular to $\mathbf{a}_3$. Let $\mathbf{v}$ be the unit vector parallel to $\mathbf{a}$; then $\mathbf{a}_3 \cdot \mathbf{v} = 0$ and $\mathbf{a} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}$. Therefore $\mathbf{a} = (\mathbf{n} \cdot \mathbf{v})\mathbf{v}$ and

$$(\mathbf{n} \cdot \mathbf{v})\mathbf{v} = (\mathbf{n} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n} \cdot \mathbf{a}_2)\mathbf{a}_2 = \mathbf{n} - (\mathbf{n} \cdot \mathbf{a}_3)\mathbf{a}_3. \tag{2}$$

If $\mathbf{v}' = -\mathbf{v}$, then $(\mathbf{n} \cdot \mathbf{v})\mathbf{v} = (\mathbf{n} \cdot \mathbf{v}')\mathbf{v}'$, i.e. $\mathbf{v}$ can always be chosen so that $\mathbf{n} \cdot \mathbf{a}_3$ and $\mathbf{n} \cdot \mathbf{v}$ have the same sign; moreover $\mathbf{v}$ is independent of a rotation of $\mathbf{a}_1, \mathbf{a}_2$ around $\mathbf{a}_3$. In the case of axial symmetry around $\mathbf{a}_3$, setting

$$C' = \mathrm{diag}(\gamma_t, \gamma_t, \gamma_l); \qquad \mathbf{a}_l = \mathbf{a}_3; \qquad \mathbf{a}_t = \mathbf{v}$$

and using Eq. (2) we obtain

$$\begin{aligned} C\mathbf{n} &= \gamma_t(\mathbf{a}_1\mathbf{a}_1^T + \mathbf{a}_2\mathbf{a}_2^T)\mathbf{n} + \gamma_l \mathbf{a}_l \mathbf{a}_l^T \mathbf{n} \\ &= \gamma_t[(\mathbf{n} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n} \cdot \mathbf{a}_2)\mathbf{a}_2] + \gamma_l(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \\ &= \gamma_t(\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t + \gamma_l(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l. \end{aligned} \tag{3}$$

Since the mathematical structure we are going to investigate will be used in modeling the heart potential field and the applications which we shall describe are related to electrocardiology, we shall recall briefly some notions about the bioelectric properties of the heart.

During a heart beat a layer of cells, changing with time, undergoes the so-called "depolarization process" i.e. a change of potential of biochemical origin which gives rise to the potential field observed in electrocardiology. The change of potential in a cell, consequent to the depolarization process, occurs in a short time interval with an approximately monotone variation from a resting value to a plateau value, and the function which describes the cell potential during this time is called action potential (see, e.g. [11]). The set of cells undergoing the depolarization process constitutes an "activating" region which moves across the myocardium; it has a thickness of about $1 \div 2$ mm (see Weidmann [29]) so that it can be approximately viewed as a moving surface called the excitation wavefront. This is equivalent to assume that the depolarization process is instantaneous and that the action potential can be represented by a step function. This idealization is adequate to describe the potential at a distance of at least few mm from any current source.

We recall that the classical way of modeling the depolarization wavefront, conceived as the seat of the equivalent source generators of the heart potential, is to represent the wavefront as a uniform and normal double layer. The elementary dipoles on the wavefront are directed toward the resting, i.e. not yet depolarized, heart tissue. Actually, another contribution to the heart potential field comes from the "repolarization process" which brings the cells back to the resting bioelectric state. To a good approximation, the repolarization process can be neglected for most of the QRS phase of the heart beat, with which we shall be mainly concerned; so, for our purposes, there will be resting and depolarized tissue, respectively ahead and behind the front.

3. Oblique Dipole Layers in a Homogeneous Isotropic Medium

We shall here present a mathematical model of a depolarization wavefront suitable for a quantitative description of the potential field generated at distance from the cardiac sources.

We assume that the activating volume comprised between the resting and the depolarized tissue is thin enough so that the depolarization wavefront spreading in the myocardium at time t, may be idealized as a moving surface $S = S(t)$. This assumption is best suited for describing the potential at some distance from the seat of the cardiac sources.

We shall assume S to be a regular and orientable (two faced) surface.

Let $\mathbf{n}$ be the unit normal to S oriented in the direction of the advancing front, i.e. toward the resting tissue. Accordingly, the region ahead and behind the front corresponds respectively to the resting and to the depolarized heart tissue. We assume the front S imbedded into an infinite homogeneous isotropic conducting medium of conductivity σ_0, occupying the whole space $\mathbb{R}^3$ thus neglecting the influence of the intra and extra cardiac tissue boundaries as well as the effects of myocardial tissue anisotropy.

We recall now the axial model of Corbin and Scher [9] which first motivated us to study oblique dipole layer models. The axial model assumes that the dipoles on the wavefront are locally oriented as the myocardium fibers. Let m_l, $\mathbf{a}_l$ denote respectively the moment and the axis of the dipole on the wavefront S where $\mathbf{a}_l$ is chosen parallel to the local fiber direction and oriented toward the resting tissue so that $\mathbf{n} \cdot \mathbf{a}_l \geqslant 0$. Then, in the axial model, the potential generated in an infinite isotropic medium with constant conductivity σ_0 is given by

$$U(x) = \frac{1}{4\pi\sigma_0} \int_S m_l(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \cdot \nabla r^{-1}\, dS, \tag{4}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$, $r = (\mathbf{r} \cdot \mathbf{r})^{1/2}$ and $\mathbf{y}$ describes S. Hence the excited fiber behaves like a small dipole oriented along the fiber axis with dipole density given by

$$m_l\, dS^* = m_l(\mathbf{n} \cdot \mathbf{a}_l)\, dS,$$

where $dS^* = (\mathbf{n} \cdot \mathbf{a}_l)\, dS$ is the projection of the surface element dS of the wavefront on the plane perpendicular to the fiber.

We remark that m_l is a kind of "intrinsic" density while $m_l(\mathbf{n} \cdot \mathbf{a}_l)$ is the usual density on S per unit area. Since the longitudinal axis of the fibers characterizes the

direction of the dipole on S the model is called "axial"; this term may be applied also to the dipole layer $m_l\mathbf{a}_l$.

We shall now investigate an oblique dipole layer model, first proposed in [7], which was shown to be adequate in simulating the potential field generated by a paced dog heart in a volume conductor. This model is here presented as a generalization of the axial model in that it takes into account a "transverse" dipole layer $m_t\mathbf{a}_t$ with the axis $\mathbf{a}_t$ perpendicular to $\mathbf{a}_l$. A different derivation of this oblique dipole layer will be obtained in a subsequent section starting from an intracellular current model. We assume axisymmetry of the bioelectric properties of the fibers around the local longitudinal axis $\mathbf{a}_l$. Because of this assumption the axis $\mathbf{a}_t$ must be chosen coplanar with $\mathbf{a}_l, \mathbf{n}$. The axis $\mathbf{a}_t$, like $\mathbf{a}_l$, points toward the resting tissue so that $\mathbf{n} \cdot \mathbf{a}_l \geqslant 0, \mathbf{n} \cdot \mathbf{a}_t \geqslant 0$. The potential generated by the superposition of the axial and transverse dipole layers $m_l\mathbf{a}_l$, $m_t\mathbf{a}_t$ in an infinite isotropic medium with conductivity σ_0 is given by

$$U(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_S [m_t(\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t + m_l(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l] \cdot \nabla r^{-1}\, dS$$

$$= \frac{1}{4\pi\sigma_0} \int_S \mathbf{n}^T [m_t\mathbf{a}_t\mathbf{a}_t^T + m_l\mathbf{a}_l\mathbf{a}_l^T] \nabla r^{-1}\, dS. \tag{5}$$

The moments m_l, m_t can, in general, be dependent on the point $\mathbf{y}$ which varies on S. We remark that $U(\mathbf{x})$ does not depend on the orientation of $\mathbf{a}_t$, $\mathbf{a}_l$.

For $m_t = 0$ and m_l constant we recover Corbin and Scher's model. If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are mutually orthogonal and $\mathbf{a}_3 = \mathbf{a}_l$ then using Eq. (2) with $\mathbf{v} = \mathbf{a}_t$ we obtain

$$(\mathbf{n} \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n} \cdot \mathbf{a}_2)\mathbf{a}_2 = (\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t$$

or equivalently

$$(\mathbf{a}_1\mathbf{a}_1^T + \mathbf{a}_2\mathbf{a}_2^T)\mathbf{n} = \mathbf{a}_t\mathbf{a}_t^T\mathbf{n}.$$

Setting

$$m_1 = m_2 = m_t, \qquad m_3 = m_l$$

and using Eqs. (1) and (2) we have

$$(m_t\mathbf{a}_t\mathbf{a}_t^T + m_l\mathbf{a}_l\mathbf{a}_l^T)\mathbf{n} = \left(\sum_{i=1}^{3} m_i\mathbf{a}_i\mathbf{a}_i^T\right)\mathbf{n} = M\mathbf{n},$$

where

$$M = AM'A^T$$

with

$$M' = \operatorname{diag}(m_1, m_2, m_3), \qquad A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3).$$

Since $m_i \geqslant 0$, $i = 1, 2, 3$ the tensor M is symmetric and positive semidefinite. Hence Eq. (5) can be written as follows:

$$U(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_S \mathbf{n}^T M \nabla r^{-1}\, dS. \tag{6}$$

Under the integral sign $\mathbf{n}$, M depend on $\mathbf{y}$ whereas r^{-1} depends both on $\mathbf{x}$ and $\mathbf{y}$; since r^{-1} is harmonic for $r \neq 0$ it follows that $U(\mathbf{x})$ is harmonic at all points $\mathbf{x}$ not on S. In particular if $m_t = 0$ we have $M = m_l \mathbf{a}_l \mathbf{a}_l^T$. We remark also that for variable fiber direction (i.e. A variable) M is generally position dependent even if M' is constant i.e. even if homogeneous bioelectric fiber properties (defined by m_1, m_2, m_3) are assumed.

In dealing with double layers on a surface S it is customary to consider the density of the layer on the surface. In the case of Eq. (6), setting $\mathbf{c} = M\mathbf{n}$, $\delta = (\mathbf{c} \cdot \mathbf{c})^{1/2}$ and $\mathbf{a} = \mathbf{c}/\delta$ we have

$$U(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_S \delta \mathbf{a} \cdot \nabla r^{-1} \, dS,$$

i.e. the density on S is δ. Since the dipole axis $\mathbf{a}$ is usually oblique to S we call $\delta\mathbf{a} = M\mathbf{n}$ an oblique dipole (or double) layer.

It seems preferable to refer to a dipole layer by means of the dipole moment tensor M which is an intrinsic quantity, it being understood that, given M, the potential (in an unbounded isotropic homogeneous medium) is defined according to Eq. (6), so that we do not specify directly the density on the surface. Since $M = \sum_{i=1}^{3} m_i \mathbf{a}_i \mathbf{a}_i^T$ then in relation to a principal axis $\mathbf{a}$ we may even further simplify the notation and speak of the dipole layer $m\mathbf{a}$, with the understanding that $M_a = m\mathbf{a}\mathbf{a}^T$. Therefore we may speak of the sum of the potentials due to the layers $m_i \mathbf{a}_i$, $i = 1, 2, 3$ with $M_i = m_i \mathbf{a}_i \mathbf{a}_i^T$ and $M = \sum_{i=1}^{3} M_i$.

With reference to Corbin and Scher's model and that defined by Eq. (5) we shall also speak of the axial layer $m_l \mathbf{a}_l$ and the transverse layer $m_t \mathbf{a}_t$. If $M' = mI$ with $m = m_l = m_t$ then $M\mathbf{n} = mAA^T\mathbf{n} = m\mathbf{n}$ and Eq. (6) yields the potential due to the normal double layer $m\mathbf{n}$.

We now point out a useful splitting of the potential given by Eq. (5).

Since $\mathbf{a}_l$, $\mathbf{a}_t$, $\mathbf{n}$ are coplanar and $\mathbf{a}_l$, $\mathbf{a}_t$ are perpendicular we have

$$\mathbf{n} = (\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t + (\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l$$

and

$$\begin{aligned} m_t(\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t + m_l(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l &= m_t[(\mathbf{n} \cdot \mathbf{a}_t)\mathbf{a}_t + (\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l] + (m_l - m_t)(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \\ &= m_t\mathbf{n} + (m_l - m_t)(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l. \end{aligned}$$

Therefore the sum of the potentials due to the axial and transverse dipole layers $m_l \mathbf{a}_l$, $m_t \mathbf{a}_t$ is equivalent to the sum of the potential due to the normal double layer $m_t\mathbf{n}$ and the axial dipole layer $(m_l - m_t)\mathbf{a}_l$, i.e.

$$\begin{aligned} U(\mathbf{x}) &= \frac{1}{4\pi\sigma_0} \int_S \mathbf{n}^T(m_t \mathbf{a}_t \mathbf{a}_t^T + m_l \mathbf{a}_l \mathbf{a}_l^T) \nabla r^{-1} \, dS \\ &= \frac{1}{4\pi\sigma_0} \int_S m_t \mathbf{n} \cdot \nabla r^{-1} \, dS + \frac{1}{4\pi\sigma_0} \int_S (m_l - m_t)(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \cdot \nabla r^{-1} \, dS. \end{aligned} \tag{7}$$

In the same way it is possible to obtain the decomposition of the oblique dipole layer into the normal dipole layer $m_l\mathbf{n}$ and the transverse dipole layer $(m_t - m_l)\mathbf{a}_t$.

The first splitting seems preferable since the axial layer $(m_l - m_t)\mathbf{a}_l$ is more directly related to the fiber structure than the transverse layer $(m_t - m_l)\mathbf{a}_t$.

From Eq. (7) it follows that the classical uniform double layer model is recovered if $m_l = m_t = \text{const}$. In general the above splitting (with the normal double layer $m_t\mathbf{n}$ or $m_l\mathbf{n}$) shows that an oblique dipole layer model is an extension of the normal double layer model obtained by adding the axial or respectively the transverse component $(m_l - m_t)\mathbf{a}_l$ or $(m_t - m_l)\mathbf{a}_t$. The splitting also facilitates the analysis of the potential field predicted by the model; for its application we refer to [7].

4. Jump Relationships of the Potential Due to an Oblique Dipole Layer

We now investigate the jump relationships of the potential given by Eq. (6) characterized by the symmetric positive semidefinite tensor M. The wavefront S, support of the oblique dipole layer, is assumed bounded and regular with continuously varying tangent plane. S may be a closed or an open surface; in the latter case the boundary ∂S is assumed to be a regular curve. A sufficient degree of regularity is also assumed for the tensor M characterizing the dipole layer (for instance we may assume for S, ∂S, M a regularity C^1). In the following, under the surface integrals, r is dependent on $\mathbf{x}$ and $\mathbf{y}$; all other quantities $\mathbf{n}$, M and later on $\alpha, \boldsymbol{\beta}, \rho, \ldots$ depend only on $\mathbf{y}$ varying on S; any exception will be explicitly stated.

Setting $\mathbf{c} = M\mathbf{n}$ and $\alpha = \mathbf{c} \cdot \mathbf{n} = \mathbf{n}^T M \mathbf{n} \geqslant 0$, we define on the surface S the vectors $\boldsymbol{\alpha} = \alpha\mathbf{n}$ and $\boldsymbol{\beta} = \mathbf{c} - \boldsymbol{\alpha}$. From the definition of $\boldsymbol{\alpha}$ it follows that $\boldsymbol{\beta} \cdot \mathbf{n} = 0$ i.e. $\boldsymbol{\beta}$ is the projection of $\mathbf{c}$ on the plane π_τ tangent to S. The vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ have the same degree of regularity of M, $\mathbf{n}$ and are respectively normal and tangent to S. Using this decomposition of $\mathbf{C}$ we obtain:

$$4\pi\sigma_0 U(\mathbf{x}) = \int_S \mathbf{n}^T M \nabla r^{-1}\, dS = \int_S (\boldsymbol{\alpha} + \boldsymbol{\beta}) \cdot \nabla r^{-1}\, dS$$
$$= 4\pi\sigma_0 [U_n(\mathbf{x}) + U_\tau(\mathbf{x})],$$

where

$$U_n(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_S \boldsymbol{\alpha} \cdot \nabla r^{-1}\, dS$$

and

$$U_\tau(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_S \boldsymbol{\beta} \cdot \nabla r^{-1}\, dS$$

on S we have also $\boldsymbol{\tau} \cdot \nabla = \boldsymbol{\tau} \cdot \nabla_S$, where ∇_S is the gradient operator on S and $\boldsymbol{\tau}$ a tangent vector to S. Therefore

$$\boldsymbol{\beta} \cdot \nabla r^{-1} = \boldsymbol{\beta} \cdot \nabla_S r^{-1} + r^{-1} \nabla_S \cdot \boldsymbol{\beta} - r^{-1} \nabla_S \cdot \boldsymbol{\beta} = \nabla_S \cdot (r^{-1}\boldsymbol{\beta}) - r^{-1} \nabla_S \cdot \boldsymbol{\beta}$$

and applying Green's formula, we obtain

$$4\pi\sigma_0 U_\tau(\mathbf{x}) = \int_{\partial S} r^{-1} \boldsymbol{\beta} \cdot \mathbf{n}_b\, ds - \int_S r^{-1} \nabla_S \cdot \boldsymbol{\beta}\, dS,$$

where $\mathbf{n}_b$ is the unit vector tangent to S and perpendicular to the boundary ∂S of S (when S is an open surface), directed outward with respect to S; if S is a closed surface the line integral is zero. Setting

$$\mu = \nabla_S \cdot \boldsymbol{\beta}, \qquad \rho = \boldsymbol{\beta} \cdot \mathbf{n}_b$$

we have then

$$4\pi\sigma_0 U_\tau(\mathbf{x}) = \int_{\partial S} \rho r^{-1}\, ds - \int_S \mu r^{-1}\, dS.$$

This decomposition shows that $U_\tau(\mathbf{x})$ has a logarithmic singularity for $\mathbf{x}$ on ∂S. In order to have a bounded potential, the line integral must be zero for any $\mathbf{x}$, i.e.

$$\rho = \boldsymbol{\beta} \cdot \mathbf{n}_b = 0 \qquad \text{on } \partial S.$$

Since the unit normal $\mathbf{n}$ to S is perpendicular to $\mathbf{n}_b$ (at $\mathbf{x} \in \partial S$) we have $\mathbf{n} \cdot \mathbf{n}_b = 0$ and $\boldsymbol{\beta} \cdot \mathbf{n}_b = 0$. But $\mathbf{c} = M\mathbf{n}$ is coplanar with $\mathbf{n}, \boldsymbol{\beta}$ therefore $\rho = 0$ implies $\mathbf{c} \cdot \mathbf{n}_b = 0$, i.e. $\mathbf{n}^T M \mathbf{n}_b = 0$. Hence the regularity requirement for U_τ is equivalent to the constraint $\mathbf{n}^T M \mathbf{n}_b = 0$ for M on ∂S.

We show that in the case of axial symmetry ($m_1 = m_2 = m_t$, $m_3 = m_l$, $\mathbf{a}_3 = \mathbf{a}_l$) the following relation holds: $\mathbf{c} \cdot \mathbf{n}_b = 0$ on ∂S if and only if either $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ or $\mathbf{a}_l$ tangent to S on ∂S.

In order to prove this statement we first remark that the eigenvectors of M are the vectors parallel or perpendicular to $\mathbf{a}_l$, therefore $\mathbf{c} = M\mathbf{n}$ is parallel to $\mathbf{n}$ if and only if $\mathbf{n}$ is parallel or perpendicular to $\mathbf{a}_l$; moreover if $\mathbf{c}$ is not parallel to $\mathbf{n}$ then $\mathbf{n}$, $\mathbf{a}_l$, $\mathbf{c}$ are distinct and coplanar. Assume now $\mathbf{c} \cdot \mathbf{n}_b = 0$ on ∂S. We distinguish the following cases:

i) $\mathbf{c}$ is parallel to $\mathbf{n}$ i.e., either $\mathbf{n}$ is parallel to $\mathbf{a}_l$, hence $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ or $\mathbf{n}$ is perpendicular to $\mathbf{a}_l$ and then $\mathbf{a}_l$ is tangent to S.

ii) $\mathbf{c}$ is not parallel to $\mathbf{n}$, then $\mathbf{n}$, $\mathbf{a}_l$, $\mathbf{c}$ are distinct and coplanar. Since $\mathbf{n}$, $\mathbf{c}$ are perpendicular to $\mathbf{n}_b$, so it is also $\mathbf{a}_l$ i.e., $\mathbf{a}_l \cdot \mathbf{n}_b = 0$.

Conversely

j) if $\mathbf{a}_l$ is tangent to S then $\mathbf{n}$ is perpendicular to $\mathbf{a}_l$, hence $\mathbf{c}$ is parallel to $\mathbf{n}$ and $\mathbf{c} \cdot \mathbf{n}_b = 0$.

jj) if $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ then this condition together with $\mathbf{n} \cdot \mathbf{n}_b = 0$ and the coplanarity of $\mathbf{n}$, $\mathbf{a}_l$, $\mathbf{c}$ ensures that $\mathbf{c} \cdot \mathbf{n}_b = 0$.

We shall show in the subsequent section 6, concerning the spreading of the wavefront, that this condition can also be motivated on physical grounds.

Therefore in the following developments of this paper we shall assume that

$$\begin{cases} M \text{ is symmetric positive semidefinite and} \\ \rho = 0 \text{ on } \partial S, \text{ i.e. } \mathbf{n}^T M \mathbf{n}_b = 0 \text{ on } \partial S. \end{cases} \tag{8}$$

To conclude we have then

$$U(\mathbf{x}) = U_n(\mathbf{x}) + U_\tau(\mathbf{x}) = \frac{1}{4\pi\sigma_0}\left\{\int_S \alpha \mathbf{n} \cdot \nabla r^{-1}\, dS - \int_S \mu r^{-1}\, dS\right\},$$

i.e. the potential generated by the oblique dipole layer with moment tensor M is

equivalent to the difference between the potential due to the normal double layer $\alpha\mathbf{n}$ and the potential of the simple layer with density $\mu = \nabla_S \cdot \boldsymbol{\beta}$.

It is now easy to establish the jump relations for the potential given by Eq. (6). Without loss of generality we may assume $\sigma_0 = 1$, i.e.

$$U(\mathbf{x}) = \frac{1}{4\pi} \int_S \mathbf{n}^T M \nabla r^{-1} \, dS, \tag{9}$$

since for any other value of σ_0 the corresponding potential differs from that of Eq. (9) by the multiplicative coefficient $1/\sigma_0$.

We also recall the notion of the positive and the negative side of a surface. If $\mathbf{x}$ is a point of S then the point $\mathbf{x} + \lambda\mathbf{n}$ tends to $\mathbf{x}$ as $\lambda \to 0$. Given the orientation of $\mathbf{n}$ on S we say that $\mathbf{x}$ is approached from the positive or negative side of S according as $\lambda \to 0+$ or $\lambda \to 0-$.

From a closed surface S with outward normal $\mathbf{n}$ the approach from the positive or negative side is equivalent to the approach from outside or inside S.

If a function w is defined on both sides of S and admits a finite limit on both sides of S, these limits (the traces of w on the sides of S) will be denoted with $w|_{S_+}$, $w|_{S_-}$. The jump of w across S is defined as $[w]_S = w|_{S_+} - w|_{S_-}$. When there is no misunderstanding we shall simply write $[w]$.

Since U_n and U_τ are respectively a double and a simple layer potential, using classical results of potential theory (see [13, 15, 18]) from Eq. (9) and assumption (8) it follows:

$$[U]_S = \alpha, \qquad \left[\frac{\partial U}{\partial n}\right]_S = \mu, \tag{10}$$

where

$$\alpha = \mathbf{n}^T M \mathbf{n}, \qquad \mu = \nabla_S \cdot \boldsymbol{\beta}.$$

5. The Integral Boundary Equation for the Potential Field in a Bounded Conductor

In this section we shall study the potential in a bounded medium applying the boundary integral method, see e.g. [12] and for electrocardiology problems [1, 2, 3, 28].

We consider now the wavefront S imbedded in a homogeneous isotropic medium with conductivity $\sigma_0 = 1$, which is bounded and surrounded by an insulating medium. The model has been used in [7] to simulate the potential field generated by paced dog hearts imbedded in a homogeneous conducting medium, neglecting the heart boundary surface. We denote with Ω this bounded open and simply connected domain and with Γ its boundary. The surface Γ is closed and it is assumed regular. The unit normal $\mathbf{n}$ to Γ is outward; the normal $\mathbf{n}$ to S is outward if S is closed, otherwise one of the two possible orientations is assigned. In the case of an excitation wavefront in the myocardium the normal $\mathbf{n}$ to S is oriented toward the resting tissue.

Given on S the oblique dipole layer characterized by the tensor M and taking into account jump relationships (10), the potential $U(\mathbf{x})$ generated in the bounded

medium Ω satisfies the following boundary value problem:

$$\begin{aligned} &\Delta U = 0 \quad \text{in } \Omega - S \qquad \text{with } \Delta = \nabla \cdot \nabla, \\ &\left.\frac{\partial U}{\partial n}\right|_{\Gamma_-} = 0, \qquad [U]_S = \alpha, \qquad \left[\frac{\partial U}{\partial n}\right]_S = \mu, \end{aligned} \tag{11}$$

where α, μ are given by the following relationships:

$$\alpha = \mathbf{n}^T M \mathbf{n}, \qquad \mu = \nabla_S \cdot \boldsymbol{\beta}.$$

For $\sigma_0 \neq 1$ the corresponding potential is obtained dividing by σ_0 the solution of problem (11).

A method for solving this problem consists in finding a solution by means of boundary integrals. More precisely we shall look for a solution of problem (11) in the class $\{\mathscr{P}\}$ of the functions defined by

$$U(\mathbf{x}) = -\frac{1}{4\pi}\int_\Gamma v\mathbf{n}^T \nabla r^{-1}\, d\Gamma + \frac{1}{4\pi}\int_S \mathbf{n}^T M \nabla r^{-1}\, dS, \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - \Gamma - S, \tag{12}$$

where v is a continuous function on Γ. It follows that a function $U(\mathbf{x})$ in the class $\{\mathscr{P}\}$ is already a solution in $\mathbb{R}^3 - \Gamma - S$ of the equation $\Delta U = 0$; moreover since the oblique dipole layer on S, using hypothesis (8), is equivalent to the difference between a double layer of density α and a simple layer of density μ, as in the previous Section 4, we obtain that $U(\mathbf{x})$ satisfies the jump relationships on S stated in problem (11).

We shall show that a function $U(\mathbf{x})$ of the class $\{\mathscr{P}\}$ satisfies the boundary condition $\partial U/\partial n|_{\Gamma_-} = 0$ if and only if the function v satisfies the following integral boundary equation of second kind on Γ:

$$\frac{1}{2}v(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma v\mathbf{n}^T \nabla r^{-1}\, d\Gamma = \frac{1}{4\pi}\int_S \mathbf{n}^T M \nabla r^{-1}\, dS, \qquad \text{for all } \mathbf{x} \text{ on } \Gamma. \tag{13}$$

We remark that, for the classical properties of a uniform double layer potential on a closed surface (see, e.g. [10, 13, 15]), we have

$$\frac{1}{4\pi}\int_\Gamma \mathbf{n}^T \nabla r^{-1}\, d\Gamma = \begin{cases} 0, & \text{in } \mathbb{R}^3 - \bar{\Omega}, \\ -\frac{1}{2}, & \text{on } \Gamma, \\ -1, & \text{in } \Omega. \end{cases}$$

If U is given by Eq. (12) and satisfies $\partial U/\partial n|_{\Gamma_-} = 0$ then, since $[\partial U/\partial n]_\Gamma = 0$, we have $\partial U/\partial n|_{\Gamma_+} = 0$ and $\Delta U = 0$ in $\mathbb{R}^3 - \bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω; the uniqueness of the classical exterior Neumann problem implies $U \equiv 0$ in $\mathbb{R}^3 - \bar{\Omega}$, hence $U|_{\Gamma_+} = 0$ but $[U]_\Gamma = -v$ and we obtain $U|_{\Gamma_-} = v$. Using the classical formula for a double layer potential, the trace $U|_{\Gamma_-}$ is given by

$$\begin{aligned} U(\mathbf{x})|_{\Gamma_-} &= \frac{1}{2}v(\mathbf{x}) - \frac{1}{4\pi}\int_\Gamma v\mathbf{n}^T \nabla r^{-1}\, d\Gamma + \frac{1}{4\pi}\int_S \mathbf{n}^T M \nabla r^{-1}\, dS \\ &= v(x), \quad \text{for all } \mathbf{x} \text{ on } \Gamma \end{aligned}$$

thus it follows that v satisfies integral equation (13). On the other hand if v is a continuous solution of Eq. (13), for the potential defined by Eq. (12) we have

$$U(\mathbf{x})|_{\Gamma_+} = -\frac{1}{2}v(\mathbf{x}) - \frac{1}{4\pi}\int_\Gamma v\mathbf{n}^T \nabla r^{-1}\, d\Gamma + \frac{1}{4\pi}\int_S \mathbf{n}^T M \nabla r^{-1}\, dS$$

$$= 0, \quad \text{for all } \mathbf{x} \text{ on } \Gamma.$$

Since $\Delta U = 0$ in $\mathbb{R}^3 - \bar{\Omega}$ from the uniqueness of the classical exterior Dirichlet problem it follows that $U \equiv 0$ in $\mathbb{R}^3 - \bar{\Omega}$ hence $\partial U/\partial n|_{\Gamma_+} = 0$; then, taking into account that $[\partial U/\partial n]_\Gamma = 0$, the potential U satisfies the boundary condition $\partial U/\partial n|_{\Gamma_-} = 0$. Moreover we remark that $[U]_\Gamma = -v$ and $U|_{\Gamma_+} = 0$ therefore we deduce $U|_{\Gamma_-} = v$. Noting that U is harmonic in $\Omega - S$ and $\partial U/\partial n|_{\Gamma_-} = 0$, due to regularity results for potential elliptic problems, it results that $v = U|_{\Gamma_-}$ is actually a function with continuous partial derivatives of any order.

Remark 5.1. We develop an equivalent way of describing the potential field in the bounded and insulated conductor Ω. We consider the potential

$$P(\mathbf{x}) = \frac{1}{4\pi}\int_S \mathbf{n}^T M \nabla r^{-1}\, dS, \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - S \tag{14}$$

due to the oblique dipole layer, in all the conductor space $\mathbb{R}^3$ and we introduce a solution $V(\mathbf{x})$ of the following Neumann problem:

$$\Delta V = 0 \quad \text{in } \Omega, \qquad \frac{\partial V}{\partial n} = \frac{\partial P}{\partial n} \quad \text{on } \Gamma. \tag{15}$$

We observe that V is defined except for an additive constant and its existence is assured if and only if the following compatibility condition is satisfied

$$\int_\Gamma \frac{\partial P}{\partial n}\, d\Gamma = 0. \tag{16}$$

We now verify this condition. Since P is a regular function outside S, changing the order of integration, we obtain

$$\int_\Gamma \frac{\partial P}{\partial n}\, d\Gamma = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_\Gamma}\left\{\int_S \alpha \frac{\partial r^{-1}}{\partial n_S}\, dS - \int_S \mu r^{-1}\, dS\right\} d\Gamma$$

$$= \frac{1}{4\pi}\int_S \alpha \frac{\partial}{\partial n_S}\left(\int_\Gamma \frac{\partial r^{-1}}{\partial n_\Gamma}\, d\Gamma\right) dS - \frac{1}{4\pi}\int_S \mu\left(\int_\Gamma \frac{\partial r^{-1}}{\partial n_\Gamma}\, d\Gamma\right) dS$$

but $(1/4\pi)\int_\Gamma (\partial r^{-1}/\partial n_\Gamma)\, d\Gamma = -1$ in Ω, for the classical properties of a uniform double layer on a closed surface, and $\mu = \nabla_S \cdot \boldsymbol{\beta}$. Consequently imposing the assumed condition (8), i.e. $\boldsymbol{\beta} \cdot \mathbf{n}_b = 0$ on ∂S, described in the previous Section 4, we get

$$\int_\Gamma \frac{\partial P}{\partial n}\, d\Gamma = \frac{1}{4\pi}\int_S \nabla_S \cdot \boldsymbol{\beta}\, dS = \frac{1}{4\pi}\int_{\partial S} \boldsymbol{\beta} \cdot \mathbf{n}_b\, ds = 0.$$

We now show that for the potential U in the insulated conductor Ω, solution of problem (11) in the class $\{\mathscr{P}\}$ of functions given by Eq. (12), we have apart from an

additive constant: $U(\mathbf{x}) = -V(\mathbf{x}) + P(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^3 - S - \Gamma$, where V is a solution of Eq. (15).

To this end it is sufficient to show that the function $V(\mathbf{x})$ can be represented by means of a double layer potential

$$V(\mathbf{x}) = \frac{1}{4\pi} \int_\Gamma v \mathbf{n}^T \nabla r^{-1} \, d\Gamma, \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - \Gamma, \tag{17}$$

where the density v is a solution of Eq. (13).

In fact if v satisfies Eq. (13), this condition implies that the potential V defined by Eq. (17) satisfies

$$\Delta V = 0 \quad \text{in } \mathbb{R}^3 - \bar{\Omega}, \qquad V|_{\Gamma_+} = P|_\Gamma.$$

Since $\Delta P = 0$ in $\mathbb{R}^3 - \bar{\Omega}$, from the uniqueness of the classical Dirichlet problem we derive $V \equiv P$ in $\mathbb{R}^3 - \bar{\Omega}$. Hence $\partial V/\partial n|_{\Gamma_+} = \partial P/\partial n|_\Gamma$, but $[\partial V/\partial n]_\Gamma = 0$, thus the potential V satisfies the boundary condition $\partial V/\partial n|_{\Gamma_-} = \partial P/\partial n|_\Gamma$. On the other hand if W is a solution of the Neumann problem (15) we have $W = V + k$, where k is a constant; since

$$\frac{1}{4\pi} \int_\Gamma \mathbf{n}^T \nabla r^{-1} \, d\Gamma = \begin{cases} 0 & \text{in } \mathbb{R}^3 - \bar{\Omega}, \\ -1 & \text{in } \Omega \end{cases}$$

and $\tilde{v} = v - k$ is also a solution of Eq. (13) we obtain

$$W(\mathbf{x}) = \frac{1}{4\pi} \int_\Gamma \tilde{v} \mathbf{n}^T \nabla r^{-1} \, d\Gamma.$$

The solvability of problem (11) for potentials U of the form given by Eq. (12) is reduced to the solvability of Eq. (13). We shall now show that the integral equation (13) admits a unique solution v determined except for an additive constant. The integral operator on the left-hand side of Eq. (13)

$$Av(\mathbf{x}) = \frac{1}{2} v(\mathbf{x}) + \frac{1}{4\pi} \int v \mathbf{n}^T \nabla r^{-1} \, d\Gamma, \qquad \text{for all } \mathbf{x} \text{ on } \Gamma$$

having a kernel with a weak singularity is a compact operator from $C(\Gamma)$ (the space of continuous function on Γ) into itself; therefore we can apply the Fredholm theory (see [15]).

Let us begin with the homogeneous equation associated to Eq. (13), i.e. $Av = 0$ and we prove that v is constant. Setting

$$W(\mathbf{x}) = \frac{1}{4\pi} \int_\Gamma v \mathbf{n}^T \nabla r^{-1} \, d\Gamma, \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - \Gamma$$

using the properties of a double layer potential we obtain

$$\Delta W = 0 \quad \text{in } \mathbb{R}^3 - \Gamma, \qquad [W]_\Gamma = v, \qquad \left[\frac{\partial W}{\partial n}\right]_\Gamma = 0$$

but $Av = W|_{\Gamma_+} = 0$ hence $W \equiv 0$ in $\mathbb{R}^3 - \bar{\Omega}$ and $\partial W/\partial n|_{\Gamma_+} = 0$. Combining these

results we have

$$\Delta W = 0 \quad \text{in } \Omega, \qquad \left.\frac{\partial W}{\partial n}\right|_{\Gamma_-} = 0, \qquad W|_{\Gamma_-} = -v$$

then from the properties of the interior homogeneous Neumann problem it follows $W = \text{constant} = W|_{\Gamma_-} = -v$. Now let us consider the homogeneous adjoint equation

$$A^T\chi(\mathbf{x}) = \frac{1}{2}\chi(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma \mathbf{n}_x^T \nabla r^{-1}\, d\Gamma = 0, \qquad \text{for all } \mathbf{x} \text{ on } \Gamma, \tag{18}$$

where $\mathbf{n}_x$ is the outward normal at point $\mathbf{x}$. Since this integral equation is the classical equation used in potential theory for solving the (homogeneous) interior Neumann problem in Ω, when the solution is represented by means of a simple layer potential, it is well known that there exists a unique non-trivial solution defined except for a multiplicative constant. Hence, using the Fredholm's alternative theorem, integral equation (13) admits a unique solution, defined except for an additive constant, if and only if the right-hand side of Eq. (13) is orthogonal to the solution of the homogeneous adjoint equation (18). In other words if v is a solution of Eq. (13) setting

$$V(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma v\mathbf{n}^T \nabla r^{-1}\, d\Gamma, \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - \Gamma$$

we have

$$\Delta V = 0 \quad \text{in } \mathbb{R}^3 - \Gamma, \qquad V|_{\Gamma_+} = Av = P|_\Gamma,$$

where P is defined by Eq. (14). Then integral equation (13) can be interpreted as the equation for solving the classical exterior Dirichlet problem when the potential V is represented by means of a double layer potential. Hence the well known results of Fredholm's theory imply that the solvability of Eq. (13) is assured if and only if the Dirichlet datum $P|_\Gamma$ satisfies the following compatibility condition:

$$\int_\Gamma P\chi\, d\Gamma = 0, \tag{19}$$

where χ is a non-trivial solution of Eq. (18). We now show that condition (19) actually holds. Setting

$$Z(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma \chi r^{-1}\, d\Gamma, \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3 - \Gamma$$

from the properties of a simple layer potential and Eq. (18) we obtain

$$\Delta Z = 0 \quad \text{in } \mathbb{R}^3 - \Gamma, \qquad \left.\frac{\partial Z}{\partial n}\right|_{\Gamma_-} = 0, \qquad \left.\frac{\partial Z}{\partial n}\right|_{\Gamma_+} = -\chi,$$

hence Z is constant in Ω. Then

$$\Delta Z = 0 \quad \text{in } \mathbb{R}^3 - \bar{\Omega}, \qquad Z|_\Gamma = \text{const.}, \qquad \left.\frac{\partial Z}{\partial n}\right|_{\Gamma_+} = -\chi.$$

Taking into account this result, the compatibility condition (19) may be expressed as

$$\int_\Gamma P \frac{\partial Z}{\partial n}\bigg|_{\Gamma_+} d\Gamma = 0,$$

where Z is the unique solution of the Dirichlet problem

$$\Delta Z = 0 \quad \text{in } \mathbb{R}^3 - \bar{\Omega}, \qquad Z|_\Gamma = 1.$$

The potential $P(\mathbf{x})$ given by Eq. (14) is a regular function outside S and satisfies $\Delta P = 0$ in $\mathbb{R}^3 - \bar{\Omega}$, hence by applying Green's formula in $\mathbb{R}^3 - \bar{\Omega}$ it follows:

$$0 = \int_{\mathbb{R}^3 - \bar{\Omega}} (P\,\Delta Z - Z\,\Delta P)\,dV = \int_\Gamma \left(\frac{\partial P}{\partial n} Z - P \frac{\partial Z}{\partial n}\bigg|_{\Gamma_+} \right) d\Gamma$$

$$= \int_\Gamma \frac{\partial P}{\partial n} d\Gamma - \int_\Gamma P \frac{\partial Z}{\partial n}\bigg|_{\Gamma_+} d\Gamma.$$

Finally collecting the results and taking into account Eq. (16) we deduce

$$\int_\Gamma P\chi\, d\Gamma = \int_\Gamma P \frac{\partial Z}{\partial n}\bigg|_{\Gamma_+} d\Gamma = \int_\Gamma \frac{\partial P}{\partial n} d\Gamma = 0$$

hence the compatibility condition holds.

Remark 5.2. In the case of an infinite homogeneous medium we derived in Section 3 the decomposition of the potential given by Eq. (7). A similar decomposition can also be derived in the case of a bounded conductor. Under the assumption of axial symmetry we know that, setting $\mathbf{a}_l = \mathbf{a}_3$, $m_l = m_3$, $m_t = m_1 = m_2$, the following identity holds:

$$\mathbf{n}^T M \nabla = m_t \mathbf{n} \cdot \nabla + (m_l - m_t)(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \cdot \nabla.$$

In the following we shall restrict ourselves to the case of m_l, m_t constant, a choice which has been made in carrying out the numerical simulations reported in [7]. Due to the linearity of Eqs. (12) and (13) it is easy to verify the following decomposition:

$$U(\mathbf{x}) = m_t U_n(\mathbf{x}) + (m_l - m_t) U_a(\mathbf{x}),$$

where, for all $\mathbf{x}$ in $\Omega - S$, U_n is the potential field due to a normal double layer of unit density:

$$U_n(\mathbf{x}) = -\frac{1}{4\pi} \int_\Gamma v_n \mathbf{n} \cdot \nabla r^{-1}\, d\Gamma + \frac{1}{4\pi} \int_S \mathbf{n} \cdot \nabla r^{-1}\, dS \tag{20}$$

and U_a is the potential field generated by an axial dipole layer with unit moment

$$U_a(\mathbf{x}) = -\frac{1}{4\pi} \int_\Gamma v_a \mathbf{n} \cdot \nabla r^{-1}\, d\Gamma + \frac{1}{4\pi} \int_S (\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l \cdot \nabla r^{-1}\, dS. \tag{21}$$

In Eqs. (20) and (21) the functions v_n and v_a, which are the traces on Γ the potentials U_n and U_a respectively, are the solutions of the following integral equations on Γ:

$$\frac{1}{2}v_n(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma v_n \mathbf{n}\cdot\nabla r^{-1}\,d\Gamma = \frac{1}{4\pi}\int_S \mathbf{n}\cdot\nabla r^{-1}\,dS,$$

$$\frac{1}{2}v_a(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma v_a \mathbf{n}\cdot\nabla r^{-1}\,d\Gamma = \frac{1}{4\pi}\int_S (\mathbf{n}\cdot\mathbf{a}_l)\mathbf{a}_l\cdot\nabla r^{-1}\,dS.$$

The description of the potential field, generated by an oblique dipole layer in an isotropic and homogeneous medium, can be extended to the case of an isotropic medium with piecewise constant conductivity. The integral equations characterizing the potential field can be derived by applying the techniques used in Barnard et al. [1], [2] and Barr et al. [3]. This extension would be suitable in the simulation of the potential in a human torso model taking into account the different conductivity of lungs and blood masses.

6. Connection Between the Oblique Dipole Layer Model and an Intracellular Current Model

The representation of the activation wavefront by means of electrical sources must be based on assumptions about the electrophysiological activity of the heart cells. For some models of cardiac sources see Plonsey [24] for a single active fiber, and Plonsey and Rudy [23] and Spach et al. [27] for a continuum of fibers.

In this section we shall establish a connection between the oblique dipole layer model defined by Eq. (6) and a limit case of the intracellular current model investigated by Spach et al. [27] in the case of a thin layer of cardiac tissue. At the same time we succeed in relating the dipole moments m_t, m_l to more definite electrophysiological quantities such as the anisotropic intracellular conductivity and the intracellular action potential jump.

We consider the heart tissue volume H as the superposition of two continuous anisotropic conducting media respectively the extracellular (e) and the intracellular (i) cardiac media; the anisotropy of these media is related to the heart fiber structure and the principal axes of the conductivity tensor are one parallel and the remaining two perpendicular to the local fiber direction. The extracardiac medium, i.e. the medium surrounding the heart H, and the extracellular cardiac medium are assumed homogeneous isotropic with the same constant conductivity σ_0, thus neglecting the influence of the heart boundary as well as the effect of the extracellular cardiac anisotropy.

These simplifying assumptions are suitable for a quantitative description of the potential field generated at distance from the cardiac sources; "local" models which describe the potential field near the sources, taking also into account the anisotropy of the (e) medium, are dealt with in Section 7.

Similarly to what has been done for the model described by Eq. (6), we shall limit the following analysis to the case of the heart imbedded in an unbounded medium.

The conductivity tensor of the anisotropic (i) medium is characterized in the local basis by $D_i' = \mathrm{diag}(\sigma_1^i, \sigma_2^i, \sigma_3^i)$ with $\sigma_k^i > 0, k = 1, 2, 3$; if $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ defines the local basis, then one of the unit vectors $\mathbf{a}_i$ (e.g. $\mathbf{a}_3$) is parallel to the local fiber direction.

In the original model used by Spach et al. [27] it is also implicitly assumed that the principal axes have constant directions (parallel fibers) so that in a suitable standard basis also the conductivity tensor D_i is diagonal. We shall drop this last assumption, i.e. we can have variable fiber direction, so that if $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ defines the local basis, the conductivity tensor D_i of the intracellular medium is given by

$$D_i = AD_i'A^T = \sigma_1^i \mathbf{a}_1 \mathbf{a}_1^T + \sigma_2^i \mathbf{a}_2 \mathbf{a}_2^T + \sigma_3^i \mathbf{a}_3 \mathbf{a}_3^T. \tag{22}$$

We note that D_i is a symmetric positive definite matrix. For uniformity we shall introduce the conductivity tensor $D_e = \sigma_0 I$ for extracellular medium. We remark that for non-constant A (variable fiber direction) D_i is variable even if D_i' is constant (i.e. anisotropic homogeneous medium). In the case of axial symmetry around $\mathbf{a}_3$ parallel to the longitudinal axis of the fiber it is

$$\mathbf{a}_3 = \mathbf{a}_l, \qquad \sigma_1^i = \sigma_2^i = \sigma_t^i, \qquad \sigma_3^i = \sigma_l^i.$$

Following Miller and Geselowitz [16], Spach et al. [27], we introduce the intra and extracellular potential distributions U_i and U_e and the related current densities

$$\mathbf{I}_i = -D_i \nabla U_i \quad \text{in } H, \qquad \mathbf{I}_e = -D_e \nabla U_e \quad \text{in } \mathbb{R}^3.$$

Utilizing the current conservation property we have

$$\begin{aligned} \nabla \cdot (\mathbf{I}_i + \mathbf{I}_e) = 0 \quad &\text{in } H, \qquad \text{i.e.} \quad \nabla \cdot (D_i \nabla U_i + D_e \nabla U_e) = 0 \quad \text{in } H, \\ \nabla \cdot \mathbf{I}_e = 0 \quad &\text{in } \mathbb{R}^3 - H, \quad \text{i.e.} \quad \nabla \cdot D_e \nabla U_e = 0 \quad \text{in } \mathbb{R}^3 - H. \end{aligned} \tag{23}$$

Moreover at the boundary ∂H the current vector $\mathbf{I}_i$ is tangent to ∂H so that

$$\mathbf{I}_i \cdot \mathbf{n}_H = 0, \qquad \text{i.e.} \qquad (D_i \nabla U_i) \cdot \mathbf{n}_H = 0, \tag{24}$$

where $\mathbf{n}_H$ is the unit normal to ∂H. The transmembrane current I_m per unit volume, i.e. the current flowing into the extracellular medium is given by

$$I_m = -\nabla \cdot \mathbf{I}_i = \nabla^T D_i \nabla U_i \qquad \text{in } H.$$

Since the extracellular medium has been assumed homogeneous with conductivity σ_0 we have

$$\sigma_0 \Delta U_e = \begin{cases} 0, & \text{in } \mathbb{R}^3 - H, \quad \text{with} \quad \Delta = \nabla \cdot \nabla, \\ -I_m, & \text{in } H. \end{cases}$$

From this relationship it follows that

$$U_e(\mathbf{x}) = \frac{1}{4\pi\sigma_0} \int_{V_a} I_m r^{-1} \, dV = \frac{1}{4\pi\sigma_0} \int_{V_a} r^{-1} \nabla^T D_i \nabla U_i \, dV. \tag{25}$$

$V_a \subset H$ is the "activating" region at time t, i.e. where the depolarization process is under way and $I_m \neq 0$.

To generate the extracellular potential U_e by means of Eq. (25) the knowledge of the intracellular potential U_i is required. We assume that the shape of the temporal intracellular action potential is the same for all cardiac cells. If for a cardiac cell the depolarization process starts at the time instant $\lambda = 0$ and ends at

the time instant $\lambda = \varepsilon$, the intracellular action potential is represented by a regular monotone function $\phi_i(\lambda)$ with $\phi_i(\lambda) = u_r$ (resting value) for $\lambda < 0$, $\phi_i(\lambda) = u_a$ (plateau value) for $\lambda > \varepsilon$ and with a monotone variation between these two values for $0 < \lambda < \varepsilon$.

Then the intracellular potential distribution U_i may be derived from the knowledge of the time of activation at each point of the active tissue. Let S_t denote the surface of all points (cells) for which the depolarization process starts at time t; as t varies we obtain a family of surfaces which are assumed regular and non-intersecting. This corresponds to the biological assumption that in the time interval considered for the spreading of the front a cell is depolarized only once. The activating region V_a is the volume comprised between the surfaces S_t and $S_{t-\varepsilon}$ which are respectively the leading and the trailing edge. S_t and $S_{t-\varepsilon}$ may be closed or open; in the latter case they intersect the endocardium or the epicardium (i.e. ∂H) and thus we must take into account also a "lateral" surface S_a on ∂H bounding V_a.

For $t - \varepsilon < \tau < t$ the surface S_τ represents the set of cells with activation time τ, which are consequently at the same phase of the depolarization process with action potential value $\phi_i(t - \tau)$. Therefore S_τ is an equipotential surface for U_i with $U_i(\mathbf{x}, t) = \phi_i(t - \tau)$. Outside V_a the potential U_i is constant, i.e. $U_i = u_r$ ahead of S_t (leading edge) and $U_i = u_a$ behind $S_{t-\varepsilon}$ (trailing edge).

We shall show that if the duration ε of the depolarization process is small and the region V_a is thin, so that for points far enough V_a may be idealized as a surface, the following approximation holds:

$$U_e(\mathbf{x}) = \frac{1}{4\pi\sigma_0}\int_{V_a} r^{-1} I_m \, dV = \frac{u_a - u_r}{4\pi\sigma_0}\int_{S_t} \mathbf{n}^T D_i \nabla r^{-1} \, dS_t + 0(\varepsilon). \tag{26}$$

To establish this result we proceed as follows. Because

$$r^{-1} I_m = r^{-1}\nabla \cdot (D_i \nabla U_i) = \nabla \cdot (r^{-1} D_i \nabla U_i) - (D_i \nabla U_i) \cdot \nabla r^{-1}$$

applying Green's formula we obtain

$$\int_{V_a} r^{-1}\nabla \cdot (D_i \nabla U_i)\, dV = \int_{S_t + S_{t-\varepsilon} + S_a} (r^{-1} D_i \nabla U_i) \cdot \mathbf{n} \, dS - \int_{V_a} (D_i \nabla I_i) \cdot \nabla r^{-1} \, dV.$$

Since U_i is a regular function, constant outside V_a, it follows that $\nabla U_i = 0$ on S_t, $S_{t-\varepsilon}$ and from Eq. (24) we have $(D_i \nabla U_i) \cdot \mathbf{n} = 0$ on $S_a \subset \partial H$, therefore

$$\int_{V_a} r^{-1} I_m \, dV = -\int_{V_a} (D_i \nabla U_i) \cdot \nabla r^{-1} \, dV. \tag{27}$$

It remains to prove that the volume integral on V_a on the second member of Eq. (27) tends to the surface integral on S_t given by Eq. (26). Let $\{C\}$ denote the family of curves orthogonal to the front surfaces S_τ with positive orientation induced by the advancing front. On a curve C the velocity of the point $\mathbf{x}$, intersection of the front S_τ with C, is the so-called normal velocity of the front at $\mathbf{x}$.

By definition this velocity $\mathbf{w}$ is orthogonal to the front i.e. $\mathbf{w} = w\mathbf{n}$ with $\mathbf{n}$ pointing in the advancing direction and, if dl is the arc of C comprised between the fronts S_τ, $S_{\tau + d\tau}$, it results $dl = w\, d\tau$.

For a point $\mathbf{y}$ of V_a let S_τ, with $t-\varepsilon<\tau<t$, be the surface passing through $\mathbf{y}$ and $\mathbf{n}$ the unit normal to S_τ at $\mathbf{y}$. Since S_τ is an equipotential surface for U_i then it results

$$\nabla U_i = \frac{dU_i}{dn}\mathbf{n} = \frac{dU_i}{dl}\mathbf{n} \qquad \text{with} \qquad \frac{dU_i}{dl} = \frac{1}{w}\frac{d\phi_i(t-\tau)}{d\tau}, \tag{28}$$

where d/dl is the directional derivative along C which is orthogonal to S_τ.

Taking into account Eq. (28) and that $dV = dS_\tau\, dl = w\, dS_\tau\, d\tau$ we obtain

$$\int_{V_a}(D_i\nabla U_i)\cdot\nabla r^{-1}\,dV = \int_{t-\varepsilon}^{t}\left(\int_{S_\tau}\frac{1}{w}\frac{d\phi_i(t-\tau)}{d\tau}\mathbf{n}^T D_i\nabla r^{-1} w\,dS_\tau\right)d\tau$$
$$= \int_{t-\varepsilon}^{t}\frac{d\phi_i(t-\tau)}{d\tau}\left(\int_{S_\tau}\mathbf{n}^T D_i\nabla r^{-1}\,dS_\tau\right)d\tau.$$

Assuming that we are far enough from V_a and that V_a is thin we have

$$\int_{S_\tau}\mathbf{n}^T D_i\nabla r^{-1}\,dS_\tau = \int_{S_t}\mathbf{n}^T D_i\nabla r^{-1}\,dS_t + 0(\varepsilon). \tag{29}$$

Therefore

$$\int_{t-\varepsilon}^{t}\frac{d\phi_i(t-\tau)}{d\tau}\left(\int_{S_\tau}\mathbf{n}^T D_i\nabla r^{-1}\,dS_\tau\right)d\tau$$
$$= \left(\int_{S_t}\mathbf{n}^T D_i\nabla r^{-1}\,dS_t\right)\left(\int_{t-\varepsilon}^{t}\frac{d\phi_i(t-\tau)}{d\tau}d\tau\right) + 0(\varepsilon)$$
$$= (u_r-u_a)\int_{S_t}\mathbf{n}^T D_i\nabla r^{-1}\,dS_t + 0(\varepsilon). \tag{30}$$

Combining Eqs. (25), (27), (29) and (30) we obtain the approximate relation (26). We remark that the second member of Eq. (26) can be formally interpreted as the limit of the first member of the same relation for $\varepsilon\to 0$, i.e. for an instantaneous depolarization process for which the action potential is represented by a step function with jump u_a-u_r.

Finally we discuss some relevant consequence of Eq. (24) when S_t is an open surface intersecting the heart surface ∂H. Equation (24) can be written as

$$\mathbf{n}_H\cdot(D_i\nabla U_i) = \mathbf{n}_H^T D_i\nabla U_i = 0,$$

beside Eq. (28) implies that ∇U_i is parallel to $\mathbf{n}$, unit normal to S_τ and taking also into account the symmetry of D_i we obtain

$$\mathbf{n}_H^T D_i\mathbf{n} = \mathbf{n}^T D_i\mathbf{n}_H = 0. \tag{31}$$

This result holds for $t-\varepsilon<\tau<t$ and, since U_i is regular, also for $\tau=t-\varepsilon$ and $\tau=t$, i.e. on the surfaces $S_{t-\varepsilon}$, S_t. When $\varepsilon\to 0$, i.e. the depolarization process becomes instantaneous, Eq. (31) must hold on the points of the intersection of S_t, representing in the limit the activating region, with ∂H.

We assume, on physiological grounds, that on ∂H the fibers are tangent to ∂H, i.e. $\mathbf{a}_l\cdot\mathbf{n}_H = 0$. Since

$$D_i\mathbf{n}_H = \sigma_t^i[(\mathbf{n}_H \cdot \mathbf{a}_1)\mathbf{a}_1 + (\mathbf{n}_H \cdot \mathbf{a}_2)\mathbf{a}_2] + \sigma_l^i(\mathbf{n}_H \cdot \mathbf{a}_3)\mathbf{a}_3$$

with $\mathbf{a}_3 = \mathbf{a}_l$, it follows that $\mathbf{n}_H \cdot \mathbf{a}_3 = 0$ and $\mathbf{n}_H$ is coplanar with the two orthogonal unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$ so that

$$D_i\mathbf{n}_H = \sigma_t^i\mathbf{n}_H.$$

Substituting into Eq. (31) we get

$$\mathbf{n} \cdot \mathbf{n}_H = 0,$$

i.e. the *two surfaces S_t and ∂H intersect perpendicularly*. From this last relation it follows also that $\mathbf{n}_H = \mathbf{n}_b$ where $\mathbf{n}_b$ is the unit vector tangent to S_t and perpendicular to $\partial S_t \subset \partial H$.

To conclude, in the approximation represented by Eq. (26), the extracellular potential may be viewed as originated by an oblique dipole layer on S_t. In fact, approximation (26) coincides with the potential due to an oblique dipole layer given by Eq. (6) if the dipole moments m_t, m_l are chosen as follows

$$m_t = (u_a - u_r)\sigma_t^i, \qquad m_l = (u_a - u_r)\sigma_l^i.$$

Hence a representation of the depolarization wavefront as the superposition of an axial and a transverse dipole layer can be derived under the assumption of an instantaneous depolarization process with a constant jump of the intracellular action potential; the anisotropy of the dipole distribution on the wavefront defined by an oblique dipole layer model can be viewed as the effect of the anisotropic structure of the intracellular medium.

Finally the oblique dipole layer model defined by $M = (u_a - u_r)D_i$ must satisfy a constraint on the boundary S_t of the wavefront since, from Eq. (31), it follows that

$$\mathbf{n}^T M\mathbf{n}_H = \mathbf{n}^T M\mathbf{n}_b = 0$$

because $\mathbf{n}_H = \mathbf{n}_b$.

We remark that this relation on ∂S_t, introduced previously on purely mathematical grounds in Section 4 as Eq. (8), is a consequence of the current intracellular conservation (Eq. (24)) and of the tangency of the fibers to the heart surface ∂H.

However if, in addition to the tangency of $\mathbf{a}_l$ to ∂H, we impose the condition $\mathbf{n}^T M\mathbf{n}_b = 0$ on ∂S instead of the intracellular current conservation (Eq. (31)) other cases, besides the orthogonality of S_t and ∂H, may occur. More precisely S_t and ∂H can be tangent or, if $\mathbf{a}_l$ is tangent to ∂S_t, the surfaces S_t and ∂H intersect at an arbitrary angle.

To prove the occurrence of these other cases we first recall, from Section 4, that $\mathbf{n}^T M\mathbf{n}_b = 0$ on ∂S_t is equivalent to either $\mathbf{a}_l$ tangent to S_t or $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ on ∂S_t.

We also introduce the unit vector $\boldsymbol{\tau}_b$ tangent to ∂S_t; this vector is tangent to both S_t and ∂H since $\partial S_t \subset \partial H$ and the vectors $\mathbf{n}$, $\mathbf{n}_b$, $\boldsymbol{\tau}_b$ are mutually orthogonal. We now distinguish the following cases:

i) $\mathbf{a}_l$ is parallel to $\boldsymbol{\tau}_b$ iff $\mathbf{n}$ is oblique to ∂H (i.e. not tangent or perpendicular to ∂H).

In fact if $\mathbf{a}_l$ is parallel to $\boldsymbol{\tau}_b$ then it is tangent to S_t and it is also $\mathbf{a}_l \cdot \mathbf{n}_b = 0$. In this case S_t and ∂H intersect at an arbitrary angle. Conversely if $\mathbf{n}$ is oblique to ∂H then,

among the vectors tangent to ∂H, only τ_b is tangent to S_t and perpendicular to $\mathbf{n}_b$. Since $\mathbf{a}_l$ is tangent to ∂H, either condition $\mathbf{a}_l$ tangent to S_t or $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ implies $\mathbf{a}_l$ parallel to τ_b.

ii) $\mathbf{a}_l$ tangent to S_t but not parallel to τ_b iff $\mathbf{n} = \mathbf{n}_H$, i.e. S_t, ∂H are tangent.

In fact since $\mathbf{a}_l$ and τ_b are distinct and both tangent to S_t and ∂H it follows that these surfaces are tangent. Conversely if S_t, ∂H are tangent then $\mathbf{a}_l$, tangent to ∂H, is also tangent to S_t.

iii) $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ and $\mathbf{a}_l$ not parallel to τ_b iff $\mathbf{n} \cdot \mathbf{n}_H = 0$, i.e. S_t, ∂H are orthogonal. In fact from $\mathbf{n} \cdot \mathbf{n}_b = 0$ it follows that $\mathbf{a}_l$, $\mathbf{n}$, τ_b are coplanar, hence $\mathbf{n}$ is tangent to ∂H. Conversely $\mathbf{n}$ and τ_b tangent to ∂H implies $\mathbf{n}_b$ perpendicular to ∂H and consequently $\mathbf{a}_l \cdot \mathbf{n}_b = 0$ since $\mathbf{a}_l$ is tangent to ∂H.

7. Oblique Dipole Layers in a Parallel Uniform Anisotropic Medium and Related Jump Relations

For an adequate description of the potential field in the heart muscle the anisotropy of the extracellular medium must be taken into account.

To simplify the mathematical developments, we shall describe two local models in which the influence of the heart boundary is neglected or equivalently the heart region H is infinite.

The activating region is still represented by a wavefront surface, i.e. these local models are suitable for predicting potentials in proximity of the sources but at distance greater than the thickness of the activating region.

To develop formulae similar to Eq. (26) for the extracellular potential when the anisotropy of the extracellular medium is also taken into account, we need the fundamental solution of the differential operator $\nabla^T D_e \nabla$. Since this solution can be written explicitly when D_e is constant we shall consider the case in which the intra and extracellular media are anisotropic, parallel and uniform. Then the direction of the principal axes is everywhere the same so that there is a common local basis and in this basis D_i', D_e' are constant. Without loss of generality we may take the standard basis coincident with the local basis, so that

$$D_i = D_i', \qquad D_e = D_e'.$$

Setting

$$D_e^* = \operatorname{diag}(\sigma_2^e \sigma_3^e, \sigma_3^e \sigma_1^e, \sigma_1^e \sigma_2^e)$$

it can be verified that $(\mathbf{r}^T D_e^* \mathbf{r})^{-1/2}$ is the fundamental solution of the operator $\nabla^T D_e \nabla$. In order to formulate a model for the extracellular potential, in terms of the intracellular potential distribution, starting from the relation $-\nabla^T D_e \nabla U_e = \nabla^T D_i \nabla U_i$, it is sufficient to substitute $(\sigma_0 r)^{-1}$ with $(\mathbf{r}^T D_e^* \mathbf{r})^{-1/2}$ in the formulae used in the preceding section for the case of the homogeneous isotropic extracellular medium. In the limit case we obtain for U_e the expression

$$U_e(\mathbf{x}) = \frac{u_a - u_r}{4\pi} \int_S \mathbf{n}^T D_i \nabla (\mathbf{r}^T D_e^* \mathbf{r})^{-1/2} \, dS. \tag{32}$$

Actually, to establish this formula at the mathematical level, we do not need the assumption of uniform anisotropy for the medium (*i*).

With a different set of assumptions we can build a model for which the jump of the potential across S coincides with that given by Roberts et al. [25].

We first remark that from

$$\nabla^T D_e \nabla U_e = -\nabla^T D_i \nabla U_i$$

adding $\nabla^T D_i \nabla U_e$ to both sides we obtain

$$\nabla^T (D_i + D_e) \nabla U_e = -\nabla^T D_i \nabla \Psi,$$

where $\Psi = U_i - U_e$ is the transmembrane potential distribution and $D_i + D_e$ can be interpreted as the conductivity tensor of the myocardial tissue.

We make now the following assumptions:

- D_i, D_e constant, i.e. anisotropic and uniform media.
- The current I_m per unit volume is given by $I_m = -\nabla^T D_i \Psi$, i.e. I_m is generated by the transmembrane instead of the intracellular potential distribution.
- The temporal shape of the transmembrane action potential is the same for all fibers with a monotone change from Ψ_r to Ψ_a where Ψ_r, Ψ_a are the transmembrane potential values of the resting and depolarized tissue.

Setting

$$D = D_i + D_e = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3) \qquad \text{with} \qquad \sigma_k = \sigma_k^i + \sigma_k^e, \qquad k = 1, 2, 3$$

since $(\mathbf{r}^T D^* \mathbf{r})^{-1/2}$ with $D^* = \operatorname{diag}(\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2)$ is the fundamental solution of the differential operator $\nabla^T D \nabla$, the potential U_e in the unbounded extracellular medium is given in the limit case, when the transmembrane action potential approaches a step function, by

$$U_e(\mathbf{x}) = \frac{\Psi_a - \Psi_r}{4\pi} \int_S \mathbf{n}^T D_i \nabla (\mathbf{r}^T D^* \mathbf{r})^{-1/2} \, dS. \tag{33}$$

The three limit equations (26), (32) and (33) can be represented by the same formula setting

$$D = \begin{cases} \sigma_0 I, \\ D_e, \\ D_i + D_e, \end{cases} \qquad \alpha = \begin{cases} u_a - u_r, \\ u_a - u_r, \\ \Psi_a - \Psi_r. \end{cases}$$

If $D = \operatorname{diag}(d_1, d_2, d_3)$ and we set $D^* = \operatorname{diag}(d_2 d_3, d_3 d_1, d_1 d_2)$ and introduce the function $\phi = (\mathbf{r}^T D^* \mathbf{r})^{1/2}$, then ϕ^{-1} is the fundamental solution of the differential operator $\nabla^T D \nabla$. In the three cases, the potential U_e is then given by

$$U_e(\mathbf{x}) = \frac{\alpha}{4\pi} \int_S \mathbf{n}^T D_i \nabla \phi^{-1} \, dS.$$

Applying formula (15.3) of Theorem 15, III, §15, Ch. II of Miranda [18] we get for U_e a jump across S expressed by

$$[U_e]_S = \alpha \frac{\mathbf{n}^T D_i \mathbf{n}}{\mathbf{n}^T D \mathbf{n}}. \tag{34}$$

We remark that, in order to derive this relation, only the assumption of D constant,

and consequently of ϕ explicitly defined, is used. Applying jump relation (34), the models defined by Eqs. (32) and (33) will give the same jump if

$$(u_a - u_r)\frac{\mathbf{n}^T D_i \mathbf{n}}{\mathbf{n}^T D_e \mathbf{n}} = (\psi_a - \psi_r)\frac{\mathbf{n}^T D_i \mathbf{n}}{\mathbf{n}^T (D_i + D_e)\mathbf{n}}.$$

This equality implies D_i proportional to D_e, i.e. $D_i = cD_e$ with the constant c satisfying the relation

$$u_a - u_r = \frac{\psi_a - \psi_r}{1 + c}.$$

From $D_i = cD_e$, it follows $\sigma_k^i/\sigma_k^e = c$, $k = 1, 2, 3$ and in the case of axial symmetry

$$\sigma_l^i/\sigma_t^e = \sigma_l^i/\sigma_l^e = c.$$

This last relation, stating the same anisotropy ratio in the intra and in the extracellular media, was a hypothesis used by Plonsey and Rudy [23]. If this hypothesis holds, then the two models predict a constant potential jump through S which may be expressed by

$$[U_e]_S = (\psi_a - \psi_r)\frac{1}{c + 1} = (u_a - u_r)\frac{1}{c}.$$

Even if the two models yield the same $[U_e]_S$, they do not coincide since they have different jumps of the potential normal derivative across S and predict different extracellular potential fields.

In the case of axial symmetry we denote as usual with $\mathbf{a}_l$ the local symmetry axis and $\mathbf{a}_t$ the transverse unit vector perpendicular to $\mathbf{a}_l$ and coplanar with $\mathbf{a}_l$, $\mathbf{n}$. The vectors $\mathbf{a}_l$, $\mathbf{a}_t$ are oriented so that $\mathbf{n} \cdot \mathbf{a}_l \geqslant 0$, $\mathbf{n} \cdot \mathbf{a}_t \geqslant 0$. In the case of parallel fibers $\mathbf{a}_l$ is taken parallel to the third axis. Then with reference to the model defined by Eq. (33), the jump U_e across S, given Eq. (34), may be written in the following form

$$\begin{aligned}[U_e]_S &= (\psi_a - \psi_r)\frac{\mathbf{n}^T D_i \mathbf{n}}{\mathbf{n}^T (D_i + D_e)\mathbf{n}} \\ &= (\psi_a - \psi_r)\frac{\sigma_l^i \cos^2\gamma + \sigma_t^i \sin^2\gamma}{(\sigma_l^i + \sigma_t^i)\cos^2\gamma + (\sigma_t^i + \sigma_t^e)\sin^2\gamma},\end{aligned} \tag{35}$$

where $\cos\gamma = \mathbf{n} \cdot \mathbf{a}_l$, $\sin\gamma = \mathbf{n} \cdot \mathbf{a}_t$, i.e. γ is the angle between the fiber direction and the normal $\mathbf{n}$ to the front S. An important feature of this last jump relation (35) is that it coincides with formula (10) of Appendix C of Roberts et al. [25].

The two models, we have thus derived, should provide a more adequate description of the local potential field near the excitation wavefront, but not too close to it since the region which is the seat of the depolarization process is idealized as a surface.

The choice between these models depends on which of the two hypotheses, intracellular or transmembrane action potential, is more adequate to represent the equivalent source generators.

Remark 7.1. The potential defined by Eqs. (32) and (33) can also be viewed as the superposition of an axial and a transverse dipole layer of density $\alpha\sigma_l^i(\mathbf{n} \cdot \mathbf{a}_l)$ and

$\alpha\sigma_t^i(\mathbf{n}\cdot\mathbf{a}_t)$ in the anisotropic extracellular medium. In fact, using Eq. (3) with $C = D_i$ we obtain

$$U_e(\mathbf{x}) = \frac{\alpha}{4\pi}\int_S \mathbf{n}^T D_i \nabla\phi^{-1}\, dS$$

$$= \frac{\alpha}{4\pi}\left\{\int_S \sigma_t^i(\mathbf{n}\cdot\mathbf{a}_t)\mathbf{a}_t\cdot\nabla\phi^{-1}\, dS + \int_S \sigma_l^i(\mathbf{n}\cdot\mathbf{a}_l)\mathbf{a}_l\nabla\phi^{-1}\, dS\right\}.$$

Moreover the splitting into a normal and an axial dipole layer potential given by Eq. (7) for the case of a uniform isotropic extracellular medium can be extended to the case of a uniform anisotropic extracellular medium (parallel fibers) assuming axial symmetry around the common fiber direction. In this case we have

$$D_i = \mathrm{diag}(\sigma_t^i, \sigma_t^i, \sigma_l^i) \qquad \text{and} \qquad D = \mathrm{diag}(d_t, d_t, d_l)$$

with $\sigma_1^i = \sigma_2^i = \sigma_t^i$, $\sigma_3^i = \sigma_l^i$ and $d_1 = d_2 = d_t$, $d_3 = d_l$. Setting $\lambda = \sigma_t^i/d_t$ we obtain

$$D_i - \lambda D = \mathrm{diag}(0, 0, \delta),$$

where $\delta = \sigma_l^i - \sigma_t^i d_l/d_t$. Hence

$$\mathbf{n}^T D_i \nabla\phi^{-1} = \lambda\mathbf{n}^T D\nabla\phi^{-1} + \delta(\mathbf{n}\cdot\mathbf{a}_l)\mathbf{a}_l\cdot\nabla\phi^{-1}.$$

Setting $\mathbf{v} = D\mathbf{n} = d_t(\mathbf{n}\cdot\mathbf{a}_t)\mathbf{a}_t + d_l(\mathbf{n}\cdot\mathbf{a}_l)\mathbf{a}_l$ we have

$$U_e(\mathbf{x}) = \frac{\alpha}{4\pi}\int_S \mathbf{n}^T D_i\nabla\phi^{-1}\, dS = \frac{\alpha\lambda}{4\pi}\int_S \mathbf{v}\cdot\nabla\phi^{-1}\, dS + \frac{\alpha\delta}{4\pi}\int_S (\mathbf{n}\cdot\mathbf{a}_l)\mathbf{a}_l\cdot\nabla\phi^{-1}\, dS.$$

We note that $\mathbf{v}$ is a vector parallel to the conormal associated to the elliptic operator $\nabla^T D\nabla$; hence when S is a closed regular surface boundary of a domain A, applying the Green's formula we have

$$\frac{1}{4\pi}\int_S \mathbf{v}\cdot\nabla\phi^{-1}\, dS = \begin{cases} 0, & x\in\mathbb{R}^3 - A, \\ -1, & x\in A. \end{cases}$$

Therefore for a closed wavefront the potential depends only on the axial component both in an isotropic and in an anisotropic medium.

The development of the model for parallel fibers was considered with the aim of deriving a manageable model to be used in applications.

From the theoretical point of view, all the arguments discussed above may be extended to the case of non-parallel fibers.

Similarly, the fundamental solution for an elliptic operator with variable coefficients will again be used, although it will not in general have a closed analytical form.

The "local" model described by Eq. (33) which we set forth in Colli-Franzone et al. [7] has been also independently investigated by Roberts et al. [26] for simulating the potential in proximity of the wavefront.

8. The Isochrone Surfaces and the Fiber Geometry

The oblique dipole model, described in the previous sections, requires the knowledge of the fiber geometry in order to define the dipole direction on the

wavefront. On the other hand, the wavefront is related to the fiber geometry; the question then arises to find if it is possible to determine the fiber geometry given the wavefront motion during the depolarization phase of the heart beat. We shall show that the question has a positive answer if we apply some results from the cable theory (see e.g. [11], [22]) to the spreading of the excitation in an anisotropic medium. The wavefront surface S_t is also called an isochrone surface since it is the set of points at which the depolarization process starts at the same time instant t. Hence if the function $\tau(\mathbf{x})$ associates to the point $\mathbf{x}$ the time instant at which the cell in $\mathbf{x}$ becomes depolarized, then the wavefront S_t is given by the surface $\tau(\mathbf{x}) = t$.

To establish the connection between the isochrones and the fiber geometry we proceed as follows.

Given a generic point $\mathbf{x}$, let $\sigma^i(\mathbf{a})$ and $\sigma^e(\mathbf{a})$ be the conductivities of the media (i) and (e) along the direction defined by the unit vector $\mathbf{a}$ in $\mathbf{x}$; we have then

$$\sigma^i(\mathbf{a}) = \mathbf{a}^T D_i \mathbf{a}, \qquad \sigma^e(\mathbf{a}) = \mathbf{a}^T D_e \mathbf{a}. \tag{36}$$

The resistivities $\rho^i(\mathbf{a})$ and $\rho^e(\mathbf{a})$ are defined by the relations

$$\rho^i(\mathbf{a}) = 1/\sigma^i(\mathbf{a}), \qquad \rho^e(\mathbf{a}) = 1/\sigma^e(\mathbf{a}). \tag{37}$$

The resistivity $\rho(\mathbf{a})$ of the composite medium (heart tissue) is then given by

$$1/\rho(\mathbf{a}) = 1/\rho^i(\mathbf{a}) + 1/\rho^e(\mathbf{a}). \tag{38}$$

We now assume that the conduction velocity in the heart tissue along the direction $\mathbf{a}$ is given by

$$v(\mathbf{a}) = K/[\rho(\mathbf{a})]^{1/2} \tag{39}$$

with K constant independent of $\mathbf{a}$ and $\mathbf{x}$. Equation (39) was obtained by Clerc [5] for the longitudinal and transverse conduction velocity in a fiber by using the cable theory and he showed that it was applicable to the heart muscle. In the form given above it was used by Roberts et al. [25] and by Muler and Markin [19, 20] for parallel axisymmetric fibers.

Substituting Eqs. (36), (37) and (38) into Eq. (39) we obtain

$$v(\mathbf{a}) = K\left[\frac{(\mathbf{a}^T D_i \mathbf{a})(\mathbf{a}^T D_e \mathbf{a})}{\mathbf{a}^T (D_i + D_e)\mathbf{a}}\right]^{1/2} \tag{40}$$

We shall now make the further assumption

$$D_e = \sigma_0 I, \qquad D_i/\sigma_0 \simeq 0. \tag{41}$$

It follows then that

$$\mathbf{a}^T D_e \mathbf{a} = \sigma_0 \mathbf{a}^T \mathbf{a} = \sigma_0$$

and

$$\mathbf{a}^T (D_i + D_r)\mathbf{a} = \sigma_0 \mathbf{a}^T (D_i/\sigma_0 + I)\mathbf{a} \simeq \sigma_0$$

hence, dropping D_i/σ_0, we obtain

$$v(\mathbf{a}) = K[\mathbf{a}^T D_i \mathbf{a}]^{1/2}. \tag{42}$$

Assumption (41) has been used in numerical simulations by Spach et al. [27]. Since

$$D_i = AD_i'A^T = \sigma_1^i \mathbf{a}_1\mathbf{a}_1^T + \sigma_2^i \mathbf{a}_2\mathbf{a}_2^T + \sigma_3^i \mathbf{a}_3\mathbf{a}_3^T,$$

where $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ defines the local basis and $D_i' = \mathrm{diag}(\sigma_1^i, \sigma_2^i, \sigma_3^i)$, Eq. (42) becomes

$$v(\mathbf{a}) = K[\sigma_1^i(\mathbf{a}\cdot\mathbf{a}_1)^2 + \sigma_2^i(\mathbf{a}\cdot\mathbf{a}_2)^2 + \sigma_3^i(\mathbf{a}\cdot\mathbf{a}_3)^2]^{1/2}. \tag{43}$$

In particular, for $\mathbf{a} = \mathbf{a}_k$, we obtain

$$v_k = v(\mathbf{a}_k) = K(\sigma_k^i)^{1/2}.$$

In the general case (40) we obtain

$$v_k = v(\mathbf{a}_k) = K\left[\frac{\sigma_k^i\sigma_k^e}{\sigma_k^i + \sigma_k^e}\right]^{1/2}.$$

Since the choice of an orthogonal set of vectors in $\mathbb{R}^3$ depends on three degrees of freedom, then Eq. (43) determines the local basis A if $v(\mathbf{a})$ is given for three directions.

In the case of axial symmetry with

$$\sigma_1^i = \sigma_2^i = \sigma_t^i, \qquad \sigma_3^i = \sigma_l^i, \qquad \mathbf{a}_3 = \mathbf{a}_l$$

the unit vectors $\mathbf{a}_1, \mathbf{a}_2$ are defined apart from a rotation around $\mathbf{a}_l$. We can choose $\mathbf{a}_1$ coplanar with $\mathbf{a}_l$ and $\mathbf{n}$ (the unit normal to the isochrone passing through $\mathbf{x}$), i.e. $\mathbf{a}_1 = \mathbf{a}_t$; we have then only two degrees of freedom (for choosing $\mathbf{a}_l$) and consequently two values of $v(\mathbf{a})$ are sufficient to determine A.

If $\mathbf{x}$ and $\mathbf{x} + \mathbf{a}\,ds$ are respectively on the wavefronts S_t and S_{t+dt} then $v(\mathbf{a})$ is given by ds/dt. An obvious choice for one of the directions $\mathbf{a}$ is $\mathbf{a} = \mathbf{n}$, i.e. $v(\mathbf{n})$ is the usual normal conduction velocity of the front. Moreover $\mathbf{n}$ is needed in the axisymmetric case in order to reduce the degrees of freedom from three to two.

In summary, the oblique dipole layer model defined by Eq. (26) requires the geometry of the front, the fiber direction, the conductivity tensor and the intracellular action potential jump. We have shown that the fiber geometry can be recovered from the knowledge of the isochrones surfaces on the basis of Eq. (43) connecting the conduction velocity and the intracellular resistivity along a generic direction.

In this sense the forward problem of electrocardiology, i.e. the prediction of the body surface potential, using a representation of the depolarization wavefront in the myocardium by means of an oblique dipole layer, could be solved on the basis of the measured activation sequence if the intracellular action potential jump and, for axisymmetric fibers, the longitudinal and transverse conductivities σ_e^i, σ_t^i are known.

9. Numerical Computation of the Potential Field

We introduce in this section a numerical procedure for computing the potential field, generated by an oblique dipole layer on a wavefront S in a bounded and insulated region Ω, by solving the integral equation (13) described in Section 5.

We consider two polyhedral surfaces S^h and Γ^h with triangular elements which approximate S and Γ. Let n_S, n_Γ and N_S, N_Γ denote respectively the number of vertices and elements on S^h, Γ^h. The vertices are usually called also nodes and will be denoted with $\mathbf{x}_j$; the elements of Γ^h, S^h will be denoted respectively with E_j and F_j. Using the finite element technique the potential v on Γ, solution of Eq. (13), is approximated by a piecewise linear continuous function $v^h(\mathbf{x})$ on Γ^h, i.e.

$$v^h(\mathbf{x}) = \sum_{j=1}^{n_\Gamma} v_j p_j(\mathbf{x}),$$

where v_j are the nodal values of $v^h(\mathbf{x})$ and $p_j(\mathbf{x})$ are piecewise linear continuous functions on Γ^h defined by the property that $p_j(\mathbf{x}_k) = \delta_{jk}$ and δ_{jk} is the Kronecker symbol. It follows that $v^h(\mathbf{x}_k) = v_k$. For theoretical aspects of the numerical approximation of boundary integral methods by means of finite element techniques see Jaswon and Symm [12], Colli-Franzone and Magenes [8], Ciarlet [6], Nedelec [21]; see also Swihart [28] for other numerical methods.

We shall denote with $\mathbf{n}^h$ the outward unit normal to the triangular element of S^h and Γ^h. However, to simplify the notation, we shall write $\partial/\partial n$ instead of $\partial/\partial n^h$ and $\mathbf{n} \cdot \nabla$ instead of $\mathbf{n}^h \cdot \nabla$. Applying the collocation method, which consists in satisfying the integral equation at the nodes of Γ^h, integral equation (13) is approximated by the following linear system

$$\omega(\mathbf{x}_k) v_k + \frac{1}{4\pi} \int_{\Gamma^h} v^h \mathbf{n} \cdot \nabla r^{-1} \, d\Gamma^h = \frac{1}{4\pi} \int_{S^h} \mathbf{n}^T M \nabla r^{-1} \, dS^h$$

for $k = 1, 2, \ldots, n_\Gamma$ and $\mathbf{r} = \mathbf{x}_k - \mathbf{y}$, the factor $\omega(\mathbf{x}_k)$ is given by

$$\omega(\mathbf{x}_k) = \frac{\alpha(\mathbf{x}_k)}{4\pi} \qquad \text{and} \qquad \alpha(\mathbf{x}) = -\int_{\Gamma^h} \mathbf{n} \cdot \nabla r^{-1} \, d\Gamma,$$

i.e. $\alpha(\mathbf{x})$ represents the solid angle under which the surface Γ^h is seen from the point $\mathbf{x}$. Now it is

$$\int_{\Gamma^h} v^h \mathbf{n} \cdot \nabla r^{-1} \, d\Gamma^h = \sum_{j=1}^{n_\Gamma} v_j \int_{\Gamma^h} p_j \mathbf{n} \cdot \nabla r^{-1} \, d\Gamma^h.$$

Setting

$$a_{kj} = \omega(\mathbf{x}_k) \delta_{kj} + \frac{1}{4\pi} \int_{\Gamma^h} p_j \mathbf{n} \cdot \nabla r^{-1} \, d\Gamma^h,$$

$$b_k = \frac{1}{4\pi} \int_{S^h} \mathbf{n}^T M \nabla r^{-1} \, dS^h \qquad \text{for} \qquad j, k = 1, 2, \ldots, n_\Gamma$$

and

$$A = (a_{kj}), \qquad \mathbf{b} = (b_1, \ldots, b_{n_\Gamma})^T, \qquad \mathbf{v} = (v_1, \ldots, v_{n_\Gamma})^T$$

the determination of the approximate solution $v^h(\mathbf{x})$ is equivalent to the solution of the linear system

$$A\mathbf{v} = \mathbf{b}. \tag{43}$$

With $\omega(\mathbf{x})$ defined as above, matrix A admits the right eigenvector $\mathbf{e} = (1, 1, \ldots, 1)^T$ associated to the zero eigenvalue, i.e. $A\mathbf{e} = 0$. For the continuous operator, of which A is a discrete approximation, zero is a simple eigenvalue and we assume that this property still holds for A. We denote with $\boldsymbol{l}$ the corresponding left eigenvector, i.e. $\boldsymbol{l}^T A = 0$.

It follows that the solution of the linear system (43) is defined up to an additive constant vector and the system has a solution if and only if it is

$$\boldsymbol{l}^T \mathbf{b} = 0, \tag{44}$$

i.e. if and only if $\mathbf{b}$ is orthogonal to the left eigenvector $\boldsymbol{l}$. This constitutes the compatibility condition for the solution of system (43).

In the continuous case the corresponding compatibility condition given by Eq. (19) is exactly satisfied; however in the discrete case, because of the approximation errors, the compatibility condition (44) is satisfied with a small error but not exactly. The vector $\mathbf{v}$ which defines the approximation $v_h(\mathbf{x})$ is determined in the following way. Let us consider the system

$$A\mathbf{v}^* = \mathbf{c},$$

where

$$\mathbf{c} = \mathbf{b} + \boldsymbol{\varepsilon} \qquad \text{and} \qquad \boldsymbol{l}^T \mathbf{c} = 0,$$

i.e. $\mathbf{c}$ satisfies the compatibility condition. It can be proved that if $\mathbf{p}$ is such the $\mathbf{p}^T\mathbf{e} \neq 0$ (e.g. $\mathbf{p}^T\mathbf{e} = 1$) then, since $\boldsymbol{l}^T\mathbf{e} \neq 0$, the matrix

$$B = A + \mathbf{e}\mathbf{p}^T$$

is non-singular and the solution of the linear system

$$B\mathbf{v}^* = \mathbf{c}$$

is also solution of the linear system

$$A\mathbf{v}^* = \mathbf{c}.$$

Moreover this solution is orthogonal to $\mathbf{p}$ that is

$$\mathbf{p}^T\mathbf{v}^* = 0.$$

This method for solving a singular system, given the right eigenvectors associated to the zero eigenvalue, is an application of the "deflation method" (see, e.g., Lynn and Timlake [14]). If, for instance, $\mathbf{p}$ is the vector whose components are all zero, except the kth component which is equal to 1, then the kth component of $\mathbf{v}^*$ is equal to zero.

Since we do not know $\mathbf{c}$, we solve instead the linear system

$$B\mathbf{v} = \mathbf{b}.$$

Thus, $\mathbf{v}$ defines an approximation $v_h(\mathbf{x})$ corresponding to that solution $v(\mathbf{x})$ of the continuous integral equation (13), subject to the constraint $v(\mathbf{x}_k) = 0$.

With reference to the computation of a_{kj}, b_k we first remark that

$$\int_{\Gamma^h} = \sum_{l=1}^{N_\Gamma} \int_{E_l} \qquad \text{and} \qquad \int_{S^h} = \sum_{l=1}^{N_S} \int_{F_l}.$$

Actually in the case of the first integral, for fixed j, the sum is restricted to those elements which have in common the node $\mathbf{x}_j$ since on all the other elements it is $p_j \equiv 0$.

The most efficient way to calculate the coefficients a_{kj} for k fixed is to scan sequentially the elements of Γ^h and, for the current element E, to compute the integrals of the form

$$\int_E p(\mathbf{y}) \frac{\partial}{\partial n_y} |\mathbf{x}_k - \mathbf{y}|^{-1} \, dE,$$

where p is the linear function defined on E and taking the value 1 on a vertex and zero on the remaining vertices. These integrals constitute partial contributions to those coefficients a_{kj} where j is associated to the vertices of the element E. The coefficients are built by adding these contributions.

For $\mathbf{x}_k$ belonging to E the integral is singular; in this case and also when $\mathbf{x}_k$ is very near to E the integral is computed analytically; otherwise the integral is computed by means of Gaussian quadrature formulae. Similar considerations apply to the computation of b_k. We remark now that for $\mathbf{x}_k$ belonging to Γ^h the matrix A and consequently the matrix B depends upon the geometry of Γ^h and that $\mathbf{b}$ depends on the geometry of Γ^h, S^h and also on M.

When investigating the dependence of v upon m_l, m_t it is better, in order to reduce the amount of computation, to calculate the inverse B^{-1} and to split $\mathbf{b}$ as indicated below. It is

$$M\mathbf{n} = m_t\mathbf{n} + (m_l - m_t)(\mathbf{n} \cdot \mathbf{a}_l)\mathbf{a}_l,$$

where $\mathbf{a}_l$ is the unit vector parallel to the local direction of the fiber. Hence

$$\int_{S^h} \mathbf{n}^T \mathbf{M} \nabla r^{-1} \, dS^h = \int_{S^h} m_t \mathbf{n}^T \nabla r^{-1} \, dS^h + \int_{S^h} (m_l - m_t)(\mathbf{n}^T \mathbf{a}_l)\mathbf{a}_l^T r^{-1} \, dS^h.$$

Under the supplementary assumption that m_l, m_t are constant we set

$$\mathbf{b}_n = \frac{1}{4\pi} \int_{S^h} \mathbf{n}^T \nabla r^{-1} \, dS^h,$$

$$\mathbf{b}_a = \frac{1}{4\pi} \int_{S^h} (\mathbf{n}^T \mathbf{a}_l)\mathbf{a}_l^T \nabla r^{-1} \, dS^h,$$

where the two integrals do not depend on the parameters m_l, m_t and kth components of $\mathbf{b}_n$ and $\mathbf{b}_a$ are related to $\mathbf{x}_k$, then we have

$$\mathbf{b} = [m_t\mathbf{b}_n + (m_l - m_t)\mathbf{b}_a]/\sigma_0.$$

Setting

$$\mathbf{v}_n = B^{-1}\mathbf{b}_n, \qquad \mathbf{v}_a = B^{-1}\mathbf{b}_a,$$

we have

$$\mathbf{v} = [m_t\mathbf{v}_n + (m_l - m_t)\mathbf{v}_a]/\sigma_0.$$

In this way it is possible to evaluate the influence of $\mathbf{v}_n$ and $\mathbf{v}_a$ on $\mathbf{v}$. Given B^{-1} and

the integrals associated with $\mathbf{b}_n$ and $\mathbf{b}_a$, we can easily calculate the corresponding approximate solutions v_n^h, v_a^h and v^h on Γ^h for different choices of m_l, m_t. The potential U^h for any point $\mathbf{x}$ inside Γ^h but not on Γ^h or S^h is given by

$$U^h(\mathbf{x}) = [m_t U_n^h(\mathbf{x}) + (m_l - m_t)U_a^h(\mathbf{x})]/\sigma_0,$$

where

$$U_n^h(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Gamma^h} v_n^h \mathbf{n}^T \nabla r^{-1}\, d\Gamma^h + \frac{1}{4\pi}\int_{S^h} \mathbf{n}^T \nabla r^{-1}\, dS^h$$

and

$$U_a^h(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Gamma^h} v_a^h \mathbf{n}^T \nabla r^{-1}\, d\Gamma^h + \frac{1}{4\pi}\int_{S^h} (\mathbf{n}^T \mathbf{a}_l)\mathbf{a}_l^T \nabla r^{-1}\, dS^h.$$

The integrals on Γ^h, S^h are the sums of the corresponding integrals on the elements E of Γ^h and F of S^h. The integration is carried out analytically when $\mathbf{x}$ is very near to an element of Γ^h or S^h and numerically otherwise. Given Γ^h and S^h, U_n^h and U_a^h are numerically computed separately by means of the methods illustrated above.

10. Conclusions

In electrocardiology there occur events which are at variance with the predictions of the classical solid angle theory, i.e. the model of the uniform and normal double layer. In fact, experiments reported in [7] have shown that a closed excitation wavefront can generate a non-uniform potential field and that, ahead of an open excitation wavefront propagating in a direction transverse to the fibers, a re-entrant current flow, i.e. directed from the resting tissue to the front, may be observed. According to the solid angle theory, one would expect zero potential in the first case and outflowing current in the second.

These facts have motivated the introduction of the oblique dipole layer concept. The models we have developed and which are generalizations of the axial model of Corbin and Scher, make the equivalent source generators on the wavefront dependent on the fiber orientation and can adequately reproduce the potential pattern for closed and open wavefronts as reported in [7]. Moreover one of our local models agrees with another experimental finding related to a variable potential jump across the wavefront, since it predicts a potential jump similar to the one studied by Roberts et al. [25]. We have also established a connection between the oblique dipole layer structure and an intracellular current model, thus providing a more adequate electrophysiological foundation for the oblique dipole concept.

In general, an oblique dipole layer model requires the knowledge of the properties of the anisotropic heart tissue, the wavefront position and the fiber geometry. We have investigated under which assumptions a knowledge of the activation sequence, i.e. of the isochrone surfaces, may yield information about the fiber geometry.

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