On the Response and Symmetry of Elastic and Hyperelastic Membrane Points

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Dedicated to J. L. Ericksen on the Occasion of his Sixtieth Birthday

1. Introduction

In continuum mechanics constitutive equations for elastic and hyperelastic (3-dim.) material points are important mathematical models with many applications. In this paper we present constitutive equations for elastic and hyperelastic (2-dim.) membrane points. As in the theory of (3-dim.) material points, the central topic for the theory of (2-dim.) membrane points are conditions of material frame-indifference, conditions of material symmetry, and representations for the response functions and for the stored energy functions.

A number of peculiar features of the 2-dimensional models are pointed out and thoroughly discussed in this paper, and many examples are treated in detail. Among the many features which are present in the 2-dimensional models, but absent in the 3-dimensional models, we mention the following three:

1) There is a non-trivial symmetry tranformation common to the symmetry groups of all response functions and the symmetry groups of all stored energy functions.

2) There are examples of elastic and hyperelastic membrane points such that the symmetry groups of the response functions and/or the symmetry groups of the stored energy functions are not contained in the unimodular group. However, we prove that, in general, the *intersection* of the symmetry group of the response function and the symmetry group of the stored energy function of any hyperelastic membrane point is always contained in the unimodular group. Thus this peculiar feature is not inconsistent with the results of [1].

3) There are examples of groups (of automorphisms of a reference configuration) which cannot be the symmetry groups of any elastic membrane point, and there are types of elastic membrane points that include no hyperelastic membrane points at all.

We formulate the constitutive equation and its basic restrictions for an elastic membrane point in general in Section 2. Examples of elastic membrane points with some interesting features are then treated in detail in Section 3. Constitutive equations for the stress tensor and for the stored energy density of a hyperelastic membrane point in general are formulated, and their restrictions, as well as their internal relations, are discussed in detail in Section 4, where representations for the stored energy functions and for the response functions of two general classes of membrane points, having features not present in any type of (3-dim.) hyperelastic material points, are derived. Finally, the general results presented in Section 4 are illustrated explicitly in Section 5 by examples of hyperelastic membrane points having many specific types of symmetries.

2. The Response Function of an Elastic Membrane Point

We assume that a membrane may be represented by a surface manifold $\mathcal{M}$ which is a 2-dimensional differentiable manifold that can be imbedded into the physical space $\mathcal{E}$ by mappings $\chi: \mathcal{M} \to \mathcal{E}$ called *configurations* of $\mathcal{M}$. For a typical configuration χ the image $\chi(\mathcal{M})$ of $\mathcal{M}$ is a smooth surface $\mathcal{S}$ in $\mathcal{E}$. Let p be a typical point in $\mathcal{M}$. Then in a configuration χ the image $\chi(p)$ of p is a point $\boldsymbol{x}$ in $\mathcal{S}$.

The induced linear map $\chi_{*p}: \mathcal{M}_p \to \mathcal{S}_x$ is a linear isomorphism of the tangent plane $\mathcal{M}_p$ of $\mathcal{M}$ at p with the tangent plane $\mathcal{S}_x$ of $\mathcal{S}$ at $\boldsymbol{x}$. We call such a linear isomorphism a *local configuration* of p, and we denote a typical local configuration of p by $\boldsymbol{\nu}$. Then

$$\boldsymbol{\nu}: \mathcal{M}_p \to \boldsymbol{\nu}(\mathcal{M}_p), \tag{2.1}$$

where $\boldsymbol{\nu}(\mathcal{M}_p)$ is a plane in $\mathcal{E}$. Every local configuration $\boldsymbol{\nu}$ may be identified as the local configuration χ_{*p} induced by a configuration χ such that $\boldsymbol{\nu}(\mathcal{M}_p) = \mathcal{S}_x$. Of course, $\boldsymbol{\nu}$ does not determine χ uniquely.

If the stress tensor $\boldsymbol{T}$ at $\boldsymbol{x}$ in all χ having common $\chi_{*p} = \boldsymbol{\nu}$ is uniquely determined by $\boldsymbol{\nu}$, *viz*

$$\boldsymbol{T} = \hat{\boldsymbol{T}}(\boldsymbol{\nu}) \tag{2.2}$$

in each local configuration $\boldsymbol{\nu}$ of p, then p is called an *elastic* membrane point, and $\hat{\boldsymbol{T}}$ is called its *response function*. Notice that the value $\hat{\boldsymbol{T}}(\boldsymbol{\nu})$ of the response function $\hat{\boldsymbol{T}}$ is necessarily a symmetric tensor acting on the tangent plane $\boldsymbol{\nu}(\mathcal{M}_p)$. When $\boldsymbol{\nu}$ varies, both $\boldsymbol{\nu}(\mathcal{M}_p)$ and $\hat{\boldsymbol{T}}(\boldsymbol{\nu})$ vary also, but always in such a way that $\hat{\boldsymbol{T}}(\boldsymbol{\nu})$ acts on $\boldsymbol{\nu}(\mathcal{M}_p)$, *viz*

$$\hat{\boldsymbol{T}}(\boldsymbol{\nu}): \boldsymbol{\nu}(\mathcal{M}_p) \to \boldsymbol{\nu}(\mathcal{M}_p) \quad \forall\, \boldsymbol{\nu}. \tag{2.3}$$

The response function $\hat{\boldsymbol{T}}$ is generally required to satisfy the following condition of material frame-indifference:

$$\hat{\boldsymbol{T}}(\boldsymbol{Q}\boldsymbol{\nu}) = \boldsymbol{Q}\,\hat{\boldsymbol{T}}(\boldsymbol{\nu})\,\boldsymbol{Q}^T \tag{2.4}$$

for all rotations $\boldsymbol{Q}$ acting on the translation space $\mathcal{V}$ of $\mathcal{E}$. The condition (2.4) is consistent with the restriction (2.3) since both sides of (2.4) are tensors acting on the tangent plane $\boldsymbol{Q}\boldsymbol{\nu}(\mathcal{M}_p)$ in $\mathcal{E}$.

Clearly, only the restriction of Q to $\boldsymbol{\nu}(\mathcal{M}_p)$ enters into the condition (2.4); the action of Q on a unit normal $\boldsymbol{n}$ of $\boldsymbol{\nu}(\mathcal{M}_p)$ in $\mathscr{V}$ need not be specified as far as the condition (2.4) is concerned. An important special case of (2.4) is furnished by a Q that corresponds to a rotation of 180° about $\boldsymbol{n}$. In this case the action of Q on $\boldsymbol{\nu}(\mathcal{M}_p)$ is just the negation operator $-\mathbf{1}_\nu$ which commutes with all tensors over $\boldsymbol{\nu}(\mathcal{M}_p)$. As a result (2.4) reduces to

$$\hat{T}(-\boldsymbol{\nu}) = \hat{T}(\boldsymbol{\nu}). \tag{2.5}$$

The constitutive equation (2.2) together with the restriction (2.4) arising from frame-indifference define the abstract mathematical model for an elastic membrane point p in general. To apply this model it is often more convenient to introduce a local reference configuration $\boldsymbol{\varkappa}$ for p. Then a local configuration $\boldsymbol{\nu}$ can be represented by a deformation gradient $\boldsymbol{F}$ of $\mathscr{V}$ such that

$$\boldsymbol{\nu} = \boldsymbol{F}\boldsymbol{\varkappa}. \tag{2.6}$$

Like the rotation Q in (2.4), $\boldsymbol{F}$ enters the representation (2.6) only through its restriction to the tangent plane $\boldsymbol{\varkappa}(\mathcal{M}_p)$; the action of $\boldsymbol{F}$ on a unit normal N of $\boldsymbol{\varkappa}(\mathcal{M}_p)$ in $\mathscr{V}$ has no effect upon the composition $\boldsymbol{F}\boldsymbol{\varkappa}$. Hence we may regard $\boldsymbol{F}$ as a two-point tensor of the form

$$\boldsymbol{F}: \boldsymbol{\varkappa}(\mathcal{M}_p) \to \boldsymbol{\nu}(\mathcal{M}_p). \tag{2.7}$$

Then there is an one-to-one correspondence between $\boldsymbol{\nu}$ and $\boldsymbol{F}$.

Using the representation (2.6) in the sense (2.7), we can rewrite (2.2) as

$$\boldsymbol{T} = \boldsymbol{H}_\varkappa(\boldsymbol{F}) \tag{2.8}$$

and (2.4) as

$$\boldsymbol{H}_\varkappa(\boldsymbol{Q}\boldsymbol{F}) = \boldsymbol{Q}\boldsymbol{H}_\varkappa(\boldsymbol{F})\,\boldsymbol{Q}^T, \tag{2.9}$$

where $\boldsymbol{F}$ acts on $\boldsymbol{\varkappa}(\mathcal{M}_p)$ while Q and $\boldsymbol{H}_\varkappa(\boldsymbol{F})$ act on $\boldsymbol{F}\boldsymbol{\varkappa}(\mathcal{M}_p)$. The condition (2.5) now takes the form

$$\boldsymbol{H}_\varkappa(-\boldsymbol{F}) = \boldsymbol{H}_\varkappa(\boldsymbol{F}), \tag{2.10}$$

which looks as if we had taken $Q = -\mathbf{1}$ in (2.9). In fact, $-\boldsymbol{F}$ is simply the (proper) deformation gradient of $\mathscr{V}$ whose restriction to $\boldsymbol{\varkappa}(\mathcal{M}_p)$ is the negation of that of $\boldsymbol{F}$. There is no need to use any improper deformation gradient of $\mathscr{V}$ in our formulation, since the actions of $-\boldsymbol{F}$ and $\boldsymbol{F}$ on N need not be specified as far as the equation (2.10) is concerned.

The deformation gradient $\boldsymbol{F}$ has a polar decomposition

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}, \tag{2.11}$$

where $\boldsymbol{U}$ is a positive-definite symmetric tensor over $\boldsymbol{\varkappa}(\mathcal{M}_p)$,

$$\boldsymbol{U}: \boldsymbol{\varkappa}(\mathcal{M}_p) \to \boldsymbol{\varkappa}(\mathcal{M}_p), \tag{2.12}$$

while $\boldsymbol{R}$ is a rotation of $\mathscr{V}$ and so may be regarded as a two-point tensor like $\boldsymbol{F}$, *viz*

$$\boldsymbol{R}: \boldsymbol{\varkappa}(\mathcal{M}_p) \to \boldsymbol{\nu}(\mathcal{M}_p). \tag{2.13}$$

Substituting (2.11) into (2.8) and then using (2.9), we obtain the representation

$$H_\varkappa(F) = R H_\varkappa(U) R^T, \tag{2.14}$$

where $H_\varkappa(U)$ acts on $\varkappa(\mathcal{M}_p)$. As usual we define

$$C = U^2 = F^T F, \tag{2.15}$$

where C acts on $\varkappa(\mathcal{M}_p)$ while F^T is a two-point tensor of the form

$$F^T : \nu(\mathcal{M}_p) \to \varkappa(\mathcal{M}_p). \tag{2.16}$$

Using the tensor C, we can rewrite the representation (2.14) as

$$H_\varkappa(F) = F A_\varkappa(C) F^T, \tag{2.17}$$

where $A_\varkappa(C)$ acts on $\varkappa(\mathcal{M}_p)$ and is related to $H_\varkappa(U)$ by

$$A_\varkappa(U^2) = U^{-1} H_\varkappa(U) U^{-1}. \tag{2.18}$$

So far, we have formulated the response function of an elastic membrane point p in general. Next we consider the material symmetry of p. Let P be a linear automorphism of $\varkappa(\mathcal{M}_p)$. Then the composition $P\varkappa$ is a local reference configuration having the same tangent plane as that of $\varkappa$, *viz*

$$P\varkappa(\mathcal{M}_p) = \varkappa(\mathcal{M}_p) \tag{2.19}$$

in the sense of set theory. We say that $\varkappa$ and $P\varkappa$ are *materially isomorphic* if their corresponding response functions are identical, *i.e.*,

$$H_\varkappa(F) = H_{P\varkappa}(F) \quad \forall F. \tag{2.20}$$

Since in general

$$H_\varkappa(F) = \hat{T}(F\varkappa) \tag{2.21}$$

for all local reference configurations $\varkappa$, the condition (2.20) is equivalent to

$$H_\varkappa(F) = H_\varkappa(FP) \quad \forall F. \tag{2.22}$$

Using the representation (2.18), we can rewrite the condition (2.22) as

$$A_\varkappa(C) = P A_\varkappa(P^T C P) P^T \quad \forall C. \tag{2.23}$$

The condition (2.23), though it looks more complicated, is actually simpler than the condition (2.22), because all tensors in (2.23) act on the same tangent plane $\varkappa(\mathcal{M}_p)$.

The set of all linear automorphisms P of $\varkappa(\mathcal{M}_p)$ satisfying the conditions (2.22) or (2.23) forms a group $\mathcal{G}_\varkappa$. We call it the *symmetry group* of p relative to $\varkappa$. Comparing (2.22) with (2.10), we see that the negation tensor $-\mathbf{1}_\varkappa$ on $\varkappa(\mathcal{M}_p)$ is always a member of $\mathcal{G}_\varkappa$ regardless of the elastic membrane point p. This fact may be seen from (2.23) also, since $-\mathbf{1}_\varkappa$ commutes with all tensors over $\varkappa(\mathcal{M}_p)$. Of course, the identity tensor $\mathbf{1}_\varkappa$ on $\varkappa(\mathcal{M}_p)$, being the identity element of the group of automorphisms of $\varkappa(\mathcal{M}_p)$, is a member of all symmetry groups $\mathcal{G}_\varkappa$ relative to $\varkappa$.

Notice that $\varkappa$ and $-\varkappa$ are both possible and distinctly different local reference configurations for the membrane point p. The preceding remark implies that the response functions $\boldsymbol{H}_\varkappa$ and $\boldsymbol{H}_{-\varkappa}$ are always the same or, equivalently, that $-\mathbf{1}_\varkappa$ is a non-trivial common symmetry belonging to all possible $\mathscr{G}_\varkappa$ relative to $\boldsymbol{\varkappa}$. This feature contrasts the response of an elastic membrane point with that of a (3-dim.) elastic material point, since the negation operator in $\mathscr{V}$ is an improper tensor on $\mathscr{V}$ and, therefore, is not appropriate as a material symmetry with respect to mechanical response.

It should be pointed out that the deformation gradient $\boldsymbol{F}$ of the form (2.7) is not assigned a determinant because the planes $\boldsymbol{\varkappa}(\mathscr{M}_p)$ and $\boldsymbol{v}(\mathscr{M}_p)$ are not oriented in $\mathscr{V}$. For tensors $\boldsymbol{P}$ satisfying (2.19), of course, we define

$$\det \boldsymbol{P} = \det [P_{\alpha\beta}], \tag{2.24}$$

where $P_{\alpha\beta}$ are the components of $\boldsymbol{P}$ relative to an orthonormal basis $\{\boldsymbol{e}_\alpha\}$ in $\boldsymbol{\varkappa}(\mathscr{M}_p)$, *viz*

$$P_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{P}\boldsymbol{e}_\beta, \; \boldsymbol{P}\boldsymbol{e}_\beta = P_{\alpha\beta}\boldsymbol{e}_\alpha. \tag{2.25}$$

As usual $\boldsymbol{P}$ is an automorphism of $\boldsymbol{\varkappa}(\mathscr{M}_p)$ if and only if

$$\det \boldsymbol{P} \neq 0. \tag{2.26}$$

Then we call $\boldsymbol{P}$ proper or improper if $\det \boldsymbol{P} > 0$ or $\det \boldsymbol{P} < 0$, respectively. However unlike an improper tensor over $\mathscr{V}$, an improper tensor over $\boldsymbol{\varkappa}(\mathscr{M}_p)$ is appropriate as a deformation of $\boldsymbol{\varkappa}(\mathscr{M}_p)$. For example, consider the improper reflection $\boldsymbol{P}_1$ with respect to $\boldsymbol{e}_1$ in $\boldsymbol{\varkappa}(\mathscr{M}_p)$ defined by

$$\boldsymbol{P}_1\boldsymbol{e}_1 = -\boldsymbol{e}_1, \quad \boldsymbol{P}_1\boldsymbol{e}_2 = \boldsymbol{e}_2. \tag{2.27}$$

We can perform the deformation $\boldsymbol{P}_1$ in $\mathscr{V}$ by simply regarding $\boldsymbol{P}_1$ as the proper tensor over $\mathscr{V}$ satisfying (2.27) and

$$\boldsymbol{P}\boldsymbol{N} = -\boldsymbol{N}. \tag{2.28}$$

By following essentially the same approach as in [2], we can derive a general representation for $\boldsymbol{A}_\varkappa$ subject to the restriction (2.23) for all $\boldsymbol{P}$ belonging to an unspecified symmetry group $\mathscr{G}_\varkappa$ in general. For each given symmetry group $\mathscr{G}_\varkappa$ an explicit general solution of the condition (2.23) is called a *representation* for the reduced response functions of the type of elastic membrane points having the common material symmetry described by the group $\mathscr{G}_\varkappa$. Of course, a representation for $\boldsymbol{A}_\varkappa$ gives rise to a representation for $\boldsymbol{H}_\varkappa$ by the general relation (2.17) which automatically satisfies the condition of material frame-indifference (2.9).

Representations for the reduced response functions of many types of elastic membrane points, including surface tension points, elastic fluid membrane points, elastic subfluid membrane points, isotropic elastic membrane points, hemitropic elastic membrane points, and orthotropic membrane points, are derived in the next section.

3. Some Examples of Elastic Membrane Points

The response of an elastic membrane differs essentially from that of a (3-dim.) elastic material in another way: for an elastic membrane it is not always appropriate to require that the symmetry group $\mathscr{G}_\varkappa$ be contained in the unimodular group $\mathscr{U}_\varkappa$ of $\varkappa(\mathscr{M}_p)$. Indeed, if the membrane is a soap film or an interface between two substances, the surface stress tensor is

$$T = H_\varkappa(F) = c\mathbf{1}_\nu, \tag{3.1}$$

where c is the surface tension which is a constant independent of the deformation. Hence the conditions (2.22) or (2.23) are satisfied by all automorphisms P of $\varkappa(\mathscr{M}_p)$. In other words, the symmetry group $\mathscr{G}_\varkappa$ is the general linear group $\mathscr{L}_\varkappa$ of $\varkappa(\mathscr{M}_p)$ in this case.

The condition $\mathscr{G}_\varkappa = \mathscr{L}_\varkappa$ is actually necessary and sufficient for (3.1). Using $U \in \mathscr{G}_\varkappa$, we get

$$H_\varkappa(F) = H_\varkappa(RU) = H_\varkappa(R) = RH_\varkappa(\mathbf{1}_\varkappa)\, R^T, \tag{3.2}$$

where we have used (2.9). Now the value $H_\varkappa(\mathbf{1}_\varkappa)$ must take the form

$$H_\varkappa(\mathbf{1}_\varkappa) = c\mathbf{1}_\varkappa, \tag{3.3}$$

since from (2.2) and (2.9)

$$H_\varkappa(Q) = H_\varkappa(\mathbf{1}_\varkappa) = QH_\varkappa(\mathbf{1}_\varkappa)\, Q^T \tag{3.4}$$

for all rotations Q of $\varkappa(\mathscr{M}_p)$. Substituting (3.3) into (3.2), we then get (3.1).

The reduced response function $A_\varkappa(C)$ associated with the special response function given by (3.1) is

$$A_\varkappa(C) = cC^{-1}. \tag{3.5}$$

This special reduced response function clearly satisfies the condition (2.23) for all $P \in \mathscr{L}_\varkappa$.

Notice that the group $\mathscr{L}_\varkappa$ has a subgroup $\mathscr{L}_\varkappa^+$ consisting in all proper linear automorphisms of $\varkappa(\mathscr{M}_p)$. Since both $\mathbf{1}_\varkappa$ and $-\mathbf{1}_\varkappa$ are contained in $\mathscr{L}_\varkappa^+$, one might wonder whether $\mathscr{L}_\varkappa^+$ could be a symmetry group. We claim that $\mathscr{L}_\varkappa^+$ cannot be a symmetry group of any elastic membrane point. Specifically, we prove that

$$\mathscr{G}_\varkappa \supset \mathscr{L}_\varkappa^+ \;\Leftrightarrow\; \mathscr{G}_\varkappa = \mathscr{L}_\varkappa. \tag{3.6}$$

Indeed, if (2.22) holds for all $P \in \mathscr{L}_\varkappa^+$, then

$$H_\varkappa(P) = H_\varkappa(\overline{P}) \tag{3.7}$$

for all $\overline{P} \in \mathscr{L}_\varkappa^+$ such that

$$\det P = \det \overline{P}. \tag{3.8}$$

In particular,

$$H_\varkappa(P) = H_\varkappa(QP) \tag{3.9}$$

for all $P \in \mathscr{L}_\varkappa^+$ and for all rotations $Q \in \mathscr{L}_\varkappa^+$. But by virtue of (2.9) the right-hand side of (3.9) is equal to $QH_\varkappa(P)\,Q^T$. As a result

$$QH_\varkappa(P) = H_\varkappa(P)\,Q, \tag{3.10}$$

which implies that

$$H_\varkappa(P) = c\mathbf{1}_\varkappa \tag{3.11}$$

for all $P \in \mathscr{L}_\varkappa^+$, where c is a constant independent of $P \in \mathscr{L}_\varkappa^+$. It follows then that

$$H_\varkappa(U) = c\mathbf{1}_\varkappa \tag{3.12}$$

for all positive-definite symmetric tensors U on $\varkappa(\mathscr{M}_p)$. Substituting (3.12) into (2.14), we obtain (3.1), which has the symmetry group $\mathscr{L}_\varkappa$. Thus (3.6) is proved.

It is not too surprising that some subgroups of $\mathscr{L}_\varkappa$, such as the subgroup $\mathscr{L}_\varkappa^+$, cannot be the symmetry group of any elastic membrane point. Indeed, the trivial subgroup consisting of the identity tensor $\mathbf{1}_\varkappa$ alone is also an example, since we have remarked that all symmetry groups of elastic membrane points relative to $\varkappa$ must contain both $\mathbf{1}_\varkappa$ and $-\mathbf{1}_\varkappa$.

If $\mathscr{G}_\varkappa$ is the unimodular group $\mathscr{U}_\varkappa$ of $\varkappa(\mathscr{M}_p)$ consisting of all $P \in \mathscr{L}_\varkappa$ such that

$$|\det P| = 1, \tag{3.13}$$

the elastic membrane point p is called a *fluid* membrane point. The constitutive equation of a fluid membrane point is

$$T = c(\varrho)\,\mathbf{1}_\nu, \tag{3.14}$$

where c is not a constant. The proof for the fact $\mathscr{G}_\varkappa = \mathscr{U}_\varkappa \Leftrightarrow$ (3.14) is similar to that for the fact $\mathscr{G}_\varkappa = \mathscr{L}_\varkappa \Leftrightarrow$ (3.1). Furthermore, we can show that the subgroup $\mathscr{U}_\varkappa^+$ consisting in all $P \in \mathscr{U}_\varkappa$ such that

$$\det P = 1 \tag{3.15}$$

cannot be the symmetry group of any elastic membrane point relative to $\varkappa$. More specifically, the result is

$$\mathscr{G}_\varkappa \supset \mathscr{U}_\varkappa^+ \;\Leftrightarrow\; \mathscr{G}_\varkappa \supset \mathscr{U}_\varkappa. \tag{3.16}$$

If $\mathscr{G}_\varkappa$ contains the orthogonal group $\mathscr{O}_\varkappa$ of $\varkappa(\mathscr{M}_p)$ we call p an *isotropic* elastic membrane point. It is well known that the condition $\mathscr{G}_\varkappa \supset \mathscr{O}_\varkappa$ is not invariant under change of local reference configurations. Hence we distinguish those $\varkappa$ for which $\mathscr{G}_\varkappa \supset \mathscr{O}_\varkappa$ as *undistorted* local reference configurations. Relative to such a reference configuration $\varkappa$ the constitutive equation of p is (*cf.* [3], [4] where the tensor $G = B^{-1}$ is used)

$$T = f_0\mathbf{1}_\nu + f_1 B, \tag{3.17}$$

where B is a positive-definite symmetric tensor acting on $\nu(\mathscr{M}_p)$,

$$B = FF^T, \tag{3.18}$$

and where f_0 and f_1 are functions of the principal invariants

$$I_B = \operatorname{tr} \boldsymbol{B}, \quad II_B = \det \boldsymbol{B} \tag{3.19}$$

of the tensor $\boldsymbol{B}$. The constitutive equation (3.17) may be expressed in terms of the reduced response function $\boldsymbol{A}_\varkappa$ by

$$\boldsymbol{A}_\varkappa(\boldsymbol{C}) = f_1 \mathbf{1}_\varkappa + f_0 \boldsymbol{C}^{-1}, \tag{3.20}$$

where the invariants of $\boldsymbol{B}$ coincide with those of $\boldsymbol{C}$. The representation (3.20) means that $\boldsymbol{A}_\varkappa(\boldsymbol{C})$ is an isotropic function, *i.e.*, $\boldsymbol{A}_\varkappa$ obeys the condition

$$\boldsymbol{A}_\varkappa(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{A}_\varkappa(\boldsymbol{C})\,\boldsymbol{Q}^T \quad \forall\, \boldsymbol{Q} \in \mathcal{O}_\varkappa, \tag{3.21}$$

which is just the condition (2.23) when $\mathcal{G}_\varkappa = \mathcal{O}_\varkappa$.

The group $\mathcal{O}_\varkappa$ has a subgroup $\mathcal{O}_\varkappa^+$ consisting of rotations of $\boldsymbol{\varkappa}(\mathcal{M}_p)$. Unlike the groups $\mathcal{L}_\varkappa^+$ and $\mathcal{U}_\varkappa^+$, the group $\mathcal{O}_\varkappa^+$ may be the symmetry group of an elastic membrane point p relative to $\boldsymbol{\varkappa}$. We call p a *hemitropic* elastic membrane point if $\mathcal{G}_\varkappa \supset \mathcal{O}_\varkappa^+$. Of course, the condition $\mathcal{G}_\varkappa \supset \mathcal{O}_\varkappa^+$ is not invariant under change of local reference configurations; we call those $\boldsymbol{\varkappa}$ for which $\mathcal{G}_\varkappa \supset \mathcal{O}_\varkappa^+$ *undistorted* local reference configurations.

Relative to an undistorted local reference configuration $\boldsymbol{\varkappa}$ the reduced response function $\boldsymbol{A}_\varkappa$ is a hemitropic function, *i.e.* $\boldsymbol{A}_\varkappa$ obeys the condition

$$\boldsymbol{A}_\varkappa(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{A}_\varkappa(\boldsymbol{C})\,\boldsymbol{Q}^T \quad \forall\, \boldsymbol{Q} \in \mathcal{O}_\varkappa^+, \tag{3.22}$$

which may be interpreted as follows: Let the spectral forms of $\boldsymbol{C}$ and $\boldsymbol{A}_\varkappa(\boldsymbol{C})$ be

$$\boldsymbol{C} = c_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + c_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \tag{3.23}$$

and

$$\boldsymbol{A}_\varkappa(\boldsymbol{C}) = t_1 \boldsymbol{f}_1 \otimes \boldsymbol{f}_1 + t_2 \boldsymbol{f}_2 \otimes \boldsymbol{f}_2, \tag{3.24}$$

where we require the orthonormal principal bases $\{\boldsymbol{e}_\alpha\}$ and $\{\boldsymbol{f}_\alpha\}$ to have the same orientation in $\boldsymbol{\varkappa}(\mathcal{M}_p)$. Applying the conjugation of $\boldsymbol{C}$ by $\boldsymbol{Q}$ to (3.23), we obtain

$$\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T = c_1 \boldsymbol{Q}\boldsymbol{e}_1 \otimes \boldsymbol{Q}\boldsymbol{e}_1 + c_2 \boldsymbol{Q}\boldsymbol{e}_2 \otimes \boldsymbol{Q}\boldsymbol{e}_2. \tag{3.25}$$

Then the condition (3.22) implies that such an operation on $\boldsymbol{C}$ gives rise to a similar operation on the value of $\boldsymbol{A}_\varkappa(\boldsymbol{C})$, *i.e.*,

$$\boldsymbol{A}_\varkappa(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{A}_\varkappa(\boldsymbol{C})\,\boldsymbol{Q}^T = t_1 \boldsymbol{Q}\boldsymbol{f}_1 \otimes \boldsymbol{Q}\boldsymbol{f}_1 + t_2 \boldsymbol{Q}\boldsymbol{f}_2 \otimes \boldsymbol{Q}\boldsymbol{f}_2. \tag{3.26}$$

Since all tensors $\boldsymbol{C}$ having common eigenvalues c_1 and c_2 are related by conjugations of $\boldsymbol{Q} \in \mathcal{O}_\varkappa^+$, the condition (3.26) means that t_1 and t_2 are determined by c_1 and c_2. Hence if we define an intermediary function $\hat{\boldsymbol{A}}_\varkappa(\boldsymbol{C})$ by the spectral form

$$\hat{\boldsymbol{A}}_\varkappa(\boldsymbol{C}) = t_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + t_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \tag{3.27}$$

where t_1 and t_2 are the eigenvalues of $A_\varkappa(C)$ as shown in (3.24), then $\hat{A}_\varkappa(C)$ is an isotropic function of C, *i.e.*, $\hat{A}_\varkappa(C)$ has the representation of the form (3.20)

$$\hat{A}_\varkappa(C) = f_1 \mathbf{1}_\varkappa + f_0 C^{-1}, \tag{3.28}$$

where f_1 and f_0 are functions of I_C and II_C as before.

Comparing (3.27) with (3.24), we see that $A_\varkappa(C)$ and $\hat{A}_\varkappa(C)$ are related by

$$A_\varkappa(C) = S\hat{A}_\varkappa(C)\, S^T, \tag{3.29}$$

where S denotes the rotation from $\{e_\alpha\}$ to $\{f_\alpha\}$, *viz*

$$Se_\alpha = f_\alpha, \quad \alpha = 1, 2. \tag{3.30}$$

Since $\mathcal{O}_\varkappa^+$ is an Abelian group, S is also the rotation from $\{Qe_\alpha\}$ to $\{Qf_\alpha\}$, *viz*

$$Qf_\alpha = QSe_\alpha = SQe_\alpha. \tag{3.31}$$

Hence from (3.25) and (3.26) we see that S depends only on the invariants of C. Substituting (3.28) into (3.29), we obtain the representation

$$A_\varkappa(C) = f_1 \mathbf{1}_\varkappa + f_0 S C^{-1} S^T, \tag{3.32}$$

where f_0, f_1, and $S \in \mathcal{O}_\varkappa^+$ are functions of I_C and II_C. Clearly (3.32) is sufficient for (3.22), since according to (3.32)

$$A_\varkappa(QCQ^T) = f_1 \mathbf{1}_\varkappa + f_0 SQC^{-1}Q^T S^T = f_1 \mathbf{1}_\varkappa + f_0 QSC^{-1}S^T Q^T \tag{3.33}$$

for all rotations $Q \in \mathcal{O}_\varkappa^+$.

Notice that, like the functions f_0 and f_1, the tensor function S is not unique. Indeed, S is entirely arbitrary when $c_1 = c_2$ and/or $t_1 = t_2$, since the principal bases $\{e_\alpha\}$ and/or $\{f_\alpha\}$ are arbitrary in these cases. The former case corresponds to

$$I_C^2 = 4II_C, \tag{3.34}$$

while the latter case corresponds to

$$f_0(I_C, II_C) = 0. \tag{3.35}$$

Also, since C is positive-definite symmetric on $\varkappa(\mathcal{M}_p)$, its invariants I_C and II_C are any positive numbers satisfying the condition

$$I_C^2 - 4II_C \geqq 0 \tag{3.36}$$

which defines the natural domain of the functions f_0, f_1, and S.

For (3-dim.) elastic materials a subfluid is defined by the condition that the symmetry group contains a dilatation group, *cf.* [5]. We can define an elastic subfluid membrane point p by a similar condition. For example, let e be a particular unit vector in $\varkappa(\mathcal{M}_p)$. We define f as the unit vector in $\nu(\mathcal{M}_p)$ in the direction of Fe, *viz*,

$$f = Fe/\|Fe\|. \tag{3.37}$$

Then we can model a film with a directional surface tension by the constitutive equation

$$T = H_\varkappa(F) = c\mathbf{1}_\nu + d f \otimes f, \tag{3.38}$$

where c and d are constants independent of F. The surface tension in $\nu(\mathcal{M}_p)$ takes the constant value $c+d$ in the direction of f and the constant value c in the direction perpendicular to f. Of course, $d \neq 0$, for otherwise (3.38) reduces to (3.1). The maximum shear stress takes the constant value $d/2$ in the direction at 45° relative to f.

By using arguments similar to those presented in [5], we can show that (3.38) represents the response of an elastic membrae point p such that the symmetry group $\mathcal{G}_\varkappa$ consists in all tensors $P \in \mathcal{L}_\varkappa$ preserving the line in $\varkappa(\mathcal{M}_p)$ generated by the unit vector e, *i.e.*, e is an eigenvector of P

$$Pe = \lambda e, \tag{3.39}$$

where $\lambda \neq 0$ since P is an automorphism of $\varkappa(\mathcal{M}_p)$. Since $-\mathbf{1}_\varkappa$ is necessarily a member of $\mathcal{G}_\varkappa$, the eigenvalue λ may be either positive or negative. Thus $\mathcal{G}_\varkappa$ preserves the line generated by e but does not preserve the direction of e.

The reduced response function $A_\varkappa(C)$ associated with (3.38) is

$$A_\varkappa(C) = cC^{-1} + \frac{d}{e \cdot Ce}\, e \otimes e. \tag{3.40}$$

We can verify easily that this $A_\varkappa$ obeys the condition (2.23) for all $P \in \mathcal{L}_\varkappa$ satisfying (3.39).

When the constitutive equation (3.38) is replaced by

$$T = H_\varkappa(F) = c(\varrho)\, \mathbf{1}_\nu + d(\varrho) f \otimes f, \tag{3.41}$$

where c and d are not both constant and where d is not identically 0, the response is similar to a type of elastic (3-dim.) subfluids (*cf.* [5, Type 2]). It can be shown that (3.41) represents the response of an elastic membrane point p such that the symmetry group $\mathcal{G}_\varkappa$ consists of all tensors $P \in \mathcal{U}_\varkappa$ preserving the line in $\varkappa(\mathcal{M}_p)$ generated by the unit vector e. Hence, P satisfies the conditions (3.13) and (3.39). The reduced form of (3.41) is (3.40) with c and d given by the functions of ϱ as shown in (3.41).

Another type of elastic subfluid membrane points may be defined as follows: Let $\mathcal{G}_\varkappa$ be the group of all tensors P in $\mathcal{L}_\varkappa$ (or $\mathcal{U}_\varkappa$) preserving two preferred lines in $\varkappa(\mathcal{M}_p)$. As in [5, Type 9] we call $\varkappa$ an undistorted reference if the preferred lines are orthogonal in $\varkappa(\mathcal{M}_p)$. In such an undistorted reference configuration $\varkappa$, there is an orthonormal basis $\{e_\alpha\}$ which generates the preferred lines. Then we define the normalized basis $\{f_\alpha\}$ in $\nu(\mathcal{M}_p)$ by (3.37), *viz*,

$$f_\alpha = Fe_\alpha / \|Fe_\alpha\|. \tag{3.42}$$

The normalized basis $\{f_\alpha\}$ generally is not orthogonal. We define

$$\gamma = f_1 \cdot f_2. \tag{3.43}$$

Using arguments similar to those presented in [5], we can show that the response function has the representation

$$T = H_\varkappa(F) = af_1 \otimes f_1 + b(f_1 \otimes f_2 + f_2 \otimes f_1) + cf_2 \otimes f_2, \tag{3.44}$$

where a and c are distinct even functions of γ while b is an odd function of γ. When $\mathscr{G}_\varkappa \subset \mathscr{U}_\varkappa$, a, b, and c also depend on the density ϱ.

Comparing (3.44) with (2.17), we see that the reduced response function $A_\varkappa$ has the representation

$$A_\varkappa(C) = \frac{a}{C_{11}} e_1 \otimes e_1 + \frac{b}{(C_{11}C_{22})^{\frac{1}{2}}}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{c}{C_{22}} e_2 \otimes e_2, \tag{3.45}$$

where the components $C_{\alpha\beta}$ of C are taken relative to the orthonormal basis $\{e_\alpha\}$, and where the argument γ of a, b, and c may be expressed in terms of C by

$$\gamma = C_{12}/(C_{11}C_{22})^{\frac{1}{2}}. \tag{3.46}$$

The symmetry group $\mathscr{G}_\varkappa$ consists in all tensors $P \in \mathscr{L}_\varkappa$ (or $\mathscr{U}_\varkappa$) such that

$$Pe_1 = \lambda_1 e_1, \quad Pe_2 = \lambda_2 e_2, \tag{3.47}$$

where λ_1 and λ_2 are arbitrary non zero numbers. We can verify easily that the representation (3.45) obeys the condition (2.23) for all such tensors P.

When the two preferred lines in the preceding type of elastic subfluid membrane points are equivalent in the sense of [5, Type 8], the result becomes another type of elastic subfluid membrane points. The response functions for this type of membrane points still take the form (3.44) and (3.45) but the functions a and c are now identical, so that the symmetry group $\mathscr{G}_\varkappa$ contains not only those $P \in \mathscr{L}_\varkappa$ (or $\mathscr{U}_\varkappa$) satisfying (3.47), but also those $P \in \mathscr{L}_\varkappa$ (or $\mathscr{U}_\varkappa$) such that

$$Pe_1 = \mu_1 e_2, \quad Pe_2 = \mu_2 e_1, \tag{3.48}$$

where μ_1 and μ_2 are arbitrary non-zero numbers. Again, we can verify that the representation (3.45) with $a = c$ obeys the condition (2.23) for those additional tensors P satisfying (3.48) or (3.47).

So far, we have defined the counterparts of (3-dim.) elastic subfluids in the context of elastic membrane points. It is important that we require the symmetry groups to consist in all tensors P belonging to $\mathscr{L}_\varkappa$ (or $\mathscr{U}_\varkappa$) and satisfying a condition of the forms (3.39), (3.47), or (3.48). If we reduce the symmetry groups to the groups consisting in all tensors P belonging to $\mathscr{L}_\varkappa^+$ (or $\mathscr{U}_\varkappa^+$) and satisfying appropriate conditions of the forms (3.39), (3.47), or (3.48), the resulting membrane points are much different from the subfluid membrane points which we have defined. In some sense, when the improper tensors are removed from the symmetry group $\mathscr{G}_\varkappa$, the representation of the response function suffers a change similar to that between (3.20) and (3.32).

For example, consider the case in which $\mathscr{G}_\varkappa$ is formed by all tensors $P \in \mathscr{L}_\varkappa^+$ such that (3.39) holds. Then for all such tensors P

$$H_\varkappa(P) = K, \tag{3.49}$$

where K is a constant symmetric tensor over $\varkappa(\mathscr{M}_p)$ which need not have e as its eigenvector. Now for any tensor $W \in \mathscr{L}_\varkappa^+$ in general we can define a rotation $S \in \mathscr{O}_\varkappa^+$ by the condition

$$Se = We/\|We\|. \tag{3.50}$$

Then $S^T W \in \mathscr{L}_\varkappa^+$ and

$$S^T W e = \|We\|\, e, \tag{3.51}$$

so that $S^T W \in \mathscr{G}_\varkappa$. Hence from (3.49)

$$H_\varkappa(S^T W) = K. \tag{3.52}$$

Using (2.9), we then have the representation

$$H_\varkappa(W) = S(W)\, K S(W)^T \tag{3.53}$$

for all $W \in \mathscr{L}_\varkappa^+$, where we have regarded S as a function of W by the condition (3.50).

Clearly the representation (3.53) is consistent with the condition (2.9) for all $Q \in \mathscr{O}_\varkappa^+$. Indeed, (3.50) implies directly that

$$S(QW) = QS(W). \tag{3.54}$$

Hence from (3.53)

$$H_\varkappa(QW) = S(QW)\, KS(QW)^T = QS(W)\, K(QS(W))^T = QS(W)\, KS(W)^T Q^T. \tag{3.55}$$

Therefore, if we use the restriction of (3.53) to positive-definite symmetric tensors $U \in \mathscr{L}_\varkappa^+$ and define the function $H_\varkappa$ for all tensors F of the form (2.7) by (2.14), *viz*

$$H_\varkappa(F) = RS(U)\, KS(U)^T R^T, \tag{3.56}$$

then the representation (3.56) is consistent with (3.53).

Now we show that the form (3.56) is not only necessary but also sufficient for the symmetry condition (2.22) for all $P \in \mathscr{L}_\varkappa^+$ satisfying (3.39). Since (3.56) is consistent with (2.9), it suffices to verify

$$H_\varkappa(UP) = H_\varkappa(U). \tag{3.57}$$

Since (3.56) is consistent with (3.53), the left-hand side of (3.57) is

$$H_\varkappa(UP) = S(UP)\, KS(UP)^T. \tag{3.58}$$

From (3.50) if the eigenvalue λ in (3.39) is positive, then

$$S(UP) = S(U). \tag{3.59}$$

On the other hand, if λ is negative, then

$$S(UP) = -S(U). \tag{3.60}$$

For both cases (3.59) and (3.60) the right-hand side of (3.58) reduces to

$$S(UP)\, KS(UP)^T = S(U)\, KS(U)^T = H_\varkappa(U). \tag{3.61}$$

Thus (3.57) holds.

Notice that the representation (3.56) reduces to the representation (3.38) when the constant tensor K takes the special form

$$K = H_\varkappa(P) = c\mathbf{1}_\varkappa + d e \otimes e, \tag{3.62}$$

which is a symmetric tensor having $\boldsymbol{e}$ as an eigenvector. For an elastic subfluid membrane point considered before, $\mathscr{G}_\varkappa$ contains improper tensors $\boldsymbol{P}$ which force the value $\boldsymbol{H}_\varkappa(\boldsymbol{P})$ to take the special form (3.62). The representation formula (3.56) shows that, when those improper tensors are removed from the symmetry group $\mathscr{G}_\varkappa$, the stress tensor $\boldsymbol{H}_\varkappa(\boldsymbol{P}) = \boldsymbol{K}$ may differ from the special form (3.62) by an arbitrary rotation, in much the same way that (3.32) is related to (3.20).

In the preceding examples, with the exception of the membrane points defined by the constitutive equations (3.32) and (3.56), all others, namely, those defined by (3.1), (3.14), (3.17), (3.38), (3.41), and (3.44) (with a, c distinct or equal) are *orthotropic* elastic membrane points, which may be defined in general as follows: We require that the symmetry group $\mathscr{G}_\varkappa$ contains the reflections with respect to certain basis vectors $\boldsymbol{e}_\alpha$ in $\varkappa(\mathscr{M}_p)$, *i.e.*, $\mathscr{G}_\varkappa \ni \boldsymbol{P}_1, \boldsymbol{P}_2$ defined as follows:

$$\boldsymbol{P}_1\boldsymbol{e}_1 = -\boldsymbol{e}_1, \quad \boldsymbol{P}_1\boldsymbol{e}_2 = \boldsymbol{e}_2, \quad \boldsymbol{P}_2\boldsymbol{e}_1 = \boldsymbol{e}_1, \quad \boldsymbol{P}_2\boldsymbol{e}_2 = -\boldsymbol{e}_2. \tag{3.63}$$

We call $\varkappa$ an undistorted reference configuration if the basis $\{\boldsymbol{e}_\alpha\}$ is orthonormal in $\varkappa(\mathscr{M}_p)$.

Let $\mathscr{P}_\varkappa$ denote the group

$$\mathscr{P}_\varkappa = \{\boldsymbol{1}_\varkappa, -\boldsymbol{1}_\varkappa, \boldsymbol{P}_1, \boldsymbol{P}_2\}. \tag{3.64}$$

We call the basis $\{\boldsymbol{e}_\alpha\}$ which enters into the definition (3.63) for $\boldsymbol{P}_\alpha$ a principal basis for $\mathscr{P}_\varkappa$. Like a principal basis for a symmetric tensor, a principal basis for $\mathscr{P}_\varkappa$ is unique to within an arbitrary change of signs and indices. We call p an orthotropic elastic solid membrane point if $\mathscr{G}_\varkappa = \mathscr{P}_\varkappa$; for an orthotropic elastic membrane point in general we require only that $\mathscr{G}_\varkappa \supset \mathscr{P}_\varkappa$.

Clearly $\mathscr{P}_\varkappa$ is a group with the following multiplication table:

	$\boldsymbol{1}_\varkappa$	$\boldsymbol{P}_1$	$\boldsymbol{P}_2$	$-\boldsymbol{1}_\varkappa$
$\boldsymbol{1}_\varkappa$	$\boldsymbol{1}_\varkappa$	$\boldsymbol{P}_1$	$\boldsymbol{P}_2$	$-\boldsymbol{1}_\varkappa$
$\boldsymbol{P}_1$	$\boldsymbol{P}_1$	$\boldsymbol{1}_\varkappa$	$-\boldsymbol{1}_\varkappa$	$\boldsymbol{P}_2$
$\boldsymbol{P}_2$	$\boldsymbol{P}_2$	$-\boldsymbol{1}_\varkappa$	$\boldsymbol{1}_\varkappa$	$\boldsymbol{P}_1$
$-\boldsymbol{1}_\varkappa$	$-\boldsymbol{1}_\varkappa$	$\boldsymbol{P}_2$	$\boldsymbol{P}_1$	$\boldsymbol{1}_\varkappa$

(3.65)

which shows that it is an Abelian group. Relative to a principal basis $\{\boldsymbol{e}_\alpha\}$ in an undistorted reference configuration $\varkappa$ the members of $\mathscr{P}_\varkappa$ have the component matrices

$$[\boldsymbol{1}_\varkappa] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\boldsymbol{P}_1] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\boldsymbol{P}_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [-\boldsymbol{1}_\varkappa] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.66}$$

which show that $\mathscr{P}_\varkappa \subset \mathscr{O}_\varkappa$, and so $\boldsymbol{1}_\varkappa, -\boldsymbol{1}_\varkappa \in \mathscr{O}_\varkappa^+$ while $\boldsymbol{P}_1, \boldsymbol{P}_2 \notin \mathscr{O}_\varkappa^+$.

From (2.22) and (2.9) the stress tensor $\boldsymbol{H}_\varkappa(\boldsymbol{1}_\varkappa)$ in an undistorted local configuration of p must commute with all members of $\mathscr{P}_\varkappa$. As a result $\boldsymbol{H}_\varkappa(\boldsymbol{1}_\varkappa)$ and $\mathscr{P}_\varkappa$

share the same principal basis $\{e_\alpha\}$, *i.e.*, $H_\varkappa(1_\varkappa)$ has the spectral form

$$H_\varkappa(1_\varkappa) = t_1 e_1 \otimes e_1 + t_2 e_2 \otimes e_2 . \tag{3.67}$$

From (2.17) the stress tensor $H_\varkappa(1_\varkappa)$ is also the value of $A_\varkappa$ at $C = 1_\varkappa$, *viz*

$$A_\varkappa(1_\varkappa) = t_1 e_1 \otimes e_1 + t_2 e_2 \otimes e_2 . \tag{3.68}$$

Furthermore, (2.23) implies that

$$A_\varkappa(P_\alpha C P_\alpha) = P_\alpha A_\varkappa(C)\, P_\alpha , \tag{3.69}$$

where we have used the fact that the P_α are symmetric.

The conditions (3.69) imply not only the spectral form (3.68) for $A_\varkappa(1_\varkappa)$ but also some severe restrictions on the gradients of $A_\varkappa$ at $C = 1_\varkappa$. For (3-dim.) orthotropic elastic materials the restrictions on the gradients of the response functions are derived in [6], which shows also that they are important in the analysis of wave propagation.

4. The Response Function of a Hyperelastic Membrane Point

In continuum mechanics an elastic material is called *hyperelastic* if the response function is derivable from a stored energy function which gives the value of the stored energy density ε in any configuration. Like the stress tensor T, the stored energy density ε is assumed to be uniquely determined by ν in all configurations χ having common induced local configurations $\chi_{*p} = \nu$, *viz*

$$\varepsilon = \hat{\varepsilon}(\nu), \tag{4.1}$$

where $\hat{\varepsilon}$ is the *stored energy function.* Like $\hat{T}$, $\hat{\varepsilon}$ is generally required to satisfy the following condition of material frame-indifference:

$$\hat{\varepsilon}(Q\nu) = \hat{\varepsilon}(\nu) \tag{4.2}$$

for all rotations Q on $\mathscr{V}$.

Using a local reference configuration $\varkappa$ for p as before, we can express (4.1) by

$$\varepsilon = \sigma_\varkappa(F) \tag{4.3}$$

and (4.2) by

$$\sigma_\varkappa(QF) = \sigma_\varkappa(F). \tag{4.4}$$

From (2.11) and (4.4) $\sigma_\varkappa$ has the reduced form

$$\sigma_\varkappa(F) = \sigma_\varkappa(U) \tag{4.5}$$

or, equivalently,

$$\sigma_\varkappa(F) = \phi_\varkappa(C), \tag{4.6}$$

where C is defined by (2.15).

For a hyperelastic membrane point p the response function $\boldsymbol{H}_\varkappa$ may be derived from the stored energy function $\sigma_\varkappa$ by (*cf.* [7])

$$\boldsymbol{H}_\varkappa(\boldsymbol{F}) = \varrho \boldsymbol{F} \left(\frac{\partial \sigma_\varkappa(\boldsymbol{F})}{\partial \boldsymbol{F}}\right)^T, \tag{4.7}$$

where ϱ denotes the density of p in the deformed configuration $\boldsymbol{\nu} = \boldsymbol{F}\boldsymbol{\varkappa}$. Since the deformation gradient $\boldsymbol{F}$ is a two-point tensor as shown in (2.7), the gradient of the stored energy function $\sigma_\varkappa$ is taken on a 6-dimensional differentiable manifold consisting in all such tensors. The representation (4.7) may be expressed in component form as follows:

Let $\{\boldsymbol{e}_\alpha, \alpha = 1, 2\}$ be an orthonormal basis for the tangent plane $\boldsymbol{\varkappa}(\mathcal{M}_p)$ in $\mathcal{V}$. We extend $\{\boldsymbol{e}_\alpha\}$ into an orthonormal basis $\{\boldsymbol{e}_i, i = 1, 2, 3\}$ for $\mathcal{V}$. Then the two-point tensor $\boldsymbol{F}$ may be characterized by the component form

$$\boldsymbol{F} = F_{i\alpha} \boldsymbol{e}_i \otimes \boldsymbol{e}_\alpha, \tag{4.8}$$

where $F_{i\alpha}$ are the components of the image vector $\boldsymbol{F}\boldsymbol{e}_\alpha$, *viz*

$$\boldsymbol{F}\boldsymbol{e}_\alpha = F_{i\beta} \boldsymbol{e}_i \otimes \boldsymbol{e}_\beta(\boldsymbol{e}_\alpha) = F_{i\beta}\, \delta_{\alpha\beta} \boldsymbol{e}_i = F_{i\alpha} \boldsymbol{e}_i. \tag{4.9}$$

We use the components $F_{i\alpha}$ of $\boldsymbol{F}$ as coordinates on the domain of the stored energy function $\sigma_\varkappa$. Then the component form of (4.7) reads

$$H_{ij}(F_{k\lambda}) = \varrho F_{i\alpha} \frac{\partial \sigma(F_{k\lambda})}{\partial F_{j\alpha}}, \tag{4.10}$$

where the action of $H_{ij}(F_{k\lambda})$ is restricted to the tangent plane $\boldsymbol{\nu}(\mathcal{M}_p)$ spanned by $\{\boldsymbol{F}\boldsymbol{e}_\alpha, \alpha = 1, 2\}$. Specifically, from (2.17), $H_{ij}(F_{k\gamma})$ may be expressed by (*cf.* [7])

$$H_{ij}(F_{k\lambda}) = F_{i\alpha} A_{\alpha\beta}(C_{\gamma\delta})\, F_{j\beta}. \tag{4.11}$$

Hence the tensor $H_{ij}(F_{k\lambda})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ may be represented by

$$H_{ij}(F_{k\lambda})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{\alpha\beta}(C_{\gamma\delta})\, (\boldsymbol{F}\boldsymbol{e}_\alpha) \otimes (\boldsymbol{F}\boldsymbol{e}_\beta), \tag{4.12}$$

where the tensor on the right-hand side acts on the tangent plane $\boldsymbol{\nu}(\mathcal{M}_p)$.

Using the argument presented in [8], we can rewrite the representation (4.10) in terms of the reduced response function $\boldsymbol{A}_\varkappa$ and the reduced stored energy function $\phi_\varkappa$ by

$$A_{\alpha\beta}(C_{\gamma\delta}) = \varrho \left(\frac{\partial \phi(C_{\gamma\delta})}{\partial C_{\alpha\beta}} + \frac{\partial \phi(C_{\gamma\delta})}{\partial C_{\beta\alpha}}\right), \tag{4.13}$$

where $\phi_\varkappa$ is extended arbitrarily to all tensors over $\boldsymbol{\varkappa}(\mathcal{M}_p)$. After the partial derivatives in (4.13) are taken, they are evaluated on the set $\mathcal{D}$ of positive-definite symmetric matrices $[C_{\gamma\delta}]$, where $\boldsymbol{A}_\varkappa$ and $\phi_\varkappa$ are naturally defined. The combination $\partial/\partial C_{\alpha\beta} + \partial/\partial C_{\beta\alpha}$ may be viewed as a tangential gradient on the set $\mathcal{D}$. Indeed, the value of the right-hand side of (4.13) on $\mathcal{D}$ depends only on the values of the function $\phi_\varkappa$ on $\mathcal{D}$.

The representation (4.13) is simpler than the representation (4.10) since (4.13) involves tensors over the fixed tangent plane $\boldsymbol{\varkappa}(\mathcal{M}_p)$ only. Also, the density ϱ

may be expressed easily as a function of $[C_{\gamma\delta}]$, *viz*

$$\varrho = \frac{\varrho_\varkappa}{(\det[C_{\gamma\delta}])^{\frac{1}{2}}}, \tag{4.14}$$

where $\varrho_\varkappa$ denotes the density of p in the local reference configuration $\varkappa$. In terms of the components $F_{k\lambda}$ of $\boldsymbol{F}$, however, the expression (4.14) for ϱ takes the complicated form

$$\varrho = \frac{\varrho_\varkappa}{[(F_{21}F_{32} - F_{31}F_{22})^2 + (F_{31}F_{12} - F_{11}F_{32})^2 + (F_{11}F_{22} - F_{21}F_{12})^2]^{\frac{1}{2}}}. \tag{4.15}$$

It can be shown that the response function $\boldsymbol{H}_\varkappa$ defined by (4.7) obeys the condition of material frame-indifference (2.9) if, and only if, the stored energy function $\sigma_\varkappa$ obeys the conditions of material frame-indifference (4.4); *cf.* [9]. For the condition of material symmetry, however, the situation is a lot more complicated. By the same idea as (2.22) we define the symmetry group $g_\varkappa$ for the stored energy function $\sigma_\varkappa$ relative to $\varkappa$ by the condition that a tensor $\boldsymbol{P} \in \mathscr{L}_\varkappa$ is a member of $g_\varkappa$ if and only if

$$\sigma_\varkappa(\boldsymbol{F}\boldsymbol{P}) = \sigma_\varkappa(\boldsymbol{F}) \quad \forall\, \boldsymbol{F}. \tag{4.16}$$

Using the reduced stored energy function $\phi_\varkappa$ defined by (4.6), we can express the preceding condition also by

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}) = \phi_\varkappa(\boldsymbol{C}) \quad \forall\, \boldsymbol{C}. \tag{4.17}$$

From (4.4) with the special choice $\boldsymbol{Q} = -\mathbf{1}_v$, we see that

$$\sigma_\varkappa(-\mathbf{1}_v\boldsymbol{F}) = \sigma_\varkappa(\boldsymbol{F}(-\mathbf{1}_\varkappa)) = \sigma_\varkappa(\boldsymbol{F}) \quad \forall\, \boldsymbol{F}. \tag{4.18}$$

Hence $g_\varkappa$, like $\mathscr{G}_\varkappa$, must always contain the common elements $\mathbf{1}_\varkappa$ and $-\mathbf{1}_\varkappa$. We can see this fact also from (4.17) since $-\mathbf{1}_\varkappa$ commutes with all positive-definite symmetric tensors $\boldsymbol{C}$ on $\varkappa(\mathscr{M}_p)$.

Substituting the representations (4.13) and (4.14) into the condition of material symmetry (2.23), we obtain

$$\frac{1}{|\det[P_{\gamma\delta}]|} P_{\mu\alpha} \left(\frac{\partial\phi(P_{\lambda\gamma}C_{\lambda\eta}P_{\eta\delta})}{\partial C_{\alpha\beta}} + \frac{\partial\phi(P_{\lambda\gamma}C_{\lambda\eta}P_{\eta\delta})}{\partial C_{\beta\alpha}}\right) P_{\zeta\beta} = \frac{\partial\phi(C_{\gamma\delta})}{\partial C_{\mu\zeta}} + \frac{\partial\phi(C_{\gamma\delta})}{\partial C_{\zeta\mu}}, \tag{4.19}$$

which is the criterion for $\boldsymbol{P} \in \mathscr{G}_\varkappa$ expressed in component form in terms of the reduced stored energy function $\phi_\varkappa$. We can rewrite the condition (4.19) as

$$\left(\frac{\partial}{\partial C_{\mu\zeta}} + \frac{\partial}{\partial C_{\zeta\mu}}\right)\left(\frac{1}{|\det[P_{\gamma\delta}]|}\phi(P_{\lambda\gamma}C_{\lambda\eta}P_{\eta\delta}) - \phi(C_{\gamma\delta})\right) = 0. \tag{4.20}$$

Hence as remarked in [8] the condition of material symmetry (2.23) means that $\boldsymbol{P} \in \mathscr{G}_\varkappa$ if and only if

$$\frac{1}{|\det \boldsymbol{P}|}\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}) = \phi_\varkappa(\boldsymbol{C}) + \frac{1}{|\det \boldsymbol{P}|}\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) - \phi_\varkappa(\mathbf{1}_\varkappa) \quad \forall\, \boldsymbol{C}. \tag{4.21}$$

This condition differs from TRUESDELL'S condition [9, eq. (85.4)] in that it has the extra factor $1/|\det \boldsymbol{P}|$ because $\mathcal{G}_\varkappa$ need not be contained in $\mathcal{U}_\varkappa$ for an elastic membrane point.

We define

$$g_\varkappa^* = g_\varkappa \cap \mathcal{G}_\varkappa. \tag{4.22}$$

Then we claim that, as in the theory of (3-dim.) hyperelastic materials *cf.* [8],

$$g_\varkappa^* \subset \mathcal{U}_\varkappa \tag{4.23}$$

provided that p is not vacuous with

$$\sigma_\varkappa(\boldsymbol{F}) = \text{constant}, \quad \boldsymbol{H}_\varkappa(\boldsymbol{F}) = \boldsymbol{0} \quad \forall\, \boldsymbol{F}, \tag{4.24}$$

in which case both $g_\varkappa$ and $\mathcal{G}_\varkappa$ are $\mathcal{L}_\varkappa$. To prove (4.23), let $\boldsymbol{P}$ obey both (4.17) and (4.21), so that $\boldsymbol{P} \in g_\varkappa^*$. Then by substituting $\boldsymbol{C} = \boldsymbol{1}_\varkappa$ in (4.17) we obtain

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) = \phi_\varkappa(\boldsymbol{1}_\varkappa). \tag{4.25}$$

Using (4.17) and (4.25), we can rewrite (4.21) as

$$\left(\frac{1}{|\det \boldsymbol{P}|} - 1\right)(\phi_\varkappa(\boldsymbol{C}) - \phi_\varkappa(\boldsymbol{1}_\varkappa)) = 0 \quad \forall\, \boldsymbol{C}. \tag{4.26}$$

But by hypothesis p is not vacuous, and so $\phi_\varkappa$ is not constant. Hence (4.26) implies

$$|\det \boldsymbol{P}| = 1. \tag{4.27}$$

Thus (4.23) is proved.

Next we claim that, with the exception of the vacuous case (4.24),

$$g_\varkappa^* = g_\varkappa \cap \mathcal{U}_\varkappa. \tag{4.28}$$

In other words, if $\boldsymbol{P}$ satisfies both (4.17) and (4.27), then it also satisfies (4.21). This fact is more or less obvious, since when (4.27) holds (4.21) reduces to TRUESDELL'S condition:

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}) = \phi_\varkappa(\boldsymbol{C}) + \phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) - \phi_\varkappa(\boldsymbol{1}_\varkappa) \quad \forall\, \boldsymbol{C}. \tag{4.29}$$

But $\boldsymbol{P}$ clearly satisfies (4.29) since by virtue of (4.17)

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) = \phi_\varkappa(\boldsymbol{1}_\varkappa). \tag{4.30}$$

Thus (4.28) is proved.

It follows directly from (4.28) and (4.22) that

$$g_\varkappa \subset \mathcal{U}_\varkappa \Rightarrow g_\varkappa \subset \mathcal{G}_\varkappa. \tag{4.31}$$

We define

$$\mathcal{G}_\varkappa^* = \mathcal{G}_\varkappa \cap \mathcal{U}_\varkappa. \tag{4.32}$$

Then from (4.22) and (4.28) we have

$$g_\varkappa^* \subset \mathcal{G}_\varkappa^*. \tag{4.33}$$

It should be noted that $\mathscr{g}_\varkappa^*$ and $\mathscr{G}_\varkappa^*$ are both groups since they are defined as intersections of groups.

We claim that $\mathscr{g}_\varkappa^*$ is a normal subgroup of $\mathscr{G}_\varkappa$, *i.e.*,

$$\boldsymbol{P}\mathscr{g}_\varkappa^*\boldsymbol{P}^{-1} = \mathscr{g}_\varkappa^* \qquad \forall\, \boldsymbol{P} \in \mathscr{G}_\varkappa. \tag{4.34}$$

Let $\boldsymbol{P}^*$ be an arbitrary element of $\mathscr{g}_\varkappa^*$. Then we must show that

$$\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1} \in \mathscr{g}_\varkappa^* \qquad \forall\, \boldsymbol{P} \in \mathscr{G}_\varkappa, \tag{4.35}$$

i.e., both (4.17) and (4.21) must hold when we replace the tensor $\boldsymbol{P}$ in them by the tensor $\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1}$.

Using the fact that $\boldsymbol{P}^{-1} \in \mathscr{G}_\varkappa$, we obtain from (4.21)

$$\begin{aligned} &|\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{*T}\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1}) \\ &\qquad = \phi_\varkappa(\boldsymbol{P}^{*T}\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{P}^*) + |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{-1}) - \phi_\varkappa(\boldsymbol{1}_\varkappa) \qquad \forall\, \boldsymbol{C}. \end{aligned} \tag{4.36}$$

Next, using the fact that $\boldsymbol{P}^* \in \mathscr{g}_\varkappa$, we obtain from (4.17)

$$\phi_\varkappa(\boldsymbol{P}^{*T}\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{P}^*) = \phi_\varkappa(\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}) \qquad \forall\, \boldsymbol{C}. \tag{4.37}$$

Finally, using the fact that $\boldsymbol{P} \in \mathscr{G}_\varkappa$, we obtain from (4.21)

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}) = |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{C}) + \phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) - |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{1}_\varkappa) \qquad \forall\, \boldsymbol{C}. \tag{4.38}$$

Substituting (4.38) and (4.37) into (4.36), we get

$$\begin{aligned} &|\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{*T}\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1}) \\ &\quad = |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{C}) + \phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) + |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{-1}) - (|\det \boldsymbol{P}| + 1)\, \phi_\varkappa(\boldsymbol{1}_\varkappa). \end{aligned} \tag{4.39}$$

But from the fact that $\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{1}_\varkappa$ and the fact that $\boldsymbol{P}^{-1} \in \mathscr{G}_\varkappa$, we obtain from (4.21)

$$\begin{aligned} |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{l}_\varkappa) &= |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{P}^{-1}) \\ &= \phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) + |\det \boldsymbol{P}|\, \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{-1}) - \phi_\varkappa(\boldsymbol{1}_\varkappa). \end{aligned} \tag{4.40}$$

Hence (4.39) reduces to

$$\phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{P}^{*T}\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1}) = \phi_\varkappa(\boldsymbol{C}) \qquad \forall\, \boldsymbol{C}, \tag{4.41}$$

which shows that $\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1} \in \mathscr{g}_\varkappa^*$. But since $\boldsymbol{P}^* \in \mathscr{g}_\varkappa^* \subset \mathscr{U}_\varkappa$ and since

$$|\det \boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1}| = |\det \boldsymbol{P}^*| = 1, \tag{4.42}$$

we see that $\boldsymbol{P}\boldsymbol{P}^*\boldsymbol{P}^{-1} \in \mathscr{g}_\varkappa^*$ by virtue of (4.28). Thus (4.35) is proved and hence (4.34).

Since $\mathscr{G}_\varkappa^*$ is a subgroup of $\mathscr{G}_\varkappa$ containing $\mathscr{g}_\varkappa^*$, $\mathscr{g}_\varkappa^*$ is then a normal subgroup of $\mathscr{G}_\varkappa^*$ also. We claim that $\mathscr{G}_\varkappa^*$ is a normal subgroup of $\mathscr{G}_\varkappa$ and that $\mathscr{g}_\varkappa^*$ is a normal subgroup of $\mathscr{g}_\varkappa$, *i.e.*,

$$\boldsymbol{P}\mathscr{G}_\varkappa^*\boldsymbol{P}^{-1} = \mathscr{G}_\varkappa^* \qquad \forall\, \boldsymbol{P} \in \mathscr{G}_\varkappa \tag{4.43}$$

and

$$\boldsymbol{P}\mathscr{g}_\varkappa^*\boldsymbol{P}^{-1} = \mathscr{g}_\varkappa^* \qquad \forall\, \boldsymbol{P} \in \mathscr{g}_\varkappa. \tag{4.44}$$

These two conditions are direct consequences of the property of the determinant function; *cf.* (4.42). Hence all inclusions among the groups $g_\varkappa$, $g_\varkappa^*$, $\mathcal{G}_\varkappa$, and $\mathcal{G}_\varkappa^*$, *viz*

$$g_\varkappa^* \subset g_\varkappa, \quad g_\varkappa^* \subset \mathcal{G}_\varkappa^* \subset \mathcal{G}_\varkappa, \tag{4.45}$$

are normal. As a result there are quotient groups

$$g_\varkappa/g_\varkappa^*, \quad \mathcal{G}_\varkappa/\mathcal{G}_\varkappa^*, \quad \mathcal{G}_\varkappa^*/g_\varkappa^*, \quad \mathcal{G}_\varkappa/g_\varkappa^*. \tag{4.46}$$

We can show that the first three quotient groups in (4.46) are Abelian; we do not know whether the fourth one is Abelian or not in general. The proof that the first two are Abelian is obvious, since it amounts to verifying

$$\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} \in g_\varkappa^* \quad \forall\, \boldsymbol{P}, \boldsymbol{Q} \in g_\varkappa \tag{4.47}$$

and

$$\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} \in \mathcal{G}_\varkappa^* \quad \forall\, \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}_\varkappa. \tag{4.48}$$

These conditions, like the previous conditions (4.43) and (4.44), reflect properties of the determinant function. The proof that the third quotient groups in (4.46) is Abelian is essentially the same as that of a similar result presented in [10] for (3-dim.) hyperelastic material points. Indeed, the proof amounts to verifying

$$\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} \in g_\varkappa^* \quad \forall\, \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}_\varkappa^*. \tag{4.49}$$

By virtue of (4.32), the preceding condition concerns only TRUESDELL's condition (4.29). Although the proof presented in reference [10] addresses only the stored energy function $\sigma_\varkappa$, not the reduced stored energy function $\phi_\varkappa$, it can be applied to (4.29) without any difficult, as we shall see in the following analysis for the fourth quotient group in (4.46).

To determine whether that quotient group be Abelian or not, we need to know whether

$$\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} \in g_\varkappa^* \quad \forall\, \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}_\varkappa? \tag{4.50}$$

By virtue of (4.28) and the obvious fact that

$$|\det \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1}| = 1, \tag{4.51}$$

the question (4.50) is equivalent to whether

$$\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1} \in g_\varkappa \quad \forall\, \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}_\varkappa? \tag{4.52}$$

Hence we need to find out whether $\forall\, \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}_\varkappa$

$$\phi_\varkappa(\boldsymbol{Q}^{-1T}\boldsymbol{P}^{-1T}\boldsymbol{Q}^T\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1}) - \phi_\varkappa(\boldsymbol{C}) = 0 \quad \forall\, \boldsymbol{C}? \tag{4.53}$$

We can calculate the difference on the left-hand side of (4.53) in the following way:

Using the fact that $\boldsymbol{Q} \in \mathcal{G}_\varkappa$, we obtain from (4.21)

$$\begin{aligned} &|\det \boldsymbol{Q}|\, \phi_\varkappa(\boldsymbol{Q}^{-1T}\boldsymbol{P}^{-1T}\boldsymbol{Q}^T\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1}) \\ &\quad = \phi_\varkappa(\boldsymbol{P}^{-1T}\boldsymbol{Q}^T\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1}) + |\det \boldsymbol{Q}|\, \phi_\varkappa(\boldsymbol{Q}^{-1T}\boldsymbol{Q}^{-1}) - \phi_\varkappa(\boldsymbol{1}_\varkappa) \quad \forall\, \boldsymbol{C}. \end{aligned} \tag{4.54}$$

Next, using the fact that $P^{-1} \in \mathscr{G}_\varkappa$, we obtain

$$\phi_\varkappa(P^{-1T}Q^T P^T CPQP^{-1}) \tag{4.55}$$
$$= \frac{1}{|\det P|}\phi_\varkappa(Q^T P^T CPQ) + \phi_\varkappa(P^{-1T}P^{-1}) - \frac{1}{|\det P|}\phi_\varkappa(\mathbf{1}_\varkappa) \quad \forall\, C.$$

Next, using the fact that $Q \in \mathscr{G}_\varkappa$, we obtain

$$\frac{1}{|\det P|}\phi_\varkappa(Q^T P^T CPQ) \tag{4.56}$$
$$= \frac{|\det Q|}{|\det P|}\phi_\varkappa(P^T CP) + \frac{1}{|\det P|}\phi_\varkappa(Q^T Q) - \frac{|\det Q|}{|\det P|}\phi_\varkappa(\mathbf{1}_\varkappa) \quad \forall\, C.$$

Finally, using the fact that $P \in \mathscr{G}_\varkappa$ we obtain

$$\frac{|\det Q|}{|\det P|}\phi_\varkappa(P^T CP) = |\det Q|\,\phi_\varkappa(C) + \frac{|\det Q|}{|\det P|}\phi_\varkappa(P^T P) - |\det Q|\,\phi_\varkappa(\mathbf{1}_\varkappa)\ \forall\, C. \tag{4.57}$$

Substituting (4.55)–(4.57) into (4.54), we obtain

$$|\det Q|\,(\phi_\varkappa(Q^{-1T}P^{-1T}Q^T P^T CPQP^{-1}Q^{-1}) - \phi_\varkappa(C))$$
$$= \frac{|\det Q|}{|\det P|}\phi_\varkappa(P^T P) + \phi_\varkappa(P^{-1T}P) + \frac{1}{|\det P|}\phi_\varkappa(Q^T Q) + |\det Q|\,\phi_\varkappa(Q^{-1T}Q^{-1})$$
$$- \left(1 + \frac{1}{|\det P|} + \frac{|\det Q|}{|\det P|} + |\det Q|\right)\phi_\varkappa(\mathbf{1}_\varkappa) \quad \forall\, C. \tag{4.58}$$

Hence whether the answer to (4.53) is yes or no depends on whether the right-hand side of (4.58), which depends only on P and Q, vanishes or not.

From the fact that $PP^{-1} = \mathbf{1}_\varkappa$ we have the condition (4.40). Similarly, from the fact that $QQ^{-1} = \mathbf{1}_\varkappa$ we have

$$\begin{aligned} |\det Q|\,\phi_\varkappa(\mathbf{1}_\varkappa) &= |\det Q|\,\phi_\varkappa(Q^{-1T}Q^T QQ^{-1}), \\ &= \phi_\varkappa(Q^T Q) + |\det Q|\,\phi_\varkappa(Q^{-1T}Q^{-1}) - \phi_\varkappa(\mathbf{1}_\varkappa). \end{aligned} \tag{4.59}$$

Using (4.40) and (4.59), we can rewrite (4.58) as

$$|\det Q|\,(\phi_\varkappa(Q^{-1T}P^{-1T}Q^T P^T CPQP^{-1}Q^{-1}) - \phi_\varkappa(C)) \tag{4.60}$$
$$= \left(\frac{|\det Q|}{|\det P|} - 1\right)\phi_\varkappa(P^T P) + \left(\frac{1}{|\det P|} - 1\right)\phi_\varkappa(Q^T Q) + \left(1 - \frac{|\det Q|}{|\det P|}\right)\phi_\varkappa(\mathbf{1}_\varkappa) \ \ \forall\, C.$$

Clearly the right-hand side of (4.60) vanishes when

$$|\det P| = |\det Q| = 1. \tag{4.61}$$

This special case gives rise to precisely the previous result (4.49), which means that the third quotient group in (4.46) is an Abelian group, as we have remarked before.

Although the right-hand side of (4.60) seems not to vanish in general, we do not have any example of $\phi_\varkappa$ for which the fourth quotient group is not Abelian either. In the next section we shall consider a number of examples of hyperelastic membrane points whose response functions possess symmetries considered in the preceding section. As we shall see, the examples belong to the following three classes only:

Class 1. This class contains all ordinary hyperelastic membrane points for which the symmetry groups $g_\varkappa$ and $\mathcal{G}_\varkappa$ are both contained in $\mathcal{U}_\varkappa$. Then (4.21) reduces to (4.29) and, therefore, as TRUESDELL remarked, $g_\varkappa$ is contained in $\mathcal{G}_\varkappa$. Hence in this case

$$g_\varkappa = g_\varkappa^* \subset \mathcal{G}_\varkappa^* = \mathcal{G}_\varkappa \subset \mathcal{U}_\varkappa. \tag{4.62}$$

As a result the first two quotient groups in (4.46) reduce to the trivial group, while the last two quotient groups coincide. This class of hyperelastic membrane points may be analyzed by following essentially the same approach as in [10], which treats (3-dim.) hyperelastic material points.

Class 2. The symmetry groups $g_\varkappa$ and $\mathcal{G}_\varkappa$ are assumed to satisfy the following condition:

$$g_\varkappa^* = g_\varkappa = \mathcal{G}_\varkappa^* \subset \mathcal{U}_\varkappa, \tag{4.63}$$

where the last inclusion becomes the equality sign for the surface tension points. Under the assumption (4.63) the first and the third quotient groups in (4.46) reduce to the trivial group, while the second and the fourth quotient groups coincide.

Class 3. The symmetry groups $g_\varkappa$ and $\mathcal{G}_\varkappa$ are assumed to satisfy the following condition:

$$g_\varkappa \neq g_\varkappa^* \subset \mathcal{G}_\varkappa^* = \mathcal{G}_\varkappa \subset \mathcal{U}_\varkappa. \tag{4.64}$$

Under this assumption the second quotient group in (4.46) reduces to the trivial group while the third and the fourth quotient groups coincide.

Classes 2 and 3 are not covered by [10]; we now analyze them.

The condition (4.63) implies that the restriction of the function $\phi_\varkappa$ to the set

$$\mathcal{G}_\varkappa^T \mathcal{G}_\varkappa = \{\boldsymbol{P}^T\boldsymbol{P},\ \boldsymbol{P} \in \mathcal{G}_\varkappa\} \tag{4.65}$$

reduces to a function of the form

$$\phi_\varkappa(\boldsymbol{P}^T\boldsymbol{P}) = f(|\det \boldsymbol{P}|) \qquad \forall\ \boldsymbol{P} \in \mathcal{G}_\varkappa. \tag{4.66}$$

To see this fact let $\boldsymbol{Q}$ be another tensor in $\mathcal{G}_\varkappa$ such that

$$|\det \boldsymbol{P}| = |\det \boldsymbol{Q}|. \tag{4.67}$$

Then the composition $Q^{-1}P$ is contained in $\mathcal{G}_\varkappa^*$ since from (4.67)

$$|\det Q^{-1}P| = 1. \tag{4.68}$$

By virtue of (4.63) we then have $Q^{-1}P \in g_\varkappa^*$. Consequently, from (4.17)

$$\phi_\varkappa(P^T Q^{-1T} C Q^{-1} P) = \phi_\varkappa(C) \quad \forall\, C. \tag{4.69}$$

Choosing $C = Q^T Q$ in (4.69), we obtain

$$\phi_\varkappa(P^T P) = \phi_\varkappa(Q^T Q). \tag{4.70}$$

Thus (4.66) is proved.

The function f defined by (4.66) must obey the following condition: Let P and Q be arbitrary members of $\mathcal{G}_\varkappa$. We put

$$x = |\det P|, \quad y = |\det Q|. \tag{4.71}$$

Then by virtue of (4.66) the condition

$$\frac{1}{|\det Q|}\phi_\varkappa(Q^T P^T P Q) = \phi_\varkappa(P^T P) + \frac{1}{|\det Q|}\phi_\varkappa(Q^T Q) - \phi_\varkappa(\mathbf{1}_\varkappa) \tag{4.72}$$

may be rewritten as

$$\frac{1}{y} f(xy) = f(x) + \frac{1}{y} f(y) - f(1). \tag{4.73}$$

This is the functional equation governing f on the quotient group $\mathcal{G}_\varkappa/\mathcal{G}_\varkappa^*$, *i.e.*, each positive number x in the domain of f represents the quotient class $\{P\}$ in $\mathcal{G}_\varkappa/\mathcal{G}_\varkappa^*$ satisfying $(4.71)_1$.

For the examples that we shall consider in the next section, the Abelian quotient group $\mathcal{G}_\varkappa/\mathcal{G}_\varkappa^*$ is represented by the multiplicative group of all positive numbers. In that case we can differentiate (4.73) with respect to x, obtaining

$$f'(xy) = f'(x) \quad \forall\, x, y \in \mathcal{R}^+, \tag{4.74}$$

which means that the derivative f' is a constant independent of x. Consequently f has the representation

$$f(x) = cx + d, \tag{4.75}$$

where c and d are constants. We can verify easily that the solution (4.75) indeed obeys the functional equation (4.73).

The solution of (4.73) is also given by (4.75) when the quotient group $\mathcal{G}_\varkappa/\mathcal{G}_\varkappa^*$ is represented by the discrete group of the form

$$\{x = \alpha^n, n = 0, \pm 1, \pm 2, \ldots\}, \tag{4.76}$$

where $0 < \alpha \neq 1$. In this case we define c and d by

$$f(1) = c + d, \quad f(\alpha) = c\alpha + d. \tag{4.77}$$

Then from (4.73) with $x = y = \alpha$ we obtain

$$f(\alpha^2) = \alpha f(\alpha) + f(\alpha) - \alpha f(1) = c\alpha^2 + d. \tag{4.78}$$

By induction if $f(\alpha^n) = c\alpha^n + d$, then using (4.73) we get

$$f(\alpha^{n+1}) = \alpha f(\alpha^n) + f(\alpha) - \alpha f(1) = c\alpha^{n+1} + d. \tag{4.79}$$

Hence the solution (4.75) is valid for all α^n with $n \geqq 0$. From (4.73) $f(\alpha^{-1})$ is related to $f(\alpha)$ by

$$f(\alpha^{-1}) = -\frac{1}{\alpha} f(\alpha) + \left(\frac{1}{\alpha} + 1\right) f(1) = c\alpha^{-1} + d. \tag{4.80}$$

Then by induction again we can show that

$$f(\alpha^{-n}) = \frac{1}{\alpha} f(\alpha^{-(n-1)}) + f(\alpha^{-1}) - \frac{1}{\alpha} f(1) = c\alpha^{-n} + d. \tag{4.81}$$

for all $n \geqq 0$. Thus (4.75) is still the general solution for (4.73) when $\mathscr{G}_\varkappa/\mathscr{G}_\varkappa^*$ is represented by (4.76).

When the quotient group $\mathscr{G}_\varkappa/\mathscr{G}_\varkappa^*$ is represented by a more complicated discrete group such as

$$\{x = \alpha^n \beta^m, \ n, m = 0, \pm 1, \pm 2, \ldots\}, \tag{4.82}$$

which is not reducible to the previous form (4.76), there are solutions of (4.73) other than the ones given by (4.75). We are not interested in those other solutions since they generally lead to discontinuous functions $\phi_\varkappa$, which are not appropriate as stored energy functions.

Having determined the explicit form for the restriction of $\phi_\varkappa$ to the set $\mathscr{G}_\varkappa^T \mathscr{G}_\varkappa$, we can use the condition (4.21) directly to define a representation for $\phi_\varkappa$, *viz*

$$\phi_\varkappa(\boldsymbol{P}^T \boldsymbol{C} \boldsymbol{P}) = |\det \boldsymbol{P}| \, (\phi_\varkappa(\boldsymbol{C}) - d) + d. \tag{4.83}$$

As usual we define an equivalence relation on the domain $\mathscr{D}$ of $\phi_\varkappa$ by

$$\boldsymbol{C} \sim \overline{\boldsymbol{C}} \quad \Leftrightarrow \quad \exists \, \boldsymbol{P} \in \mathscr{G}_\varkappa : \overline{\boldsymbol{C}} = \boldsymbol{P}^T \boldsymbol{C} \boldsymbol{P}. \tag{4.84}$$

If $\overline{\boldsymbol{C}}$ is equivalent to $\boldsymbol{C}$ by (4.84), then the condition (4.83) implies that the value $\phi_\varkappa(\overline{\boldsymbol{C}})$ must be related to the value $\phi_\varkappa(\boldsymbol{C})$ by

$$\phi_\varkappa(\overline{\boldsymbol{C}}) = |\det \boldsymbol{P}| \, (\phi_\varkappa(\boldsymbol{C}) - d) + d. \tag{4.85}$$

On the other hand, if $\overline{\boldsymbol{C}}$ is not equivalent to $\boldsymbol{C}$, then the condition (4.83) does not restrict the values $\phi_\varkappa(\overline{\boldsymbol{C}})$ and $\phi_\varkappa(\boldsymbol{C})$ in any way at all.

For each equivalence class in $\mathscr{D}$ we select a representative element, say $\boldsymbol{C}_0$, and we assign an arbitrary value $\phi_\varkappa(\boldsymbol{C}_0)$. Then for any $\boldsymbol{C}$ equivalent to $\boldsymbol{C}_0$, say by

$$\boldsymbol{C} = \boldsymbol{P}_0^T \boldsymbol{C}_0 \boldsymbol{P}_0, \quad \boldsymbol{P}_0 \in \mathscr{G}_\varkappa, \tag{4.86}$$

we define

$$\phi_\varkappa(\boldsymbol{C}) = |\det \boldsymbol{P}_0| \, (\phi_\varkappa(\boldsymbol{C}_0) - d) + d. \tag{4.87}$$

The value $\phi_\varkappa(\boldsymbol{C})$ is well defined by (4.87) since if $\boldsymbol{C}$ is related to $\boldsymbol{C}_0$ also by

$$\boldsymbol{C} = \boldsymbol{P}_1^T \boldsymbol{C}_0 \boldsymbol{P}_1, \ \boldsymbol{P}_1 \in \mathscr{G}_\varkappa, \tag{4.88}$$

then by taking the determinant of (4.86) and (4.88), we see that

$$|\det \boldsymbol{P}_0| = |\det \boldsymbol{P}_1|. \tag{4.89}$$

Hence the right-hand side of (4.87) does not depend on the choice of $\boldsymbol{P}_0$ satisfying (4.86). As a result, $\phi_\varkappa$ is defined on $\mathscr{D}$ by piecing together its definition (4.87) on each equivalence class in $\mathscr{D}$.

Having obtained the representation (4.87) for the reduced stored energy functions $\phi_\varkappa$ for hyperelastic membrane points belonging to Class 2, we now turn our attention to Class 3. For this class the symmetry group $\mathscr{g}_\varkappa$ of the stored energy function is not contained in the symmetry group $\mathscr{G}_\varkappa$ of the response function; this condition sets the assumption (4.64) apart from the assumption (4.62).

Assumption (4.64) makes all tensors $\boldsymbol{P}$ belonging to $\mathscr{G}_\varkappa$ unimodular, so that the condition (4.21) reduces to TRUESDELL's condition (4.29). This fact, however, does not preclude the latter from having solutions $\boldsymbol{P}$ which are not unimodular and, therefore, do not belong to $\mathscr{G}_\varkappa$. Indeed, all non-unimodular members $\boldsymbol{P}$ in $\mathscr{g}_\varkappa$ (*i.e.* all $\boldsymbol{P}$ belonging to the complement of $\mathscr{g}_\varkappa^*$ in $\mathscr{g}_\varkappa$) obey (4.29) since from (4.17) they clearly satisfy (4.30) and, therefore, (4.29) because it reduces to (4.17). This argument is essentially the same as TRUESDELL's proof that $\mathscr{g}_\varkappa$ is a subgroup of $\mathscr{G}_\varkappa$ when both are assumed to be contained in $\mathscr{U}_\varkappa$. We must be careful not to interpret TRUESDELL's result erroneously as the assertion that if $\mathscr{G}_\varkappa$ is contained in $\mathscr{U}_\varkappa$, then $\mathscr{g}_\varkappa$ is contained in $\mathscr{G}_\varkappa$. The correct generalization of TRUESDELL's result is this statement: Define the group $\tilde{\mathscr{G}}_\varkappa$ as the group of all tensors $\boldsymbol{P}$, unimodular or otherwise, which satisfy TRUESDELL's condition (4.29). Then

$$\mathscr{g}_\varkappa \subset \tilde{\mathscr{G}}_\varkappa. \tag{4.90}$$

Since all unimodular tensors $\boldsymbol{P}$ satisfying (4.29) *ipso facto* also satisfy (4.21), in general we always have

$$\mathscr{G}_\varkappa^* = \tilde{\mathscr{G}}_\varkappa \cap \mathscr{U}_\varkappa. \tag{4.91}$$

Under the assumption (4.64), we then have

$$\mathscr{G}_\varkappa = \tilde{\mathscr{G}}_\varkappa \cap \mathscr{U}_\varkappa. \tag{4.92}$$

Beyond its parts $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ the group $\tilde{\mathscr{G}}_\varkappa$ does not really have any physical significance, since TRUESDELL's condition (4.29) is not the symmetry condition for the response function when $\boldsymbol{P}$ is not a unimodular tensor. It is important to introduce the group $\tilde{\mathscr{G}}_\varkappa$ for the Class 3, however, since in this case the condition (4.29) is the overall restriction on the reduced stored energy function $\phi_\varkappa$. Indeed, both (4.17) and (4.21) are now special cases of (4.29) for this particular class of hyperelastic membrane points.

Although $\mathscr{g}_\varkappa$ and $\tilde{\mathscr{G}}_\varkappa$ are not contained in $\mathscr{U}_\varkappa$, their relations with respect to TRUESDELL's condition (4.29) are exactly the same as those for $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ in [10], so that we still have the following results:

i) $\mathscr{g}_\varkappa$ is a normal subgroup of $\tilde{\mathscr{G}}_\varkappa$.

ii) The quotient group $\tilde{\mathscr{G}}_\varkappa/\mathscr{g}_\varkappa$ is Abelian.

iii) The restriction of $\phi_\varkappa$ to the set $\tilde{\mathscr{G}}_\varkappa^T\tilde{\mathscr{G}}_\varkappa$ is an Abelian function with respect to the Abelian group $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$.

Indeed, result i) means that

$$P g_\varkappa P^{-1} = g_\varkappa \quad \forall\, P \in \tilde{\mathscr{G}}_\varkappa, \tag{4.93}$$

which may be proved by following exactly the same steps (4.36)–(4.41) in the proof of (4.34), except that we now replace the factors $|\det \boldsymbol{P}|$ in (4.36)–(4.40) all by 1. Next, result ii) means that

$$PQP^{-1}Q^{-1} \in g_\varkappa \quad \forall\, P, Q \in \tilde{\mathscr{G}}_\varkappa, \tag{4.94}$$

which may be proved by following exactly the same steps (4.54)–(4.60) in the proof of (4.49) except that we now replace the factors $|\det \boldsymbol{P}|$ and $|\det \boldsymbol{Q}|$ in (4.54)–(4.60) all by 1, as shown in (4.61). Finally, result iii) means that

iii)a If $P, Q \in \tilde{\mathscr{G}}_\varkappa$ belong to the same quotient in $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$, *i.e.*, $P^{-1}Q \in g_\varkappa$, then

$$\phi_\varkappa(P^TP) = \phi_\varkappa(Q^TQ), \tag{4.95}$$

and so the restriction of $\phi_\varkappa$ to $\tilde{\mathscr{G}}_\varkappa^T\tilde{\mathscr{G}}_\varkappa$ may be viewed as a function on the quotient group $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$.

iii)b For all $P, Q \in \tilde{\mathscr{G}}_\varkappa$

$$\phi_\varkappa(P^TQ^TQP) - \phi_\varkappa(\mathbf{1}_\varkappa) = (\phi_\varkappa(P^TP) - \phi_\varkappa(\mathbf{1}_\varkappa)) + (\phi_\varkappa(Q^TQ) - \phi_\varkappa(\mathbf{1}_\varkappa)). \tag{4.96}$$

The proofs of both (4.95) and (4.96) are obvious, since when $P^{-1}Q \in g_\varkappa$ the condition (4.17) implies directly

$$\phi_\varkappa(P^TP) = \phi_\varkappa(Q^TP^{-1T}P^TPP^{-1}Q) = \phi_\varkappa(Q^TQ), \tag{4.97}$$

while (4.96) is just another way of writing (4.29) with $C = Q^TQ$.

For the example that we shall consider in the next section, the Abelian quotient group $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$ is represented by the group of all positive numbers with respect to multiplication. Let the quotient class of $\boldsymbol{P}$ in $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$ be represented by the number x and that of $\boldsymbol{Q}$ the number y. Then the condition (4.96) may be rewritten as the functional equation

$$f(xy) - f(1) = (f(x) - f(1)) + (f(y) - f(1)). \tag{4.98}$$

It is well known that the general solution of (4.98) is

$$f(x) = c \log x + d, \tag{4.99}$$

where c and d are constants, provided that f is a continuous function. It is easily verified that the general solution of (4.98) is given by (4.99) when $\tilde{\mathscr{G}}_\varkappa/g_\varkappa$ is represented by the discrete group (4.76).

Having determined the restriction of $\phi_\varkappa$ to the set $\tilde{\mathscr{G}}_\varkappa^T\tilde{\mathscr{G}}_\varkappa$, we can now use the condition (4.29) directly to define a representation for $\phi_\varkappa$, *viz*,

$$\phi_\varkappa(P^TCP) = c \log x + \phi_\varkappa(C), \tag{4.100}$$

where x represents the quotient of P in $\tilde{\mathscr{G}}_\varkappa/\mathscr{g}_\varkappa$. As before, we define an equivalence relation on the domain $\mathscr{D}$ of $\phi_\varkappa$ by

$$C \sim \overline{C} \quad \Leftrightarrow \quad \exists\, P \in \tilde{\mathscr{G}}_\varkappa : \overline{C} = P^T C P. \tag{4.101}$$

If $\overline{C}$ is equivalent to C by (4.101), then (4.100) implies that the values $\phi_\varkappa(\overline{C})$ and $\phi_\varkappa(C)$ are related by

$$\phi_\varkappa(\overline{C}) = c \log x + \phi_\varkappa(C). \tag{4.102}$$

On the other hand, if $\overline{C}$ is not equivalent to C, then the condition (4.100) does not restrict the values $\phi_\varkappa(\overline{C})$ and $\phi_\varkappa(C)$ in any way.

For each equivalence class in $\mathscr{D}$ we select a representative element, say C_0, and we assign an arbitrary value $\phi_\varkappa(C_0)$. Then for any C equivalent to C_0, say by

$$C = P_0^T C_0 P_0, \qquad P_0 \in \tilde{\mathscr{G}}_\varkappa, \tag{4.103}$$

we define

$$\phi_\varkappa(C) = c \log x_0 + \phi_\varkappa(C_0), \tag{4.104}$$

where x_0 represents the quotient of P_0 in $\tilde{\mathscr{G}}_\varkappa/\mathscr{g}_\varkappa$. The fact that $\phi_\varkappa(C)$ is well defined by (4.104) may be proved as follows:

When the constant c is zero, the restriction of $\phi_\varkappa$ to $\tilde{\mathscr{G}}_\varkappa^T \tilde{\mathscr{G}}_\varkappa$ is a constant d, *cf.* (4.99). In that case (4.29) reduces to (4.17), so that

$$\mathscr{g}_\varkappa = \tilde{\mathscr{G}}_\varkappa. \tag{4.105}$$

Then the only restriction on $\phi_\varkappa$ is (4.17), which means that $\phi_\varkappa$ takes on a constant value on each equivalence class in $\mathscr{D}$, *i.e.*, (4.104) reduces to

$$\phi_\varkappa(C) = \phi_\varkappa(C_0); \tag{4.106}$$

evidently, $\phi_\varkappa(C)$ is well defined. Next, when the constant c is not zero, we claim that if

$$P^T C_0 P = C_0 \tag{4.107}$$

for any $P \in \tilde{\mathscr{G}}_\varkappa$, then P must be contained in $\mathscr{g}_\varkappa$. Indeed, substituting (4.107) into (4.100), we obtain

$$\phi_\varkappa(C_0) = c \log x + \phi_\varkappa(C_0). \tag{4.108}$$

Thus

$$x = 1, \tag{4.109}$$

which means precisely that P belongs to $\mathscr{g}_\varkappa$ since 1 represents $\mathscr{g}_\varkappa$ in the quotient $\tilde{\mathscr{G}}_\varkappa/\mathscr{g}_\varkappa$. By virtue of the preceding result, if C is related to C_0 by both (4.103) and

$$C = P_1^T C_0 P_1, \qquad P_1 \in \tilde{\mathscr{G}}_\varkappa, \tag{4.110}$$

then P_0 and P_1 must belong to the same quotient in $\tilde{\mathscr{G}}_\varkappa/\mathscr{g}_\varkappa$. As a result, P_0 and P_1 are represented by the same number x_0. Consequently the right-hand side of

(4.104) does not depend on the choice of P_0 satisfying the condition (4.103). Thus $\phi_\varkappa(C)$ is well defined by (4.104).

It should be pointed out that there is a basic difference between representations for Classes 1, 2, or 3. For Class 1 a representation gives the general solution for the conditions (4.29) for a given symmetry group $\mathscr{G}_\varkappa$ under the assumption that the symmetry group $g_\varkappa$ is a subgroup of $\mathscr{G}_\varkappa$. Such a representation can be derived by following essentially the same approach as in [10]. In this approach the symmetry group $g_\varkappa$ is not assigned *a priori*, but is determined by the form of the representation. For Class 2, however, $g_\varkappa$ is determined at the outset by the assumption (4.63). Then the representation (4.87) is derived under the assumption (4.63). As a result, if the stored energy function of a given hyperelastic membrane point fails to satisfy the assumption (4.63), then the representation (4.87) cannot be applied. Since we do not know *a priori* whether a hyperelastic membrane point with symmetry group $\mathscr{G}_\varkappa$ actually has a stored energy function with symmetry group $g_\varkappa$ given by (4.63), the representation (4.87) gives us not all possible stored energy functions, but only those whose symmetry group $g_\varkappa$ satisfies (4.63). For Class 3, the situation is even more complex than for Class 2, since neither $g_\varkappa$ nor $\tilde{\mathscr{G}}_\varkappa$ is determined by $\mathscr{G}_\varkappa$. We can apply the representation (4.104) only in the following sense.

Let $\mathscr{G}_\varkappa$ be given as a subgroup of $\mathscr{U}_\varkappa$. Then we follow the procedure developed in [10] to find a representation for the stored energy functions when the corresponding response functions have symmetry groups containing $\mathscr{G}_\varkappa$. Most of these stored energy functions have symmetry groups $g_\varkappa$ contained in $\mathscr{G}_\varkappa$; they define hyperelastic membrane points of Class 1. The representation may prescribe, however, also some special stored energy functions whose symmetry groups $g_\varkappa$ are not contained in $\mathscr{G}_\varkappa$. These special stored energy functions then define hyperelastic membrane points of Class 3. Since in this case the form of the stored energy functions is already known, we can determine the groups $g_\varkappa$ and $\tilde{\mathscr{G}}_\varkappa$ explicitly. Then we can identify the stored energy functions as the ones given by (4.104). Hence (4.104) merely summarizes the common form of the stored energy functions for hyperelastic membrane points of Class 3. We cannot actually use (4.104), as we can use (4.87), to determine the stored energy functions.

5. Some Examples of Hyperelastic Membrane Points

From the representation (4.13) it is explained in [8] that the reduced response function $A_\varkappa$ must obey the compatibility condition:

$$\left(\frac{\partial}{\partial C_{\alpha\beta}}+\frac{\partial}{\partial C_{\beta\alpha}}\right)\left(\frac{1}{\varrho}A_{\xi\eta}(C_{\gamma\delta})\right)=\left(\frac{\partial}{\partial C_{\xi\eta}}+\frac{\partial}{\partial C_{\eta\xi}}\right)\left(\frac{1}{\varrho}A_{\alpha\beta}(C_{\gamma\delta})\right), \qquad (5.1)$$

for all positive-definite symmetric matrices $[C_{\gamma\delta}]$. Of course, the function ϱ in (5.1) is given by (4.14).

In Section 3 we have defined the reduced response function $A_\varkappa$ for a surface tension point by (3.5). We claim that such a function obeys (5.1). Indeed, in component form (3.5) implies

$$\frac{1}{\varrho} A_{\alpha\beta}(C_{\gamma\delta}) = \frac{c}{\varrho_\varkappa} (\det [C_{\gamma\delta}])^{\frac{1}{2}} C^{-1}_{\alpha\beta}(C_{\gamma\delta}). \tag{5.2}$$

Hence we can calculate directly

$$\left(\frac{\partial}{\partial C_{\xi\eta}} + \frac{\partial}{\partial C_{\eta\xi}}\right) \left(\frac{1}{\varrho} A_{\alpha\beta}(C_{\gamma\delta})\right) = \frac{c}{\varrho} (C^{-1}_{\xi\eta} C^{-1}_{\alpha\beta} - C^{-1}_{\alpha\xi} C^{-1}_{\eta\beta} - C^{-1}_{\alpha\eta} C^{-1}_{\xi\beta}) (C_{\gamma\delta}), \tag{5.3}$$

where we have used the symmetry of $C^{-1}_{\xi\eta}$ on the domain $\mathscr{D}$ consisting of positive-definite symmetric matrices $[C_{\gamma\delta}]$. Clearly the right-hand side of (5.3) is symmetric in $(\alpha\beta)$, $(\xi\eta)$, so that the condition (5.1) are satisfied. As a result, a surface tension point is hyperelastic.

Substituting (3.5) into (4.13) and using the formula for the derivative of the determinant function, we get

$$\left(\frac{\partial}{\partial C_{\alpha\beta}} + \frac{\partial}{\partial C_{\beta\alpha}}\right) \left(\phi(C_{\gamma\delta}) - \frac{c}{\varrho}\right) = 0 \tag{5.4}$$

for all $[C_{\gamma\delta}] \in \mathscr{D}$. As a result, $\phi_\varkappa$ is given by

$$\phi_\varkappa(C) = \frac{c}{\varrho} + d = \frac{c}{\varrho_\varkappa} (\det C)^{\frac{1}{2}} + d, \tag{5.5}$$

where c and d are constants. Of course, $c \neq 0$; otherwise, p is vacuous, *i.e.*,

$$\phi_\varkappa(C) = \text{constant}, \quad A_\varkappa(C) = \mathbf{0} \quad \forall\, C, \tag{5.6}$$

which is equivalent to (4.24). It is customary to set the constant d to 0. Then (5.5) means that

$$\varrho\varepsilon = c, \tag{5.7}$$

so that c is just the stored energy per unit area associated with the surface tension.

Substituting (5.5) into (4.17), we see clearly that $P \in g_\varkappa$ if and only if it is unimodular. Hence in this case

$$g^*_\varkappa = g_\varkappa = \mathscr{U}_\varkappa. \tag{5.8}$$

Also, since $\mathscr{G}_\varkappa$ is $\mathscr{L}_\varkappa$, from (4.32)

$$\mathscr{G}^*_\varkappa = \mathscr{U}_\varkappa. \tag{5.9}$$

Thus the condition (4.63) is satisfied: a surface tension point is of Class 2.

There is only one equivalence class on $\mathscr{D}$ as defined by (4.84) with respect to $\mathscr{G}_\varkappa = \mathscr{L}_\varkappa$, since every $C \in \mathscr{D}$ may be represented by

$$C = U^T \mathbf{1}_\varkappa U = U^2, \tag{5.10}$$

so that $C \sim \mathbf{1}_\varkappa$. Hence the representation (5.5) amounts to choosing

$$\phi_\varkappa(\mathbf{1}_\varkappa) = \frac{c}{\varrho_\varkappa} + d \tag{5.11}$$

in (4.83), *viz*

$$\phi_\varkappa(C) = \phi_\varkappa(U^T \mathbf{1}_\varkappa U) = \det U \left(\frac{c}{\varrho_\varkappa} + d - d\right) + d = \frac{c}{\varrho_\varkappa} (\det C)^{\frac{1}{2}} + d. \tag{5.12}$$

Next, we claim that an elastic fluid membrane point with constitutive equation (3.14) is also hyperelastic. The reduced response function $A_\varkappa$ is given by (3.5) with $c = c(\varrho)$, so that (5.3) is replaced by

$$\left(\frac{\partial}{\partial C_{\xi\eta}} + \frac{\partial}{\partial C_{\eta\xi}}\right) \left(\frac{1}{\varrho} A_{\alpha\beta}(C_{\gamma\delta})\right) \tag{5.13}$$
$$= \frac{c}{\varrho} \left[C_{\xi\eta}^{-1} C_{\alpha\beta}^{-1} - \left(1 + \frac{\varrho c'}{c}\right) (C_{\alpha\xi}^{-1} C_{\eta\beta}^{-1} + C_{\alpha\eta}^{-1} C_{\xi\beta}^{-1})\right] (C_{\gamma\delta}),$$

where $[C_{\gamma\delta}] \in \mathscr{D}$. Clearly the right-hand side of (5.13) is still symmetric in $(\alpha\beta)$, $(\xi\eta)$. As a result an elastic fluid membrane point is hyperelastic.

Substituting (3.5) with $c = c(\varrho)$ into (4.13) and using the formula for the derivative of the determinant function as before, we get

$$\left(\frac{\partial}{\partial C_{\alpha\beta}} + \frac{\partial}{\partial C_{\beta\alpha}}\right) \left(\phi(C_{\gamma\delta}) + \int^{\cdot} \frac{c(\varrho)}{\varrho^2} d\varrho\right) = 0. \tag{5.14}$$

Hence $\phi_\varkappa$ is given by

$$\phi_\varkappa(C) = -\int^{\cdot} \frac{c(\varrho)}{\varrho^2} d\varrho, \tag{5.15}$$

where ϱ is regarded as a function of C by (4.14). From (5.15) we see that $\mathscr{g}_\varkappa$ is $\mathscr{U}_\varkappa$. Thus the condition (4.62) is satisfied: a hyperelastic fluid membrane point is of Class 1. The representation (5.15) for the stored energy function of an elastic fluid membrane point has the same form as that of a (3-dim.) elastic fluid; *cf.* [9].

Next we consider the case $\mathscr{G}_\varkappa \supset \mathscr{O}_\varkappa$. When $P \in \mathscr{G}_\varkappa$ is contained in $\mathscr{U}_\varkappa$, the condition (4.21) reduces to TRUESDELL's condition (4.29). As TRUESDELL pointed out, $\mathscr{g}_\varkappa$ must contain all orthogonal tensors in $\mathscr{G}_\varkappa$. Hence from (4.17) $\phi_\varkappa$ is an isotropic function, *viz,*

$$\phi_\varkappa(Q^T C Q) = \phi_\varkappa(C) \quad \forall\, Q \in \mathscr{O}_\varkappa. \tag{5.16}$$

The equivalence relation (4.84) becomes

$$C \sim \overline{C} \quad \Leftrightarrow \quad \exists\, Q \in \mathscr{O}_\varkappa : \overline{C} = Q^T C Q, \tag{5.17}$$

and it is well known that

$$C \sim \overline{C} \quad \Leftrightarrow \quad I_C = I_{\overline{C}}, \quad II_C = II_{\overline{C}}, \tag{5.18}$$

where

$$I_C = \operatorname{tr} C, \quad II_C = \det C. \tag{5.19}$$

Hence $\phi_\varkappa$ has the representation

$$\phi_\varkappa(C) = f(I_C, II_C). \tag{5.20}$$

Since the invariants obey the conditions

$$\operatorname{tr} C = \operatorname{tr} C^T, \quad \det C = \det C^T \tag{5.21}$$

for all C over $\varkappa(\mathcal{M}_p)$, the representation (4.13) reduces to

$$A_{\alpha\beta}(C_{\gamma\delta}) = 2\varrho \frac{\partial \phi(C_{\gamma\delta})}{\partial C_{\alpha\beta}}, \tag{5.22}$$

as we have explained in [8]. Substituting (5.20) into (5.22), and comparing the result with (3.20), we obtain

$$f_1 = 2\varrho \frac{\partial f}{\partial I_C}, \quad f_0 = 2\varrho \frac{\partial f}{\partial II_C} II_C. \tag{5.23}$$

Hence the compatibility condition is

$$\frac{\partial}{\partial II_C}\left(\frac{f_1}{\varrho}\right) = \frac{\partial}{\partial I_C}\left(\frac{f_0}{\varrho II_C}\right). \tag{5.24}$$

For the case $\mathcal{G}_\varkappa \supset \mathcal{O}_\varkappa^+$ TRUESDELL's remark implies again that $g_\varkappa \supset \mathcal{O}_\varkappa^+$, so that (4.84) becomes

$$C \sim \overline{C} \quad \Leftrightarrow \quad \exists\, Q \in \mathcal{O}_\varkappa^+ : \overline{C} = Q^T C Q. \tag{5.25}$$

However, it is well known that this equivalence relation can be characterized by exactly the same conditions as (5.18), *viz*

$$C \sim \overline{C} \quad \Leftrightarrow \quad I_C = I_{\overline{C}}, \quad II_C = II_{\overline{C}}. \tag{5.26}$$

Consequently the group $\mathcal{O}_\varkappa^+$ itself cannot be the symmetry group of any stored energy function. In other words, we have the result

$$g_\varkappa \supset \mathcal{O}_\varkappa^+ \quad \Leftrightarrow \quad g_\varkappa \supset \mathcal{O}_\varkappa, \tag{5.27}$$

which is similar to the previous result (3.16).

There is an interesting type of isotropic elastic membrane points whose symmetry group $\mathcal{G}_\varkappa$ relative to an undistorted reference configuration $\varkappa$ is such that

$$\mathcal{G}_\varkappa \supset \{qQ,\, q \neq 0,\, Q \in \mathcal{O}_\varkappa\}. \tag{5.28}$$

Since $\mathcal{G}_\varkappa$ clearly contains $\mathcal{O}_\varkappa$, the reduced response functions $A_\varkappa$ for this type of elastic membrane points are special cases of the representation (3.20). Indeed, using $P = qQ$, we see that the form $A_\varkappa$ given by (3.20) obeys the condition (2.23) if and only if the functions f_0 and f_1 satisfy

$$f_0(I_C, II_C) = f_0(q^2 I_C, q^4 II_C), \quad f_1(I_C, II_C) = q^2 f_1(q^2 I_C, q^4 II_C) \tag{5.29}$$

for all I_C, II_C in the domain characterized by (3.36) and for all nonzero numbers q. By virtue of (5.29) the reduced response functions for this type of isotropic elastic membrane points take the form

$$A_\varkappa(C) = \frac{1}{II_C^{\frac{1}{2}}} h_1 \mathbf{1}_\varkappa + h_0 C^{-1}, \tag{5.30}$$

where h_0 and h_1 are functions of the single variable

$$\theta = \frac{I_C}{II_C^{\frac{1}{2}}} \geqq \frac{1}{2}, \tag{5.31}$$

where we have used the condition (3.36).

Substituting (5.30) into the compatibility condition (5.24), we obtain the condition

$$\tfrac{1}{2}\theta h_1'(\theta) + h_0'(\theta) = 0 \quad \forall\, \theta \geqq \tfrac{1}{2}, \tag{5.32}$$

which is necessary and sufficient for the elastic membrane point to be hyperelastic. Integrating the system (5.23) with f_1 and f_0 given by (5.30), we obtain

$$\phi_\varkappa(C) = f(I_C, II_C) = \frac{1}{\varrho} h(\theta) + d, \tag{5.33}$$

where θ is defined by (5.31) and where d is a constant. Substituting the representation (5.33) into (5.23), we obtain

$$h_1(\theta) = 2h'(\theta), \qquad h_0(\theta) = h(\theta) - \theta h'(\theta). \tag{5.34}$$

Clearly the functions h_0 and h_1 defined by (5.34) for any function h obey the compatibility condition (5.32). The system (5.34) is the special case of the system (5.23) when the symmetry group $\mathscr{G}_\varkappa$ satisfies the condition (5.28).

Notice that the representation (5.33) reduces to (5.5) when the function h is a constant c. For this case (5.30) reduces to (3.5) since from (5.34) h_1 vanishes while h_0 reduces to the constant c. Then $\mathscr{G}_\varkappa = \mathscr{L}_\varkappa$ and $g_\varkappa = \mathscr{U}_\varkappa$ as we have remarked before. We claim that when h is arbitrary, $\mathscr{G}_\varkappa$ coincides with the group on the right-hand side of (5.28), *viz*

$$\mathscr{G}_\varkappa = \{qQ,\, q \neq 0,\, Q \in \mathscr{O}_\varkappa\} \tag{5.35}$$

while

$$g_\varkappa = \mathscr{O}_\varkappa. \tag{5.36}$$

Indeed, when h is arbitrary, a tensor $\boldsymbol{P} \in \mathscr{L}_\varkappa$ preserves (5.33) in the sense of (4.17) if and only if $\boldsymbol{P}$ preserves both θ and ϱ in the same sense. By using component forms we can verify that

$$\frac{I_{P^TCP}}{(II_{P^TCP})^{\frac{1}{2}}} = \frac{I_C}{(II_C)^{\frac{1}{2}}} \quad \forall\, C \in \mathscr{D} \tag{5.37}$$

if and only if P belongs to the group $\mathscr{G}_\varkappa$ given by (5.35). Specifically, when we use a principal basis for C, the equation (5.37) takes the component form

$$(P_{11}^2 + P_{12}^2)\, c_1 + (P_{21}^2 + P_{22}^2)\, c_2 = (P_{11}P_{22} - P_{12}P_{21})\,(c_1 + c_2), \qquad (5.38)$$

where c_1 and c_2 are the eigenvalues of C. Since c_1 and c_2 are arbitrary, (5.38) implies

$$P_{11}^2 + P_{12}^2 = P_{21}^2 + P_{22}^2 = P_{11}P_{22} - P_{12}P_{21} = q^2, \qquad (5.39)$$

where we have denoted the common value by q^2. Then

$$\begin{aligned} 0 &= P_{11}^2 + P_{12}^2 + P_{21}^2 + P_{22}^2 - 2P_{11}P_{22} + 2P_{12}P_{21}, \\ &= (P_{11} - P_{22})^2 + (P_{12} + P_{21})^2. \end{aligned} \qquad (5.40)$$

Hence

$$P_{11} = P_{22}, \quad P_{12} = -P_{21} \qquad (5.41)$$

and, therefore,

$$P_{11}P_{12} + P_{22}P_{21} = 0. \qquad (5.42)$$

It follows from (5.39) and (5.42) that if we write the tensor P as

$$P = qQ, \qquad (5.43)$$

then Q is orthogonal, *i.e.*, $Q \in \mathscr{O}_\varkappa$. Now since P must also preserve ϱ in the sense of (4.17) with ϱ given by (4.14), (5.36) is proved. The condition (5.35) may be verified similarly by substituting (5.30) into (2.23).

From (5.36) and (5.35) we see that for this type of hyperelastic membrane points

$$g_\varkappa^* = g_\varkappa = \mathscr{G}_\varkappa^* = \mathscr{O}_\varkappa \subset \mathscr{G}_\varkappa. \qquad (5.44)$$

Hence the condition (4.63) is satisfied: the points are of Class 2. The relation between the representation (5.33) and the general representation (4.87) may be explained as follows.

As before we define an equivalence relation on the domain $\mathscr{D}$ of $\phi_\varkappa$ by (4.84) with $\mathscr{G}_\varkappa$ given by (5.35), *viz*

$$C \sim \overline{C} \quad \Leftrightarrow \quad \overline{C} = q^2 Q^T C Q, \;\; q \neq 0, \;\; Q \in \mathscr{O}_\varkappa. \qquad (5.45)$$

We claim that

$$C \sim \overline{C} \quad \Leftrightarrow \quad \overline{\theta} = \frac{I_{\overline{C}}}{(II_{\overline{C}})^{\frac{1}{2}}} = \frac{I_C}{(II_C)^{\frac{1}{2}}} = \theta. \qquad (5.46)$$

This fact is more or less obvious, since from (5.45) C and $\overline{C}$ are equivalent if and only if their eigenvalues have the same ratio, *i.e.*,

$$\frac{c_1}{c_2} = \frac{\bar{c}_1}{\bar{c}_2} \quad \text{or} \quad \frac{c_1}{c_2} = \frac{\bar{c}_2}{\bar{c}_1}. \qquad (5.47)$$

Indeed, the ratio c_1/c_2 and its reciprocal c_2/c_1 are the roots of the equation

$$x^2 + (2 - \theta^2)\, x + 1 = 0. \tag{5.48}$$

Hence the parameter θ characterizes the equivalence classes with respect to (5.45).

Now for the equivalence class in $\mathscr{D}$ corresponding to the parameter θ_0 we choose a particular representative element C_0 such that

$$\det C_0 = 1. \tag{5.49}$$

Then we assign the value

$$\phi_\varkappa(C_0) = \frac{1}{\varrho_\varkappa} h(\theta_0) + d, \tag{5.50}$$

where h is an arbitrary function of θ_0. For a tensor C equivalent to C_0 (5.46) implies

$$\frac{I_C}{(II_C)^{\frac{1}{2}}} = \theta_0 \tag{5.51}$$

and (5.45) implies

$$C = P_0^T C_0 P_0 \tag{5.52}$$

for some $P_0 \in \mathscr{G}_\varkappa$. Taking the determinant of (5.52) and using (5.49), we see that

$$|\det P_0| = II_C^{\frac{1}{2}}. \tag{5.53}$$

According to the general representation formula (4.87) the value $\phi_\varkappa(C)$ is then given by

$$\phi_\varkappa(C) = II_C^{\frac{1}{2}} \left(\frac{1}{\varrho_\varkappa} h(\theta_0)\right) + d = \frac{1}{\varrho} h(\theta_0) + d, \tag{5.54}$$

which is just the representation formula (5.33) by virtue of (5.51).

An interesting special case in this type of isotropic hyperelastic membrane points can be defined by choosing

$$h(\theta) = c\theta, \tag{5.55}$$

where c is a constant. When h takes this special form, (5.34) reduces to

$$h_1(\theta) = 2c, \quad h_0(\theta) = 0, \tag{5.56}$$

so that (5.30) reduces to

$$A_\varkappa(C) = \frac{2c}{II_C^{\frac{1}{2}}} \mathbf{1}_\varkappa = \frac{2c}{\varrho_\varkappa} \varrho \mathbf{1}_\varkappa. \tag{5.57}$$

From (2.17) the response function $H_\varkappa$ is then given by

$$H_\varkappa(F) = \frac{2c}{\varrho_\varkappa} \varrho B, \tag{5.58}$$

where B is given by (3.18). Substituting (5.55) into (5.33) and using the definition (5.31) for θ, we see that the reduced stored energy function $\phi_\varkappa$ for this special case is given by

$$\phi_\varkappa(C) = \frac{c}{\varrho_\varkappa} \operatorname{tr} C + d, \tag{5.59}$$

where we have used $(3.19)_1$.

Having considered isotropic hyperelastic membrane points, we now turn our attention to some anisotropic hyperelastic membrane points. First, we claim that a directional surface tension point defined by the constitutive equation (3.38) is not hyperelastic. Indeed, using an orthonormal basis $\{e_\alpha\}$, where e_1 coincides with the unit vector e in the preferred line, we can express the representation formula (3.40) for $A_\varkappa$ in component form by

$$A_{\alpha\beta}(C_{\gamma\delta}) = c C_{\alpha\beta}^{-1}(C_{\gamma\delta}) + \frac{d}{C_{11}} \delta_{\alpha 1}\delta_{\beta 1}. \tag{5.60}$$

We have verified that the leading term on the right-hand side of (5.55) obeys the compatibiity condition (5.1), *cf.* (5.3). The last term of (5.60), however, does not satisfy (5.1). When we divide (5.60) by ϱ and use (4.14), *i.e.*

$$\frac{1}{\varrho} A_{\alpha\beta}(C_{\gamma\delta}) = \frac{c}{\varrho_\varkappa} (\det C)^{\frac{1}{2}} C_{\alpha\beta}^{-1}(C_{\gamma\delta}) + \frac{d}{\varrho_\varkappa} \frac{(\det C)^{\frac{1}{2}}}{C_{11}} \delta_{\alpha 1}\delta_{\beta 1}, \tag{5.61}$$

we see clearly that the last term depends on C_{22}. But since that term fails to vanish only if $\alpha = \beta = 1$, it cannot possibly satisfy (5.1) with $\alpha = \beta = 1$ and $\xi = \eta = 2$. Hence the reduced response function given by (3.40), like that given by (3.32), cannot be obtained from any function $\phi_\varkappa$ through the representation (4.13).

By the same token, if the response function is given by (3.41), where c and d are functions of the density ϱ, then the leading term on the right-hand side of (5.60) again satisfies the compatibility conditions (5.1); *cf.* (5.13). The last term of (5.60), however, will satisfy the compatibility conditions if and only if

$$\frac{d(\varrho)}{\varrho} = b, \tag{5.62}$$

where b is a constant. A similar result is known for (3-dim.) hyperelastic subfluids having a single preferred line; *cf.* [5]. Substituting (5.62) into (5.60), we see that the reduced response functions $A_\varkappa$ for this type of hyperelastic subfluid membrane points take the form

$$A_\varkappa(C) = c(\varrho)\, C^{-1} + \frac{b\varrho}{e \cdot Ce} e \otimes e \tag{5.63}$$

and the corresponding response functions $H_\varkappa(F)$ take the form

$$H_\varkappa(F) = c(\varrho)\, \mathbf{1}_\nu + b\varrho\, f \otimes f, \tag{5.64}$$

where f is defined by (3.37). Substituting (5.62) into (5.60) and then integrating the representation (4.13), we get

$$\phi_\varkappa(C) = \frac{1}{2} b \log (e \cdot Ce) - \int \frac{c(\varrho)}{\varrho^2} d\varrho. \tag{5.65}$$

Representation formulas similar to (5.64) and (5.65) are known for (3-dim.) hyperelastic subfluids; *cf.* [5, Type 2]. The proofs of (5.64) and (5.65) here are much simpler than the ones in [5], however, since we work with the frame-indifferent reduced response function $A_\varkappa$ and the reduced stored energy function $\phi_\varkappa$ rather than the response function $H_\varkappa$ and the stored energy function $\sigma_\varkappa$. In our analysis here we do not have to worry about the conditions of material frame-indifference (2.9) and (4.4), since they are automatically satisfied by the reduced forms (2.17) and (4.6).

The representation (5.65) reduces to (5.15) when the constant b vanishes. In that case the subfluid membrane point reduces to a surface tension point when $c(\varrho)$ is a non-zero constant c or to a fluid membrane point when $c(\varrho)$ is not a constant. When the constant b is not zero, a tensor $P \in \mathscr{L}_\varkappa$ preserves the leading term on the right-hand side of (5.65) in the sense of (4.17) if and only if

$$Pe = \pm e. \tag{5.66}$$

The last term of (5.65) is a function of ϱ (including the special case $c(\varrho) = c \neq 0$) and, therefore, it is preserved by P if and only if P is unimodular. The only exception to this assertion is when $c(\varrho)$ vanishes for all ϱ. Then the last term in (5.65) should really be replaced by a constant d, *viz*

$$\phi_\varkappa(C) = \frac{1}{2} b \log (e \cdot Ce) + d. \tag{5.67}$$

We did not bother to include the constant d in (5.65) because it could be absorbed into the indefinite integral in the general case when $c(\varrho) \neq 0$.

In view of (5.63), (5.65), and (5.67) we see that when $b \neq 0$ there are two possibilities:

First, when $c(\varrho) \neq 0$ (including, but not limited to the special case $c(\varrho) = c \neq 0$) the symmetry groups $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ are given by

$$\begin{aligned} \mathscr{g}_\varkappa &= \{P : |\det P| = 1, Pe = \pm e\}, \\ \mathscr{G}_\varkappa &= \{P : |\det P| = 1, Pe = \lambda e, |\lambda| > 0\}. \end{aligned} \tag{5.68}$$

Evidently this case belongs to Class 1, *i.e.*, $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ obey (4.62).

Second, when $c(\varrho) = 0 \quad \forall\, \varrho$, the symmetry groups $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ are given by

$$\begin{aligned} \mathscr{g}_\varkappa &= \{P : |\det P| > 0, Pe = \pm e\}, \\ \mathscr{G}_\varkappa &= \{P : |\det P| = 1, Pe = \lambda e, |\lambda| > 0\}. \end{aligned} \tag{5.69}$$

This case belongs to Class 3, *i.e.*, $\mathscr{g}_\varkappa$ and $\mathscr{G}_\varkappa$ obey (4.64).

In the preecding section we did not bother to analyze the reduced stored energy functions of Class 1 because the results are essentially the same as those derived in [10]. We now explain how to visualize the representation formula (5.67) as a special case of the general representation formula (4.104) for reduced stored energy functions of Class 3.

First, from (5.67) it is easily verified that the group $\tilde{\mathscr{G}}_\varkappa$ based on TRUESDELL's condition (4.29) is given by

$$\tilde{\mathscr{G}}_\varkappa = \{\boldsymbol{P} : |\det \boldsymbol{P}| > 0, \boldsymbol{P}\boldsymbol{e} = \lambda \boldsymbol{e}, |\lambda| > 0\}. \tag{5.70}$$

Next, it can be verified that the quotient class of any $\boldsymbol{P} \in \tilde{\mathscr{G}}_\varkappa$ is represented by the positive number

$$x = \|\boldsymbol{P}\boldsymbol{e}\|. \tag{5.71}$$

Finally, there is only one equivalence class and it is the whole domain $\mathscr{D}$, *i.e.*, every positive-definite symmetric tensor $\boldsymbol{C}$ may be represented by

$$\boldsymbol{C} = \boldsymbol{P}^T \boldsymbol{1}_\varkappa \boldsymbol{P} = \boldsymbol{P}^T \boldsymbol{P} \tag{5.72}$$

for some (maybe more than one) $\boldsymbol{P} \in \tilde{\mathscr{G}}_\varkappa$.

We pick the identity tensor $\boldsymbol{1}_\varkappa$ as the representative element of the equivalence class $\mathscr{D}$, and we assign an arbitrary value

$$\phi_\varkappa(\boldsymbol{1}_\varkappa) = d. \tag{5.73}$$

Then the constant b in (5.67) is just the constant c in (4.104), since from (5.71) (5.72), (5.73), and (4.104)

$$\phi_\varkappa(\boldsymbol{C}) = c \log \|\boldsymbol{P}\boldsymbol{e}\| + d = c \log (\boldsymbol{e} \cdot \boldsymbol{C}\boldsymbol{e})^{\frac{1}{2}} + d = \tfrac{1}{2} c \log (\boldsymbol{e} \cdot \boldsymbol{C}\boldsymbol{e}) + d, \tag{5.74}$$

where we have used (2.15) and

$$\|\boldsymbol{P}\boldsymbol{e}\| = (\boldsymbol{P}\boldsymbol{e} \cdot \boldsymbol{P}\boldsymbol{e})^{\frac{1}{2}} = (\boldsymbol{e} \cdot \boldsymbol{P}^T\boldsymbol{P}\boldsymbol{e})^{\frac{1}{2}} = (\boldsymbol{e} \cdot \boldsymbol{C}\boldsymbol{e})^{\frac{1}{2}}. \tag{5.75}$$

Representations for reduced stored energy functions of the two types of hyperelastic subfluid membrane points with reduced response functions of the form (3.45) may be derived by following essentially the same approach as in [5]. Since there is an extra state variable γ defined by (3.46) in the arguments of the functions a, b, and c, the relation between $\phi_\varkappa$ and $\boldsymbol{A}_\varkappa$ is much more complicated than the simple relation between (5.65) and (5.63) for the type of hyperelastic subfluid membrane points which we just analyzed. Hence, we omit the treatment of those two types of hyperelastic subfluid membrane points from this paper.

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