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Infinite subharmonic bifurcation in an SEIR epidemic model

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Abstract. The existence of both periodic and aperiodic behavior in recurrent epidemics is now well-documented. In this paper, it is proven that for epidemic models that incur permanent immunity with seasonal variations in the contact rate, there exists an infinite number of stable subharmonic solutions. Random effects in the environment could perturb the state of the dynamics from the domain of attraction from one subharmonic to another, thus producing aperiodic levels of incidence.

Key words: Epidemic modelling — Measles — Subharmonic bifurcation — Infectious diseases — Chaos — Mathematical modelling

Introduction

Prior to the introduction of vaccines, several infectious diseases such as measles, chicken pox, mumps and poliomyelitis were both endemic and exhibited oscillatory levels of incidence in large population centers of the US. Data from New York City for the period between 1948 and 1964 show annual outbreaks of chicken pox and mumps. Measles, however, exhibits a biennial cycle; i.e. alternating years of high and low incidence [17], [24]. Anderson and May [2] report data from England and Wales showing an average four year cycle in poliomyelitis incidence.

Although measles has outbreaks of varying size in late winter and early spring, both longer and shorter periods have been observed. Emerson [9] observes a pattern of measles outbreaks of approximately three years in Baltimore. Furthermore, from 1928–1958, large outbreaks of measles occurred every first, second, or third year with no apparent pattern [17].

Mathematical models of the spread of an infectious disease in large populations have been based on the law of mass action whereby the rate at which the subpopulation of susceptible individuals become infected is assumed to be

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proportional to the product of the fraction of susceptible individuals and the number of infectious persons in the population. The constant of proportionality in the mass action relation is usually referred to as the contact or transmission rate of the infection, or the coefficient of infectivity. Deterministic mathematical models of diseases that incur permanent immunity typically predict damped oscillations about some endemic level when the contact rate (as well as other parameters) is held constant in time [3, 7, 12, 17, 23]. However, as stated above, the observations of incidence point to the contrary. London and Yorke [17] attempt to infer the contact rate from existing data. Their work demonstrates that contact rates vary seasonally for chicken pox, mumps, and measles. More recently, Fine and Clarkson [10] give a more detailed analysis of the contact rate for measles in England and Wales. They find that while measles oscillates biennially, the contact rate varies annually with a high correlation to the opening and closing of school terms. (See also [24].)

London and Yorke [17] formulated several mathematical models in which they used their calculated contact rates to simulate yearly outbreaks of chicken pox and mumps and the biennial pattern of measles. One of their models involves differential delay equations which, in general, are difficult to analyze and consequently their work consists primarily of numerical simulations. Very little mathematical analysis has been done on the London–Yorke differential-delay model.

Both London and Yorke [17] and Dietz [7] introduced mathematical models involving ordinary differential equations in which the contact rate was assumed to vary seasonally. This class of models has proved more amenable to mathematical analysis. Numerical simulations of the model in [7] (in which the latent period was ignored) indicate the possible existence of periodic solutions having periods one and two for suitable values of the parameters. Longer periods of 3, 4, and 6 years are conjectured. In Schwartz [20], periods of one, two, and three were found numerically to coexist for measles parameters in models incorporating latency.

Grossman, Gumowski, and Dietz [11] formally obtained periodic solutions of period two years by using perturbation methods. (See also Grossman [12] where delays are introduced.) Smith gave a rigorous proof of the existence of period two year solutions of the Dietz model in [21]. In later work [22] it was shown that periodic solutions of period n years exist for many values of n and are simultaneously stable. All the work on the Dietz model indicated above ignores the latent period during which the infected individual is not yet infectious. However for most diseases and certainly for chicken pox, mumps and measles, the latent period is typically as long as the period of infectiousness or even longer [2, 17]. For example, for measles the infectious period is between 3–4 days while the latent period is 10–12 days.

In this paper we extend the results in [22] to the more general Dietz model with periodic contact rate which includes a latent period [7]. In particular, stable subharmonic solutions of period n for many values of n are proved to coexist simultaneously. Numerical calculations for parameter values relevant to measles suggest that in addition to the period two subharmonic there exist subharmonics of period n, n an integer greater than 2.

The significance of multiple stable subharmonics of various periods may lie in the possibility that random effects in the environment could perturb the state of the system from the domain of attraction of one subharmonic to that of another, the effect of which could be to produce aperiodic looking levels of incidence. In fact, data in [17] and [9] suggest that measles has not always exhibited a regular biennial pattern. Moreover, the structure of long term outbreaks is such that it is possible to have a very low number of cases for several years, followed by a large increase in the number of cases over a time interval of approximately one year.

In Sect. 1 the main results are described and their significance discussed. The proof of the main theorem is outlined in Appendix A. Section 2 contains the numerical results of our computations of subharmonic solutions. Included is a discussion of the stability of the orbits as well as a discussion of the dependence of solutions on the forcing amplitude of the contact rate.

1. The model and main results

Assume a given population may be divided into the following categories:
Susceptibles — those capable of contracting the disease
Infective — those capable of transmitting the disease
Exposed — those who are infected but not yet infectious
Recovered — those who are immune.
We follow Dietz [7] in making the following assumptions:

1. The population consists of susceptibles, infected but not yet infectious, infectious, and immunes. The population size is constant.

2. The disease is not lethal; equal and constant birth and death rates μ are assumed.

3. The population is homogeneous and uniformly mixing.

The contact rate, β, is defined as the average number of effective contacts with other individuals per infective per unit time. An exposure or effective contact of a susceptible by an infective is an encounter in which the infection is transmitted.

4. An exposed individual's probability of becoming infectious in a specified time interval is independent of time after initial contact; hence the probability of still remaining in the exposed class at time τ after initial contact is $e^{-\alpha\tau}$, where $1/\alpha$ is the mean latent period.

5. After an individual enters the infectious class, the probability of that individual recovering at time τ is given by $e^{-\gamma\tau}$, where $1/\gamma$ is the mean infectious period, and γ is called the recovery rate.

6. Recovered individuals are permanently immune.

Letting S, E, I, R denote respectively the fraction of the population which is susceptible, exposed, infectious, and recovered, the above assumptions lead to the following system of ordinary differential equations:

$$S' = \mu - \mu S - \beta IS \tag{1.1}$$

$$E' = \beta IS - (\mu + \alpha)E$$

$$I' = \alpha E - (\mu + \gamma)I$$

$$R' = \gamma I - \mu R.$$

Notice that the first three equations are sufficient for the description of the model since $S+E+I+R=1$.

We begin by reviewing what is known about system (1.1) when β is constant (see [6]). There are two steady states: $(S, E, I, R) = (1, 0, 0, 0)$ and $(S, E, I, R) = (S_0, E_0, I_0, R_0)$ where

$$S_0 = \frac{(\mu+\alpha)(\mu+\gamma)}{\beta\alpha} \tag{1.2}$$

$$E_0 = \frac{\mu+\gamma}{\alpha} I_0$$

$$I_0 = \frac{\mu(Q-1)}{\beta}.$$

The latter steady state lies in the positive octant provided

$$Q \equiv \frac{\beta\alpha}{(\mu+\alpha)(\mu+\gamma)} > 1. \tag{1.3}$$

Steady state (1.2) is usually called the endemic steady state. Q is biologically interpreted as the reproductive rate for the infection; i.e. the number of secondary cases produced by a single infectious individual in a population of susceptibles in one infectious period. Therefore an infectious disease can be endemic only if the reproduction rate exceeds unity.

It is not difficult to show by the Routh-Hurwitz test [6] that when (1.3) holds the endemic steady state is asymptotically stable, and the trivial steady state is unstable. The question of whether the endemic steady state is globally stable for all initial conditions in the interior of the first octant appears to be an open question [15]. It will be assumed throughout that (1.3) holds. Estimated values of Q for mumps, chicken pox and measles are roughly 7, 9, and 16 respectively [2].

We are interested in the case that the contact rate is periodic of period one year:

$$\beta(t) = \beta_0(1 + \delta \cos 2\pi t). \tag{1.4}$$

For this reason it is convenient to take a year as our unit of time.[1] With this convention, the birth and death rate, μ, will be such that $1/\mu$ is 50 years, the average latent period prior to becoming infections, $1/(\mu+\alpha)$, is typically a few days to a week, the same is true for the average infectious period $1/(\mu+\gamma)$. We exploit the fact that μ, $1/\mu+\alpha$, $1/\mu+\gamma$ are $0(10^{-2})$ by introducing a small parameter ε as follows:

$$\beta_0 I_0 = \mu(Q-1) = \varepsilon \tag{1.6}$$

[1] Although the periodic term in the contact rate is idealized, it approximates the (monthly) measles contact rate computed in (17) for New York City data. However, it does not approximate as well the data (weekly) in (10), although their data do appear to be seasonal.

$$\mu+\alpha=\frac{\Delta_2}{\varepsilon},\qquad 0<\varepsilon\ll 1.$$
$$\mu+\gamma=\frac{\Delta_3}{\varepsilon},$$

New variables (x, y, z) are introduced by setting

$$S=S_0(1+x),\ E=E_0(1+y),\ I=I_0(1+z). \tag{1.7}$$

Incorporating (1.4), (1.6) and (1.7) in (1.1) yields the following system of equations which has the property that when $\delta=0$, the endemic equilibrium becomes $(x, y, z)=0$:

$$\begin{aligned} x'&=-\varepsilon[(\eta+\delta\cos 2\pi t)x+(1+\delta\cos 2\pi t)z+\delta\cos 2\pi t\\ &\quad+xz(1+\delta\cos 2\pi t)]\\ y'&=\frac{\Delta_2}{\varepsilon}[\delta\cos 2\pi t+x(1+\delta\cos 2\pi t)+z(1+\delta\cos 2\pi t)\\ &\quad-y+xz(1+\delta\cos 2\pi t)]\\ z'&=\frac{\Delta_3}{\varepsilon}[y-z]\quad\text{where }\eta\equiv\frac{Q}{Q-1}>1. \end{aligned} \tag{1.8}$$

Before proceeding further in analyzing the periodic system (1.8) we need information concerning the eigenvalues of the linearized system about the endemic steady state when $\delta=0$. For $\delta=0$, the endemic steady state is the origin and we have the following lemma, the proof of which will be left to the reader as it involves routine algebra and a straightforward application of the Implicit Function Theorem.

Lemma 1.1. *The eigenvalues corresponding to the linearized system* $(\delta=0)$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}'=\begin{pmatrix}-\varepsilon\eta & 0 & -\varepsilon\\ \Delta_2/\varepsilon & -\Delta_2/\varepsilon & \Delta_2/\varepsilon\\ 0 & \Delta_3/\varepsilon & -\Delta_3/\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

are given by λ_+, λ_-, λ_3 *below*: $\lambda_\pm(\varepsilon)=\varepsilon r\pm i\nu+0(\varepsilon^2)$, *where*

$$\nu=\sqrt{\frac{\Delta_2\Delta_3}{\Delta_2+\Delta_3}}=\sqrt{\frac{\mu[Q-1]}{1/(\mu+\alpha)+1/(\mu+\gamma)}} \tag{1.9}$$
$$r=\frac{\Delta_2\Delta_3-(\Delta_3+\Delta_2)^2\eta}{2(\Delta_3+\Delta_2)^2}<0$$
$$\lambda_3(\varepsilon)=-\frac{(\Delta_2+\Delta_3)}{\varepsilon}+0(\varepsilon).$$

Observe that the endemic steady state is asymptotically stable but the attraction is weak. There is a rapid relaxation onto a center manifold on which orbits slowly spiral into the origin. The underlying mechanism we will exploit is the weakly damped oscillation on the center manifold. To leading order, the frequency of this oscillation, ν, depends on the reproduction rate, Q, the birth rate, μ, and the sum of the latent and infectious periods [2, 7].

Our aim at this point is to make a change of variables in (1.8) so that when $\delta = 0$ the resulting linear part is close to its Jordan form. In addition, further analysis will require the assumption of a small amplitude periodic perturbation in (1.8). Inspection of (1.8) shows that this will be the case only if δ/ε is small. With these remarks as motivation we proceed as follows. Let

$$\begin{aligned}
\bar{\delta} &= \delta/\varepsilon \\
\bar{x} &= \nu\left[\frac{x}{\varepsilon} - \varepsilon\frac{\Delta_3}{(\Delta_2+\Delta_3)^2}(z-y)\right] \\
\bar{y} &= \frac{\Delta_3 y + \Delta_2 z}{\Delta_2+\Delta_3} \\
\bar{z} &= z - y,
\end{aligned} \tag{1.10}$$

where ν is given in Lemma 1.1. While this transformation appears to mix up the various epidemiological classes, a glance at (1.8) indicates that one should expect $z-y$ to be small, say order ε, hence (1.10) should take the particularly simple form

$$\begin{aligned}
x &= \frac{\varepsilon\bar{x}}{\nu} + 0(\varepsilon^3) \\
y &= \bar{y} + 0(\varepsilon) \\
z &= \bar{y} - 0(\varepsilon).
\end{aligned} \tag{1.11}$$

In fact we will show later that this is the case. Epidemiologically, Eq. (1.11) has the interpretation that the ratio of infectives to exposed individuals is γ/α to order ε.

Putting (1.10) in (1.8) and ignoring the bar over δ we obtain

$$\begin{aligned}
\bar{x}' &= -\nu\bar{y} + \varepsilon f_1(\bar{x}, \bar{y}, \bar{z}, t, \varepsilon, \delta) \\
\bar{y}' &= \nu\bar{x}(1+\bar{y}) + \nu\frac{\Delta_3}{\Delta_2+\Delta_3}\bar{x}\bar{z} \\
&\quad + \nu^2\delta\cos 2\pi t\left(1+\bar{y}+\frac{\Delta_3\bar{z}}{\Delta_2+\Delta_3}\right) \\
&\quad + \varepsilon f_2(\bar{x}, \bar{y}, \bar{z}, t, \varepsilon, \delta) \\
\varepsilon\bar{z}' &= -(\Delta_2+\Delta_3)\bar{z} + \varepsilon f_3(\bar{x}, \bar{y}, \bar{z}, t, \varepsilon, \delta),
\end{aligned} \tag{1.12}$$

where

$$\begin{aligned}
&f_i(x, y, z, t+1, \varepsilon, \delta) = f_i(x, y, z, t, \varepsilon, \delta) \\
&f_i(0, 0, 0, t, 0, 0) = 0, \qquad i = 1, 2, 3.
\end{aligned}$$

The f_i are explicitly written out in the appendix (see (A.2)).

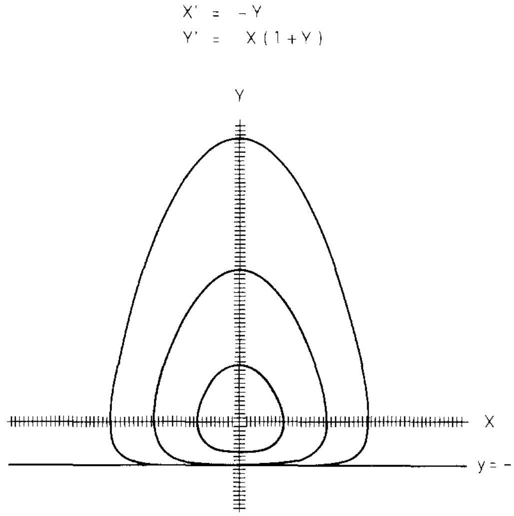

Fig. 1. A plot of the phase portrait of the case when $\varepsilon = \delta = 0$ using Eq. (1.13)

We will treat ε and δ as small parameters in (1.12). Setting $\varepsilon = \delta = 0$ in (1.12) we obtain the reduced equations

$$\begin{aligned} \bar{x}' &= -\nu\bar{y} \\ \bar{y}' &= \nu\bar{x}(1+\bar{y}) \\ \bar{z} &= 0. \end{aligned} \tag{1.13}$$

This reduced two-dimensional system is precisely the one encountered in some earlier work on a related $S \to I \to R$ model [21, 22]. The system (1.13) is conservative with first integral given by $V = x^2 + 2y - 2\ln|1+y|$. In Fig. 1 above we give the phase portrait of (1.13).

It is interesting to observe that if we write $u = \ln(1+y)$ and use (1.13) to write a second order differential equation for u we obtain $u'' + \nu^2(e^u - 1) = 0$. This equation was first derived over fifty years ago by H. E. Soper [15] as the limiting case of a difference equation model which he derived for the spread of an infection.

The essential feature of Fig. 1 is that the origin is a center surrounded by periodic orbits with periods ranging between $2\pi/\nu$ near the origin to $+\infty$ near the invariant line $y = -1$. In particular, for every integer n, $2\pi/\nu < n < \infty$, there is a periodic solution of (1.13), $(\bar{x}_n(t), \bar{y}_n(t))$, of least period n. These solutions of the reduced equation may be excited by period one forcing at the correct amplitude. Hence we are led to expect n-periodic solutions of (1.12) near $\Gamma_n = \{(x, y, z)\colon (x, y, z) = (\bar{x}_n(t), \bar{y}_n(t), 0), 0 \leq t \leq n\}$ at least for suitable small values of ε and δ. To illustrate this, we state the following theorem which is proved in Appendix A.

Theorem 1.2. *Let $(\bar{x}_n(t), \bar{y}_n(t) = (\bar{x}_n(t+n), \bar{y}_n(t+n)$ denote a periodic solution of the reduced equation* (1.13), *where $n > 2\pi/\nu$. Let*

$$\gamma_2 \equiv \nu^2 \int_0^n \bar{y}_n(t)\cos 2\pi t\,dt \neq 0 \tag{1.14}$$

$$\gamma_1 \equiv \frac{-2r}{\nu} \qquad \text{(Area interior to } \Gamma_n\text{)},$$

and for $\alpha \varepsilon\ [0, n)$, $|\varepsilon| \ll 1, |\delta| \ll 1$, *let*

$$B(\alpha, \varepsilon, \delta) = -\gamma_1 \varepsilon + \gamma_2 \delta \cos 2\pi\alpha + 0(|\varepsilon| + |\delta|)^2. \tag{1.15}$$

If $(\bar{x}_n', \bar{y}_n')$ spans the n-periodic solutions of the variational equations of (1.13) about $(\bar{x}_n, \bar{y}_n)$, *and* $(\alpha, \varepsilon, \delta)$ is such that $B(\alpha, \varepsilon, \delta) = 0$, then equation (1.12) has an n-periodic solution $(\bar{x}, \bar{y}, \bar{z})$ given by

$$\bar{x}(t) = \bar{x}_n(t+\alpha) + 0(|\varepsilon|(1+|\delta|)) \tag{1.16}$$

$$\bar{y}(t) = \bar{y}_n(t+\alpha) + 0(|\varepsilon| + |\delta|)$$

$$\bar{z}(t) = -\frac{\varepsilon \Delta_2}{\nu^2(\Delta_2 + \Delta_3)} \bar{y}'(t+\alpha) + 0(|\varepsilon| + |\delta|)^2.$$

The requirement that $\gamma_2 \neq 0$ restricts the set of integers $n > 2\pi/\nu$ for which the *method* yields n-periodic solutions (1.16) (see Table 1 for some computed values). In [22] it was shown that there is an infinite subset, P, of the positive integers containing 1, 2, and 3 such that if $n \in P$ there is an open dense set of ν, $2\pi/\nu < n$, for which both $\gamma_2 \neq 0$ and the nondegeneracy conditions on the variational equation are satisfied.

The approach we have taken to find subharmonic oscillations follows closely recent work of J. Hale and P. Taboas [14] and Hale, S. Chow and J. Mallet-Paret [5] (See also [22]) where we encountered Eq. (1.15) in a similar context. Since the following description of the solution of (1.15) appears in [22] only a brief account will be given. The Equation $B(\alpha, \varepsilon, \delta) = 0$ is the so-called bifurcation equation (see the appendix for more detail). It is to be solved for the phase angle $\alpha \in [0, n)$ for small (ε, δ). In [22] we show that there are two smooth curves described by functions $\varepsilon_+(\delta)$ and $\varepsilon_-(\delta)$ for small δ, $\varepsilon_\pm(0) = 0$ and these curves are tangent at the origin to the lines $\varepsilon = \pm(\gamma_2/\gamma_1)\delta$, respectively. The two curves separate a neighborhood of the origin in the (ε, δ) plane into two regions as depicted in Fig. 2.

Referring to Fig. 2, for (ε, δ) in region S^c there are $2n$ solutions α of (1.15) corresponding to $2n$ periodic solutions of period n of the form (1.16). More precisely, for $(\varepsilon, \delta) \in S^c$ there exists $\alpha_i \in [0, 1)$, $i = 1, 2$, such that $(\alpha_i + j, \varepsilon, \delta)$ solves (1.15) for $j = 0, 1, \ldots, n-1$, $i = 1, 2$. The periodic solutions (1.16) corresponding to $\alpha_1 + j$ are just translations by $t \to t + j$ of a single n periodic solution, similarly for α_2. If we identify these translates then there are really just 2 distinct n periodic solutions of the form (1.16) for (ε, δ) in S^c. One solution is a saddle (unstable with one dimensional stable manifold) and the other is a node which is stable for $\varepsilon > 0$ and unstable for $\varepsilon < 0$ (see [22] or [5]). In region S there are no n-periodic solutions near Γ_n. Except for the assertions concerning stability, most of the above description of solutions of $B = 0$ is easy to see by simply ignoring $0(|\varepsilon| + |\delta|)^2$ in (1.15) and solving.

Let us examine the implications of Theorem 1.2 and the preceding discussion to Eq. (1.1) in terms of the original variables S, E, I, R and the original parameter δ appearing in (1.4). Recalling (1.11), we find that for those n for which $\gamma_2 \neq 0$, if $2\pi/\nu < n$, there is a stable periodic solution of period n of (1.1) (with β given by (1.4)) of the form

$$S = S_0 + \frac{\varepsilon}{\nu}\bar{x}_n(t+\alpha) + 0(\varepsilon^2 + \varepsilon|\delta|) \tag{1.17}$$

$$E = E_0\left[1 + \bar{y}_n(t+\alpha) + 0\left(\varepsilon + \frac{|\delta|}{\varepsilon}\right)\right]$$

$$I = I_0\left[1 + \bar{y}_n(t+\alpha) + 0\left(\varepsilon + \frac{|\delta|}{\varepsilon}\right)\right].$$

In (1.17), $\varepsilon = \beta_0 I_0$ is positive and recall from (1.2) that $S_0 = 1/Q$, $I_0 = 0(\varepsilon^2)$ and $E_0 = 0(\varepsilon^2)$, the latter estimates follow from $S_0 = 0(1)$. Assuming δ is positive, (1.17) is valid for $0 < \varepsilon$, $\delta/\varepsilon < \rho$ for some sufficiently small ρ (depending on n) and only if $\varepsilon < \varepsilon_+(\delta/\varepsilon)$. Recalling the fact that $\varepsilon = \bar{\delta} \equiv \delta/\varepsilon = 0$, the requirement that $\varepsilon < \varepsilon_+(\delta/\varepsilon)$ can be crudely expressed as $\delta > (\gamma_1/\gamma_2)\ \varepsilon^2$ (we have approximated ε_+ by $(\gamma_2/\gamma_1)\ \delta/\varepsilon$). Hence (1.17) is valid only if the amplitude δ exceeds a threshold value given approximately by $(\gamma_1/\gamma_2)\ \varepsilon^2$. Of course, the requirement that $0 < \varepsilon$, $\delta/\varepsilon < \rho$ for some unknown number ρ is the usual price to be paid for employing Implicit Function Theorem techniques.

The ratio γ_1/γ_2 is a rough measure of the size of the parameter region in which the solution (1.17) of period n exists. If γ_1/γ_2 is small the region is large and the solution is likely to be observed while if γ_1/γ_2 is large the region is small and it may be difficult to observe the solution. In Table 1 we display some computations of the ratio γ_1/γ_2 for the subharmonics of period n for parameter values relevant to measles listed in Table 2 below. (Notice that $2\pi/\nu \in (2, 3)$.)

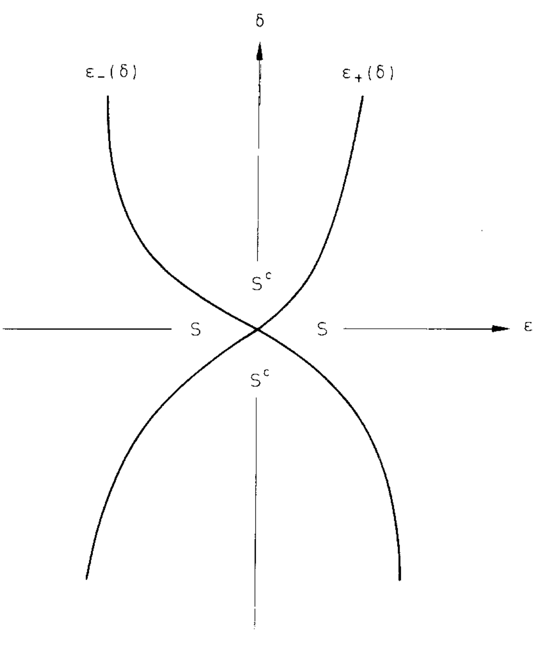

Fig. 2. A schematic of the local region in (ε, δ) space is depicted. In the region S^c, there are $2n$ solutions of the bifurcation equation 1.15. See text for further details

Table 1. γ_1/γ_2 vs. period (n)

n	3	4	5
γ^1/γ^2	0.79	1.27	1.94

Notice that lower order subharmonics are more likely to be observed; i.e. the domains of existence for lower order subharmonics may be larger than for higher order subharmonics.

2. Numerical simulations

Here we describe the behavior of the periodic solutions of equation (1.8) as a function of the forcing amplitude, δ. First, we restrict our analysis to that of measles. The parameters for measles are listed in Table 2. (See London and Yorke [17], Schwartz [19], Yorke et al. [20].)

Since $\varepsilon = \mu(Q-1)$, if we fix the reproductive rate of infection, Q, ε is fixed. We take δ, the forcing amplitude in Eq. (1.4), to be our variable parameter. The techniques used for computing periodic orbits are presented in Appendix B, along with an analysis of how we compute the Floquet multipliers.

When $\delta = 0$, all solutions are seen to exhibit damped oscillations that converge to the origin as t approaches infinity. If δ is positive but small (say less than 0.075), periodic orbits having period 1 bifurcate from the origin. Figure 3 is a projection onto the $x-z$ plane of periodic orbits having period 1 for several values of δ between 0.0 and 0.15.[2]

To plot the norm of a periodic solution as a function of δ we let $\phi^\delta(\cdot\,; 0, x_u)$ denote the solution of Eq. (1.8) such that $\phi^\delta(0; 0, x_0) = x_0$ and $\phi^\delta(T; 0, x_0) = x_0$; i.e. ϕ^δ is a periodic orbit having period T passing through x_0 at $t = 0$ for a fixed value of δ. We define the norm of ϕ^δ as

$$\|\phi^\delta\| = \left\{\frac{1}{T}\int_0^T \phi^\delta(t; 0, x_0)\cdot\phi^\delta(t; 0, x_0)\,dt\right\}^{1/2},$$

where the $(\cdot)$ denotes the usual l_2 inner product. Figure 4 depicts the norm of ϕ as a monotonic increasing function of δ. It is found numerically that at a value

Table 2.

$\mu = 0.02\ (\text{year})^{-1}$
$\alpha = 1/0.0279\ (\text{year})^{-1}$
$\gamma = 1/0.01\ (\text{year})^{-1}$
$\beta_0 = 1575.0\ (\text{year})^{-1}$
$Q \cong 15.73807$
$\varepsilon \cong 0.29476$

[2] The smooth orbit appears as a polygon due to choice of step size in plotting.

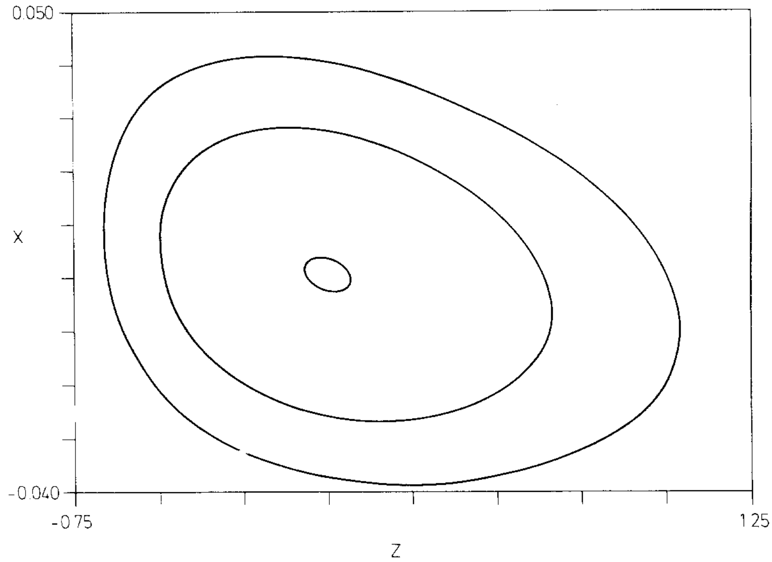

Fig. 3. A plot of periodic orbits of period 1 projected onto the Z–X plane. The parameter ε was fixed at 0.29476137 while δ was varied

of $\delta_2 \cong 0.11479$ one of the values of the Floquet multipliers is -1, signalling the presence of a period doubling orbit bifurcating off the period 1 branch. It is further observed that if one continues to follow the period 1 solutions for $\delta > \delta_2$, the period 1 solutions become unstable. Figure 5a depicts the stable branch of

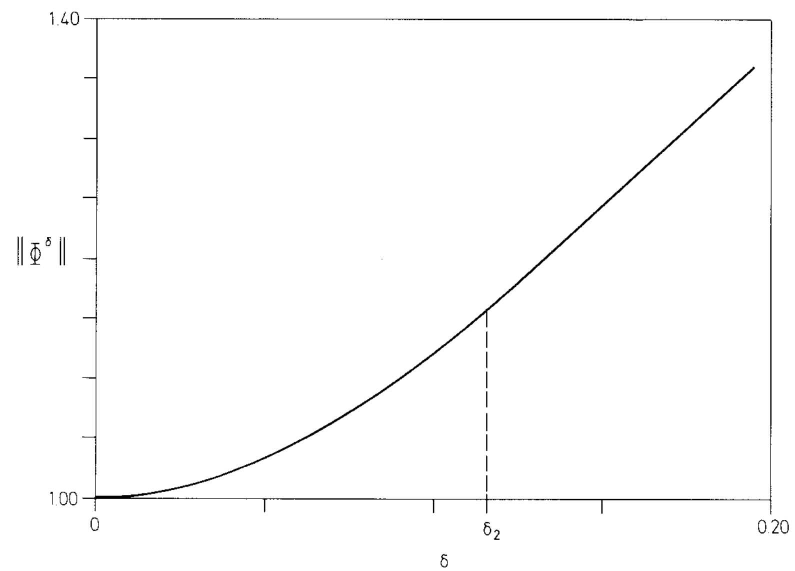

Fig. 4. A plot of the norm of the period 1 orbits as a function of δ illustrates the monotonic behavior of the norm of the solution. For values of $\delta < \delta_2$, the period 1 orbits are stable, while for $\delta > \delta_2$, the orbits are unstable

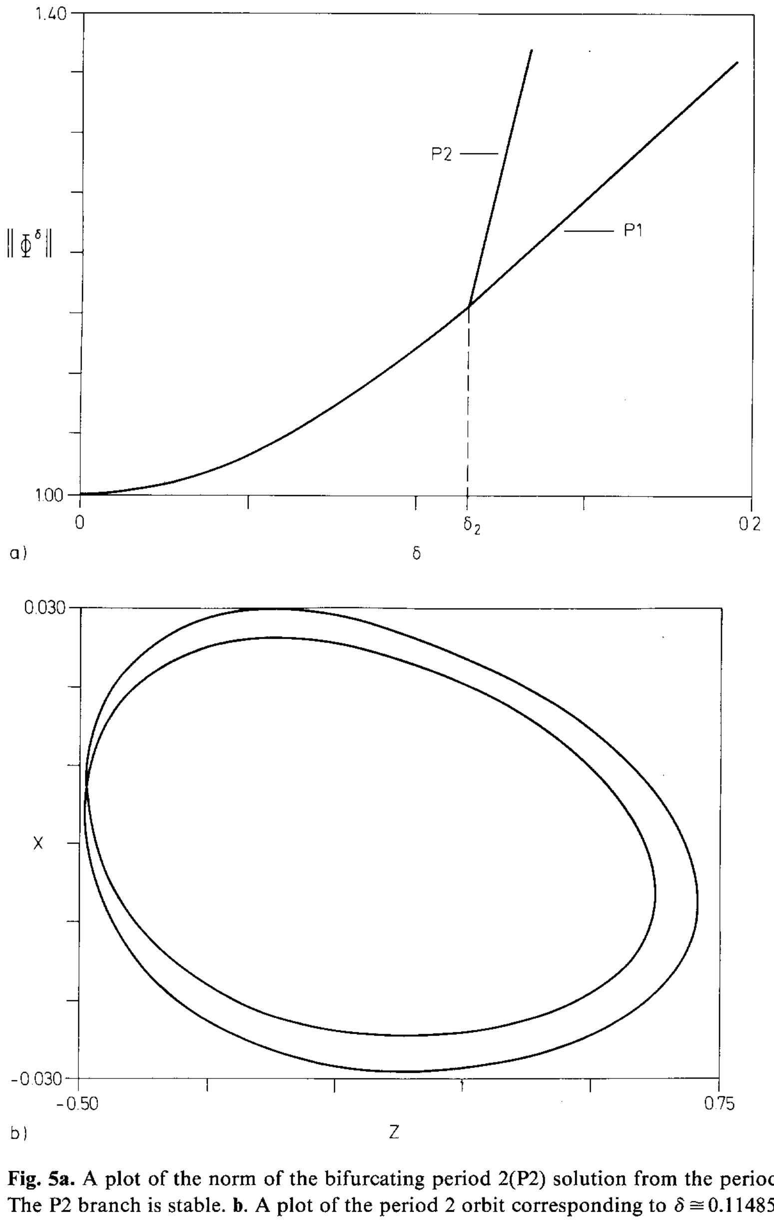

Fig. 5a. A plot of the norm of the bifurcating period 2(P2) solution from the period 1 (P1) solution. The P2 branch is stable. **b.** A plot of the period 2 orbit corresponding to $\delta \cong 0.114856$ projected onto the X–Z plane

period 2 solutions (P2) bifurcating from the period 1 (P1) branch at $\delta = \delta_2$. A plot of a period 2 orbit is shown in Fig. 5b.

Theorem 1.2 implies that in addition to a subharmonic period 2 orbit, there exist subharmonic solutions that bifurcate from large amplitude solutions appearing in the reduced model (1.14) when $\varepsilon = \delta = 0$. Keeping ε fixed at the value given in Table 2, we begin to follow the period 3 orbits as a function of δ with the period 3 orbit found in Schwartz [20]. The theory in section 1 predicts stability of the bifurcating solutions is such that we expect them to appear in saddle node pairs; i.e. there should exist for each value of δ, one stable periodic orbit and one unstable periodic orbit. Both period 3 and 4 solutions exhibit coexistence of saddle-node pairs of solutions. The bifurcation diagram for the period 3 orbits is illustrated in Fig. 6a. Our numerical analysis is kept local since we are only

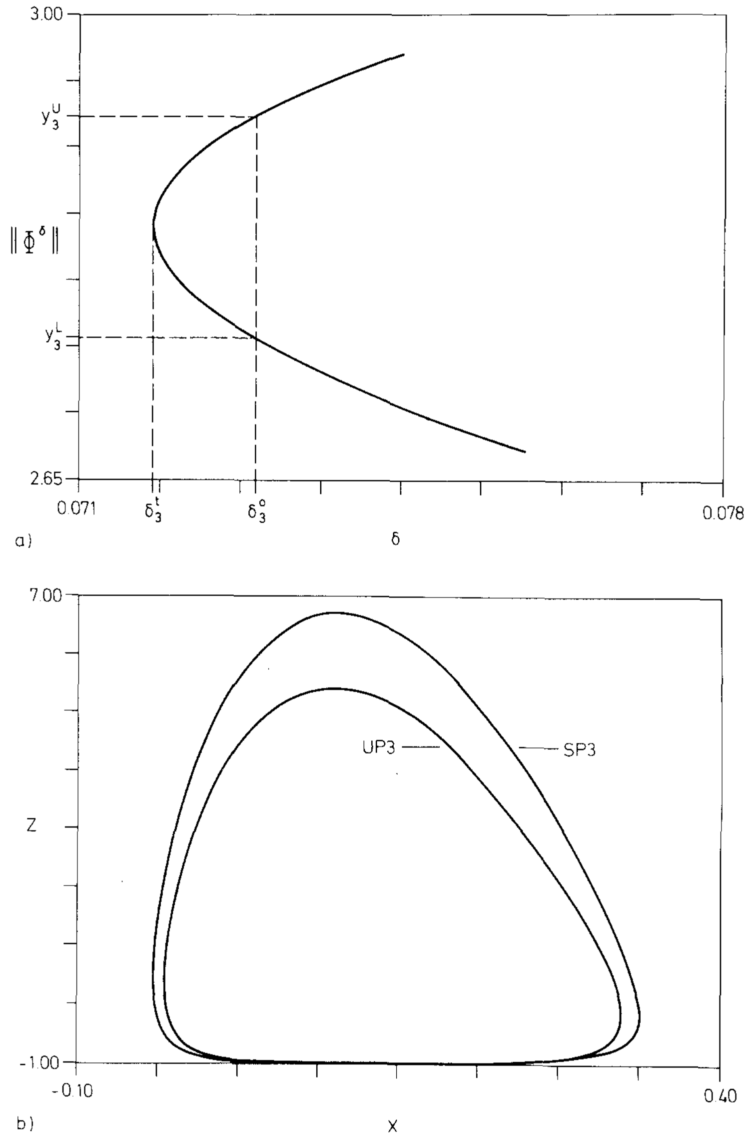

Fig. 6a. A plot of the norm of the period 3 solutions as a function of δ for the set of measles parameters given in table 2. δ_3^t denotes the turning point at which the onset of bifurcation occurs. For a fixed value $\delta = \delta_3^0 > \delta_3^t$, there exists a saddle-node pair of period 3 orbits. **b.** A plot of two period 3 solutions. SP3 corresponds to a stable orbit from the upper branch, while UP3 corresponds to an unstable orbit from the lower branch

interested in the neighborhood about the bifurcations. The turning point, δ_3^t, marks the coordinate (ε, δ) in parameter space where the bifurcation occurs. For $\delta > \delta_3^t$, but close to δ_3^t, there are two period 3 solutions.

At $\delta = \delta_3^0$ in Fig. 6a, there correspond two periodic solutions with norms equal to y_3^u and y_3^l, where y_3^u is the solution corresponding to the upper branch and y_3^l is the solution corresponding to the lower branch. All solutions on the

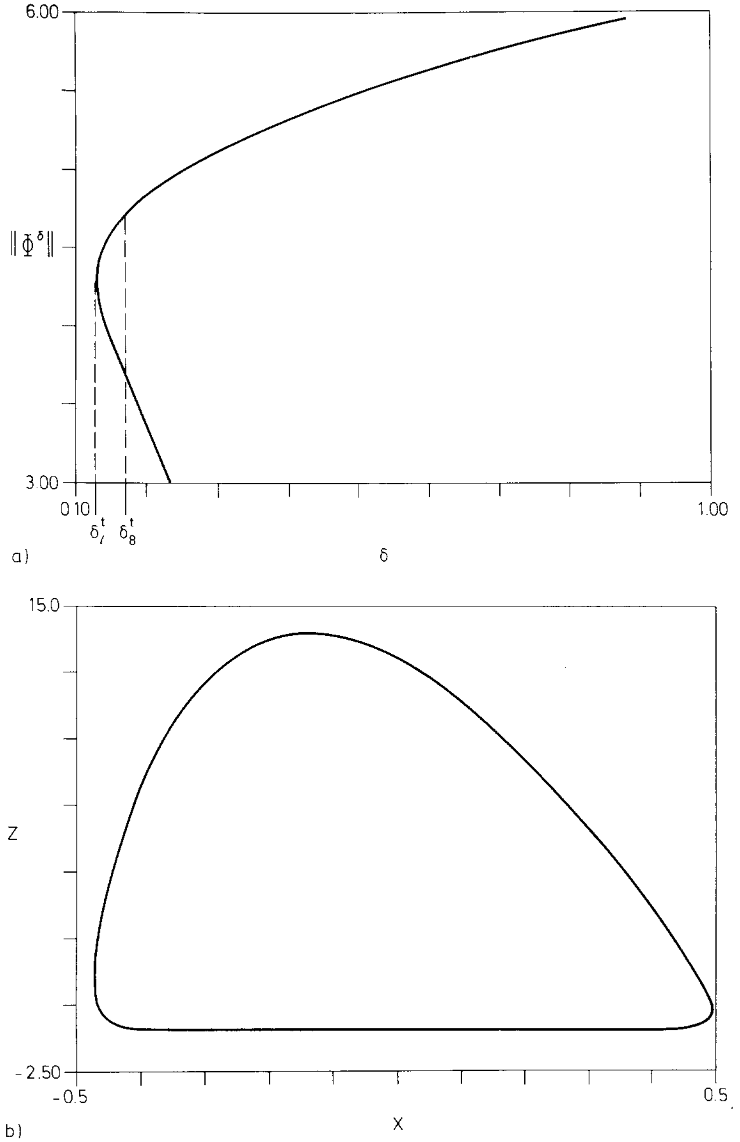

Fig. 7a. A plot of the period 4 solutions as a function of δ. For $\delta > \delta_4^t$, two period 4 solutions appear as saddle-node pairs. The lower branch is unstable for $\delta > \delta_4^t$. The upper branch is stable for $\delta \in (\delta_4^t, \delta_8)$. For $\delta > \delta_8$, a stable period 8 orbit bifurcates from the branch of period 4 orbits, and the period 4 branch becomes unstable. **b.** A plot of the period 4 orbit at $\delta = \delta_4^t$ projected onto the X–Z plane

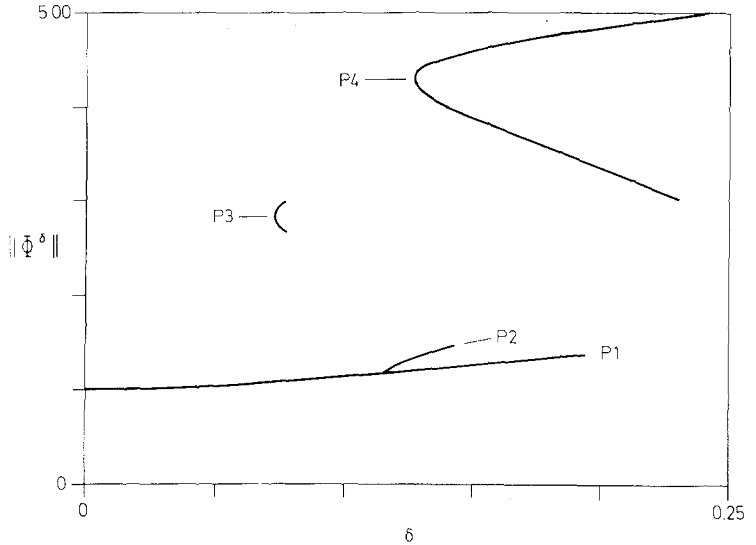

Fig. 8. A summary of the bifurcation analysis

upper branch correspond to stable period 3 solutions, while all solutions on the lower branch correspond to unstable solutions. Figure 6b illustrates a stable solution (SP3) from the upper branch at $\delta \cong 7.5003 \times 10^{-2}$. The unstable solution (UP3) from the lower branch was found at $\delta = 7.6152 \times 10^{-2}$.

For the period 4 solutions, a similar numerical result holds. Figure 7a pictures the turning point at $\delta = \delta_4^t$. For $\delta > \delta_4^t$, two period 4 solutions appear in saddle node pairs. The unstable solution is on the lower branch and has a 1-dimensional unstable manifold. On the upper branch of solutions we find that for $\delta \in (\delta_4^t, \delta_8)$, the period 4 orbits are stable. However, at $\delta = \delta_8$, one of the Floquet multipliers is -1, signalling a bifurcation of a period 8 orbit from the period 4. We have not followed the period 8 orbit. When $\delta > \delta_8$, the period 4 solution on the upper branch becomes unstable. Figure 7b depicts the period 4 orbit at $\delta = \delta_4^t$. A summary of the previous bifurcation figures is given in Fig. 8.

If one reduces ε, and then computes the value of δ at that value of ε for which a turning point occurs, one can check to see if the threshold predicted by Theorem 1.2 holds for parameters near measles. Table 3 exhibits some of these computations. For the given value of ε, the value of δ in the left column is the value at which a turning point occurs. For δ greater than this threshold value, two period 3 orbits appear. From Table 1, locally (near $\varepsilon = \delta = 0$), we predict

Table 3.

δ	ε	δ/ε^2
7.18687×10^{-2}	2.9466×10^{-1}	0.82775
7.139003×10^{-2}	2.9311×10^{-1}	0.83095
6.75575×10^{-2}	2.7992×10^{-1}	0.8621958

that if $\delta/\varepsilon^2 > \gamma_1/\gamma_2$, where $\gamma_1/\gamma_2 \cong 0.79$, bifurcation of period 3 orbits occurs. Although the ratio of δ/ε^2 is slightly larger in Table 3 than γ_1/γ_2, nevertheless, the calculated values are all within 10% of the predicted value. The global properties of this bifurcation manifold will be discussed in detail in a later paper.

The model produces oscillations that have very low minima of infective incidence. However, the addition of noise upon the contact rate may tend to keep the number of infectives large enough for the epidemic to remain recurrent in addition to having aperiodic levels of incidence.

Appendix A

In this appendix we give a proof of Theorem 1.2 concerning the differential equation (1.12). The plan of the proof is to use results of N. Fenichel [25] concerning the existence of a center manifold for singularly perturbed ordinary differential equations in order to reduce the dimension of the problem from three to two. Thus, we will reduce the study of equation (1.12) to the study of a perturbation of the planer conservative system (1.13). Then, we will apply the methods of J. Hale and P. Taboas [14] and Chow, Hale, and Mallet–Paret [5] to the resulting equation as in [25].

We begin by rewriting (1.12) in terms of the fast time $\tau = t/\varepsilon$ and making the equation autonomous by the addition of a trivial dependent variable as follows.

$$\begin{aligned}
\dot{\bar{x}} &= \varepsilon[-\nu\bar{y} + \varepsilon f_1(\bar{x}, \bar{y}, \bar{z}, \theta, \varepsilon, \delta)] \\
\dot{\bar{y}} &= \varepsilon\left[\nu\bar{x}(1+\bar{y}) + \nu\frac{\Delta_3}{\Delta_2+\Delta_3}\bar{x}\bar{z} + \nu^2\delta\cos 2\pi\theta\left(1+\bar{y}+\frac{\Delta_3}{\Delta_2+\Delta_3}\bar{z}\right) + \varepsilon f_2\right] \\
\dot{\bar{z}} &= -(\Delta_2+\Delta_3)\bar{z} + \varepsilon f_3 \\
\dot{\delta} &= 0 \\
\dot{\theta} &= \varepsilon
\end{aligned} \tag{A1}$$

In (A1) "$\cdot$" $= d/d\tau$, and the f_i are given to lowest order by

$$f_1(\bar{x}, \bar{y}, \bar{z}, \theta, \varepsilon, \delta) = -\bar{x}\left(\eta - \frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3)^2}\right) - \bar{x}\left(\bar{y} + \frac{\Delta_3\bar{z}}{\Delta_2+\Delta_3}\right)\left(1 - \frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3)^2}\right) + 0(|\varepsilon|+|\delta|) \tag{A2}$$

$$f_2(\bar{x}, \bar{y}, \bar{z}, \theta, \varepsilon, \delta) = \nu^2\frac{\Delta_3}{(\Delta_2+\Delta_3)^2}\bar{z}\left(1+\bar{y}+\frac{\Delta_3\bar{z}}{\Delta_2+\Delta_3}\right) + 0(|\varepsilon|+|\delta|)$$

$$f_3(\bar{x}, \bar{y}, \bar{z}, \theta, \varepsilon, \delta) = -\Delta_2\nu^{-1}\bar{x}\left(1+\bar{y}+\frac{\Delta_3\bar{z}}{\Delta_2+\Delta_3}\right) + 0(|\delta|+|\varepsilon|)$$

We may view (A1) as a one-parameter family of vector fields, X^ε, on $R^4 \times S^1$ since the right hand side is periodic in θ. X° possesses a manifold of equilibria given by

$$\xi: \bar{z} = 0.$$

Given the compact set $K_b = \{(\bar{x}, \bar{y}, 0, \delta, \theta) \in R^4 \times S^1 : |\bar{x}| \leq b, |\bar{y}| \leq b, |\delta| \leq b\}$ on ξ, Theorem 9.1, [25], implies the existence of a global center manifold for the vector field $X^\varepsilon \times 0$ ((A1) with the addition of $\varepsilon' = 0$) near K_b.

The center manifold can be represented by a smooth function

$$\bar{z} = \varepsilon h(\bar{x}, \bar{y}, \delta, \theta, \varepsilon), \qquad (\bar{x}, \bar{y}, \delta, \theta, \varepsilon) \in K_b x(-\varepsilon_0, \varepsilon_0) \tag{A3}$$

for some $\varepsilon_0 > 0$. We emphasize that the locally invariant center manifold given in (A3) is global in that it is defined for $(x, y, \delta, \theta) \in K_b$ for sufficiently small ε. In particular h is periodic in θ of period one.

The flow (A1), restricted to the center manifold, is given by

$$\begin{aligned}\dot{\bar{x}} &= \varepsilon[-\nu\bar{y}+\varepsilon\tilde{f}_1]\\ \dot{\bar{y}} &= \varepsilon[\nu\bar{x}(1+\bar{y})+\nu^2\delta\cos 2\pi\theta(1+\bar{y})+\varepsilon\tilde{f}_2]\\ \dot{\delta} &= 0\\ \dot{\theta} &= \varepsilon\end{aligned} \tag{A4}$$

where

$$\tilde{f}_1(\bar{x},\bar{y},\theta,\varepsilon,\delta)=f_1(\bar{x},\bar{y},\varepsilon h,\theta,\varepsilon,\delta)$$

$$\tilde{f}_2(\bar{x},\bar{y},\theta,\varepsilon,\delta)=\nu\frac{\Delta_3}{\Delta_1+\Delta_3}\bar{x}h+\nu^2\frac{\Delta_3}{\Delta_2+\Delta_3}\delta\cos 2\pi\theta h+f_2(\bar{x},\bar{y},\varepsilon h,\theta,\varepsilon,\delta).$$

The local invariance of the center manifold implies that

$$h(\bar{x},\bar{y},\delta,\theta,0)=\frac{1}{\Delta_2+\Delta_3}f_3(\bar{x},\bar{y},0,\theta,0,\delta). \tag{A5}$$

The theorem of Fenichel implies that all solutions of (A1) which exist for all $\tau\in R$ and remain in some neighborhood of K_b in $R^4\times S^1$ lie on the center manifold. Thus we have reduced our problem to the study of system (A4) in which we now return to the fast time t:

$$\begin{aligned}\bar{x}' &= -\nu\bar{y}+\varepsilon\bar{x}(2r-\xi_1\bar{y})\\ \bar{y}' &= \nu\bar{x}(1+\bar{y})+\nu^2\delta\cos 2\pi t(1+\bar{y})-\varepsilon\xi_2x^2(1+\bar{y}).\end{aligned} \tag{A6}$$

In obtaining (A6) we have made use of (A5), ignored terms which are quadratic or higher order in (ε,δ), and introduced the notational simplification:

$$\xi_2=\frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3)^2},\quad r<0 \text{ as in (1.9)},\qquad \xi_1=1-\xi_2.$$

Since the higher order terms in (ε,δ) do not affect the subsequent analysis it is permissible to ignore them.

Equation (A6) is a perturbation of the conservative system (1.13). We study (A6) for the existence of subharmonic solutions of period n near $\Gamma_n=\{(\bar{x}_n(t),\bar{y}_n(t)): 0<t<n\}$ following the work in [22] which is based on [5] and [14]. Before we begin, some remarks are necessary. First, the compact set K_b must be chosen such that Γ_n is contained in the ball of radius b about the origin. We assume this has been done. Secondly, it follows from the theorem of Fenichel that if (A6) possesses an asymptotically stable subharmonic of order n, $(\bar{x}(t),\bar{y}(t))$, near Γ_n, then $(x(t),y(t),z(t))$ will be an asymptotically stable solution of (1.12), where

$$z(t)=\varepsilon h(x(t),y(t),\delta,t,\varepsilon). \tag{A7}$$

Indeed, in [25] it is proved that solutions collapse onto the center manifold at the exponential rate, $0(e^{-kt/\varepsilon})$, where $k=\Delta_2+\Delta_3$.

Let us now consider the problem of the existence of n-period solutions of (A6) near $\Gamma_n=\{(\bar{x}_n(t),\bar{y}_n(t)): 0<t<n\}$ where $(x_n(t),y_n(t))$ is an n-periodic solution of the unperturbed equation (1.13). Since the following analysis is entirely similar to that employed in [22] we give here only an abbreviated version.

Assume that $\bar{x}_n(0)=0$, $\bar{y}_n(0)>0$ so that $\bar{x}_n$ is an odd function of t and $\bar{y}_n$ is an even function. if $(u_1(t),u_2(t))$ is an n-periodic solution of (A6) near Γ_n for all t, then we may write

$$\begin{vmatrix}u_1(t-\alpha)\\ u_2(t-\alpha)\end{vmatrix}=\begin{vmatrix}\bar{x}_n(t)\\ \bar{y}_n(t)\end{vmatrix}+\begin{vmatrix}v_1(t)\\ v_2(t)\end{vmatrix},$$

for some α, $0<\alpha<n$. The phase α is introduced to account for the arbitrary normalization of the

phase $(\bar{x}_n(t), \bar{y}_n(t))$. The perturbation v satisfies $v_1(0)=0$ and the equation

$$v' = A(t)v + \varepsilon\begin{pmatrix} \bar{x}_n(t)(2r - \xi_1\bar{y}_n(t)) \\ \xi_2\bar{x}_n^2(t)(1+\bar{y}_n(t)) \end{pmatrix} + \delta\begin{pmatrix} 0 \\ \nu^2 \cos 2\pi(t-\alpha)(1+\bar{y}_n(t)) \end{pmatrix} \tag{A8}$$

where

$$A(t) = \nu\begin{pmatrix} 0 & -1 \\ (1+\bar{y}_n(t)) & \bar{x}_n(t) \end{pmatrix}$$

and terms (ε, δ, v) which are of quadratic or higher order have been dropped. The equation (A8) is an inhomogeneous linear equation for which $A(t)$ and the inhomogeneous term are n-periodic in t. We seek an n-periodic solution $v(t)$ giving the order ε^1 and δ^1 terms in a perturbation series for u. One of the assumptions made in Theorem 1.2 is that (x_n', y_n') spans the n-periodic solutions of (A6) with $\varepsilon = \delta = 0$. Thus the adjoint equation

$$W' = -A(t)^t W$$

has one n-periodic solution up to a constant multiple. As in [22], it is easily checked that this solution may be taken to be $(\bar{x}_n(t), \bar{y}_n(t)/1+\bar{y}_n(t))$. The Fredholm alternative gives a necessary and sufficient condition for the solvability of (A8), namely,

$$\int_0^n \left\{ \varepsilon\begin{pmatrix} \bar{x}_n(2r-\xi_1\bar{y}_n) \\ \xi_2\bar{x}_n^2(1-\bar{y}_n) \end{pmatrix} + \delta\begin{pmatrix} 0 \\ \nu^2\cos 2\pi(t-\alpha)(1+\bar{y}_n) \end{pmatrix} \right\} \cdot \begin{pmatrix} \bar{x}_n \\ \bar{y}_n/(1+\bar{y}_n) \end{pmatrix} dt = 0. \tag{A9}$$

If (A9) holds, there will be exactly one n-periodic solution of (A8) satisfying $v_1(0)=0$. Equation (A9) represents the lowest order terms in the bifurcation equation. In order to see that (A9) coincides with the first two terms in (1.15) note that y_n is even in t and

$$\int_0^n \bar{x}_n^2\bar{y}_n\, dt = -\frac{1}{\nu}\int_0^n \bar{x}_n^2\bar{x}_n'\, dt = 0$$

so that

$$\begin{aligned}\int_0^n ((2r-\xi_1\bar{y}_n)\bar{x}_n^2 + \xi_2\bar{x}_n^2\bar{y}_n)\, dt &= 2r\int_0^n \bar{x}_n^2(1+\bar{y}_n)\, dt \\ &= \frac{-2r}{\nu}\int_0^n \bar{x}_n\bar{y}_n'\, dt \\ &= \frac{-2r}{\nu}\int_{\Gamma_n} x\, dy \\ &= \frac{-2r}{\nu}\int_{\text{int}}\int_{\Gamma_n} dx\, dy.\end{aligned}$$

The above nonrigorous argument can be made rigorous by application of the Implicit Function theorem as in [22]. Note that (1.16) follows from (A5) and (A7).

Appendix B. Computation of Periodic Orbits of Periodically Forced Differential Equations

Here we briefly describe the techniques used in computing the periodic orbits for system (1.8) and their Floquet multipliers. The basic idea is to embed the non-autonomous differential equation into a higher dimensional autonomous differential equation and then use standard homotopy (continuation) techniques to compute periodic orbits as a function of a parameter for autouous systems. (See, for example, Doedel [8] and Rheinboldt [18].)

Let $f: R^n \times R \times R \to R^n$ be a smooth function such that $f(x, t, \lambda) = f(x, t+1, \lambda)$ for all (x, t, λ) in the domain of f. We consider the problem of computing periodic solutions of period p for

$$x' = f(x, t, \lambda). \tag{B1}$$

Equation (B1) has a periodic solution of period p for a fixed λ if and only if there is a vector $x(0) = x_0$ such that $x(0) = x(p)$. It has a minimal period p if

$$x(0) \neq x(1) \neq \cdots \neq x(p-1).$$

Therefore, if $\phi_\lambda(\cdot, 0, x_0)$ is the unique solution of (B1) for λ fixed such that $\phi_\lambda(0; 0, x_0) = x_0$, then computing periodic orbits of (B1) is equivalent to finding those vectors x_0 satisfying $\phi_\lambda(p; 0, x_0) - x_0 = 0$. If $T_\lambda p: R^n \to R^n$ is defined by $T_\lambda p(x_0) = \phi_\lambda(p; 0, x_0)$, then $\phi_\lambda(\cdot, 0, x_0)$ is a periodic solution having period p if and only if x_0 is a fixed point of $T_\lambda p$; i.e. $T_\lambda p(x_0) = x_0$. In order to compute unstable periodic solutions as well as stable ones, one needs to compute the linear variational equations along a periodic orbit. In particular, if $Y(t)$ is a matrix such that

$$Y'(t) = \frac{\partial f}{\partial x}(t, \phi_\lambda(t; 0, x_0), \lambda) Y(t), \; Y(0) = I, \tag{B2}$$

then the derivative of $T_\lambda p$ is given by

$$dT_\lambda P(x_0) = \frac{\partial \phi_\lambda}{x_0}(p; 0, x_0),$$

since $\partial\phi_\lambda/\partial x_0(\cdot, 0, x_0)$ satisfies (B2). If x^* is a fixed point of $T_\lambda p$, then the Floquet multipliers are the eigenvalues of $dT_\lambda p(x)$.

The initial computation of a periodic orbit is done as follows. Let $[o, p]$ be partitioned in the following way:

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = p.$$

For each $j = 0, 1, \ldots, m-1$, let $\phi_\lambda(t_j; t_j, x_j) = x_j$.

Let $\pi_j^\lambda: R^n \to R^n$, be defined by

$$\pi_j^\lambda(x_j) = \phi_\lambda(t_{j+1}; t_j, x_j).$$

If $x_0 \in R^n$ is a fixed point of $T_\lambda p$, it is necessary and sufficient that the following continuity and boundary conditions hold for the nm-dimensional vector $x = (x_0, x_1, \ldots, x_{m-1})$:

$$x_{j+1} = \pi_j^\lambda(xj), \qquad j = 0, 1, \ldots, m-1 \tag{B1a}$$

$$x_0 = x_m.$$

The problem of finding a fixed point of $T_\lambda p$ is equivalent for finding an $x \in R^{nm}$ satisfying (B1a). Equation (B1a) may be solved by a Newton iterative technique, and it is described in detail in Schwartz [20].

The advantage of decomposing [0, p] into m subintervals is that it allows us to make computations of unstable periodic orbits as well as those of stable periodic orbits. Furthermore, it is less sensitive to initial conditions than computing the fixed points of $T_\lambda p$ since the Lipschitz constant of the vector field is effectively reduced by a factor of m.

We now consider the technique of embedding equation (B1) into a higher dimensional autonomous system. Let $q:[-1, 1]x[-1, 1] \to R$, and let

$$f(x, t, \lambda) = f_0(x) + \lambda q(\sin 2\pi t, \cos 2\pi t) f_1(x).$$

Consider the following reduced 2-dimensional system:

$$y' = Ay + g(y), \tag{B3}$$

where

$$A = \begin{pmatrix} 1 & 2\pi \\ -2\pi & 1 \end{pmatrix}, \qquad g(y) = \begin{pmatrix} -y_1(y_1^2 + y_2^2) \\ -y_2(y_1^2 + y_2^2) \end{pmatrix}.$$

Equation (B3) has $(y_1, y_2)(t) = (\sin 2\pi t, \cos 2\pi t)$ as its solutions passing through $(y_1, y_2)(0) = (0, 1)$. Furthermore, the solutions can be shown to be isolated.

Let $z = (x, y)$ be a vector in $R^n \times R^2$, and let $h: R^n \times R^2 \times R \to R^{n+2}$ be given by

$$h(z, \lambda) = \begin{pmatrix} f_0(x) + \lambda q(y_1, y_2) f_1(x) \\ Ay + g(y) \end{pmatrix}.$$

Then the expanded autonomous system for (B1) and (B3) is given by

$$z' = h(z, \lambda). \tag{B4}$$

We now use the continuation methods described in [8] to follow a branch of periodic orbits as a function of parameter. The method used to compute periodic orbits at a particular parameter value was that of collocation. (See, for example, Keller [16] and Ascher [1].)

In order to determine the Floquet multipliers of the periodic orbit, we need to know the Floquet multipliers of Eq. (B3). Since $(\sin 2\pi t, \cos 2\pi t)$ is a periodic solution of period 1 to Eq. (B3), it follows that one of its Floquet multipliers is equal to 1. Let ρ_1 and ρ_2 denote the Floquet multipliers to Eq. (B3). If we are at a p-periodic orbit of system (B4), and since $(\sin 2\pi t, \cos 2\pi t)$ is a p-periodic orbit of (B3), then Abel's formula [Hale [13], page 82] applied to system (B3) implies

$$\rho_1 \rho_2 = e^{-2p}.$$

But $\rho_1 = 1$ implies $\rho_2 = e^{-2p}$, which in turn implies that the solutions to (B3) are stable. Furthermore, this can be used as a check on the accuracy of the computation of the Floquet multipliers.

Now that two of the Floquet multipliers are known, the remaining n Floquet multipliers can be determined and the stability of the original equation can be inferred. (Note that these n remaining Floquet multipliers are exactly the eigenvalues to $dT_\lambda p(x^*)$ given above).

As a final note on the numerical computations, we remark that all computations were done on a VAX 11/780 which has a 32 bit word. Double precision was used throughout our calculations. The accuracy of the computed periodic orbits was monitored by comparing the results of collocation to those of multipoint shooting for the same set of parameter values. In addition, the Floquet multipliers of Eq. (B3) were used to monitor the accuracy of the linear equation solvers.

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