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Stability in a Class of Cyclic Epidemic Models with Delay

Harlan Stech and Michael Williams

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Abstract. A detailed analysis of a general class of *SIRS* epidemic models is given. Sufficient conditions are derived which guarantee the global stability of the endemic equilibrium solution. Further conditions are found which ensure instability for the equilibrium. Finally, the dependence of the stability on the contact number and the ratio of the mean length of infection to the mean removed time is considered.

Key words: Epidemiology – SIRS – Deterministic models – Distributed delays – Stability

§1. Introduction

The study of time-lag induced oscillations in cyclic epidemic models has been the subject of considerable interest. The purpose of this paper is to give a more detailed analysis of a general class of *SIRS* models derived in [4] and to determine to what extent the special cases of this model discussed there are representative of the model as a whole.

In the *SIRS* model under consideration the population (assumed to be uniform, homogeneously mixed and of constant size) is subdivided into subclasses of susceptibles, infectives and removed individuals, who are, respectively, capable of becoming infected, infectious and isolated or immune as a result of having been infectious. The respective fractions of the entire population in each of these classes at time t will be denoted by $S(t)$, $I(t)$ and $R(t)$. Individuals are assumed to pass from the susceptible class to the infective class to the removed class and back again to the susceptible class with the expected length of time of traversal for such a "cycle" finite. The transfer rate from the susceptible class, S, to the infective class, I, is assumed to be proportional to product SI with the proportionality constant, $\beta > 0$, interpreted as the mean number of effective contacts per unit time per infective. Individuals enter the removed class from I at a rate proportional to I; the proportionality constant, $\gamma > 0$, is the reciprocal of the mean infective time. To describe the transfer rate from R to S, we define $P(t)$ for $t \geqslant 0$ to be the probability that an individual who became removed at time 0 is still removed at time t. $P(t)$ is non-negative, non-increasing, $P(0^+) = P(0) = 1$ and $P(\infty) = 0$. We assume that the average length of immunity, $\omega = \int_0^\infty P(t)\,dt$, is finite. Additional requirements will be imposed on P later.

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With the preceding delineations, the *SIRS* model is described by the system of equations,

$$I(t) = I_0 e^{-\gamma t} + \beta \int_0^t S(x) I(x) e^{-\gamma(t-x)}\, dx, \tag{1.1}$$

$$R(t) = R_0(t) + \gamma \int_0^t I(x) P(t-x)\, dx, \tag{1.2}$$

and

$$S(t) + I(t) + R(t) = 1 \qquad \text{for} \qquad t \geqslant 0. \tag{1.3}$$

$I(0) = I_0 > 0$ is the initial fraction infected and $R_0(t) \geqslant 0$ is the fraction of the population initially removed and still removed at time t. Clearly $R_0(t) \leqslant 1$ and is non-increasing, and moreover, we assume $R_0(t)$ tends to 0 as $t \to +\infty$. It is shown in [4] that (1.1)–(1.2)–(1.3) admits a unique solution defined for all $t \geqslant 0$ and that the model is "well-posed" in the sense that $S(t)$, $I(t)$, $R(t) > 0$ for all $t > 0$. Moreover, when the contact number $\sigma \equiv \beta/\gamma \leqslant 1$ all solutions, $(S(t), I(t), R(t))$ of (1.1)–(1.2)–(1.3) approach $(1, 0, 0)$ as $t \to +\infty$.

Equations (1.1)–(1.2)–(1.3) may be reformulated as an equivalent scalar equation,

$$I'(t) = -\gamma I + \beta I \left[1 - I - R_0(t) - \gamma \int_0^t I(t-u) P(u)\, du \right] \tag{1.4}$$

with initial condition $I(0) = I_0$. When $\sigma > 1$ (1.4) has a positive "endemic" equilibrium solution defined by $I_e(t) \equiv (1 - (1/\sigma))/(1 + \omega\gamma)$ which satisfies (1.4) when $R_0(t) = \gamma \int_t^\infty I_e P(u)\, du$.

If the removed period is assumed to have a constant duration (i.e. $P(t) = 1$ for $0 \leqslant t \leqslant \omega$; $P(t) = 0$ for $t > \omega$), the equilibrium solution, I_e, may be locally asymptotically stable or unstable depending on the parameters (γ, σ). Using Hopf bifurcation techniques it is shown in [4] that the model supports stable, non-trivial periodic solutions for parameter values (γ, σ) near those at which the endemic equilibrium changes stability. Similar results hold when the removed class is divided into $n \geqslant 3$ subclasses, where the transfers $R_i \to R_{i+1}$ $(i = 1, \ldots, n-1)$ and $R_n \to S$ all occur at a rate ε, proportional to $R_i(t)$ and $R_n(t)$, respectively.

In section 2 we consider the general model (1.4) and derive sufficient conditions on $P(t)$ so that the endemic equilibrium is the asymptotic limit of all positive solutions of (1.4). Section 3 is devoted to determining qualitative conditions on $P(t)$ that ensure that the endemic equilibrium is unstable for some choice of σ and γ. In the final section we consider the dependence of the stability of the endemic equilibrium on the parameters, $(\omega\gamma, \sigma)$.

§2. Global Stability

The goal of this section is to derive conditions on the probability $P(t)$ which imply global stability of the endemic equilibrium state for the model (1.1)–(1.2)–(1.3). A general, yet unintuitive, condition is provided by the following theorem.

Theorem 2.1. *Assume*

$$\int_0^\infty \int_s^\infty P(u)\,du\,ds < \infty \tag{2.1}$$

and

$$1 + \gamma \int_0^\infty \cos(\nu s)P(s)\,ds > 0 \qquad \textit{for all} \qquad \nu \in \mathbb{R}. \tag{2.2}$$

Then all solutions of (1.1)–(1.2)–(1.3) satisfy

$$\lim_{t\to\infty} (S(t), I(t), R(t)) = \left(\frac{1}{\sigma}, I_e, \omega\gamma I_e\right).$$

Proof. The limits of $S(t)$ and $R(t)$ will follow from (1.2) and (1.3) once the limit of $I(t)$ is established. Thus it suffices to consider (1.4) alone.

Since $I(t) > 0$ for all $t \geqslant 0$ we may make the change of variable

$$z(t) = \ln(I(t)/I_e). \tag{2.3}$$

In terms of the new variable $z(t)$, (1.4) becomes

$$z'(t) = -\int_0^t g(z(t-u))\,d\mu(u) + f(t) \tag{2.4}$$

and

$$z(0) = \ln(I_0/I_e)$$

where

$$g(z) = e^z - 1, \qquad \mu(0) = 0,$$

$$\mu(u) = \beta I_e + \gamma\beta I_e \int_0^u P(v)\,dv \qquad \text{for} \qquad u > 0,$$

and

$$f(t) = -\beta R_0(t) + \gamma\beta I_e \int_t^\infty P(u)\,du \qquad \text{for} \qquad t \geqslant 0.$$

Clearly $f(t) \to 0$ as $t \to \infty$ and the measure μ has finite total variation on $0 \leqslant u \leqslant \infty$ equal to $\gamma(\sigma - 1)$.

Equation (2.4) is now in the form studied by Londen; in order to make use of his results there we must verify: (i) solutions of (2.4) are bounded and (ii) $\operatorname{Re}\hat{\mu}(\nu) > 0$, where $\hat{\mu}$ denotes the Fourier transform of the measure μ. An argument by contradiction may be used to verify (i). Details are omitted. (ii) follows equally easily from (2.2) since, for all ν,

$$\operatorname{Re}\hat{\mu}(\nu) = \beta I_e\left[1 + \gamma \int_0^\infty \cos(\nu s)P(s)\,ds\right].$$

Thus, Theorem 3 of [6] applies and we may conclude $g(z(t)) \to 0$ as $t \to \infty$, and the theorem is established.

Remark 2.2. If the length of time an individual spends in the removed class is denoted by the random variable τ, it is easily seen that (2.1) is equivalent to requiring τ to have two moments. Also, (2.2) holds whenever $\omega = \int_0^\infty P(t)\,dt < 1/\gamma$, i.e. whenever the mean removed time is less than the mean infective time.

Remark 2.3. Both (2.1) and (2.2) are implied by

$$P(t+s) \leqslant P(t)P(s) \qquad \text{for all} \quad t,s \geqslant 0. \tag{2.5}$$

Indeed, $h(t) \equiv \ln P(t)$ is a non-positive subadditive function for $t \geqslant 0$. By Theorem 7.6.1 of [5]

$$\alpha \equiv \lim_{t\to\infty} \frac{h(t)}{t} = \inf_{t>0} \frac{h(t)}{t}$$

is finite. Since $P(t) \to 0$ as $t \to \infty$, we conclude $\alpha < 0$ so that for all large t, $h(t) < (\alpha/2)t$ and $P(t) < e^{(\alpha/2)t}$; (2.1) follows since $\alpha < 0$.

For (2.2) it suffices to note from (2.5) that for $\nu \geqslant 0$ and each $n = 0,1,2,\ldots$

$$\int_{n(2\pi/\nu)}^{(n+1)(2\pi/\nu)} \cos(\nu s)P(s)\,ds = \int_{n(2\pi/\nu)}^{(n+\frac{1}{2})(2\pi/\nu)} \cos(\nu s)\left[P(s) - P\left(s+\frac{\pi}{\nu}\right)\right]ds \geqslant 0.$$

Note that $P(t+s)/P(s)$ may be given the interpretation as the conditional probability that an individual will remain immune t more units given that he has already been immune s units. Thus, (2.5) asserts that the probability that an individual remains immune an additional t units is a non-increasing function of time, that is, short removed periods in a sense are more likely than long ones. In discussing a non-cyclic *SIR* model with delay in the infective class, Wang [10] makes use of the assumption (2.5) with a similar interpretation to obtain estimates on the maximum number of infectives.

Example 2.4. Consider the *SIRS* model in which the class R is divided into two subclasses, R_1 and R_2. Denote by $R_1(t)$ and $R_2(t)$ the fractions of the population in each of these classes and assume the transfer rate between R_1 and R_2 is proportional to $R_1(t)$, the transfer rate from R_2 to S is proportional to $R_2(t)$ and the transfer rates from S to I and I to R_1 are as described earlier. The model $S \to I \to R_1 \to R_2 \to S$ is governed by the equations

$$I' = -\gamma I + \beta IS; \qquad I(0) = I_0 > 0, \tag{2.6}$$

$$R_1' = -\varepsilon_1 R_1 + \gamma I; \qquad R_1(0) = R_{1,0} \geqslant 0, \tag{2.7}$$

$$R_2' = -\varepsilon_2 R_2 + \varepsilon_1 R_1; \qquad R_2(0) = R_{2,0} \geqslant 0 \tag{2.8}$$

and

$$S(t) + I(t) + R_1(t) + R_2(t) = 1; \qquad t \geqslant 0 \tag{2.9}$$

with ε_1, ε_2 positive proportionality constants.

Equations (2.7) and (2.8) may be integrated so that

$$R(t) = R_1(t) + R_2(t) = R_0(t) + \gamma \int_0^t I(t-s)P(s)\,ds$$

where

$$R_0(t) = R_1(0)e^{-\varepsilon_1 t} + R_2(0)e^{-\varepsilon_2 t} + R_1(0)\varepsilon_1 e^{-\varepsilon_2 t}\int_0^t e^{(\varepsilon_2-\varepsilon_1)u}\,du$$

and

$$P(t) = e^{-\varepsilon_1 t} + \varepsilon_1 e^{-\varepsilon_2 t}\int_0^t e^{(\varepsilon_2-\varepsilon_1)u}\,du.$$

With P and R_0 as above, each solution of (2.6) – (2.7) – (2.8) – (2.9) may be identified with a solution of (1.1) – (1.2) – (1.3). Since $\varepsilon_1, \varepsilon_2 > 0$, (2.1) is satisfied by P and $R_0(t) \to 0$ as $t \to \infty$. For (2.2), an elementary computation reveals

$$1 + \gamma\int_0^\infty \cos(vs)P(s)\,ds = 1 + \gamma\frac{\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)}{(\varepsilon_1^2+v^2)(\varepsilon_2^2+v^2)} > 0 \qquad \text{for all } v.$$

Therefore, by the theorem all solutions of (2.6) – (2.7) – (2.8) – (2.9) satisfy

$$(S(t), I(t), R_1(t), R_2(t)) \to \left(\frac{1}{\sigma}, I_e, \frac{\gamma}{\varepsilon_1}I_e, \frac{\gamma}{\varepsilon_2}I_e\right)$$

where

$$I_e = \left(1 - \frac{1}{\sigma}\right)\Big/\left(1 + \frac{\gamma}{\varepsilon_1} + \frac{\gamma}{\varepsilon_2}\right).$$

Remark 2.5. For the more general model $S \to I \to R_1 \to R_2 \to \cdots \to R_n \to S$ where $n > 2$, it is known that the endemic equilibrium ceases to be globally attracting for all values of the parameters, in fact, in the case $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \varepsilon$, the model possesses (for various σ, γ and ε) an unstable endemic equilibrium and a stable periodic solution (see [4] and Example 3.3). For the case $n = 1$ global stability of the endemic equilibrium was shown in [3].

§3. An Instability Criterion

As discussed in the remarks following Theorem 2.1, the endemic equilibrium for (1.1) – (1.2) – (1.3) is globally attracting provided the removed time is in some sense "small". We will now show, on the other hand, that when the removed time is "large" the endemic equilibrium can always be made unstable for appropriate choices of σ and γ.

Theorem 3.1. *Assume* (2.1) *and*

$$1 + \gamma\int_0^\infty \cos(vs)P(s)\,ds < 0 \tag{3.1}$$

for some $v, \gamma > 0$. Then for some $\sigma > 1$ the "equilibrium solution" $I(t) \equiv I_e$ of (1.4) *is unstable. That is, there exists a $\delta > 0$ and sequences $\{I_m(0)\}$, $\{R_{0,m}(\cdot)\}$ and $\{t_m\}$ such that $t_m > 0$, $I_m(0) \to I_e$, $\|R_0(t) - \gamma I_e \int_t^\infty P(u)\,du\|_\infty \to 0$ and $|I(t_m, I_m(0), R_{0,m}) - I_e| \geqslant \delta$. ($\|\cdot\|_\infty$ denotes the usual L^∞-norm on $\mathbb{R}^+$.)*

Proof. Under the change of variables (2.3), it suffices to show that the zero solution of (2.4) (where $f(t) \equiv 0$ and $z(0) = 0$) is unstable. By [7] we need only show that for some $\gamma > 0$ and $\sigma > 1$ there is a zero, λ, of

$$\Delta(\lambda) = \Delta(\lambda; \sigma, \gamma) \equiv \lambda + \frac{(\sigma - 1)\gamma}{1 + \omega\gamma}\left[1 + \gamma \int_0^\infty e^{-\lambda s} P(s)\, ds\right] \tag{3.2}$$

with $\operatorname{Re}\lambda > 0$. Let $\lambda = \mu + i\nu$ and consider real and imaginary parts of $\Delta(\lambda) = 0$:

$$0 = \mu + \frac{(\sigma - 1)\gamma}{1 + \omega\gamma}\left[1 + \gamma \int_0^\infty \cos(\nu s) e^{-\mu s} P(s)\, ds\right] \tag{3.3}$$

$$0 = \nu - \frac{(\sigma - 1)\gamma^2}{1 + \omega\gamma} \int_0^\infty \sin(\nu s) e^{-\mu s} P(s)\, ds. \tag{3.4}$$

By (2.1) the function $\nu \to \int_0^\infty \cos(\nu s) P(s)\, ds$ is continuously differentiable and is non-constant by the Riemann-Lebesgue lemma. Therefore by (3.1) we may find $\nu_0 > 0$ so that $\int_0^\infty \cos(\nu_0 s) P(s)\, ds < 0$ and

$$\beta \equiv (d/d\nu) \int_0^\infty \cos(\nu s) P(s)\, ds \bigg|_{\nu = \nu_0} = -\int_0^\infty \sin(\nu_0 s) s P(s)\, ds \neq 0.$$

Let $\gamma_0^{-1} \equiv -\int_0^\infty \cos(\nu_0 s) P(s)\, ds$ and $\sigma_0 > 1$ be defined so that (3.4) holds when $\mu = 0$, $\nu = \nu_0$ and $\gamma = \gamma_0$.

Define F: $\mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(\mu; \nu, \sigma) = \begin{bmatrix} \operatorname{Re}\Delta(\mu + i\nu; \sigma, \gamma_0) \\ \operatorname{Im}\Delta(\mu + i\nu; \sigma, \gamma_0) \end{bmatrix}.$$

By the choice of γ_0, σ_0 and ν_0, $F(0; \nu_0, \sigma_0) = (0, 0)$. By (2.1), F can be extended to a continuously differentiable function in a neighborhood of $(0, \nu_0, \sigma_0)$ and, in fact,

$$\det D_{(\nu,\sigma)} F(0, \nu_0, \sigma_0) = -\frac{\gamma_0^2 \beta \nu_0}{1 + \omega\gamma_0} \neq 0.$$

The result now follows from the Implicit Function Theorem.

Remark 3.2. The hypothesis, (3.1), of the theorem is satisfied if for some $s > 0$

$$1 - P(2s) > 4(1 - P(s)). \tag{3.5}$$

We demonstrate that (3.5) implies the existence of a ν such that $\int_0^\infty \cos(\nu s) P(s)\, ds < 0$ (which in turn implies (3.1) for some γ). Assume the contrary and define Q to be the even extension of P to $\mathbb{R}$. The Fourier transform of Q satisfies

$$\hat{Q}(\nu) \equiv \int_{-\infty}^\infty e^{-i\nu s} Q(s)\, ds = 2\int_0^\infty \cos(\nu s) P(s)\, ds \geqslant 0 \qquad \text{for all } \nu.$$

A standard fact from harmonic analysis (e.g. [9], p. 15) asserts that since $Q \in L^1(\mathbb{R})$, $\hat{Q} \geqslant 0$ and Q is continuous at 0, then $\hat{Q} \in L^1(\mathbb{R})$, the inversion formula holds and $Q(0) = \int_{-\infty}^\infty \hat{Q}(x)\, dx$. Thus integrating the trigonometric inequality

$$1 - \cos(2\nu s) = 2(1 - \cos^2(\nu s)) \leqslant 4(1 - \cos(\nu s))$$

against $\hat{Q}(v)$, we obtain

$$[1 - Q(2s)] \leqslant 4[1 - Q(s)].$$

Since $Q(s) = P(s)$ for $s \geqslant 0$, (3.5) is contradicted, establishing the implication.

As before $(1 - P(s))/(1 - P(2s))$ may be interpreted as the conditional probability that an individual's immune period is less than s units given that it is less than $2s$ units. Thus (3.5) suggests that instability of the endemic equilibrium results when long removed periods are more likely than short ones. Condition (3.5) is clearly satisfied when the removed period is assumed to have constant length ω. (See section 1.)

Example 3.3. For the general multiple removed class model, $S \to I \to R_1 \to R_2 \to \cdots \to R_n \to S$, of Remark 2.5 (with $\varepsilon = \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n$), the associated probability is

$$P(t) = \left[1 + (\varepsilon t) + \frac{(\varepsilon t)^2}{2!} + \cdots + \frac{(\varepsilon t)^{n-1}}{(n-1)!}\right] e^{-\varepsilon t}.$$

One obtains easily that

$$\lim_{t \to 0} \frac{1 - P(t)}{1 - P(2t)} = \frac{1}{2^n}.$$

Thus, (3.5) is satisfied for $n > 2$; for $n = 1, 2$ (3.5) must obviously fail to hold in light of Example 2.4 and Remark 2.5.

§4. Dependence of Stability on σ and $\omega\gamma$

We conclude with a brief discussion of the stability of the endemic equilibrium as a function of the ratio of the mean removed time to the mean length of infection and the contact number. The following lemma easily adapted from [1, 2] provides a useful way of determining local asymptotic stability of the endemic equilibrium.

Lemma 4.1. *Assume that*

$$\Delta(\lambda) \neq 0 \qquad \textit{for all} \qquad \operatorname{Re} \lambda \geqslant 0. \tag{4.1}$$

Then the endemic equilibrium solution of (1.4) *is locally asymptotically stable in the sense that given* $\varepsilon > 0$, *there exists an* $\eta > 0$ *such that if* $\|R_0(t) - \gamma I_e \int_t^\infty P(u)\,du\|_\infty < \eta$ *and* $|I_0 - I_e| < \eta$ *then* $\|I(t, I_0, R_0(\cdot)) - I_e\|_\infty < \varepsilon$ *and* $I(t, I_0, R_0(\cdot)) \to I_e$ *as* $t \to +\infty$.

We now proceed to show that (1.1)–(1.2)–(1.3) has a locally asymptotically stable endemic equilibrium if either (i) the ratio of the second moments of the infective time to the removed time is sufficiently large compared to the contact number or (ii) the ratio of the mean removed time to the mean infective time is sufficiently small compared to the contact number.

Theorem 4.2. *Assume* (2.1) *and* $\omega\gamma \geqslant 1$. *If either*

(i) $$\sigma - 1 < \frac{1 + \omega\gamma}{\gamma^2 m}$$

or

(ii) $$\sigma - 1 > 2(1 + \omega\gamma)$$

holds where $m = \int_0^\infty sP(s)\,ds$ *is finite by* (2.1), *then the endemic equilibrium is locally asymptotically stable.*

Proof. For (i), we apply a result of Stech [8, Theorem 1] (where "α" $= (\sigma - 1)\gamma$ and "$\eta(s)$" $= \mu(-s)/(\gamma(\sigma - 1)))$ and conclude that all zeros of Δ have non-positive real parts when $(\sigma - 1)\gamma(m\gamma/(1 + \omega\gamma)) < 1$. Moreover, since $\operatorname{Im}\Delta(iv) = 0$ implies

$$\frac{1 + \gamma\omega}{(\sigma - 1)\gamma^2} = \int_0^\infty \frac{\sin(vs)}{vs} sP(s)\,ds \leqslant m,$$

purely imaginary zeros are also impossible.

Suppose (ii) holds; we first show that there are no purely imaginary roots. Since $\lambda = 0$ is never a root, we may integrate by parts to obtain

$$\int_0^\infty e^{-\lambda s}P(s)\,ds = \frac{1}{\lambda}\left[1 + \int_0^\infty e^{-\lambda s}\,dP(s)\right].$$

Using this relation in $\Delta(iv) = 0$ and separating real and imaginary parts, we arrive at

$$\int_0^\infty \sin(vs)\,dP(s) = \frac{v}{\gamma} \tag{4.2}$$

and

$$\int_0^\infty \cos(vs)\,dP(s) = \frac{1 + \omega\gamma}{(\sigma - 1)\gamma^2}v^2 - 1. \tag{4.3}$$

Squaring each equation and adding,

$$\left(\frac{v}{\gamma}\right)^2 + \left(\frac{(1 + \omega\gamma)}{(\sigma - 1)\gamma^2}v^2 - 1\right)^2 = \left|\int_0^\infty e^{ivs}\,dP(s)\right|^2 \leqslant 1,$$

which gives

$$0 < \left(\frac{(1 + \omega\gamma)v}{(\sigma - 1)\gamma}\right)^2 \leqslant 2\left(\frac{1 + \omega\gamma}{\sigma - 1}\right) - 1$$

and the assertion.

To show that there are no roots with $\operatorname{Re}\lambda > 0$ we fix σ_0 and γ_0 so that (ii) holds and assume $\Delta(\lambda_0; \sigma_0, \gamma_0) = 0$ with $\operatorname{Re}\lambda_0 > 0$. For $\delta \geqslant 0$ define $P_\delta(s) = e^{-\delta s}P(s)$ and denote by ω_δ and Δ_δ the corresponding entities obtained by replacing P with P_δ in their definitions. Routine estimates show that Δ_δ (analytic for $\operatorname{Re}\lambda > -\delta$) converges uniformly on compact subsets of $\operatorname{Re}\lambda > 0$ as $\delta \to 0$ and with $\gamma \in [0, \gamma_0]$. Rouche's theorem applied to a small disk centered at λ_0 implies that $\Delta_{\delta_0}(\lambda; \sigma_0, \gamma_0)$ has a root $\tilde{\lambda}$ with $\operatorname{Re}\tilde{\lambda} > 0$ near λ_0 if $\delta_0 > 0$ is chosen sufficiently small. Also, choose δ_0 so small that (ii) holds for $\omega = \omega_{\delta_0}$, $\sigma = \sigma_0$ and $\gamma = \gamma_0$. Now, the zeros of $\Delta_{\delta_0}(\cdot\,; \sigma_0, \gamma)$ must leave $\operatorname{Re}\lambda > 0$ as γ decreases from γ_0 to less than $\omega_{\delta_0}^{-1}$ (by Remark 2.2). An easy estimate of (3.2) shows that any zero, $\tilde{\lambda}$, of Δ_{δ_0} with $\operatorname{Re}\tilde{\lambda} \geqslant 0$ must satisfy $|\tilde{\lambda}| \leqslant (\sigma_0 - 1)\gamma_0$. Thus $\Delta_\delta(\lambda)$ must have a purely imaginary zero for some $\gamma \in [\omega_{\delta_0}^{-1}, \gamma_0]$. This is a contradiction, proving the final assertion.

To summarize, when the mean removed time, ω, is less than the mean infective time, $1/\gamma$, the endemic equilibrium is always globally attracting (see Remark 2.2). Theorem 4.2 implies that when the ratio $\omega\gamma$ exceeds one there is at most a bounded interval of contact numbers, given by

$$\frac{2(1+\omega\gamma)}{\gamma^2(\Sigma^2+\omega^2)} \leqslant \sigma - 1 \leqslant 2(1+\omega\gamma); \qquad m = \frac{1}{2}(\Sigma^2+\omega^2)$$

(where Σ^2 is the variance of τ) on which the endemic equilibrium can fail to be locally asymptotically stable. For fixed variance, this interval increases in size as the ratio $\omega\gamma$ increases. This was observed in the special case of constant removed period in [4].

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