Integrity Bases for Vectors — The Crystal Classes

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Contents

1. Introduction

In a previous paper (SMITH, SMITH & RIVLIN (1963)), irreducible integrity bases for a symmetric three-dimensional tensor and absolute vector under the transformation groups describing each of the 32 crystal symmetries have been obtained and their minimality demonstrated. In the present paper, rather similar methods are used to determine irreducible integrity bases for an arbitrary number of absolute vectors under the transformation groups describing 31 of the 32 crystal symmetries. The remaining crystal class, which is the gyroidal class in the cubic system, has so far proven intractable for technical reasons although the methods used for the four classes of the cubic system, for which irreducible integrity bases have been found, are in principle applicable.

The fundamental theorems in invariant theory which are used in generating the integrity bases are given in § 2. In the case of all the crystal classes, except those of the cubic system, these theorems are used to generate integrity bases which in most cases are highly-redundant. The redundant elements are then eliminated and the irreducibility of the integrity basis so obtained is then proven.

In the case of the four classes of the cubic system considered, a different procedure is used. The elements of lowest degree in an irreducible integrity basis are first generated and then those of increasingly high degree, one of the theorems in § 2 being used to show when this procedure may be safely terminated.

For groups which consist only of proper transformations, an irreducible integrity basis for absolute vectors is also an irreducible integrity basis for polar vectors. This is, however, not the case for groups which contain improper transformations. In § 10, we discuss how the integrity basis for a number of absolute vectors, under a transformation group which lacks a center of symmetry, can, in certain cases, be used to obtain an integrity basis for a system of absolute and polar vectors under the group formed by adjoining the central inversion transformation to the original group of transformations.

The integrity basis under a group of transformations for an arbitrary number of absolute vectors can be used to obtain an irreducible tensor basis for the group of transformations. This is done in § 11 for each of the thirty-one groups considered in this paper.

Finally, in § 12, we derive certain theorems which may be of use, in certain cases, in generating invariants of tensors under a group of transformations which is a subgroup of the orthogonal group.

2. Some theorems concerning integrity bases

In deriving the integrity bases for n vectors, the following theorems will be used. Theorems 1, 2, 3 and 4 are the same as theorems 1, 2, 3 and 5 given by SMITH, SMITH & RIVLIN (1963), but a different notation, more convenient for the present paper, is used.

Theorem 1. *An integrity basis for polynomials* $P(X_1^{(1)}, X_2^{(1)}, X_1^{(2)}, X_2^{(2)}, \ldots, X_1^{(n)}, X_2^{(n)})$ *which are invariant under interchange of the subscripts* 1 *and* 2 *is formed by the* $\binom{n}{1}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x $(i = 1, \ldots, n)$, *the* $\binom{n}{2}$ *quantities obtained by substituting* $X^{(i)}$ *for* x, $X^{(j)}$ *for* y $(i, j = 1, \ldots, n;\ i<j)$ *in the quantities*

$$x_1 + x_2, \quad x_1 x_2 \tag{2.1}$$

and

$$x_1 y_2 + x_2 y_1$$

respectively.

Theorem 2. *An integrity basis for polynomials* $P(X_1^{(1)}, X_2^{(1)}, X_3^{(1)}, X_1^{(2)}, X_2^{(2)}, X_3^{(2)}, \ldots, X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ *which are invariant under all permutations of the subscripts is formed by the* $\binom{n}{1}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x $(i = 1, \ldots, n)$, *the* $\binom{n}{2}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x, $X^{(j)}$ *for* y $(i, j = 1, \ldots, n;\ i<j)$, *the* $\binom{n}{3}$ *quantities obtained by substituting* $X^{(i)}$ *for* x, $X^{(j)}$ *for* y, $X^{(k)}$ *for* z $(i, j, k = 1, \ldots, n;\ i<j<k)$ *in*

$$\begin{aligned} &x_1 + x_2 + x_3, \quad x_1 x_2 + x_2 x_3 + x_3 x_1, \quad x_1 x_2 x_3; \\ &x_1 y_1 + x_2 y_2 + x_3 y_3, \quad x_1 x_2 y_3 + x_2 x_3 y_1 + x_3 x_1 y_2, \quad y_1 y_2 x_3 + y_2 y_3 x_1 + y_3 y_1 x_2; \\ &x_1 y_1 z_1 + x_2 y_2 z_2 + x_3 y_3 z_3 \end{aligned} \tag{2.2}$$

respectively.

Theorem 3. *An integrity basis for polynomials* $P(X_1^{(1)}, X_2^{(1)}, X_3^{(1)}, X_1^{(2)}, X_2^{(2)}, X_3^{(2)}, \ldots, X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ *which are invariant under cyclic permutations of the subscripts is formed by the* $\binom{n}{1}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x $(i=1, \ldots, n)$, *the* $\binom{n}{2}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x, $X^{(j)}$ *for* y $(i, j=1, \ldots, n;\ i<j)$, *the* $\binom{n}{3}$ *sets of quantities obtained by substituting* $X^{(i)}$ *for* x, $X^{(j)}$ *for* y, $X^{(k)}$ *for* z $(i, j, k=1, \ldots, n;\ i<j<k)$ *in*

$$\begin{aligned}
&x_1+x_2+x_3,\ x_1x_2+x_2x_3+x_3x_1,\ x_1x_2x_3,\\
&x_1x_2(x_1-x_2)+x_2x_3(x_2-x_3)+x_3x_1(x_3-x_1);\\
&x_1y_1+x_2y_2+x_3y_3,\ x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2),\\
&x_1x_2y_3+x_2x_3y_1+x_3x_1y_2,\ y_1y_2x_3+y_2y_3x_1+y_3y_1x_2,\\
&x_1x_2(y_1-y_2)+x_2x_3(y_2-y_3)+x_3x_1(y_3-y_1),\\
&y_1y_2(x_1-x_2)+y_2y_3(x_2-x_3)+y_3y_1(x_3-x_1);\\
&\quad x_1y_1z_1+x_2y_2z_2+x_3y_3z_3,\\
&x_1y_1(z_2-z_3)+x_2y_2(z_3-z_1)+x_3y_3(z_1-z_2)
\end{aligned} \tag{2.3}$$

respectively.

Theorem 4. *An integrity basis for polynomials in the variables* $X_1, \ldots, X_q$, $I_1, I_2, \ldots, I_r$ *which are invariant under a group of transformations for which* $I_1, I_2, \ldots, I_r$ *are invariants, is formed by adjoining to the quantities* $I_1, I_2, \ldots, I_r$ *an integrity basis for polynomials in the variables* $X_1, \ldots, X_q$ *which are invariant under the given group of transformations.*

Theorem 5. *If the total degree of the elements of the irreducible integrity basis for polynomials in* n *vectors, under the group of transformations* $\mathcal{G}$ *of degree** n *is at most* N, *then the degree of the elements of the irreducible integrity basis for polynomials in* m $(>n)$ *vectors, under the group* $\mathcal{G}$, *is also at most* N.

Theorem 6. *If* $\det[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(n)}]$, *where* $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(n)}$ *are* n *vectors, is invariant under the group* $\mathcal{G}$ *of degree* n *and the degree of the elements of the integrity basis for polynomials in* $n-1$ *vectors, under* $\mathcal{G}$, *is at most* N, *then the degree of the elements of the integrity basis for polynomials in* m $(>n-1)$ *vectors, under* $\mathcal{G}$, *is also at most* N.

Theorems 5 and 6 follow immediately from a more general result in the theory of vector invariants (see WEYL (1946), pp. 39—44).

3. Notation

In accord with the notation employed in the previous section, we shall use the following notation for an integrity basis for polynomials in n vectors $\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \ldots, \boldsymbol{A}^{(n)}$:

$$\begin{aligned}
&L_1(\boldsymbol{x}), \ldots, L_p(\boldsymbol{x});\\
&M_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, M_q(\boldsymbol{x}, \boldsymbol{y});\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&N_1(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}), \ldots, N_r(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}).
\end{aligned} \tag{3.1}$$

* If in the matrix representation of the group $\mathcal{G}$, the matrices transforming the vectors are $n \times n$ matrices, the degree of the group is n.

The quantities in the first line of (3.1) represent the $\binom{n}{1}$ sets of quantities obtained from these by substituting $\boldsymbol{A}^{(i)}$ $(i = 1, 2, \ldots, n)$ for $\boldsymbol{x}$; these are, of course, the elements of the integrity basis which involve only a single vector. The quantities in the second line of (3.1) represent the $\binom{n}{2}$ sets of quantities obtained from these by substituting $\boldsymbol{A}^{(i)}$ for $\boldsymbol{x}$, $\boldsymbol{A}^{(j)}$ for $\boldsymbol{y}$ $(i, j = 1, \ldots, n;\ i < j)$; these are, of course, the elements of the integrity basis which involve two vectors. The remaining lines of (3.1) are interpreted in an analogous manner.

4. The transformation of vectors

We suppose that a three-dimensional vector has components A_i $(i = 1, 2, 3)$ in a particular rectangular Cartesian coordinate system and components $\bar{A}_i$ $(i = 1, 2, 3)$ in the coordinate system into which this is transformed by the transformation $(\boldsymbol{\alpha} = \|\alpha_{ij}\|)$. In Table 1 we give the relation between $\bar{A}_i$ and A_i for each of the transformations which occur in the groups describing the crystal classes. $\boldsymbol{I}$ is the identity transformation; $\boldsymbol{C}$ is the central inversion transformation; $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3$ are reflection transformations and $\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3$,

Table 1

α	$\boldsymbol{I}$	$\boldsymbol{C}$	$\boldsymbol{R}_1$	$\boldsymbol{R}_2$	$\boldsymbol{R}_3$	$\boldsymbol{D}_1$	$\boldsymbol{D}_2$	$\boldsymbol{D}_3$
$\bar{A}_1$	A_1	$-A_1$	$-A_1$	A_1	A_1	A_1	$-A_1$	$-A_1$
$\bar{A}_2$	A_2	$-A_2$	A_2	$-A_2$	A_2	$-A_2$	A_2	$-A_2$
$\bar{A}_3$	A_3	$-A_3$	A_3	A_3	$-A_3$	$-A_3$	$-A_3$	A_3

α	$\boldsymbol{T}_1$	$\boldsymbol{C}\,\boldsymbol{T}_1$	$\boldsymbol{R}_1\,\boldsymbol{T}_1$	$\boldsymbol{R}_2\,\boldsymbol{T}_1$	$\boldsymbol{R}_3\,\boldsymbol{T}_1$	$\boldsymbol{D}_1\,\boldsymbol{T}_1$	$\boldsymbol{D}_2\,\boldsymbol{T}_1$	$\boldsymbol{D}_3\,\boldsymbol{T}_1$
$\bar{A}_1$	A_1	$-A_1$	$-A_1$	A_1	A_1	A_1	$-A_1$	$-A_1$
$\bar{A}_2$	A_3	$-A_3$	A_3	$-A_3$	A_3	$-A_3$	A_3	$-A_3$
$\bar{A}_3$	A_2	$-A_2$	A_2	A_2	$-A_2$	$-A_2$	$-A_2$	A_2

α	$\boldsymbol{T}_2$	$\boldsymbol{C}\,\boldsymbol{T}_2$	$\boldsymbol{R}_1\,\boldsymbol{T}_2$	$\boldsymbol{R}_2\,\boldsymbol{T}_2$	$\boldsymbol{R}_3\,\boldsymbol{T}_2$	$\boldsymbol{D}_1\,\boldsymbol{T}_2$	$\boldsymbol{D}_2\,\boldsymbol{T}_2$	$\boldsymbol{D}_3\,\boldsymbol{T}_2$
$\bar{A}_1$	A_3	$-A_3$	$-A_3$	A_3	A_3	A_3	$-A_3$	$-A_3$
$\bar{A}_2$	A_2	$-A_2$	A_2	$-A_2$	A_2	$-A_2$	A_2	$-A_2$
$\bar{A}_3$	A_1	$-A_1$	A_1	A_1	$-A_1$	$-A_1$	$-A_1$	A_1

α	$\boldsymbol{T}_3$	$\boldsymbol{C}\,\boldsymbol{T}_3$	$\boldsymbol{R}_1\,\boldsymbol{T}_3$	$\boldsymbol{R}_2\,\boldsymbol{T}_3$	$\boldsymbol{R}_3\,\boldsymbol{T}_3$	$\boldsymbol{D}_1\,\boldsymbol{T}_3$	$\boldsymbol{D}_2\,\boldsymbol{T}_3$	$\boldsymbol{D}_3\,\boldsymbol{T}_3$
$\bar{A}_1$	A_2	$-A_2$	$-A_2$	A_2	A_2	A_2	$-A_2$	$-A_2$
$\bar{A}_2$	A_1	$-A_1$	A_1	$-A_1$	A_1	$-A_1$	A_1	$-A_1$
$\bar{A}_3$	A_3	$-A_3$	A_3	A_3	$-A_3$	$-A_3$	$-A_3$	A_3

α	$\boldsymbol{M}_1$	$\boldsymbol{C}\,\boldsymbol{M}_1$	$\boldsymbol{R}_1\,\boldsymbol{M}_1$	$\boldsymbol{R}_2\,\boldsymbol{M}_1$	$\boldsymbol{R}_3\,\boldsymbol{M}_1$	$\boldsymbol{D}_1\,\boldsymbol{M}_1$	$\boldsymbol{D}_2\,\boldsymbol{M}_1$	$\boldsymbol{D}_3\,\boldsymbol{M}_1$
$\bar{A}_1$	A_2	$-A_2$	$-A_2$	A_2	A_2	A_2	$-A_2$	$-A_2$
$\bar{A}_2$	A_3	$-A_3$	A_3	$-A_3$	A_3	$-A_3$	A_3	$-A_3$
$\bar{A}_3$	A_1	$-A_1$	A_1	A_1	$-A_1$	$-A_1$	$-A_1$	A_1

α	$\boldsymbol{M}_2$	$\boldsymbol{C}\,\boldsymbol{M}_2$	$\boldsymbol{R}_1\,\boldsymbol{M}_2$	$\boldsymbol{R}_2\,\boldsymbol{M}_2$	$\boldsymbol{R}_3\,\boldsymbol{M}_2$	$\boldsymbol{D}_1\,\boldsymbol{M}_2$	$\boldsymbol{D}_2\,\boldsymbol{M}_2$	$\boldsymbol{D}_3\,\boldsymbol{M}_2$
$\bar{A}_1$	A_3	$-A_3$	$-A_3$	A_3	A_3	A_3	$-A_3$	$-A_3$
$\bar{A}_2$	A_1	$-A_1$	A_1	$-A_1$	A_1	$-A_1$	A_1	$-A_1$
$\bar{A}_3$	A_2	$-A_2$	A_2	A_2	$-A_2$	$-A_2$	$-A_2$	A_2

Table 1 (Continued)

α	S_1	$C\,S_1$	$R_1\,S_1$	$R_2\,S_1$	$R_3\,S_1$	$D_1\,S_1$	$D_2\,S_1$	$D_3\,S_1$
$\bar{A}_1$	α_1	$-\alpha_1$	$-\alpha_1$	α_1	α_1	α_1	$-\alpha_1$	$-\alpha_1$
$\bar{A}_2$	α_2	$-\alpha_2$	α_2	$-\alpha_2$	α_2	$-\alpha_2$	α_2	$-\alpha_2$
$\bar{A}_3$	A_3	$-A_3$	A_3	A_3	$-A_3$	$-A_3$	$-A_3$	A_3

α	S_2	$C\,S_2$	$R_1\,S_2$	$R_2\,S_2$	$R_3\,S_2$	$D_1\,S_2$	$D_2\,S_2$	$D_3\,S_2$
$\bar{A}_1$	β_1	$-\beta_1$	$-\beta_1$	β_1	β_1	β_1	$-\beta_1$	$-\beta_1$
$\bar{A}_2$	β_2	$-\beta_2$	β_2	$-\beta_2$	β_2	$-\beta_2$	β_2	$-\beta_2$
$\bar{A}_3$	A_3	$-A_3$	A_3	A_3	$-A_3$	$-A_3$	$-A_3$	A_3

M_1, M_2, S_1, S_2 are rotation transformations. Explicit expressions for the transformation matrices are given by SMITH, SMITH & RIVLIN (1963). The notation

$$\alpha_1 = -\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2, \quad \beta_1 = -\tfrac{1}{2}A_1 - \tfrac{\sqrt{3}}{2}A_2,$$
$$\alpha_2 = -\tfrac{\sqrt{3}}{2}A_1 - \tfrac{1}{2}A_2, \quad \beta_2 = \tfrac{\sqrt{3}}{2}A_1 - \tfrac{1}{2}A_2$$

is used in Table 1.

5. The triclinic, monoclinic, rhombic and tetragonal systems

(a) *General description of procedure*

For each of the classes of these systems, we determine integrity bases for polynomials in n vectors $A_i^{(r)}$ $(r=1,\ldots,n;\ i=1,2,3)$ under the transformations describing the symmetry of the class. These transformations follow the name of the class, in each case, in the list given below. In applying theorem 1, we first obtain, from Table 1, the quantities $\bar{A}_i^{(r)}$ into which $A_i^{(r)}$ $(r=1,\ldots,n)$ are changed by the first (non-identity) transformation. We note that invariants must be symmetric with respect to the pairs $\bar{A}_i^{(r)}, A_i^{(r)}$. Using theorem 1, we obtain a set of polynomials in terms of which any polynomial, invariant under the transformation, must be expressible. We omit the redundant elements of this set and find, again using Table 1, the manner in which the remaining polynomials of the set are changed by the second transformation. We again apply theorem 1 to obtain the limitations on the form of a polynomial invariant under the first two (non-identity) transformations. Proceeding in this way for each transformation of the class, we obtain an integrity basis for polynomials in the vectors $A_i^{(r)}$ $(r=1,\ldots,n)$.

These are listed below, using the notation of § 3. It is immediately obvious, for the classes of the triclinic, monoclinic and rhombic systems, that the integrity bases are irreducible. The irreducibility of the integrity bases for the classes of the tetragonal system is demonstrated in § 6.

(b) *Triclinic system*

Pedial class (**I**)

$$x_1,\ x_2,\ x_3. \tag{5.1}$$

Pinacoidal class (**I**, **C**)

$$\begin{aligned} &x_1^2,\ x_2^2,\ x_3^2,\ x_1x_2,\ x_2x_3,\ x_3x_1;\\ &x_1y_1,\ x_2y_2,\ x_3y_3,\ x_1y_2,\ x_1y_3,\ x_2y_1,\ x_2y_3,\ x_3y_1,\ x_3y_2. \end{aligned} \tag{5.2}$$

(c) Monoclinic system

Domatic class $(\boldsymbol{I}, \boldsymbol{R}_1)$

$$x_1^2,\ x_2,\ x_3;\quad x_1 y_1. \tag{5.3}$$

Sphenoidal class $(\boldsymbol{I}, \boldsymbol{D}_1)$

$$x_1,\ x_2^2,\ x_3^2,\ x_2 x_3;\quad x_2 y_2,\ x_2 y_3,\ x_3 y_2,\ x_3 y_3. \tag{5.4}$$

Prismatic class $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{D}_1)$

$$x_1^2,\ x_2^2,\ x_3^2,\ x_2 x_3;\quad x_1 y_1,\ x_2 y_2,\ x_3 y_3,\ x_2 y_3,\ x_3 y_2. \tag{5.5}$$

(d) Rhombic system

Rhombic-pyramidal class $(\boldsymbol{I}, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1)$

$$x_1,\ x_2^2,\ x_3^2;\quad x_2 y_2,\ x_3 y_3. \tag{5.6}$$

Rhombic-disphenoidal class $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)$

$$\begin{gathered} x_1^2,\ x_2^2,\ x_3^2,\ x_1 x_2 x_3; \\ x_1 y_1,\ x_2 y_2,\ x_3 y_3,\ x_1 x_2 y_3,\ x_2 x_3 y_1,\ x_3 x_1 y_2,\ y_1 y_2 x_3,\ y_2 y_3 x_1,\ y_3 y_1 x_2; \\ x_1 y_2 z_3,\ x_2 y_3 z_1,\ x_3 y_1 z_2,\ x_1 y_3 z_2,\ x_2 y_1 z_3,\ x_3 y_2 z_1. \end{gathered} \tag{5.7}$$

Rhombic-dipyramidal class $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)$

$$x_1^2,\ x_2^2,\ x_3^2;\quad x_1 y_1,\ x_2 y_2,\ x_3 y_3. \tag{5.8}$$

(e) Tetragonal system

Tetragonal-pyramidal class $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3)$.

$A_3^{(1)}, A_3^{(2)}, \ldots, A_3^{(n)}$ together with

$$\begin{gathered} x_1^2 + x_2^2,\ x_1 x_2 (x_1^2 - x_2^2),\ x_1^2 x_2^2; \\ x_1 y_1 + x_2 y_2,\ x_1 y_2 - x_2 y_1,\ x_1 x_2 y_1 y_2,\ x_1 x_2 (y_1^2 - y_2^2), \\ x_1 x_2 (x_1 y_2 + x_2 y_1),\ x_1 x_2 (x_1 y_1 - x_2 y_2),\ y_1 y_2 (x_1 y_2 + x_2 y_1), \\ y_1 y_2 (x_1 y_1 - x_2 y_2); \\ x_1 x_2 (y_1 z_2 + y_2 z_1),\ x_1 x_2 (y_1 z_1 - y_2 z_2),\ y_1 y_2 (x_1 z_2 + x_2 z_1), \\ y_1 y_2 (x_1 z_1 - x_2 z_2),\ z_1 z_2 (x_1 y_2 + x_2 y_1),\ z_1 z_2 (x_1 y_1 - x_2 y_2); \\ x_1 y_1 z_1 u_1 + x_2 y_2 z_2 u_2,\ x_1 y_1 z_1 u_2 - x_2 y_2 z_2 u_1. \end{gathered} \tag{5.9}$$

Tetragonal-disphenoidal class $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{D}_1 \boldsymbol{T}_3, \boldsymbol{D}_2 \boldsymbol{T}_3)$.

The quantities (5.9) together with

$$\begin{gathered} x_3^2,\ x_1 x_2 x_3,\ x_3 (x_1^2 - x_2^2); \\ x_3 y_3,\ x_3 (x_1 y_1 - x_2 y_2),\ x_3 (x_1 y_2 + x_2 y_1),\ y_1 y_2 x_3, \\ x_3 (y_1^2 - y_2^2),\ y_3 (x_1 y_1 - x_2 y_2),\ y_3 (x_1 y_2 + x_2 y_1),\ x_1 x_2 y_3,\ y_3 (x_1^2 - x_2^2); \\ x_3 (y_1 z_1 - y_2 z_2),\ x_3 (y_1 z_2 + y_2 z_1),\ y_3 (x_1 z_1 - x_2 z_2), \\ y_3 (x_1 z_2 + x_2 z_1),\ z_3 (x_1 y_1 - x_2 y_2),\ z_3 (x_1 y_2 + x_2 y_1). \end{gathered} \tag{5.10}$$

Tetragonal-dipyramidal class ($\boldsymbol{I}$, $\boldsymbol{C}$, $\boldsymbol{R}_3$, $\boldsymbol{D}_3$, $\boldsymbol{R}_1\boldsymbol{T}_3$, $\boldsymbol{R}_2\boldsymbol{T}_3$, $\boldsymbol{D}_1\boldsymbol{T}_3$, $\boldsymbol{D}_2\boldsymbol{T}_3$).
The quantities (5.9) together with

$$\begin{aligned} &x_3^2; \\ &x_3 y_3. \end{aligned} \tag{5.11}$$

Ditetragonal-pyramidal class ($\boldsymbol{I}$, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{D}_3$) · ($\boldsymbol{I}$, $\boldsymbol{T}_3$).
$A_3^{(1)}, \ldots, A_3^{(n)}$ together with

$$\begin{aligned} &x_1^2 + x_2^2,\ x_1^2 x_2^2; \\ &x_1 y_1 + x_2 y_2,\ x_1 x_2 y_1 y_2,\ x_1 x_2 (x_1 y_2 + x_2 y_1),\ y_1 y_2 (x_1 y_2 + x_2 y_1); \\ &x_1 x_2 (y_1 z_2 + y_2 z_1),\ y_1 y_2 (x_1 z_2 + x_2 z_1),\ z_1 z_2 (x_1 y_2 + x_2 y_1); \\ &x_1 y_1 z_1 u_1 + x_2 y_2 z_2 u_2. \end{aligned} \tag{5.12}$$

Tetragonal-scalenohedral class ($\boldsymbol{I}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$) · ($\boldsymbol{I}$, $\boldsymbol{T}_3$).
The quantities (5.12) together with

$$\begin{aligned} &x_3^2,\ x_1 x_2 x_3; \\ &x_3 y_3,\ x_3 (x_1 y_2 + x_2 y_1),\ y_3 (x_1 y_2 + x_2 y_1),\ x_1 x_2 y_3,\ y_1 y_2 x_3; \\ &x_3 (y_1 z_2 + y_2 z_1),\ y_3 (x_1 z_2 + x_2 z_1),\ z_3 (x_1 y_2 + x_2 y_1). \end{aligned} \tag{5.13}$$

Ditetragonal-dipyramidal class ($\boldsymbol{I}$, $\boldsymbol{C}$, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$) · ($\boldsymbol{I}$, $\boldsymbol{T}_3$).
The quantities (5.12) together with

$$\begin{aligned} &x_3^2; \\ &x_3 y_3. \end{aligned} \tag{5.14}$$

Tetragonal-trapezohedral class ($\boldsymbol{I}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$, $\boldsymbol{C}\boldsymbol{T}_3$, $\boldsymbol{R}_1\boldsymbol{T}_3$, $\boldsymbol{R}_2\boldsymbol{T}_3$, $\boldsymbol{R}_3\boldsymbol{T}_3$).
The quantities (5.12) together with

$$\begin{aligned} &x_3^2,\ x_1 x_2 x_3 (x_1^2 - x_2^2); \\ &x_3 y_3,\ x_3 (x_1 y_2 - x_2 y_1),\ x_3 y_1 y_2 (y_1^2 - y_2^2),\ x_3 x_1 x_2 (y_1^2 - y_2^2), \\ &x_3 x_1 x_2 (x_1 y_1 - x_2 y_2),\ x_3 y_1 y_2 (x_1 y_1 - x_2 y_2) \end{aligned}$$

and the quantities obtained by interchanging x and y;

$$\begin{aligned} &x_3 (y_1 z_2 - y_2 z_1),\ x_3 z_1 z_2 (y_1^2 - y_2^2),\ x_3 z_1 z_2 (y_1 z_1 - y_2 z_2), \\ &x_3 y_1 y_2 (y_1 z_1 - y_2 z_2),\ x_3 x_1 x_2 (y_1 z_1 - y_2 z_2),\ x_3 z_1 z_2 (x_1 y_1 - x_2 y_2), \\ &x_3 y_1 y_2 (x_1 z_1 - x_2 z_2) \end{aligned} \tag{5.15}$$

and the quantities obtained by cyclic permutation of x, y, z;

$$\begin{aligned} &x_3 (x_1 y_1 z_1 u_2 - x_2 y_2 z_2 u_1),\ x_3 y_1 y_2 (z_1 u_1 - z_2 u_2), \\ &x_3 z_1 z_2 (y_1 u_1 - y_2 u_2),\ x_3 u_1 u_2 (y_1 z_1 - y_2 z_2) \end{aligned}$$

and the quantities obtained by cyclic permutation of x, y, z, u;

$$x_3 (y_1 z_1 u_1 v_2 - y_2 z_2 u_2 v_1)$$

and the quantities obtained by cyclic permutation of x, y, z, u, v.

6. Irreducibility of the integrity bases for the tetragonal system

The irreducibility of the integrity bases for n vectors, derived in § 5, for the classes of the tetragonal system can be proven in a manner somewhat similar to that employed by SMITH, SMITH & RIVLIN (1963) to demonstrate the irreducibility of the integrity bases for a single vector and symmetric tensor.

We note from § 5 that the elements of the integrity bases for the various classes of the tetragonal system are either independent of x_3 (*i.e.* of $A_3^{(1)}, \ldots, A_3^{(n)}$), or are linear in these quantities, or have one of the forms x_3^2, $x_3 y_3$. For each class it is apparent by inspection that the irreducibility of the set of elements which is linear in x_3 and the irreducibility of the set which is independent of x_3 may be considered separately. Furthermore, it is also apparent that in the cases in which the integrity basis given in § 5 has x_3^2 or $x_3 y_3$, or both sets, as elements, none of these elements are redundant.

Accordingly, we define a polynomial of degree $i_1 i_2 \ldots i_n$ in the components $(x_1, x_2), (y_1, y_2), \ldots, (z_1, z_2)$ of the n vectors $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$ as a polynomial of partial degrees i_r $(r=1, \ldots, n)$ in $(x_1, x_2), (y_1, y_2), \ldots, (z_1, z_2)$. Let $P_{i_1 i_2 \ldots i_n}$ be the number of linearly independent invariants, which do not involve $x_3, y_3, \ldots, z_3$, of degree $i_1 i_2 \ldots i_n$. Let $\gamma_{i_1 i_2 \ldots i_n}$ be the number of invariants of degree $i_1 i_2 \ldots i_n$ in an irreducible integrity basis, which do not involve $x_3, y_3, \ldots, z_3$. Let $\beta_{i_1 i_2 \ldots i_n}$ be the number of invariants of degree $i_1 i_2 \ldots i_n$, which do not involve $x_3, y_3, \ldots, z_3$ in the integrity basis under consideration. Let $J_1, J_2, \ldots, J_\nu$ be the elements of an irreducible integrity basis of degree 0 or 1 in $x_3, y_3, \ldots, z_3$, of total degree in $(x_1, x_2), (y_1, y_2), \ldots, (z_1, z_2)$ less than $i_1 + i_2 + \cdots + i_n$. Let $\vartheta_{i_1 i_2 \ldots i_n}$ be the number of terms of the form $J_1^{\alpha_1} J_2^{\alpha_2} \ldots J_\nu^{\alpha_\nu}$, where the α's are positive integers or zero, which are of partial degrees $i_1, i_2, \ldots, i_n$ in $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$, and do not involve $x_3, y_3, \ldots, z_3$.

It then follows [see, for example, SMITH (1960)] that

$$P_{i_1 i_2 \ldots i_n} - \vartheta_{i_1 i_2 \ldots i_n} \leqq \gamma_{i_1 i_2 \ldots i_n} \leqq \beta_{i_1 i_2 \ldots i_n}. \tag{6.1}$$

If it is possible to establish $\eta_{i_1 i_2 \ldots i_n}$ independent syzygies of degree $i_1 i_2 \ldots i_n$ among the invariants $J_1, J_2, \ldots, J_\nu$, the inequality (6.1) may be strengthened to

$$P_{i_1 i_2 \ldots i_n} - \vartheta_{i_1 i_2 \ldots i_n} + \eta_{i_1 i_2 \ldots i_n} \leqq \gamma_{i_1 i_2 \ldots i_n} \leqq \beta_{i_1 i_2 \ldots i_n}. \tag{6.2}$$

We now define $\bar{P}_{i_1 i_2 \ldots i_n}, \bar{\gamma}_{i_1 i_2 \ldots i_n}, \bar{\beta}_{i_1 i_2 \ldots i_n}$ analogously to $P_{i_1 i_2 \ldots i_n}, \gamma_{i_1 i_2 \ldots i_n}, \beta_{i_1 i_2 \ldots i_n}$ for the elements of the integrity basis which involve $x_3, y_3, \ldots, z_3$ linearly. Thus, for example, $\bar{P}_{i_1 i_2 \ldots i_n}$ is the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ which are linear in $A_3^{(1)}$ (say). Let $\bar{\vartheta}_{i_1 i_2 \ldots i_n}$ be the number of terms of the form $J_1^{\alpha_1} J_2^{\alpha_2} \ldots J_\nu^{\alpha_\nu}$, where the α's are positive integers or zero, which are of partial degrees $i_1, i_2, \ldots, i_n$ in $(x_1, x_2), (y_1, y_2), \ldots, (z_1, z_2)$ and are linear in $A_3^{(1)}$ (say). Then, analogously to (6.1),

$$\bar{P}_{i_1 i_2 \ldots i_n} - \bar{\vartheta}_{i_1 i_2 \ldots i_n} \leqq \bar{\gamma}_{i_1 i_2 \ldots i_n} \leqq \bar{\beta}_{i_1 i_2 \ldots i_n}. \tag{6.3}$$

Again, if it is possible to establish $\bar{\eta}_{i_1 i_2 \ldots i_n}$ independent syzygies of degree $i_1 i_2 \ldots i_n$ and linear in $A_3^{(1)}$ (say) among the invariants $J_1, J_2, \ldots, J_\nu$, the inequality (6.3) may be strengthened to

$$\bar{P}_{i_1 i_2 \ldots i_n} - \bar{\vartheta}_{i_1 i_2 \ldots i_n} + \bar{\eta}_{i_1 i_2 \ldots i_n} \leqq \bar{\gamma}_{i_1 i_2 \ldots i_n} \leqq \bar{\beta}_{i_1 i_2 \ldots i_n}. \tag{6.4}$$

Let $\boldsymbol{L}_\alpha$ be the 2×2 matrices formed by the elements of the first two rows and columns of the matrices $\boldsymbol{\alpha}$ of the three-dimensional matrix representation of the group characterizing the crystal class considered. Let M_α be the element in the third row and third column of $\boldsymbol{\alpha}$. Then, $P_{i_1 i_2 \dots i_n}$ and $\bar{P}_{i_1 i_2 \dots i_n}$ are given by

$$\begin{aligned} P_{i_1 i_2 \dots i_n} &= \frac{1}{N}\sum_\alpha \operatorname{tr} \boldsymbol{L}_\alpha^{(i_1)} \operatorname{tr} \boldsymbol{L}_\alpha^{(i_2)} \dots \operatorname{tr} \boldsymbol{L}_\alpha^{(i_n)}, \\ \bar{P}_{i_1 i_2 \dots i_n} &= \frac{1}{N}\sum_\alpha M_\alpha \operatorname{tr} \boldsymbol{L}_\alpha^{(i_1)} \operatorname{tr} \boldsymbol{L}_\alpha^{(i_2)} \dots \operatorname{tr} \boldsymbol{L}_\alpha^{(i_n)}, \end{aligned} \tag{6.5}$$

where N is the number of transformations in the group considered. $\boldsymbol{L}_\alpha^{(i_r)}$ is the symmetrized Kronecker i_r-th power of $\boldsymbol{L}_\alpha$ and $\operatorname{tr} \boldsymbol{L}_\alpha^{(i_r)}$ is given by

$$\operatorname{tr} \boldsymbol{L}_\alpha^{(i_r)} = \sum \frac{1}{\mu_1!\,\mu_2!\dots\mu_{i_r}!}\left(\frac{S_1}{1}\right)^{\mu_1}\left(\frac{S_2}{2}\right)^{\mu_2}\cdots\left(\frac{S_{i_r}}{i_r}\right)^{\mu_{i_r}}, \tag{6.6}$$

where

$$S_{i_r} = \operatorname{tr} \boldsymbol{L}_\alpha^{i_r}, \tag{6.7}$$

and the summation is carried out over the set of all positive integers $\mu_1, \mu_2, \dots, \mu_{i_r}$ such that

$$\mu_1 + 2\mu_2 + \cdots + i_r \mu_{i_r} = i_r.$$

Thus,

$$\begin{aligned} 1!\operatorname{tr} \boldsymbol{L}_\alpha^{(1)} &= S_1, \\ 2!\operatorname{tr} \boldsymbol{L}_\alpha^{(2)} &= S_1^2 + S_2, \\ 3!\operatorname{tr} \boldsymbol{L}_\alpha^{(3)} &= S_1^3 + 3S_1S_2 + 2S_3, \\ 4!\operatorname{tr} \boldsymbol{L}_\alpha^{(4)} &= S_1^4 + 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 + 6S_4, \\ 5!\operatorname{tr} \boldsymbol{L}_\alpha^{(5)} &= S_1^5 + 10S_1^3S_2 + 20S_1^2S_3 + 15S_1S_2^2 + 30S_1S_4 + 20S_2S_3 + 24S_5, \\ 6!\operatorname{tr} \boldsymbol{L}_\alpha^{(6)} &= S_1^6 + 15S_1^4S_2 + 40S_1^3S_3 + 45S_1^2S_2^2 + 90S_1^2S_4 + \\ &\quad + 120S_1S_2S_3 + 144S_1S_5 + 15S_2^3 + 90S_2S_4 + 40S_3^2 + 120S_6. \end{aligned} \tag{6.8}$$

The values of M_α and $\operatorname{tr} \boldsymbol{L}_\alpha^{(1)}, \operatorname{tr} \boldsymbol{L}_\alpha^{(2)}, \dots, \operatorname{tr} \boldsymbol{L}_\alpha^{(6)}$ are given in Tables 2 and 3 for each of the transformations of the groups describing the symmetry of the classes in the tetragonal and hexagonal systems. The values for the transformations of the hexagonal system are included since they will be required later in § 8.

Table 2

Transformation $\boldsymbol{\alpha}$	M_α
$(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3)\cdot(\boldsymbol{I}, \boldsymbol{T}_3, \boldsymbol{S}_1, \boldsymbol{S}_2)$	1
$(\boldsymbol{C}, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2)\cdot(\boldsymbol{I}, \boldsymbol{T}_3, \boldsymbol{S}_1, \boldsymbol{S}_2)$	-1

Table 3

Transformations $\boldsymbol{\alpha}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(1)}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(2)}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(3)}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(4)}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(5)}$	$\operatorname{tr} \boldsymbol{L}_\alpha^{(6)}$
$\boldsymbol{I}, \boldsymbol{R}_3$	2	3	4	5	6	7
$\boldsymbol{C}, \boldsymbol{D}_3$	-2	3	-4	5	-6	7
$\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_1, \boldsymbol{D}_2$	0	1	0	1	0	1
$\boldsymbol{T}_3, \boldsymbol{C}\boldsymbol{T}_3, \boldsymbol{R}_3\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3$	0	1	0	1	0	1
$\boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3$	0	-1	0	1	0	-1
$\boldsymbol{S}_1, \boldsymbol{R}_3\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{R}_3\boldsymbol{S}_2$	-1	0	1	-1	0	1
$\boldsymbol{C}\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{C}\boldsymbol{S}_2, \boldsymbol{D}_3\boldsymbol{S}_2$	1	0	-1	-1	0	1
$\boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_2\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_2\boldsymbol{S}_1$	0	1	0	1	0	1
$\boldsymbol{R}_1\boldsymbol{S}_2, \boldsymbol{R}_2\boldsymbol{S}_2, \boldsymbol{D}_1\boldsymbol{S}_2, \boldsymbol{D}_2\boldsymbol{S}_2$	0	1	0	1	0	1

In order to prove that the integrity bases given in § 5 for the various classes of the tetragonal system are irreducible, we first prove the irreducibility for the tetragonal-pyramidal class.

For the tetragonal-pyramidal class, apart from the invariants of the type x_3, which are evidently not redundant, we are concerned only with invariants which are independent of $x_3, y_3, \ldots$. We therefore do not need to calculate $\bar{P}_{i_1 i_2 \ldots i_n}$, $\bar{\vartheta}_{i_1 i_2 \ldots i_n}$, $\bar{\beta}_{i_1 i_2 \ldots i_n}$ in this case. Also, it is evident by inspection that none of the invariants in (5.9), which involve x_1, x_2 only, is redundant. Accordingly, $P_{i_1 i_2 \ldots i_n}$, $\vartheta_{i_1 i_2 \ldots i_n}$, $\beta_{i_1 i_2 \ldots i_n}$ are calculated, for the various values of $i_1 i_2 \ldots i_n$ represented in the integrity basis for the tetragonal-pyramidal class, except those corresponding to invariants of the type x_3 and to invariants which involve x_1, x_2 only. The values of $P_{i_1 i_2 \ldots i_n}$, $\vartheta_{i_1 i_2 \ldots i_n}$, $\beta_{i_1 i_2 \ldots i_n}$ are given in Table 4. The corresponding values of $P_{i_1 i_2 \ldots i_n} - \vartheta_{i_1 i_2 \ldots i_n}$ $(= Q_{i_1 i_2 \ldots i_n})$ are also given. In the last line of Table 4 we give the number of independent syzygies $\eta_{i_1 i_2 \ldots i_n}$ which must be demonstrated in order to make the upper and lower bounds on $\gamma_{i_1 i_2 \ldots i_n}$ in (6.2) equal.

Table 4

$i_1 i_2 \ldots i_n$	11	22	31	211	1111
$P_{i_1 i_2 \ldots i_n}$	2	5	4	6	8
$\vartheta_{i_1 i_2 \ldots i_n}$	0	4	2	6	12
$Q_{i_1 i_2 \ldots i_n}$	2	1	2	0	−4
$\beta_{i_1 i_2 \ldots i_n}$	2	2	2	2	2
$\eta_{i_1 i_2 \ldots i_n}$	0	1	0	2	6

These syzygies are

$i_1 i_2 \ldots i_n = 22$:

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 y_1 + x_2 y_2)^2 - (x_1 y_2 - x_2 y_1)^2 = 0.$$

$i_1 i_2 \ldots i_n = 211$:

$$(x_1^2 + x_2^2)(y_1 z_1 + y_2 z_2) - (x_1 y_1 + x_2 y_2)(x_1 z_1 + x_2 z_2) - \\ - (x_1 y_2 - x_2 y_1)(x_1 z_2 - x_2 z_1) = 0.$$

$$(x_1 y_1 + x_2 y_2)(x_1 z_2 - x_2 z_1) - (x_1 z_1 + x_2 z_2)(x_1 y_2 - x_2 y_1) - \\ - (x_1^2 + x_2^2)(y_1 z_2 - y_2 z_1) = 0.$$

$i_1 i_2 \ldots i_n = 1111$:

$$(x_1 y_1 + x_2 y_2)(z_1 u_1 + z_2 u_2) - (x_1 z_2 - x_2 z_1)(y_1 u_2 - y_2 u_1) - \\ - (x_1 u_1 + x_2 u_2)(y_1 z_1 + y_2 z_2) = 0.$$

$$(x_1 y_1 + x_2 y_2)(z_1 u_1 + z_2 u_2) - (y_1 z_2 - y_2 z_1)(x_1 u_2 - x_2 u_1) - \\ - (y_1 u_1 + y_2 u_2)(x_1 z_1 + x_2 z_2) = 0.$$

$$(x_1 u_1 + x_2 u_2)(y_1 z_1 + y_2 z_2) + (z_1 u_2 - z_2 u_1)(x_1 y_2 - x_2 y_1) - \\ - (y_1 u_1 + y_2 u_2)(x_1 z_1 + x_2 z_2) = 0.$$

$$(x_1 u_1 + x_2 u_2)(y_1 z_2 - y_2 z_1) - (x_1 z_1 + x_2 z_2)(y_1 u_2 - y_2 u_1) + \\ + (x_1 y_1 + x_2 y_2)(z_1 u_2 - z_2 u_1) = 0.$$

$$(x_1 y_2 - x_2 y_1)(z_1 u_1 + z_2 u_2) + (x_1 z_1 + x_2 z_2)(y_1 u_2 - y_2 u_1) - \\ - (x_1 u_2 - x_2 u_1)(y_1 z_1 + y_2 z_2) = 0.$$

$$(x_1 z_2 - x_2 z_1)(y_1 u_1 + y_2 u_2) + (x_1 y_1 + x_2 y_2)(z_1 u_2 - z_2 u_1) - \\ - (x_1 u_2 - x_2 u_1)(y_1 z_1 + y_2 z_2) = 0.$$

It is immediately evident that the integrity basis given in § 5 for the tetragonal-dipyramidal class is irreducible, since it differs from that for the tetragonal-pyramidal class only in the invariants involving x_3, y_3 only and these are evidently not redundant.

Again, since the integrity basis given in § 5 for the ditetragonal-pyramidal class is contained in that for the tetragonal-pyramidal class, it must be irreducible.

The integrity basis for the tetragonal-scalenohedral class given in § 5 differs from that for the ditetragonal-pyramidal class only in invariants which involve x_3, y_3 or z_3. By inspection it is seen that none of the invariants of this type which are listed are redundant. It follows that the integrity basis for the tetragonal-scalenohedral class is irreducible. By a similar argument, it is seen that the integrity basis for the ditetragonal-dipyramidal class is irreducible.

The integrity basis given in § 5 for the tetragonal-trapezohedral class differs from that for the tetragonal-scalenohedral class only in the elements which involve x_3, y_3, ... linearly. Since it has already been shown that the integrity basis for the tetragonal-scalenohedral class is irreducible, in order to prove the irreducibility of the integrity basis for the tetragonal-trapezohedral class, we need only prove that none of the invariants (5.15) which is linear in x_3, y_3, ... is redundant. Accordingly, we compute $\overline{P}_{i_1 i_2 \ldots i_n}$, $\overline{\vartheta}_{i_1 i_2 \ldots i_n}$, $\overline{\beta}_{i_1 i_2 \ldots i_n}$ for the various values of $i_1 i_2 \ldots i_n$ which are valid for the terms in (5.15) which are linear in x_3, y_3, ... The values obtained are given in Table 5. The corresponding values of $\overline{Q}_{i_1 i_2 \ldots i_n}$ $(= \overline{P}_{i_1 i_2 \ldots i_n} - \overline{\vartheta}_{i_1 i_2 \ldots i_n})$ are also given. Also in the last line of Table 5, we give the number of independent syzygies $\overline{\eta}_{i_1 i_2 \ldots i_n}$ which must be demonstrated in order to make the upper and lower bounds on $\overline{\gamma}_{i_1 i_2 \ldots i_n}$ equal.

Table 5

$i_1 i_2 \ldots i_n$	11	04	22	31	211	1111
$\overline{P}_{i_1 i_2 \ldots i_n}$	1	1	2	2	3	4
$\overline{\vartheta}_{i_1 i_2 \ldots i_n}$	0	0	1	1	3	6
$\overline{Q}_{i_1 i_2 \ldots i_n}$	1	1	1	1	0	-2
$\overline{\beta}_{i_1 i_2 \ldots i_n}$	1	1	1	1	1	1
$\overline{\eta}_{i_1 i_2 \ldots i_n}$	0	0	0	0	1	3

The syzygies are:

$i_1 i_2 \ldots i_n = 211$:

$$x_3(y_1 z_2 - y_2 z_1)(x_1^2 + x_2^2) + x_3(x_1 y_2 - x_2 y_1)(x_1 z_1 + x_2 z_2) - \\ - x_3(x_1 z_2 - x_2 z_1)(x_1 y_1 + x_2 y_2) = 0.$$

$i_1 i_2 \ldots i_n = 1111$:

$$x_3(y_1 z_2 - y_2 z_1)(x_1 u_1 + x_2 u_2) + x_3(x_1 y_2 - x_2 y_1)(z_1 u_1 + z_2 u_2) - \\ - x_3(x_1 z_2 - x_2 z_1)(y_1 u_1 + y_2 u_2) = 0.$$

$$x_3(y_1 u_2 - y_2 u_1)(x_1 z_1 + x_2 z_2) + x_3(x_1 y_2 - x_2 y_1)(z_1 u_1 + z_2 u_2) - \\ - x_3(x_1 u_2 - x_2 u_1)(y_1 z_1 + y_2 z_2) = 0.$$

$$x_3(z_1 u_2 - z_2 u_1)(x_1 y_1 + x_2 y_2) + x_3(x_1 z_2 - x_2 z_1)(y_1 u_1 + y_2 u_2) - \\ - x_3(x_1 u_2 - x_2 u_1)(y_1 z_1 + y_2 z_2) = 0.$$

We note that the integrity basis for the tetragonal-disphenoidal class, derived in § 5, consists of x_3^2, elements which are independent of x_3, y_3, ... and elements which are linear in $x_3, y_3, \ldots$. The elements which are independent of $x_3, y_3, \ldots$ are the same as those in the integrity basis derived in § 5 for the tetragonal-pyramidal class, which has been proven to be irreducible. The element x_3^2 is clearly not redundant and it is evident by inspection that none of the elements linear in $x_3, y_3, \ldots$ is redundant. It follows that the integrity basis for the tetragonal-disphenoidal class is irreducible.

7. The hexagonal system

In this section the notation of § 3 together with the following notation will be used:

$$\begin{aligned}
I_1(x) &= x_1^3 - 3x_1x_2^2,\\
I_2(x) &= x_2^3 - 3x_2x_1^2,\\
J_1(x,y) &= x_1y_1^2 - x_1y_2^2 - 2y_1y_2x_2,\\
J_2(x,y) &= x_2y_2^2 - x_2y_1^2 - 2y_1y_2x_1,\\
K_1(x,y,z) &= x_1y_1z_1 - x_1y_2z_2 - y_1x_2z_2 - z_1x_2y_2,\\
K_2(x,y,z) &= x_2y_2z_2 - x_2y_1z_1 - y_2x_1z_1 - z_2x_1y_1.
\end{aligned} \tag{7.1}$$

Trigonal-pyramidal class $(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

Any polynomial in $A_i^{(r)}$ $(r = 1, 2, \ldots, n)$ is expressible as a polynomial in $A_3^{(r)}$ and the quantities $X_i^{(r)}$ $(r = 1, 2, \ldots, n)$ defined by

$$X_1^{(r)} = A_1^{(r)}, \qquad X_2^{(r)} = -\tfrac{1}{2}A_1^{(r)} + \frac{\sqrt{3}}{2}A_2^{(r)},$$
$$X_3^{(r)} = -\tfrac{1}{2}A_1^{(r)} - \frac{\sqrt{3}}{2}A_2^{(r)}. \tag{7.2}$$

If this polynomial is invariant under the transformations $\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2$, it must be invariant under cyclic permutation of the subscripts on $X_i^{(r)}$. Then, with theorems 3 and 4, we see that an integrity basis for the vectors $A_i^{(r)}$, for the trigonal-pyramidal class, is formed by the quantities $A_3^{(r)}$ $(r = 1, 2, \ldots, n)$ together with

$$\begin{aligned}
&x_1^2 + x_2^2,\ I_1(x),\ I_2(x);\\
&x_1y_1 + x_2y_2,\ x_1y_2 - x_2y_1,\ J_1(x,y),\ J_2(x,y),\ J_1(y,x),\ J_2(y,x);\\
&K_1(x,y,z),\ K_2(x,y,z).
\end{aligned} \tag{7.3}$$

Trigonal-dipyramidal class $(\boldsymbol{I}, \boldsymbol{R}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It has been seen in the case of the trigonal-pyramidal class that if a polynomial in $A_i^{(r)}$ is invariant under the transformations $\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2$, it is expressible as a polynomial in $A_3^{(r)}$ and the quantities (7.3). It is readily seen, with theorem 1, that if this polynomial is invariant under the transformations $\boldsymbol{R}_3 \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it is expressible as a polynomial in the quantities (7.3) together with

$$x_3^2;\ x_3y_3, \tag{7.4}$$

and this set of quantities therefore forms an integrity basis for the vectors $A_i^{(r)}$ for the trigonal-dipyramidal class.

Hexagonal-pyramidal class $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It is seen in the case of the trigonal-pyramidal class that if a polynomial in $A_i^{(r)}$ is invariant under the transformations $\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2$, it is expressible as a polynomial in $A_3^{(r)}$ and the quantities (7.3). We note that $A_3^{(r)}$ is invariant under the transformations $\boldsymbol{D}_3 \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$. We denote by N_α $(\alpha = 1, \ldots, \mu)$ those elements of (7.3) which are invariant under these transformations and by Q_α $(\alpha = 1, \ldots, \nu)$ the remaining elements. Thus, the quantities N_α are

$$x_1^2 + x_2^2;$$
$$x_1 y_1 + x_2 y_2, \; x_1 y_2 - x_2 y_1;$$

and the quantities Q_α are

$$I_1(x), I_2(x);$$
$$J_1(x, y), J_2(x, y), J_1(y, x), J_2(y, x);$$
$$K_1(x, y, z), K_2(x, y, z).$$

From theorem 1, we see that an integrity basis for the vectors $A_i^{(r)}$, for the hexagonal-pyramidal class, is expressible as a polynomial in $A_3^{(r)}$, N_α and $Q_\alpha Q_\beta$ $(\alpha, \beta = 1, \ldots, \nu;\ \alpha \leqq \beta)$. This basis is highly redundant. We therefore proceed in the following manner.

We consider first an integrity basis for the n vectors $A_i^{(n)}$ under the group of transformations $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$. We note that $A_3^{(r)}$, N_α $(\alpha = 1, \ldots, \mu)$ are invariant under the transformations $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$. Since $Q_\alpha Q_\beta$ is of total degree 6, it is apparent that an integrity basis for n vectors $A_i^{(r)}$ $(r = 1, \ldots, n)$, under this group, is formed by $A_3^{(r)}, N_\alpha$ $(\alpha = 1, \ldots, \mu)$ and quantities of total degree not less than 6. We now consider the integrity basis derived in § 5 for the tetragonal-pyramidal class, for which the appropriate transformation group is $\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3$. Then, we consider the additional restrictions imposed on a polynomial in the elements of this integrity basis by the further requirement that it be invariant under the transformations $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{S}_1, \boldsymbol{S}_2)$. In this way it is easily seen that an integrity basis, for the n vectors under the group $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, which contains no elements of total degree 6, can be constructed. It follows that an integrity basis for n vectors under the group $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ can be constructed, which consists of $A_3^{(r)}$, N_α $(\alpha = 1, \ldots, \mu)$ and possibly additional invariants, independent of x_3, of total degree greater than 6.

Using equations (6.5), ..., (6.8), we now calculate the number, $P_{i_1 i_2 \ldots i_n}$, of linearly independent invariants under the group $(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ of partial degrees $i_1, i_2, \ldots, i_n$ in the vectors $A_i^{(r)}$ $(r = 1, \ldots, n)$ and total degree 6, which are independent of $A_3^{(r)}$. These values of $P_{i_1 i_2 \ldots i_n}$ are given in Table 6. We note that these linearly independent invariants of degree 6 must be expressible as monomials in N_α $(\alpha = 1, \ldots, \mu)$ and represents the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ for the group $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ which are derivable from invariants of lower order.

Again, using equations (6.5), ..., (6.8), we may calculate the number, $P^*_{i_1 i_2 \ldots i_n}$ (say), of linearly independent invariants under the group $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ of partial degrees $i_1, i_2, \ldots, i_n$ in the vectors $A_i^{(r)}$ $(r = 1, \ldots, n)$ and total degree 6, which are independent of $A_3^{(r)}$. These values are also given in Table 6.

We note that for each choice of $i_1 i_2 \ldots i_n$, $P^*_{i_1 i_2 \ldots i_n} - P_{i_1 i_2 \ldots i_n} = 2$. It follows that an integrity basis for the vectors $A_i^{(n)}$ for the hexagonal-pyramidal class will be formed by $A_3^{(r)}$, N_α $(\alpha = 1, \ldots, \mu)$, together with two invariants of the

Table 6

$i_1 i_2 \dots i_n$	51	42	33	411	321	222	3111	2211	21111	111111
$P_{i_1 i_2 \dots i_n}$	2	3	4	4	6	7	8	10	14	20
$P^*_{i_1 i_2 \dots i_n}$	4	5	6	6	8	9	10	12	16	22

form $Q_\alpha Q_\beta$, for each value of $i_1 i_2 \dots i_n$ represented in Table 6. The additional invariants of the form $Q_\alpha Q_\beta$ are chosen in such a way that they are independent of each other. It is obvious that they cannot be expressed as linear combinations of monomials in N_α $(\alpha = 1, \dots, \mu)$, since N_α is invariant under the transformations $\boldsymbol{R}_1 \boldsymbol{T}_3$ and $\boldsymbol{R}_2 \boldsymbol{T}_3$, while none of the invariants $Q_\alpha Q_\beta$ is invariant under these transformations.

Thus an integrity basis for the vectors $A_i^{(r)}$ $(r = 1, \dots, n)$ for the hexagonal-pyramidal class is formed by $A_3^{(r)}$ $(r = 1, \dots, n)$ together with the quantities

$$
\begin{aligned}
& x_1^2 + x_2^2,\ I_1^2(x),\ I_1(x)\, I_2(x); \\
& x_1 y_1 + x_2 y_2,\ x_1 y_2 - x_2 y_1,\ I_1(x)\, J_1(y, x), \\
& I_1(x)\, J_2(y, x),\ I_1(y)\, J_1(x, y),\ I_1(y)\, J_2(x, y), \\
& I_1(x)\, J_1(x, y),\ I_1(x)\, J_2(x, y),\ I_1(y)\, J_1(y, x), \\
& I_1(y)\, J_2(y, x),\ I_1(x)\, I_1(y),\ I_1(x)\, I_2(y); \\
& I_1(x)\, K_1(x, y, z),\ I_1(x)\, K_2(x, y, z),\ I_1(x)\, J_1(y, z), \\
& I_1(x)\, J_2(y, z),\ I_1(x)\, J_1(z, y),\ I_1(x)\, J_2(z, y), \\
& K_1^2(x, y, z),\ K_1(x, y, z)\, K_2(x, y, z)
\end{aligned} \tag{7.5}
$$

and the distinct invariants obtained from these by cyclic permutation of x, y and z;

$$
\begin{aligned}
& I_1(x)\, K_1(y, z, u),\ I_1(x)\, K_2(y, z, u) \\
& K_1(x, y, z)\, K_1(x, y, u),\ K_1(x, y, z)\, K_2(x, y, u) \\
& K_1(x, z, y)\, K_1(x, z, u),\ K_1(x, z, y)\, K_2(x, z, u)
\end{aligned}
$$

and the distinct invariants obtained from these by cyclic permutation of x, y, z and u;

$$K_1(x, y, z)\, K_1(x, u, v),\ K_1(x, y, z)\, K_2(x, u, v)$$

and the invariants obtained by cyclic permutation of x, y, z, u and v;

$$K_1(x, y, z)\, K_1(u, v, w),\ K_1(x, y, z)\, K_2(u, v, w).$$

Hexagonal-dipyramidal class $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_3, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

The restrictions imposed on a polynomial in $A_i^{(r)}$ $(r = 1, 2, \dots, n)$ by invariance under the transformations $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ have been determined in the case of the hexagonal-pyramidal class. If the polynomial is further invariant under the transformations $(\boldsymbol{C}, \boldsymbol{R}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows from theorem 1, that it is expressible as a polynomial in the quantities (7.5) together with

$$x_3^2;\ x_3 y_3, \tag{7.6}$$

Thus, (7.5) and (7.6) form an integrity basis for the n vectors $A_i^{(r)}$ for the hexagonal-dipyramidal class.

Rhombohedral class $(\boldsymbol{I}, \boldsymbol{C}) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It is seen from Table 1 that the effect of the transformations $(\boldsymbol{I}, \boldsymbol{C}) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ on the vector components $A_1^{(r)}, A_2^{(r)}$ $(r = 1, \ldots, n)$ is the same as that of the transformations $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ associated with the hexagonal-pyramidal class. Hence the elements of the integrity bases which do not involve $A_3^{(r)}$ $(r = 1, \ldots, n)$ will be the same for the rhombohedral and hexagonal-pyramidal classes. We therefore need only determine the elements of the integrity basis for the rhombohedral class which involve $A_3^{(r)}$.

We have seen that if a polynomial in $A_i^{(r)}$ is invariant under the transformations $\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2$, it is expressible as a polynomial in $A_3^{(r)}$ and the quantities (7.3). If this polynomial is further invariant under the transformations $\boldsymbol{C} \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows from theorem 1 that the elements of an integrity basis for the vectors $A_i^{(r)}$, for the rhombohedral class, which involve $A_3^{(r)}$ are:

$$\begin{aligned}
&x_3^2,\ x_3 I_1(x),\ x_3 I_2(x);\\
&x_3 y_3,\ x_3 I_1(y),\ x_3 I_2(y),\ x_3 J_1(x, y),\\
&x_3 J_2(x, y),\ x_3 J_1(y, x),\ x_3 J_2(y, x)
\end{aligned}$$

together with the invariants obtained from these by interchanging x and y;

$$\begin{aligned}
&x_3 J_1(y, z),\ x_3 J_2(y, z),\ x_3 J_1(z, y),\ x_3 J_2(z, y),\\
&x_3 K_1(x, y, z),\ x_3 K_2(x, y, z)
\end{aligned} \tag{7.7}$$

together with the invariants obtained from these by cyclic permutation of x, y and z;

$$x_3 K_1(y, z, u),\ x_3 K_2(y, z, u)$$

together with the invariants obtained from these by cyclic permutation of x, y, z and u.

Thus, an integrity basis for the vectors $A_i^{(r)}$ for the rhombohedral class is formed by the quantities listed above together with the quantities (7.5).

Ditrigonal-pyramidal class $(\boldsymbol{I}, \boldsymbol{R}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

Any polynomial in the vectors $A_i^{(r)}$ $(r = 1, \ldots, n)$ is expressible as a polynomial in the quantities $A_3^{(r)}$ $(r = 1, \ldots, n)$ and the quantities $X_i^{(r)}$ defined by

$$X_1^{(r)} = A_2^{(r)}, \qquad X_2^{(r)} = -\tfrac{1}{2} A_2^{(r)} - \frac{\sqrt{3}}{2} A_1^{(r)}, \qquad X_3^{(r)} = -\tfrac{1}{2} A_2^{(r)} + \frac{\sqrt{3}}{2} A_1^{(r)}. \tag{7.8}$$

If this polynomial is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be invariant under all permutations of the subscripts on $X^{(r)}$ $(r = 1, \ldots, n)$. Then, it follows from theorems 2 and 4, together with equations (7.8), that an integrity basis for the vectors $A_i^{(r)}$, for the ditrigonal-pyramidal class, is formed by

$$\begin{aligned}
&x_3,\ x_1^2 + x_2^2,\ I_2(x);\\
&x_1 y_1 + x_2 y_2,\ J_2(x, y),\ J_2(y, x);\\
&K_2(x, y, z).
\end{aligned} \tag{7.9}$$

Ditrigonal-dipyramidal class $(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_3, \boldsymbol{D}_2) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It has already been shown, in the case of the ditrigonal-pyramidal class, that if a polynomial in $A_i^{(r)}$ is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

it must be expressible as a polynomial in the quantities (7.9). If this polynomial is further invariant under the transformations $(\boldsymbol{R}_3, \boldsymbol{D}_2) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows, from theorem 1, that it must be expressible as a polynomial in

$$\begin{aligned}&x_3^2,\ x_1^2+x_2^2,\ I_2(x);\\ &x_3 y_3,\ x_1 y_1+x_2 y_2,\ J_2(x,y),\ J_2(y,x);\\ &K_2(x,y,z).\end{aligned} \tag{7.10}$$

These quantities therefore form an integrity basis for the vectors $A_i^{(r)}$, for the ditrigonal-dipyramidal class.

Trigonal-trapezohedral class $(\boldsymbol{I}, \boldsymbol{D}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It has been shown, in the case of the trigonal-pyramidal class, that if a polynomial is invariant under the transformations $\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2$, it must be expressible as a polynomial in the quantities

$$\begin{aligned}&x_3,\ I_2(x);\\ &x_1 y_2-x_2 y_1,\ J_2(x,y),\ J_2(y,x);\\ &K_2(x,y,z),\end{aligned}$$

which change sign under the transformations $\boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2$, and the quantities

$$\begin{aligned}&x_1^2+x_2^2,\ I_1(x);\\ &x_1 y_1+x_2 y_2,\ J_1(x,y),\ J_1(y,x);\\ &K_1(x,y,z),\end{aligned}$$

which remain invariant under the transformations $\boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2$. It follows, from theorem 1, that an integrity basis for the vectors $A_i^{(r)}$ $(r=1,\ldots,n)$, for the trigonal-trapezohedral class, is formed by

$$\begin{aligned}&x_3^2,\ x_1^2+x_2^2,\ I_1(x),\ x_3 I_2(x);\\ &x_3 y_3,\ x_1 y_1+x_2 y_2,\ J_1(x,y),\ J_1(y,x),\ x_3(x_1 y_2-x_2 y_1),\\ &x_3 J_2(x,y),\ x_3 J_2(y,x),\ x_3 I_2(y)\end{aligned}$$

together with the distinct invariants obtained from these by interchanging x and y;

$$K_1(x,y,z),\ x_3 J_2(y,z),\ x_3 J_2(z,y),\ x_3 K_2(x,y,z),\ x_3(y_1 z_2-y_2 z_1) \tag{7.11}$$

together with the distinct invariants obtained from these by cyclic permutation of x, y and z;

$$x_3 K_2(y,z,u)$$

together with the invariants from this by cyclic permutation of x, y, z and u.

It is noted that there are no basis elements of degree higher than three in $x_1, x_2, y_1, y_2, \ldots, z_1, z_2$ alone. This follows immediately from theorem 2 and the fact that any polynomial in $x_1, x_2, y_1, y_2, \ldots, z_1, z_2$ is expressible as a polynomial in the quantities $X_i^{(1)}, \ldots, X_i^{(n)}$ defined by (7.2) which is invariant under all permutations of the subscripts on $X^{(1)}, \ldots, X^{(n)}$.

Dihexagonal-pyramidal class $(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It has been shown, in the case of the ditrigonal-pyramidal class, that if a polynomial in $A_i^{(r)}$ $(r=1, \ldots, n)$ is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be expressible as a polynomial in the quantities (7.9). If, further, this polynomial is invariant under the transformations $(\boldsymbol{R}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows from theorem 1 that it must be expressible as a polynomial in a set of invariants which contain only elements of degrees 1, 2 and 6.

Again, it has been shown, in the case of the hexagonal-pyramidal class, that if a polynomial in $A_i^{(r)}$ $(r=1, \ldots, n)$ is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be expressible as a polynomial in $A_3^{(r)}$ $(r=1, \ldots, n)$ and the quantities (7.5). If, further, this polynomial is invariant under the transformations $(\boldsymbol{R}_1, \boldsymbol{R}_2) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ it follows, from theorem 1 and the fact that it must be expressible as a polynomial in invariants of degrees 1, 2 and 6 only, that it must be expressible as a polynomial in $A_3^{(r)}$ $(r=1, \ldots, n)$ together with

$$\begin{aligned}
&x_1^2 + x_2^2,\ I_1^2(x);\\
&x_1 y_1 + x_2 y_2,\ I_1(x) J_1(y, x),\ I_1(x) J_1(x, y),\\
&I_1(y) J_1(x, y),\ I_1(y) J_1(y, x),\ I_1(x) I_1(y);\\
&I_1(x) K_1(x, y, z),\ I_1(x) J_1(y, z),\ I_1(x) J_1(z, y),\\
&K_1^2(x, y, z)
\end{aligned}$$

and the distinct invariants obtained from these by cyclic permutation of x, y and z;

$$I_1(x) K_1(y, z, u),\ K_1(x, y, z) K_1(x, y, u),\ K_1(x, z, y) K_1(x, z, u) \tag{7.12}$$

and the distinct invariants obtained from these by cyclic permutation of x, y, z and u;

$$K_1(x, y, z)\, K_1(x, u, v)$$

and the invariants obtained from this by cyclic permutation of x, y, z, u and v;

$$K_1(x, y, z)\, K_1(u, v, w).$$

Thus $A_3^{(r)}$ $(r=1, \ldots, n)$ and the quantities (7.12) form an integrity basis for the vectors $A_i^{(r)}$ $(r=1, \ldots, n)$, for the dihexagonal-pyramidal class.

Dihexagonal-dipyramidal class $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

It has been shown, in the case of the dihexagonal-pyramidal class, that if a polynomial in $A_i^{(r)}$ $(r=1, 2, \ldots, n)$ is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be expressible as a polynomial in $A_3^{(r)}$ $(r=1, 2, \ldots, n)$ and the quantities (7.12). If further this polynomial is invariant under the transformations $(\boldsymbol{C}, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows from theorem 1 that it must be expressible as a polynomial in the quantities (7.12) together with

$$x_3^2 \text{ and } x_3 y_3. \tag{7.13}$$

Hexagonal-scalenohedral class $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{D}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

We note that the components $A_1^{(r)}$, $A_2^{(r)}$ of the vectors $A_i^{(r)}$ $(r=1, \ldots, n)$ transform, under the transformations characterizing the hexagonal-scalenohedral class, in precisely the same way as they do under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ which characterize the dihexagonal-pyramidal class. Thus, the elements of the integrity bases for the vectors $A_i^{(r)}$, which do not involve $A_3^{(r)}$, will be the same for the dihexagonal-pyramidal and hexagonal-scalenohedral classes and are given by (7.12). It remains therefore to determine the elements of the integrity basis for the hexagonal-scalenohedral class which involve $A_3^{(r)}$ $(r=1, \ldots, n)$.

It has been shown, in the case of the ditrigonal-pyramidal class, that if a polynomial is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be expressible as a polynomial in the quantities (7.9). If, further, this polynomial is invariant under the transformations $(\boldsymbol{C}, \boldsymbol{D}_1) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ it follows from theorem 1 that it must be expressible as a polynomial in the invariants (7.12) together with

$$x_3^2,\ x_3 I_2(x);$$

$$x_3 y_3,\ x_3 J_2(x, y),\ x_3 J_2(y, x),\ x_3 I_2(y)$$

and the invariants obtained from these by interchanging x and y;

$$x_3 J_2(y, z),\ x_3 J_2(z, y),\ x_3 K_2(x, y, z) \tag{7.14}$$

and the invariants obtained from these by cyclic permutation of x, y and z;

$$x_3 K_2(y, z, u)$$

and the invariants obtained from this by cyclic permutation of x, y, z and u. Thus, an integrity basis for the vectors $A_i^{(r)}$ $(r=1, \ldots, n)$ for the hexagonal-scalenohedral class is formed by the invariants (7.12) and (7.14).

Hexagonal-trapezohedral class $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$

We note that the components $A_1^{(r)}$, $A_2^{(r)}$ of the vectors $A_i^{(r)}$ $(r=1, \ldots, n)$ transform, under the transformations characterizing the hexagonal-trapezohedral class, in precisely the same way as they do under the transformations $(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$ which characterize the dihexagonal-pyramidal class. Thus, the elements of the integrity bases for the vectors $A_i^{(r)}$, which do not involve $A_3^{(r)}$, will be the same for the dihexagonal-pyramidal and hexagonal-trapezohedral classes and are given by (7.12). It remains therefore to determine the elements of the integrity basis for the hexagonal-trapezohedral class which involve $A_3^{(r)}$ $(r=1, \ldots, n)$.

It has been shown, in the case of the hexagonal-pyramidal class, that if a polynomial is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it must be expressible as a polynomial in the quantities (7.5) and $A_3^{(r)}$. If further this polynomial is invariant under the transformations $(\boldsymbol{D}_1, \boldsymbol{D}_2) \cdot (\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, it follows, from theorem 1, that it must be expressible as a polynomial in the quantities (7.12), together with the following invariants:

$$x_3^2,\ x_3 I_1(x) I_2(x);$$

$$x_3 y_3,\ x_3(x_1 y_2 - x_2 y_1),$$

$$x_3 I_1(x) J_2(x, y),\ x_3 I_1(x) J_2(y, x),$$

$$x_3 I_1(y) J_2(x, y),\ x_3 I_1(y) J_2(y, x),$$

$$x_3 I_1(x) I_2(y),\ x_3 I_1(y) I_2(y)$$

and the invariants obtained from these by interchanging x and y;

$$\begin{aligned}
&x_3(y_1z_2 - y_2z_1),\ x_3I_1(y)\,J_2(y,z),\\
&x_3I_1(y)\,J_2(z,y),\ x_3I_1(z)\,J_2(y,z),\\
&x_3I_1(z)\,J_2(z,y),\ x_3I_1(y)\,I_2(z),\\
&x_3I_1(x)\,K_2(x,y,z),\ x_3I_1(y)\,K_2(y,z,x),\\
&x_3I_1(z)\,K_2(z,x,y),\ x_3I_1(x)\,J_2(y,z),\\
&x_3I_1(x)\,J_2(z,y),\ x_3I_1(y)\,J_2(x,z),\\
&x_3I_1(y)\,J_2(z,x),\ x_3I_1(z)\,J_2(x,y),\\
&x_3I_1(z)\,J_2(y,x),\ x_3K_1(x,y,z)\,K_2(x,y,z)
\end{aligned}$$

and the distinct invariants obtained from these by cyclic permutation of x, y and z;

$$\begin{aligned}
&x_3I_1(y)\,K_2(y,z,u),\ x_3I_1(z)\,K_2(y,z,u),\\
&x_3I_1(u)\,K_2(y,z,u),\ x_3I_1(y)\,J_2(z,u),\\
&x_3I_1(y)\,J_2(u,z),\ x_3I_1(z)\,J_2(y,u),\\
&x_3I_1(z)\,J_2(u,y),\ x_3I_1(u)\,J_2(y,z),\\
&x_3I_1(u)\,J_2(z,y),\ x_3I_1(x)\,K_2(y,z,u),\\
&x_3I_1(y)\,K_2(x,z,u),\ x_3I_1(z)\,K_2(x,y,u),\\
&x_3I_1(u)\,K_2(x,y,z),\ x_3K_1(x,y,z)\,K_2(x,y,u),\\
&x_3K_1(x,z,y)\,K_2(x,z,u),\ x_3K_1(x,u,y)\,K_2(x,u,z),\\
&x_3K_1(y,z,x)\,K_2(y,z,u),\ x_3K_1(y,u,x)\,K_2(y,u,z),\\
&x_3K_1(z,u,x)\,K_2(z,u,y),\ x_3K_1(y,z,u)\,K_2(y,z,u)
\end{aligned} \tag{7.15}$$

and the distinct invariants obtained from these by cyclic permutation of x, y, z and u;

$$\begin{aligned}
&x_3I_1(y)\,K_2(z,u,v),\ x_3I_1(z)\,K_2(y,u,v),\\
&x_3I_1(u)\,K_2(y,z,v),\ x_3I_1(v)\,K_2(y,z,u),\\
&x_3K_1(y,z,u)\,K_2(y,z,v),\ x_3K_1(y,u,z)\,K_2(y,u,v),\\
&x_3K_1(y,v,z)\,K_2(y,v,u),\ x_3K_1(z,u,y)\,K_2(z,u,v),\\
&x_3K_1(z,v,y)\,K_2(z,v,u),\ x_3K_1(u,v,y)\,K_2(u,v,z),\\
&x_3K_1(x,y,z)\,K_2(x,u,v),\ x_3K_1(y,z,u)\,K_2(y,v,x),\\
&x_3K_1(z,u,v)\,K_2(z,x,y),\ x_3K_1(u,v,x)\,K_2(u,y,z),\\
&x_3K_1(v,x,y)\,K_2(v,z,u)
\end{aligned}$$

and the distinct invariants obtained from these by cyclic permutation of x, y, z, u and v;

$$\begin{aligned}
&x_3K_1(y,z,u)\,K_2(y,v,w),\ x_3K_1(z,u,v)\,K_2(z,w,y),\\
&x_3K_1(u,v,w)\,K_2(u,y,z),\ x_3K_1(v,w,y)\,K_2(v,z,u),\\
&x_3K_1(w,y,z)\,K_2(w,u,v),\ x_3K_1(x,y,z)\,K_2(u,v,w)
\end{aligned}$$

and the distinct invariants obtained from these by cyclic permutation of x, y, z, u, v and w;

$$x_3K_1(y,z,u)\,K_2(v,w,t)$$

and the invariants obtained from this by cyclic permutation of $x, y, \ldots, t$.

Thus, an integrity basis for the vectors $A_i^{(r)}$ $(r = 1, \ldots, n)$ for the hexagonal-trapezohedral class is formed by the invariants (7.12) and (7.15).

8. Irreducibility of the integrity bases for the hexagonal system

We first consider the integrity bases derived in § 7 for the trigonal-pyramidal class. We note that the elements $A_3^{(r)}$ are not redundant and that the remaining elements involve $(x_1, x_2), (y_1, y_2), \ldots, (z_1, z_2)$ only. We accordingly employ the notation of § 6 and calculate $P_{i_1 i_2 \ldots i_n}$ for the values of $i_1 i_2 \ldots i_n$ represented in (7.3). We also calculate $\vartheta_{i_1 i_2 \ldots i_n}$, $\beta_{i_1 i_2 \ldots i_n}$ and $Q_{i_1 i_2 \ldots i_n}$ $(= P_{i_1 i_2 \ldots i_n} - \vartheta_{i_1 i_2 \ldots i_n})$. These values are given in Table 7, together with the numbers $\eta_{i_1 i_2 \ldots i_n}$ of independent syzygies of degree $i_1 i_2 \ldots i_n$ which must be demonstrated in order to prove the irreducibility of the integrity basis. Since for each of the values of $i_1 i_2 \ldots i_n$ represented in (7.3), $\eta_{i_1 i_2 \ldots i_n} = 0$, we do not have to demonstrate any syzygies, and the integrity basis given in § 7 for the trigonal-pyramidal class is irreducible.

Table 7

$i_1 i_2 \ldots i_n$	11	21	111
$P_{i_1 i_2 \ldots i_n}$	2	2	2
$\vartheta_{i_1 i_2 \ldots i_n}$	0	0	0
$Q_{i_1 i_2 \ldots i_n}$	2	2	2
$\beta_{i_1 i_2 \ldots i_n}$	2	2	2
$\eta_{i_1 i_2 \ldots i_n}$	0	0	0

The integrity basis given in § 7 for the trigonal-dipyramidal class differs from that for the trigonal-pyramidal class only in the elements involving x_3, y_3 only. These are evidently not redundant. Consequently, the integrity basis given for the trigonal-dipyramidal class is irreducible.

It is evident from the manner in which it is derived that the integrity basis obtained in § 7 for the hexagonal-pyramidal class is irreducible.

The integrity basis obtained in § 7 for the hexagonal-dipyramidal class differs from that for the hexagonal-pyramidal class only in the elements which involve x_3, y_3 only. Since these are evidently not redundant, it follows that the integrity basis for the hexagonal-dipyramidal class is irreducible.

The integrity basis derived in § 7 for the rhombohedral class differs from that for the hexagonal-pyramidal class only in the elements involving $x_3, y_3, \ldots$. Consequently, since it has already been shown that the integrity basis for the hexagonal-pyramidal class is irreducible, it follows that none of the elements which do not involve $x_3, y_3, \ldots$, in the integrity basis for the rhombohedral class is redundant. It is also evident that none of the elements of the type x_3^2 and $x_3 y_3$ are redundant. We need therefore examine only the remaining elements in (7.7) which are linear in x_3. We note that these are of the form $x_3 P$, where P is an element, independent of $x_3, y_3, \ldots$, of the irreducible integrity basis for the trigonal-pyramidal class. Since x_3 is not an invariant for the rhombohedral class, it is apparent that none of the elements (7.7) is redundant and consequently the integrity basis for the rhombohedral class is irreducible.

Since the integrity basis (7.9) for the ditrigonal-pyramidal class is contained in the irreducible integrity basis for the trigonal-pyramidal class, it is irreducible. Similarly, since the integrity basis (7.10) for the ditrigonal-dipyramidal class is contained in the irreducible integrity basis for the trigonal-dipyramidal class, it is irreducible.

The elements of the integrity basis (7.11) for the trigonal-trapezohedral class, which are independent of $x_3, y_3, \ldots$, are contained in the irreducible integrity basis for the trigonal-pyramidal class and consequently none of these elements is redundant. The elements which involve $x_3, y_3, \ldots$ only are of second degree and are, by inspection, not redundant. The remaining elements, which are of first degree in x_3, are of the form $x_3 P$, where P is an element, independent of $x_3, y_3, \ldots$, of the irreducible integrity basis for the trigonal-pyramidal class. Since x_3 is not an invariant for the trigonal-trapezohedral class, it is apparent that none of the elements in (7.11) which are linear in x_3 is redundant. It follows that the integrity basis (7.11) for the trigonal-trapezohedral class is irreducible.

Since the integrity basis derived in § 7 for the dihexagonal-pyramidal class is contained in the irreducible integrity basis for the hexagonal-pyramidal class, it is irreducible. Similarly, since the integrity basis derived in § 7 for the dihexagonal-dipyramidal class is contained in the irreducible integrity basis for the hexagonal-dipyramidal class, it is irreducible.

The elements of the integrity basis for the hexagonal-scalenohedral class, derived in § 7, which do not involve $x_3, y_3, \ldots$, are the same as those in the irreducible integrity basis for the dihexagonal-pyramidal class, and consequently none of these elements are redundant. The elements which involve $x_3, y_3, \ldots$ only are evidently not redundant, by inspection, and are of second degree. The remaining elements are linear in x_3 and are of the form $x_3 P$, where P is an element, independent of $x_3, y_3, \ldots$, of the irreducible integrity basis for the trigonal-pyramidal class. Since x_3 is not an invariant for the hexagonal-scalenohedral class, it follows that none of these elements linear in x_3 is redundant. Consequently, the integrity basis given in § 7 for the hexagonal-scalenohedral class is irreducible.

The irreducibility of the integrity basis for the hexagonal-trapezohedral class, derived in § 7, follows from an argument analogous to that employed for the hexagonal-scalenohedral class. The elements independent of $x_3, y_3, \ldots$ are the same as those in the irreducible integrity basis for the dihexagonal-pyramidal class; the non-redundancy of the elements involving $x_3, y_3, \ldots$ only is evident by inspection; the remaining elements are of the form $x_3 P$ where P is an element independent of $x_3, y_3, \ldots$ of the irreducible integrity basis for the hexagonal-pyramidal class.

9. The cubic system

(a) General description of procedure

The procedure which we adopt for finding irreducible integrity bases for the classes of the cubic system is different from that used for the crystal classes previously discussed. Instead of generating highly-redundant integrity bases, then eliminating the redundant elements and finally proving the irreducibility of the integrity bases so obtained, we instead generate irreducible integrity bases by an iterative process in which the elements of lowest degree are first generated and then those of successively higher degrees, the theorems of § 2 being used to indicate the total degree at which it is safe to terminate this synthesis.

A polynomial of partial degrees $i_1, i_2, \ldots, i_n$ in the n vectors $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$ will be said to have degree $i_1 i_2 \ldots i_n$ in $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$. Let $P_{i_1 i_2 \ldots i_n}$ be the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$. Let $J_1, J_2, \ldots, J_\nu$ be the elements of an irreducible integrity basis, for the vectors $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$ under the transformation group considered, which are of total degree in $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$ less than $i_1 + i_2 + \cdots + i_n$. Let $\vartheta_{i_1 i_2 \ldots i_n}$ be the number of distinct invariants of the form $J_1^{\alpha_1} J_2^{\alpha_2} \ldots J_\nu^{\alpha_\nu}$ where the α's are positive integers or zero, which are of degree $i_1 i_2 \ldots i_n$ in $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$. $P_{i_1 i_2 \ldots i_n}$ is given by

$$P_{i_1 i_2 \ldots i_n} = \frac{1}{N} \sum_{\alpha=1}^{N} \operatorname{tr} \boldsymbol{T}_\alpha^{(i_1)} \operatorname{tr} \boldsymbol{T}_\alpha^{(i_2)} \ldots \operatorname{tr} \boldsymbol{T}_\alpha^{(i_n)},$$

where $\boldsymbol{T}_\alpha$ ($\alpha = 1, \ldots, N$) are the transformations of the three-dimensional matrix representation of the group describing the crystal symmetry considered. $\operatorname{tr} \boldsymbol{T}_\alpha^{(i_r)}$ is given by equations similar to (6.7) and (6.8), in which $\boldsymbol{L}_\alpha$ is replaced by $\boldsymbol{T}_\alpha$. For the various transformations $\boldsymbol{T}_\alpha$ occurring in the groups describing the symmetry of the classes of the cubic system, the values of $\operatorname{tr} \boldsymbol{T}_\alpha^{(1)}, \operatorname{tr} \boldsymbol{T}_\alpha^{(2)}, \ldots, \operatorname{tr} \boldsymbol{T}_\alpha^{(5)}$ are given in Table 8.

Table 8

Transformation $\boldsymbol{T}_\alpha$	$\operatorname{tr} \boldsymbol{T}_\alpha^{(1)}$	$\operatorname{tr} \boldsymbol{T}_\alpha^{(2)}$	$\operatorname{tr} \boldsymbol{T}_\alpha^{(3)}$	$\operatorname{tr} \boldsymbol{T}_\alpha^{(4)}$	$\operatorname{tr} \boldsymbol{T}_\alpha^{(5)}$
$\boldsymbol{I}$	3	6	10	15	21
$\boldsymbol{C}$	-3	6	-10	15	-21
$\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{T}_1, \boldsymbol{T}_2$	1	2	2	3	3
$\boldsymbol{T}_3, \boldsymbol{D}_1 \boldsymbol{T}_1, \boldsymbol{D}_2 \boldsymbol{T}_2, \boldsymbol{D}_3 \boldsymbol{T}_3$	1	2	2	3	3
$\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{C} \boldsymbol{T}_1, \boldsymbol{R}_1 \boldsymbol{T}_1$	-1	2	-2	3	-3
$\boldsymbol{C} \boldsymbol{T}_2, \boldsymbol{R}_2 \boldsymbol{T}_2, \boldsymbol{C} \boldsymbol{T}_3, \boldsymbol{R}_3 \boldsymbol{T}_3$	-1	2	-2	3	-3
$\boldsymbol{R}_2 \boldsymbol{T}_1, \boldsymbol{R}_3 \boldsymbol{T}_1, \boldsymbol{R}_1 \boldsymbol{T}_2$	1	0	0	1	1
$\boldsymbol{R}_3 \boldsymbol{T}_2, \boldsymbol{R}_1 \boldsymbol{T}_3, \boldsymbol{R}_2 \boldsymbol{T}_3$	1	0	0	1	1
$\boldsymbol{D}_2 \boldsymbol{T}_1, \boldsymbol{D}_3 \boldsymbol{T}_1, \boldsymbol{D}_1 \boldsymbol{T}_2$	-1	0	0	1	-1
$\boldsymbol{D}_3 \boldsymbol{T}_2, \boldsymbol{D}_1 \boldsymbol{T}_3, \boldsymbol{D}_2 \boldsymbol{T}_3$	-1	0	0	1	-1
$\boldsymbol{M}_1, \boldsymbol{D}_1 \boldsymbol{M}_1, \boldsymbol{D}_2 \boldsymbol{M}_1, \boldsymbol{D}_3 \boldsymbol{M}_1$	0	0	1	0	0
$\boldsymbol{C} \boldsymbol{M}_1, \boldsymbol{R}_1 \boldsymbol{M}_1, \boldsymbol{R}_2 \boldsymbol{M}_1, \boldsymbol{R}_3 \boldsymbol{M}_1$	0	0	-1	0	0
$\boldsymbol{M}_2, \boldsymbol{D}_1 \boldsymbol{M}_2, \boldsymbol{D}_2 \boldsymbol{M}_2, \boldsymbol{D}_3 \boldsymbol{M}_2$	0	0	1	0	0
$\boldsymbol{C} \boldsymbol{M}_2, \boldsymbol{R}_1 \boldsymbol{M}_2, \boldsymbol{R}_2 \boldsymbol{M}_2, \boldsymbol{R}_3 \boldsymbol{M}_2$	0	0	-1	0	0

The integrity bases for the classes of the cubic system may then be generated by the following procedure.

$P_{i_1 i_2 \ldots i_n}$, the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ in $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{z}$, is first computed. The $\vartheta_{i_1 i_2 \ldots i_n}$ invariants of degree $i_1 i_2 \ldots i_n$ obtained from the elements of an irreducible integrity basis of lower total degree than $i_1 + i_2 + \cdots + i_n$ are listed. Let us denote them by $A_1, A_2, \ldots, A_Q$. These invariants are then expressed as linear combinations of a set of $P_{i_1 i_2 \ldots i_n}$ linearly independent invariants, $B_1, B_2, \ldots, B_R$, say, thus:

$$A_L = \sum_{M=1}^{P_{i_1 i_2 \ldots i_n}} \alpha_{LM} B_M \qquad (L = 1, 2, \ldots, Q). \tag{9.1}$$

The rank $R_{i_1 i_2 \ldots i_n}$ of the matrix $\|\alpha_{LM}\|$ is equal to the number of linearly independent invariants in the set of invariants $A_1, A_2, \ldots, A_Q$. We choose from the invariants $B_1, B_2, \ldots, B_R$, by inspection, $P_{i_1 i_2 \ldots i_n} - R_{i_1 i_2 \ldots i_n}$ invariants which, together with the invariants A_P form a linearly independent set. These may be taken as the elements of degree $i_1 i_2 \ldots i_n$ of an irreducible integrity basis and, together with the elements corresponding to all permutations of $i_1 i_2 \ldots i_n$, will be denoted $\{C_{i_1 i_2 \ldots i_n}\}$.

The theorems of § 2 can be used to limit the values of $i_1+i_2+\cdots+i_n$ which can apply to the elements of an irreducible integrity basis. Suppose these are $q_1, q_2, \ldots$, where $q_1<q_2<\cdots$. We determine $\{C_{i_1}\}$ successively for $i_1=q_1, q_2, \ldots$; *i.e.* we determine the elements of an irreducible integrity basis which involve only one vector. Next, we determine $\{C_{i_1 i_2}\}$ for all $i_1 i_2$ such that $i_1+i_2=q_1, q_2, \ldots$, *i.e.* the elements which involve two vectors. We repeat this process for the elements of the irreducible integrity basis which involve three, four, ... vectors, until we reach a number of vectors equal to the maximum q.

In carrying out these calculations for the various classes of the cubic system, it is convenient to use the notation $\sum$ to indicate the sum of the three quantities obtained by cyclic permutation of the subscripts on the summand, thus:

$$\sum x_1=\sum x_2=\sum x_3=x_1+x_2+x_3,$$
$$\sum x_1 y_2=\sum x_2 y_3=\sum x_3 y_1=x_1 y_2+x_2 y_3+x_3 y_1.$$

(*b*) *Hexoctahedral class* $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)\cdot(\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$

It has been shown in the case of the rhombic-dipyramidal class, that if a polynomial in the vectors $\boldsymbol{A}^{(r)}$ $(r=1, \ldots, n)$ is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)$, it must be expressible as a polynomial in x_1^2, x_2^2, x_3^2, $x_1 y_1$, $x_2 y_2$, $x_3 y_3$. The further requirement that this polynomial be invariant under the transformations $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)\cdot(\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$ implies that it must be invariant under all permutations on the subscripts. It then follows immediately from theorem 2 that an integrity basis for the n vectors $\boldsymbol{A}^{(r)}$ $(r=1, \ldots, n)$, for the hexoctahedral class, may be constructed with elements of total degrees two, four and six only.

The elements of this integrity basis involving only one vector are clearly, from theorem 2,
$$\sum x_1^2, \quad \sum x_1^2 x_2^2 \quad \text{and} \quad x_1^2 x_2^2 x_3^2$$
and it is evident that none of these elements is redundant.

In Table 9 are given the values of $P_{i_1 i_2 \ldots i_n}$ for all possible values of $i_1 i_2 \ldots i_n$, such that $i_1+i_2+\cdots+i_n=2$, 4, or 6 (apart from the cases when all except one of the i's is zero). We note that when $i_1 \ldots i_n=11$, $P_{i_1 i_2 \ldots i_n}=1$. Also, $\vartheta_{11}=0$, since there are no invariants of total degree 1. Thus, in order to obtain the elements $\{C_{11}\}$ of an irreducible integrity basis of degree 11, we must choose one invariant of degree 11. From theorem 2, it is seen that $\{C_{11}\}$ may be taken as $\sum x_1 y_1$.

Table 9

$i_1 i_2 \ldots i_n$	11	22	31	33	42	51	211	411	321	222	1111	2211	3111	21111	111111
$P_{i_1 i_2 \ldots i_n}$	1	3	2	6	6	4	3	7	9	12	4	15	12	21	31
$\vartheta_{i_1 i_2 \ldots i_n}$	0	2	1	5	5	3	2	6	8	11	3	14	11	20	30
$R_{i_1 i_2 \ldots i_n}$	0	2	1	5	5	3	2	6	8	11	3	14	11	20	30

We now note that $P_{22}=3$. The ϑ_{22} monomial invariants of degree 22 which can be formed from the elements of the irreducible integrity basis of lower degree are $(\sum x_1 y_1)^2=A_1$ (say) and $\sum x_1^2 \sum y_1^2=A_2$ (say). Thus $\vartheta_{22}=2$. These can be shown to be linearly independent. For, it is easily seen that

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}=\begin{pmatrix} 1, & 1, & 0 \\ 0, & 1, & 2 \end{pmatrix}\begin{pmatrix} B_3 \\ B_1 \\ B_2 \end{pmatrix},$$

where

$$B_1 = \sum x_1^2 y_1^2, \qquad B_2 = \sum x_1 x_2 y_1 y_2, \qquad B_3 = \sum x_1^2 (y_2^2 + y_3^2)$$

and we note that B_1, B_2, B_3 are linearly independent invariants, and the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ has rank $R_{22} = 2$. It follows that there are $P_{22} - R_{22} = 1$ elements of degree 22 in the irreducible integrity basis. Thus we may take $\{C_{22}\} = \sum x_1 x_2 y_1 y_2$, since it is easily shown that this is linearly independent of A_1, A_2.

Proceeding in this way for the various values of $i_1 i_2 \ldots i_n$ given in Table 9, we obtain the following results. For $i_1 i_2 \ldots i_n = 31$:

$$A_1 = (1, 1) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $A_1 = \sum x_1^2 \sum x_1 y_1$, $B_1 = \sum x_1^3 y_1$, $B_2 = \sum x_1^2 (x_2 y_2 + x_3 y_3)$, and we may take $\{C_{31}\} = \sum x_1^2 (x_2 y_2 + x_3 y_3)$, $\sum y_1^2 (x_2 y_2 + x_3 y_3)$.

For $i_1 i_2 \ldots i_n = 33$:

$$\begin{pmatrix} A_1 \\ A_4 \\ A_5 \\ A_3 \\ A_2 - A_1 + A_4 - 3A_3 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 1, & 1, & 1, & 0 \\ & 1, & 1, & 1, & 0, & 0 \\ & & 1, & 1, & 1, & 0 \\ & & & 1, & 0, & 3 \\ & & & & -1, & -3 \end{pmatrix} \begin{pmatrix} B_1 \\ B_4 \\ B_5 \\ B_2 \\ B_3 \\ B_6 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= \sum x_1^2 \sum y_1^2 \sum x_1 y_1, & A_2 &= (\sum x_1 y_1)^3, \\ A_3 &= \sum x_1 x_2 y_1 y_2 \sum x_1 y_1, & A_4 &= \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum y_1^2, \\ A_5 &= \sum y_1^2 (x_2 y_2 + x_3 y_3) \sum x_1^2, & & \\ B_1 &= \sum x_1^3 y_1^3, & B_2 &= \sum x_1^2 y_1^2 (x_2 y_2 + x_3 y_3), \\ B_3 &= \sum x_1^3 y_1 (y_2^2 + y_3^2), & B_4 &= \sum x_1^2 (x_2 y_2^3 + x_3 y_3^3), \\ B_5 &= \sum x_1^2 y_2 y_3 (x_3 y_2 + x_2 y_3), & B_6 &= x_1 x_2 x_3 y_1 y_2 y_3, \end{aligned}$$

and we may take $\{C_{33}\} = x_1 x_2 x_3 y_1 y_2 y_3$.

For $i_1 i_2 \ldots i_n = 42$:

$$\begin{pmatrix} A_2 \\ A_3 \\ A_1 \\ A_4 \\ A_5 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 2, & 2, & 0, & 0 \\ & 1, & 0, & 1, & 2, & 2 \\ & & 1, & 1, & 0, & 0 \\ & & & 1, & 1, & 2 \\ & & & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_4 \\ B_3 \\ B_2 \\ B_1 \\ B_5 \\ B_6 \end{pmatrix},$$

where

$$\begin{aligned} & A_1 = \sum x_1^2 x_2^2 \sum y_1^2, \qquad A_2 = (\sum x_1^2)^2 \sum y_1^2, \qquad A_3 = (\sum x_1 y_1)^2 \sum x_1^2, \\ & A_4 = \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum x_1 y_1, \qquad A_5 = \sum x_1 x_2 y_1 y_2 \sum x_1^2, \\ & B_1 = \sum x_1^2 y_1^2 (x_2^2 + x_3^2), \qquad B_2 = \sum x_2^2 x_3^2 y_1^2, \\ & B_3 = \sum x_1^4 y_1^2, \qquad B_4 = \sum x_1^4 (y_2^2 + y_3^2), \\ & B_5 = \sum x_1^3 y_1 (x_2 y_2 + x_3 y_3), \qquad B_6 = x_1 x_2 x_3 \sum x_1 y_2 y_3, \end{aligned}$$

and we may take $\{C_{42}\} = x_1 x_2 x_3 \sum x_1 y_2 y_3$, $y_1 y_2 y_3 \sum x_2 x_3 y_1$.

For $i_1 i_2 \dots i_n = 51$:

$$\begin{pmatrix} A_2 \\ A_1 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 2, & 2 \\ & 1, & 1, & 2 \\ & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_3 \\ B_1 \\ B_2 \\ B_4 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum x_1^2, \qquad A_2 = \sum x_1 y_1 (\sum x_1^2)^2,$$
$$A_3 = \sum x_1 y_1 \sum x_2^2 x_3^2,$$
$$B_1 = \sum x_1^4 (x_2 y_2 + x_3 y_3), \qquad B_2 = \sum x_1^3 y_1 (x_2^2 + x_3^2),$$
$$B_3 = \sum x_1^5 y_1, \qquad B_4 = x_1 x_2 x_3 \sum x_2 x_3 y_1,$$

and we may take $\{C_{51}\} = x_1 x_2 x_3 \sum x_1 x_2 y_3,\ y_1 y_2 y_3 \sum y_1 y_2 x_3$.

For $i_1 i_2 \dots i_n = 211$:

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 0 \\ & 1, & 1 \end{pmatrix} \begin{pmatrix} B_2 \\ B_1 \\ B_3 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 \sum y_1 z_1, \qquad A_2 = \sum x_1 y_1 \sum x_1 z_1,$$
$$B_1 = \sum x_1^2 y_1 z_1, \qquad B_2 = \sum x_1 y_1 (x_2 z_2 + x_3 z_3),$$
$$B_3 = \sum x_1^2 (y_2 z_2 + y_3 z_3),$$

and we may take $\{C_{211}\} = \sum x_1^2 (y_2 z_2 + y_3 z_3),\ \sum y_1^2 (z_2 x_2 + z_3 x_3),$

$$\sum z_1^2 (x_2 y_2 + x_3 y_3).$$

For $i_1 i_2 \dots i_n = 411$:

$$\begin{pmatrix} A_2 \\ A_1 \\ A_3 \\ A_4 \\ A_5 - A_2 \\ A_6 - A_3 - A_4 + A_5 - A_2 \end{pmatrix} = \begin{pmatrix} 1, & 2, & 0, & 0, & 1, & 2, & 0 \\ & 1, & 0, & 0, & 0, & 1, & 0 \\ & & 1, & 0, & 0, & 1, & 1 \\ & & & 1, & 0, & 1, & 1 \\ & & & & -1, & -1, & 0 \\ & & & & & -2, & -1 \end{pmatrix} \begin{pmatrix} B_4 \\ B_2 \\ B_5 \\ B_6 \\ B_3 \\ B_1 \\ B_7 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 x_2^2 \sum y_1 z_1, \qquad A_2 = (\sum x_1^2)^2 \sum y_1 z_1,$$
$$A_3 = \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum x_1 z_1, \qquad A_4 = \sum x_1^2 (x_2 z_2 + x_3 z_3) \sum x_1 y_1,$$
$$A_5 = \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum x_1^2, \qquad A_6 = \sum x_1 y_1 \sum x_1 z_1 \sum x_1^2,$$
$$B_1 = \sum x_1^2 y_1 z_1 (x_2^2 + x_3^2), \qquad B_2 = \sum x_2^2 x_3^2 y_1 z_1,$$
$$B_3 = \sum x_1^4 y_1 z_1, \qquad B_4 = \sum x_1^4 (y_2 z_2 + y_3 z_3),$$
$$B_5 = \sum x_1^3 z_1 (x_2 y_2 + x_3 y_3), \qquad B_6 = \sum x_1^3 y_1 (x_2 z_2 + x_3 z_3),$$
$$B_7 = x_1 x_2 x_3 \sum x_1 (y_2 z_3 + y_3 z_2),$$

and we may take $\{C_{411}\} = x_1 x_2 x_3 \sum x_1 (y_2 z_3 + y_3 z_2),\ y_1 y_2 y_3 \sum x_1 (y_2 z_3 + y_3 z_2),$

$$z_1 z_2 z_3 \sum x_1 (y_2 z_3 + y_3 z_2).$$

For $i_1 i_2 \ldots i_n = 321$:

$$\begin{pmatrix} A_8 \\ A_6 \\ A_7 \\ A_3 \\ A_2 - A_8 + A_7 - A_3 \\ A_4 \\ A_1 - A_6 - A_4 - A_2 + A_8 \\ A_5 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 1, & 1, & 1, & 1, & 0, & 0, & 0 \\ & 1, & 1, & 0, & 0, & 1, & 1, & 1, & 0 \\ & & 1, & 1, & 0, & 0, & 0, & 2, & 2 \\ & & & 1, & 1, & 1, & 0, & 0, & 0 \\ & & & & -1, & -2, & 0, & 2, & 2 \\ & & & & & 1, & 1, & 1, & 0 \\ & & & & & & -1, & -2, & 0 \\ & & & & & & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_5 \\ B_1 \\ B_8 \\ B_4 \\ B_6 \\ B_2 \\ B_3 \\ B_7 \\ B_9 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum x_1 y_1, & A_2 &= \sum y_1^2 (z_2 x_2 + z_3 x_3) \sum x_1^2, \\ A_3 &= \sum x_1^2 (x_2 z_2 + x_3 z_3) \sum y_1^2, & A_4 &= \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum y_1 z_1, \\ A_5 &= \sum x_1 y_1 x_2 y_2 \sum x_1 z_1, & A_6 &= \sum x_1 y_1 \sum y_1 z_1 \sum x_1^2, \\ A_7 &= (\sum x_1 y_1)^2 \sum x_1 z_1, & A_8 &= \sum x_1 z_1 \sum x_1^2 \sum y_1^2, \\ B_1 &= \sum x_1^3 y_1 (y_2 z_2 + y_3 z_3), & B_2 &= \sum x_1^2 (x_2 y_2^2 z_2 + x_3 y_3^2 z_3), \\ B_3 &= \sum x_1^2 y_2 y_3 (x_2 z_3 + x_3 z_2), & B_4 &= \sum x_1^2 y_1^2 (x_2 z_2 + x_3 z_3), \\ B_5 &= \sum x_1^3 z_1 (y_2^2 + y_3^2), & B_6 &= \sum x_1^2 (x_2 y_3^2 z_2 + x_3 y_2^2 z_3), \\ B_7 &= \sum x_1^2 y_1 z_1 (x_2 y_2 + x_3 y_3), & B_8 &= \sum x_1^3 y_1^2 z_1, \\ B_9 &= x_1 x_2 x_3 \sum y_2 y_3 z_1, \end{aligned}$$

and we may write $\{C_{321}\} = x_1 x_2 x_3 \sum y_2 y_3 z_1,\ y_1 y_2 y_3 \sum z_2 z_3 x_1,\ z_1 z_2 z_3 \sum x_2 x_3 y_1,$
$x_1 x_2 x_3 \sum y_1 z_2 z_3,\ y_1 y_2 y_3 \sum z_1 x_2 x_3,\ z_1 z_2 z_3 \sum x_1 y_2 y_3.$

For $i_1 i_2 \ldots i_n = 222$:

$$\begin{pmatrix} A_{11} \\ A_{10} \\ A_5 \\ A_7 \\ A_6 \\ A_2 - A_7 + A_6 \\ A_4 \\ A_3 - A_5 + A_4 \\ A_1 \\ A_8 \\ A_9 - A_1 - A_8 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0, & 0, & 1, & 0, & 1, & 0, & 0, & 1, & 0, & 1 \\ & 1, & 1, & 1, & 0, & 0, & 0, & 0, & 1, & 0, & 0, & 1 \\ & & 1, & 0, & 0, & 0, & 2, & 2, & 0, & 0, & 0, & 1 \\ & & & 1, & 2, & 2, & 0, & 0, & 0, & 0, & 0, & 1 \\ & & & & 1, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ & & & & & 1, & 0, & 0, & 0, & 0, & 0, & -1 \\ & & & & & & 1, & 1, & 0, & 0, & 0, & 0 \\ & & & & & & & 1, & 0, & 0, & 0, & -1 \\ & & & & & & & & 1, & 1, & 2, & 0 \\ & & & & & & & & & 1, & 1, & 0 \\ & & & & & & & & & & -1, & 1 \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{10} \\ B_8 \\ B_5 \\ B_4 \\ B_6 \\ B_7 \\ B_9 \\ B_2 \\ B_1 \\ B_3 \\ B_{12} \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum y_1 z_1, & A_2 &= \sum y_1^2 (x_2 z_2 + x_3 z_3) \sum x_1 z_1, \\ A_3 &= \sum z_1^2 (x_2 y_2 + x_3 y_3) \sum x_1 y_1, & A_4 &= \sum x_1 x_2 y_1 y_2 \sum z_1^2, \\ A_5 &= (\sum x_1 y_1)^2 \sum z_1^2, & A_6 &= \sum x_1 x_2 z_1 z_2 \sum y_1^2, \\ A_7 &= (\sum x_1 z_1)^2 \sum y_1^2, & A_8 &= \sum y_1 y_2 z_1 z_2 \sum x_1^2, \\ A_9 &= (\sum y_1 z_1)^2 \sum x_1^2, & A_{10} &= \sum x_1^2 \sum y_1^2 \sum z_1^2, \\ A_{11} &= \sum x_1 y_1 \sum x_1 z_1 \sum y_1 z_1, \end{aligned}$$

$$
\begin{aligned}
B_1 &= \sum x_1^2 y_1 z_1 (y_2 z_2 + y_3 z_3), & B_2 &= \sum x_1^2 (y_2^2 z_2^2 + y_3^2 z_3^2), \\
B_3 &= \sum x_1^2 y_2 y_3 z_2 z_3, & B_4 &= \sum x_1 y_1^2 z_1 (x_2 z_2 + x_3 z_3), \\
B_5 &= \sum x_1^2 z_1^2 (y_2^2 + y_3^2), & B_6 &= \sum y_1^2 x_2 x_3 z_2 z_3, \\
B_7 &= \sum x_1 y_1 z_1^2 (x_2 y_2 + x_3 y_3), & B_8 &= \sum x_1^2 y_1^2 (z_2^2 + z_3^2), \\
B_9 &= \sum z_1^2 x_2 x_3 y_2 y_3, & B_{10} &= \sum x_1^2 (y_2^2 z_3^2 + y_3^2 z_2^2), \\
B_{11} &= \sum x_1 y_1 z_2 z_3 (x_2 y_3 + x_3 y_2), & B_{12} &= \sum x_1^2 y_1^2 z_1^2,
\end{aligned}
$$

and we may take $\{C_{222}\} = \sum x_1^2 y_1^2 z_1^2$.

For $i_1 i_2 \ldots i_n = 1111$:

$$
\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0, & 1 \\ & 1, & 0, & 1 \\ & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix},
$$

where

$$
\begin{aligned}
A_1 &= \sum x_1 y_1 \sum z_1 u_1, & A_2 &= \sum x_1 z_1 \sum y_1 u_1, \quad A_3 = \sum x_1 u_1 \sum y_1 z_1, \\
B_1 &= \sum x_1 y_1 z_1 u_1, & B_2 &= \sum x_1 y_1 (z_2 u_2 + z_3 u_3), \\
B_3 &= \sum x_1 z_1 (y_2 u_2 + y_3 u_3), & B_4 &= \sum x_1 u_1 (y_2 z_2 + y_3 z_3),
\end{aligned}
$$

and we may take $\{C_{1111}\} = \sum x_1 y_1 z_1 u_1$.

For $i_1 i_2 \ldots i_n = 2211$:

$$
\begin{bmatrix} A_{14} \\ A_{13} \\ A_{11} \\ A_5 \\ A_4 - A_{11} + A_5 \\ A_{10} \\ A_2 \\ A_3 - A_{10} + A_2 \\ A_{12} \\ A_7 \\ A_6 - A_{12} + A_7 \\ A_9 \\ A_8 \\ A_1 \end{bmatrix} =
\begin{bmatrix}
1, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 1 \\
 & 1, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 1 \\
 & & 1, & 1, & 1, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 1 \\
 & & & 1, & 1, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\
 & & & & 1, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & -1 \\
 & & & & & 1, & 1, & 1, & 0, & 1, & 0, & 0, & 0, & 0, & 1 \\
 & & & & & & 1, & 1, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\
 & & & & & & & 1, & 0, & 1, & 0, & 0, & 0, & 0, & -1 \\
 & & & & & & & & 1, & 1, & 1, & 1, & 0, & 0, & 1 \\
 & & & & & & & & & 1, & 1, & 1, & 0, & 0, & 0 \\
 & & & & & & & & & & 1, & 1, & 0, & 0, & -1 \\
 & & & & & & & & & & & 1, & 2, & 2, & 1 \\
 & & & & & & & & & & & & 1, & 1, & 0 \\
 & & & & & & & & & & & & & 1, & 1
\end{bmatrix}
\begin{bmatrix} B_{15} \\ B_{14} \\ B_7 \\ B_{10} \\ B_9 \\ B_6 \\ B_3 \\ B_5 \\ B_8 \\ B_4 \\ B_{12} \\ B_{11} \\ B_{13} \\ B_2 \\ B_1 \end{bmatrix},
$$

where

$$
\begin{aligned}
A_1 &= \sum x_1 y_1 z_1 u_1 \sum x_1 y_1, & A_2 &= \sum y_1^2 (x_2 z_2 + x_3 z_3) \sum x_1 u_1, \\
A_3 &= \sum y_1^2 (x_2 u_2 + x_3 u_3) \sum x_1 z_1, & A_4 &= \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum y_1 u_1, \\
A_5 &= \sum x_1^2 (y_2 u_2 + y_3 u_3) \sum y_1 z_1, & A_6 &= \sum x_1^2 (z_2 u_2 + z_3 u_3) \sum y_1^2, \\
A_7 &= \sum y_1^2 (z_2 u_2 + z_3 u_3) \sum x_1^2, & A_8 &= \sum x_1 x_2 y_1 y_2 \sum z_1 u_1, \\
A_9 &= (\sum x_1 y_1)^2 \sum z_1 u_1, & A_{10} &= \sum x_1 u_1 \sum x_1 z_1 \sum y_1^2, \\
A_{11} &= \sum y_1 u_1 \sum x_1^2 \sum y_1 z_1, & A_{12} &= \sum z_1 u_1 \sum x_1^2 \sum y_1^2, \\
A_{13} &= \sum x_1 y_1 \sum x_1 z_1 \sum y_1 u_1, & A_{14} &= \sum x_1 y_1 \sum x_1 u_1 \sum y_1 z_1,
\end{aligned}
$$

$$B_1 = \sum x_1^2 y_1^2 z_1 u_1, \qquad B_2 = \sum x_1 y_1 z_1 u_1 (x_2 y_2 + x_3 y_3),$$
$$B_3 = \sum x_1 y_1^2 u_1 (x_2 z_2 + x_3 z_3), \qquad B_4 = \sum x_1^2 z_1 u_1 (y_2^2 + y_3^2),$$
$$B_5 = \sum x_2 x_3 y_1^2 (z_2 u_3 + z_3 u_2), \qquad B_6 = \sum x_1 y_1^2 z_1 (x_2 u_2 + x_3 u_3),$$
$$B_7 = \sum x_1^2 y_1 u_1 (y_2 z_2 + y_3 z_3), \qquad B_8 = \sum x_1^2 (y_2^2 z_2 u_2 + y_3^2 z_3 u_3),$$
$$B_9 = \sum x_1^2 y_2 y_3 (z_2 u_3 + z_3 u_2), \qquad B_{10} = \sum x_1^2 y_1 z_1 (y_2 u_2 + y_3 u_3),$$
$$B_{11} = \sum x_1^2 y_1^2 (z_2 u_2 + z_3 u_3), \qquad B_{12} = \sum x_1^2 (y_2^2 z_3 u_3 + y_3^2 z_2 u_2),$$
$$B_{13} = \sum x_2 x_3 y_2 y_3 z_1 u_1, \qquad B_{14} = \sum x_2 x_3 y_1 u_1 (y_2 z_3 + y_3 z_2),$$
$$B_{15} = \sum x_2 x_3 y_1 z_1 (y_2 u_3 + y_3 u_2),$$

and we may take $\{C_{2211}\} = \sum x_1^2 y_1^2 z_1 u_1,\ \sum x_1^2 z_1^2 y_1 u_1,\ \sum x_1^2 u_1^2 y_1 z_1,$
$\sum y_1^2 z_1^2 x_1 u_1,\ \sum y_1^2 u_1^2 x_1 z_1,\ \sum z_1^2 u_1^2 x_1 y_1.$

For $i_1 i_2 \ldots i_n = 3111$:

$$\begin{pmatrix} A_{11} \\ A_8 \\ A_9 \\ A_{10} \\ A_7 \\ A_6 \\ A_5 \\ A_4 + A_5 - A_8 \\ A_3 + A_6 - A_9 \\ A_2 + A_7 - A_{10} \\ A_1 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0, & 0, & 1, & 1, & 1, & 0, & 0, & 0, & 0, & 1 \\ & 1, & 0, & 0, & 0, & 0, & 1, & 1, & 0, & 0, & 1, & 1 \\ & & 1, & 0, & 0, & 1, & 0, & 0, & 1, & 0, & 1, & 1 \\ & & & 1, & 1, & 0, & 0, & 0, & 0, & 1, & 1, & 1 \\ & & & & 1, & 0, & 0, & 0, & 0, & 1, & 1, & 0 \\ & & & & & 1, & 0, & 0, & 1, & 0, & 1, & 0 \\ & & & & & & 1, & 1, & 0, & 0, & 1, & 0 \\ & & & & & & & 1, & 0, & 0, & 1, & -1 \\ & & & & & & & & 1, & 0, & 1, & -1 \\ & & & & & & & & & 1, & 1, & -1 \\ & & & & & & & & & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_{12} \\ B_7 \\ B_5 \\ B_3 \\ B_{11} \\ B_{10} \\ B_9 \\ B_8 \\ B_6 \\ B_4 \\ B_2 \\ B_1 \end{pmatrix},$$

where

$$A_1 = \sum x_1 y_1 z_1 u_1 \sum x_1^2, \qquad A_2 = \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum x_1 u_1,$$
$$A_3 = \sum x_1^2 (y_2 u_2 + y_3 u_3) \sum x_1 z_1, \qquad A_4 = \sum x_1^2 (z_2 u_2 + z_3 u_3) \sum x_1 y_1,$$
$$A_5 = \sum x_1^2 (x_2 y_2 + x_3 y_3) \sum z_1 u_1, \qquad A_6 = \sum x_1^2 (x_2 z_2 + x_3 z_3) \sum y_1 u_1,$$
$$A_7 = \sum x_1^2 (x_2 u_2 + x_3 u_3) \sum y_1 z_1, \qquad A_8 = \sum x_1 y_1 \sum z_1 u_1 \sum x_1^2,$$
$$A_9 = \sum x_1 z_1 \sum y_1 u_1 \sum x_1^2, \qquad A_{10} = \sum x_1 u_1 \sum y_1 z_1 \sum x_1^2,$$
$$A_{11} = \sum x_1 y_1 \sum x_1 z_1 \sum x_1 u_1,$$
$$B_1 = \sum x_1^3 y_1 z_1 u_1, \qquad B_2 = \sum x_1^2 (x_2 y_2 z_2 u_2 + x_3 y_3 z_3 u_3),$$
$$B_3 = \sum x_1^3 u_1 (y_2 z_2 + y_3 z_3), \qquad B_4 = \sum x_1^2 (x_3 y_2 z_2 u_3 + x_2 y_3 z_3 u_2),$$
$$B_5 = \sum x_1^3 z_1 (y_2 u_2 + y_3 u_3), \qquad B_6 = \sum x_1^2 (x_3 y_2 z_3 u_2 + x_2 y_3 z_2 u_3),$$
$$B_7 = \sum x_1^3 y_1 (z_2 u_2 + z_3 u_3), \qquad B_8 = \sum x_1^2 (x_3 y_3 z_2 u_2 + x_2 y_2 z_3 u_3),$$
$$B_9 = \sum x_1^2 z_1 u_1 (x_2 y_2 + x_3 y_3), \qquad B_{10} = \sum x_1^2 y_1 u_1 (x_2 z_2 + x_3 z_3),$$
$$B_{11} = \sum x_1^2 y_1 z_1 (x_2 u_2 + x_3 u_3), \qquad B_{12} = x_1 x_2 x_3 \sum y_1 (z_2 u_3 + z_3 u_2),$$

and we may take $\{C_{3111}\} = \sum x_1^3 y_1 z_1 u_1,\ \sum y_1^3 x_1 z_1 u_1,\ \sum z_1^3 x_1 y_1 u_1,\ \sum u_1^3 x_1 y_1 z_1.$

For $i_1 i_2 \dots i_n = 21111$:

$$\begin{pmatrix} A_{20} \\ A_{19} \\ A_{18} \\ A_{17} \\ A_{16} \\ A_{15} \\ A_{14} \\ A_{13} \\ A_{12} \\ A_{11} \\ A_{10} \\ A_9+A_{10}-A_{13} \\ A_8 \\ A_7+A_{11}-A_{14} \\ A_6+A_8-A_{12} \\ A_5 \\ A_4 \\ A_3 \\ A_2 \\ A_1 \end{pmatrix} = \begin{pmatrix}
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&1 \\
&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&1 \\
&&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&1 \\
&&&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1 \\
&&&&1&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&1 \\
&&&&&1&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1 \\
&&&&&&1&0&0&1&0&0&0&1&0&0&0&0&0&1&1 \\
&&&&&&&1&0&0&1&1&0&0&0&0&0&0&0&1&1 \\
&&&&&&&&1&0&0&0&1&0&1&0&0&0&0&1&1 \\
&&&&&&&&&1&0&0&0&1&0&0&0&0&0&1&0 \\
&&&&&&&&&&1&1&0&0&0&0&0&0&0&1&0 \\
&&&&&&&&&&&1&0&0&0&0&0&0&0&1&-1 \\
&&&&&&&&&&&&1&0&1&0&0&0&0&1&0 \\
&&&&&&&&&&&&&1&0&0&0&0&0&1&-1 \\
&&&&&&&&&&&&&&1&0&0&0&0&1&-1 \\
&&&&&&&&&&&&&&&1&0&0&0&0&1 \\
&&&&&&&&&&&&&&&&1&0&0&0&1 \\
&&&&&&&&&&&&&&&&&1&0&0&1 \\
&&&&&&&&&&&&&&&&&&1&0&1 \\
&&&&&&&&&&&&&&&&&&&1&1
\end{pmatrix} \begin{pmatrix} B_{18} \\ B_{17} \\ B_{12} \\ B_{21} \\ B_{20} \\ B_{19} \\ B_9 \\ B_{13} \\ B_7 \\ B_{16} \\ B_{15} \\ B_{14} \\ B_{11} \\ B_{10} \\ B_8 \\ B_6 \\ B_5 \\ B_4 \\ B_3 \\ B_2 \\ B_1 \end{pmatrix}$$

where

$$A_1 = \sum x_1^2 \sum y_1 z_1 u_1 v_1, \qquad A_2 = \sum x_1 y_1 z_1 u_1 \sum x_1 v_1,$$

$$A_3 = \sum x_1 y_1 z_1 v_1 \sum x_1 u_1, \qquad A_4 = \sum x_1 y_1 u_1 v_1 \sum x_1 z_1,$$

$$A_5 = \sum x_1 z_1 u_1 v_1 \sum x_1 y_1, \qquad A_6 = \sum x_1^2 (y_2 z_2 + y_3 z_3) \sum u_1 v_1,$$

$$A_7 = \sum x_1^2 (z_2 u_2 + z_3 u_3) \sum y_1 v_1, \qquad A_8 = \sum x_1^2 (u_2 v_2 + u_3 v_3) \sum y_1 z_1,$$

$$A_9 = \sum x_1^2 (z_2 v_2 + z_3 v_3) \sum y_1 u_1, \qquad A_{10} = \sum x_1^2 (y_2 u_2 + y_3 u_3) \sum z_1 v_1,$$

$$A_{11} = \sum x_1^2 (y_2 v_2 + y_3 v_3) \sum z_1 u_1, \qquad A_{12} = \sum x_1^2 \sum u_1 v_1 \sum y_1 z_1,$$

$$A_{13} = \sum x_1^2 \sum z_1 v_1 \sum y_1 u_1, \qquad A_{14} = \sum x_1^2 \sum z_1 u_1 \sum y_1 v_1,$$

$$A_{15} = \sum x_1 y_1 \sum x_1 z_1 \sum u_1 v_1, \qquad A_{16} = \sum x_1 y_1 \sum x_1 u_1 \sum z_1 v_1,$$

$$A_{17} = \sum x_1 z_1 \sum x_1 u_1 \sum y_1 v_1, \qquad A_{18} = \sum x_1 u_1 \sum x_1 v_1 \sum y_1 z_1,$$

$$A_{19} = \sum x_1 z_1 \sum x_1 v_1 \sum y_1 u_1, \qquad A_{20} = \sum x_1 y_1 \sum x_1 v_1 \sum z_1 u_1,$$

$$B_1 = \sum x_1^2 y_1 z_1 u_1 v_1, \qquad B_2 = \sum x_1^2 (y_2 z_2 u_2 v_2 + y_3 z_3 u_3 v_3),$$

$$B_3 = \sum x_1 y_1 z_1 u_1 (x_2 v_2 + x_3 v_3), \qquad B_4 = \sum x_1 y_1 z_1 v_1 (x_2 u_2 + x_3 u_3),$$

$$B_5 = \sum x_1 y_1 u_1 v_1 (x_2 z_2 + x_3 z_3), \qquad B_6 = \sum x_1 z_1 u_1 v_1 (x_2 y_2 + x_3 y_3),$$

$$B_7 = \sum x_1^2 u_1 v_1 (y_2 z_2 + y_3 z_3), \qquad B_8 = \sum x_1^2 (y_2 z_2 u_3 v_3 + y_3 z_3 u_2 v_2),$$

$$B_9 = \sum x_1^2 y_1 v_1 (z_2 u_2 + z_3 u_3), \qquad B_{10} = \sum x_1^2 (y_2 z_3 u_3 v_2 + y_3 z_2 u_2 v_3),$$

$$B_{11} = \sum x_1^2 y_1 z_1 (u_2 v_2 + u_3 v_3), \qquad B_{12} = \sum x_1 v_1 (x_2 y_3 z_3 u_2 + x_3 y_2 z_2 u_3),$$

$$B_{13} = \sum x_1^2 y_1 u_1 (z_2 v_2 + z_3 v_3), \qquad B_{14} = \sum x_1^2 (y_2 z_3 u_2 v_3 + y_3 z_2 u_3 v_2),$$
$$B_{15} = \sum x_1^2 z_1 v_1 (y_2 u_2 + y_3 u_3), \qquad B_{16} = \sum x_1^2 z_1 u_1 (y_2 v_2 + y_3 v_3),$$
$$B_{17} = \sum x_1 v_1 (x_2 y_3 z_2 u_3 + x_3 y_2 z_3 u_2), \qquad B_{18} = \sum x_1 v_1 (x_2 y_2 z_3 u_3 + x_3 y_3 z_2 u_2),$$
$$B_{19} = \sum x_1 y_1 (x_2 z_2 u_3 v_3 + x_3 z_3 u_2 v_2), \qquad B_{20} = \sum x_1 y_1 (x_2 z_3 u_2 v_3 + x_3 z_2 u_3 v_2),$$
$$B_{21} = \sum x_1 z_1 (x_2 y_3 u_2 v_3 + x_3 y_2 u_3 v_2),$$

and we may take $\{C_{21111}\} = \sum x_1^2 y_1 z_1 u_1 v_1,\ \sum y_1^2 x_1 z_1 u_1 v_1,\ \sum z_1^2 x_1 y_1 u_1 v_1,$
$\sum u_1^2 x_1 y_1 z_1 v_1,\ \sum v_1^2 x_1 y_1 z_1 u_1.$

For $i_1 i_2 \dots i_n = 111111$:

$$\begin{pmatrix} A_{30} \\ A_{29} \\ A_{28} \\ A_{27} \\ A_{26} \\ A_{25} \\ A_{24} \\ A_{23} \\ A_{22} \\ A_{21} \\ A_{20} \\ A_{19} \\ A_{18} \\ A_{17} \\ A_{16} \\ A_{15} \\ A_{14} \\ A_{13} \\ A_{12} \\ A_{11} \\ A_{10} \\ A_9 \\ A_8 \\ A_7 \\ A_6 \\ A_5 \\ A_4 \\ A_3 \\ A_2 \\ A_1 \end{pmatrix} = \left(\begin{array}{r}
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,1,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,1,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,1,1 \\
1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,0,0,0,1,1 \\
1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,1 \\
1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,1 \\
1,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,1,0,0,0,0,1 \\
1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,0,1 \\
1,0,0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,0,1 \\
1,0,0,1,0,0,0,0,0,0,0,0,0,0,1,1,0,0,1 \\
1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,1,0,1 \\
1,1,0,0,0,0,0,0,0,0,1,0,0,0,0,1,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,0,1 \\
1,0,0,0,0,0,1 \\
1,0,0,0,0,1 \\
1,0,0,0,1 \\
1,0,0,1 \\
1,0,1 \\
1,1
\end{array}\right) \begin{pmatrix} B_{31} \\ B_{30} \\ B_{29} \\ B_{28} \\ B_{27} \\ B_{26} \\ B_{25} \\ B_{24} \\ B_{23} \\ B_{22} \\ B_{21} \\ B_{20} \\ B_{19} \\ B_{18} \\ B_{17} \\ B_{16} \\ B_{15} \\ B_{14} \\ B_{13} \\ B_{12} \\ B_{11} \\ B_{10} \\ B_9 \\ B_8 \\ B_7 \\ B_6 \\ B_5 \\ B_4 \\ B_3 \\ B_2 \\ B_1 \end{pmatrix},$$

where

$$A_1 = \sum x_1 y_1 z_1 u_1 \sum v_1 w_1, \qquad A_2 = \sum x_1 y_1 z_1 v_1 \sum u_1 w_1,$$
$$A_3 = \sum x_1 y_1 z_1 w_1 \sum u_1 v_1, \qquad A_4 = \sum x_1 y_1 u_1 v_1 \sum z_1 w_1,$$
$$A_5 = \sum x_1 y_1 u_1 w_1 \sum z_1 v_1, \qquad A_6 = \sum x_1 y_1 v_1 w_1 \sum z_1 u_1,$$
$$A_7 = \sum x_1 z_1 u_1 v_1 \sum y_1 w_1, \qquad A_8 = \sum x_1 z_1 u_1 w_1 \sum y_1 v_1,$$
$$A_9 = \sum x_1 z_1 v_1 w_1 \sum y_1 u_1, \qquad A_{10} = \sum x_1 u_1 v_1 w_1 \sum y_1 z_1,$$
$$A_{11} = \sum y_1 z_1 u_1 v_1 \sum x_1 w_1, \qquad A_{12} = \sum y_1 z_1 u_1 w_1 \sum x_1 v_1,$$
$$A_{13} = \sum y_1 z_1 v_1 w_1 \sum x_1 u_1, \qquad A_{14} = \sum y_1 u_1 v_1 w_1 \sum x_1 z_1,$$
$$A_{15} = \sum z_1 u_1 v_1 w_1 \sum x_1 y_1, \qquad A_{16} = \sum x_1 y_1 \sum z_1 u_1 \sum v_1 w_1,$$
$$A_{17} = \sum x_1 y_1 \sum z_1 v_1 \sum u_1 w_1, \qquad A_{18} = \sum x_1 y_1 \sum z_1 w_1 \sum u_1 v_1,$$

$$A_{19}=\sum x_1u_1\sum y_1v_1\sum z_1w_1, \qquad A_{20}=\sum x_1u_1\sum y_1w_1\sum z_1v_1,$$
$$A_{21}=\sum x_1v_1\sum y_1u_1\sum z_1w_1, \qquad A_{22}=\sum x_1z_1\sum u_1v_1\sum y_1w_1,$$
$$A_{23}=\sum x_1z_1\sum u_1w_1\sum y_1v_1, \qquad A_{24}=\sum x_1z_1\sum v_1w_1\sum y_1u_1,$$
$$A_{25}=\sum x_1u_1\sum v_1w_1\sum y_1z_1, \qquad A_{26}=\sum y_1z_1\sum u_1v_1\sum x_1w_1,$$
$$A_{27}=\sum y_1z_1\sum u_1w_1\sum x_1v_1, \qquad A_{28}=\sum y_1w_1\sum z_1u_1\sum x_1v_1,$$
$$A_{29}=\sum y_1u_1\sum z_1v_1\sum x_1w_1, \qquad A_{30}=\sum y_1v_1\sum z_1u_1\sum x_1w_1,$$
$$B_1=\sum x_1y_1z_1u_1v_1w_1, \qquad B_2=\sum x_1y_1z_1u_1(v_2w_2+v_3w_3),$$
$$B_3=\sum x_1y_1z_1v_1(u_2w_2+u_3w_3), \qquad B_4=\sum x_1y_1z_1w_1(u_2v_2+u_3v_3),$$
$$B_5=\sum x_1y_1u_1v_1(z_2w_2+z_3w_3), \qquad B_6=\sum x_1y_1u_1w_1(z_2v_2+z_3v_3),$$
$$B_7=\sum x_1y_1v_1w_1(z_2u_2+z_3u_3), \qquad B_8=\sum x_1z_1u_1v_1(y_2w_2+y_3w_3),$$
$$B_9=\sum x_1z_1u_1w_1(y_2v_2+y_3v_3), \qquad B_{10}=\sum x_1z_1v_1w_1(y_2u_2+y_3u_3),$$
$$B_{11}=\sum x_1u_1v_1w_1(y_2z_2+y_3z_3), \qquad B_{12}=\sum y_1z_1u_1v_1(x_2w_2+x_3w_3),$$
$$B_{13}=\sum y_1z_1u_1w_1(x_2v_2+x_3v_3), \qquad B_{14}=\sum y_1z_1v_1w_1(x_2u_2+x_3u_3),$$
$$B_{15}=\sum y_1u_1v_1w_1(x_2z_2+x_3z_3), \qquad B_{16}=\sum z_1u_1v_1w_1(x_2y_2+x_3y_3),$$
$$B_{17}=\sum x_1y_1(z_2u_2v_3w_3+z_3u_3v_2w_2), \qquad B_{18}=\sum x_1y_1(z_2u_3v_2w_3+z_3u_2v_3w_2),$$
$$B_{19}=\sum x_1y_1(z_2u_3v_3w_2+z_3u_2v_2w_3), \qquad B_{20}=\sum x_1u_1(y_2z_3v_2w_3+y_3z_2v_3w_2),$$
$$B_{21}=\sum x_1u_1(y_2z_3v_3w_2+y_3z_2v_2w_3), \qquad B_{22}=\sum x_1v_1(y_2z_3u_2w_3+y_3z_2u_3w_2),$$
$$B_{23}=\sum x_1z_1(y_2u_3v_3w_2+y_3u_2v_2w_3), \qquad B_{24}=\sum x_1z_1(y_2u_3v_2w_3+y_3u_2v_3w_2),$$
$$B_{25}=\sum x_1z_1(y_2u_2v_3w_3+y_3u_3v_2w_2), \qquad B_{26}=\sum x_1u_1(y_2z_2v_3w_3+y_3z_3v_2w_2),$$
$$B_{27}=\sum y_1z_1(x_2u_3v_3w_2+x_3u_2v_2w_3), \qquad B_{28}=\sum y_1z_1(x_2u_3v_2w_3+x_3u_2v_3w_2),$$
$$B_{29}=\sum y_1w_1(x_2z_3u_3v_2+x_3z_2u_2v_3), \qquad B_{30}=\sum y_1u_1(x_2z_3v_3w_2+x_3z_2v_2w_3),$$
$$B_{31}=\sum y_1v_1(x_2z_3u_3w_2+x_3z_2u_2w_3),$$

and we may take $\{C_{111111}\}=\sum x_1y_1z_1u_1v_1w_1$.

Summarizing the conclusions reached above, we see that an irreducible integrity basis for the n vectors $\boldsymbol{A}^{(r)}$ $(r=1,\ldots,n)$ for the hexoctahedral class is formed by

$$\sum x_1^2,\ \sum x_1^2x_2^2,\ x_1^2x_2^2x_3^2;$$
$$\sum x_1y_1,\ \sum x_1x_2y_1y_2,\ \sum x_1^2(x_2y_2+x_3y_3),\ \sum y_1^2(x_2y_2+x_3y_3),$$
$$x_1x_2x_3y_1y_2y_3,\ x_1x_2x_3\sum x_1x_2y_3,\ x_1x_2x_3\sum y_1y_2x_3,$$
$$y_1y_2y_3\sum x_1x_2y_3,\ y_1y_2y_3\sum y_1y_2x_3;$$
$$\sum x_1^2(y_2z_2+y_3z_3),\ x_1x_2x_3\sum x_1(y_2z_3+y_3z_2),$$
$$x_1x_2x_3\sum y_1y_2z_3,\ x_1x_2x_3\sum z_1z_2y_3,\ \sum x_1^2y_1^2z_1^2$$

together with the distinct invariants obtained from these by cyclic permutation of x, y and z;

$$\sum x_1y_1z_1u_1,\quad \sum x_1^2y_1^2z_1u_1,\quad \sum x_1^2z_1^2y_1u_1,$$
$$\sum x_1^2u_1^2y_1z_1,\quad \sum y_1^2z_1^2x_1u_1,\quad \sum y_1^2u_1^2x_1z_1,$$
$$\sum z_1^2u_1^2x_1y_1,\quad \sum x_1^3y_1z_1u_1,\quad \sum y_1^3z_1u_1x_1,$$
$$\sum z_1^3u_1x_1y_1,\quad \sum u_1^3x_1y_1z_1;$$
$$\sum x_1^2y_1z_1u_1v_1 \tag{9.2}$$

together with the invariants obtained from this by cyclic permutation of x, y, z, u and v;

$$\sum x_1 y_1 z_1 u_1 v_1 w_1.$$

(c) *Diploidal class* $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2)$

As in the case of the hexoctahedral class, we see that since any polynomial invariant for the vectors $\boldsymbol{A}^{(r)}$ $(r = 1, \ldots, n)$ for the diploidal class must be invariant under the transformations $\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3$, it must be expressible as a polynomial in x_1^2, x_2^2, x_3^2; $x_1 y_1, x_2 y_2, x_3 y_3$. The further requirement that this polynomial be invariant under the transformations $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{M}_1, \boldsymbol{M}_2)$ implies that it must be invariant under cyclic permutation of the subscripts. It follows from theorem 3 that an integrity basis for the n vectors $\boldsymbol{A}^{(r)}$ $(r = 1, \ldots, n)$, for the diploidal class, may be constructed with elements of total degree two, four and six only. It is also clear from theorem 3 that the elements of the integrity basis which involve only one vector are

$$\sum x_1^2, \ \sum x_1^2 x_2^2, \ x_1^2 x_2^2 x_3^2 \ \text{ and } \ \sum x_1^2 x_2^2 (x_1^2 - x_2^2).$$

In constructing an irreducible integrity basis for the diploidal class, we note that all of the elements of the irreducible integrity basis derived for the hexoctahedral class are also invariants for the diploidal class. We therefore construct additional invariants for the diploidal class, which are not invariants for the hexoctahedral class, but which together with the irreducible integrity basis for the hexoctahedral class form an irreducible integrity basis for the diploidal class. We denote the sets of elements of total degrees two, four and six in the irreducible integrity basis for the hexoctahedral class by $\{H(2)\}$, $\{H(4)\}$ and $\{H(6)\}$ respectively. Also, we denote the additional elements of total degrees two, four and six, which must be added in order to form an integrity basis for the diploidal class, by $\{D(2)\}$, $\{D(4)\}$, $\{D(6)\}$ respectively.

The possibility exists that one or more of the elements in $\{H(2)\}$, $\{H(4)\}$ and $\{H(6)\}$ may be expressed as a polynomial in other of these elements together with elements from $\{D(2)\}$, $\{D(4)\}$ and $\{D(6)\}$. This is, however, not the case. For, it is easily seen from theorem 3 that there are no elements in the set $\{D(2)\}$. Therefore, we cannot eliminate any of the elements $\{H(4)\}$. Furthermore, if an element of $\{H(6)\}$ is to be expressible as a polynomial in the elements of $\{H(2)\}$, $\{H(4)\}$ and $\{D(4)\}$, it would involve the elements of $\{D(4)\}$ linearly. It could not then be invariant under all the transformations of the hexoctahedral class. It follows, therefore, that we may take the elements of the hexoctahedral class derived in § 9(b) as elements of an irreducible integrity basis for the diploidal class.

We define $P^*_{i_1 i_2 \ldots i_n}$ as the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ for the diploidal class, which are not invariants for the hexoctahedral class. We note that $P^*_{i_1 i_2 \ldots i_n} = P^{(1)}_{i_1 i_2 \ldots i_n} - P^{(2)}_{i_1 i_2 \ldots i_n}$, where $P^{(1)}_{i_1 i_2 \ldots i_n}$ is the total number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ for the diploidal class and $P^{(2)}_{i_1 i_2 \ldots i_n}$ is the number of linearly independent invariants of degree $i_1 i_2 \ldots i_n$ for the hexoctahedral class. $P^{(2)}_{i_1 i_2 \ldots i_n}$ are therefore the quantities $P_{i_1 i_2 \ldots i_n}$ given in Table 9. The values of $P^*_{i_1 i_2 \ldots i_n}$ for all possible values of $i_1 i_2 \ldots i_n$, such that

$i_1+i_2+\cdots+i_n=2, 4$ or 6 and at least two of the i's are non-zero, are given in Table 10.

We define $\vartheta^*_{i_1 i_2 \ldots i_n}$ as the number of monomial invariants of degree $i_1 i_2 \ldots i_n$ for the diploidal class, which are not invariants for the hexoctahedral class, that can be formed from the elements of an irreducible integrity basis for the diploidal class of lower degree than $i_1 i_2 \ldots i_n$. In constructing relations of the type (9.1), the A's and B's will, of course, refer to invariants which are invariant under the diploidal transformations, but not under all the hexoctahedral transformations. $R^*_{i_1 i_2 \ldots i_n}$ is then the rank of the matrix relating A's and B's of degree $i_1 i_2 \ldots i_n$. Also $\{C^*_{i_1 i_2 \ldots i_n}\}$ will denote the elements of the irreducible integrity basis for the diploidal class of degrees $i_1 i_2 \ldots i_n$ and all permutations of $i_1 i_2 \ldots i_n$ which are not also elements of the hexoctahedral integrity basis. Bearing this in mind, we proceed in the manner described in § 9(a) and obtain the following results.

Table 10

$i_1 i_2 \ldots i_n$	31	22	51	42	33	211	411	321	222	1111	3111	2211	21111	111111
$P^*_{i_1 i_2 \ldots i_n}$	1	1	2	3	4	2	5	7	8	3	11	13	20	30
$\vartheta^*_{i_1 i_2 \ldots i_n}$	0	0	1	2	3	0	4	7	9	0	12	16	27	45
$R^*_{i_1 i_2 \ldots i_n}$	0	0	1	2	3	0	4	6	7	0	10	12	19	29

For $i_1 i_2 \ldots i_n=31$: $P^*_{31}=1$, $\vartheta^*_{31}=0$, hence $\{C^*_{31}\}$ may be taken as $\sum x_1^2(x_2y_2-x_3y_3)$, $\sum y_1^2(x_2y_2-x_3y_3)$.

For $i_1 i_2 \ldots i_n=22$: $P^*_{22}=1$, $\vartheta^*_{22}=0$, hence $\{C^*_{22}\}$ may be taken as $\sum x_1^2(y_2^2-y_3^2)$.

For $i_1 i_2 \ldots i_n=51$:
$$A_1=(1,\,-1)\begin{pmatrix}B_1\\B_2\end{pmatrix},$$
where
$$A_1=\sum x_1^2\sum x_1^2(x_2y_2-x_3y_3),$$
$$B_1=\sum x_1^4(x_2y_2-x_3y_3), \qquad B_2=\sum x_1^2x_2^2(x_1y_1-x_2y_2),$$
and $\{C^*_{51}\}$ may be taken as $\sum x_1^2x_2^2(x_1y_1-x_2y_2)$, $\sum y_1^2y_2^2(x_1y_1-x_2y_2)$.

For $i_1 i_2 \ldots i_n=42$:
$$\begin{pmatrix}A_1\\A_2\end{pmatrix}=\begin{pmatrix}1, & 0, & -1\\ & 1, & -1\end{pmatrix}\begin{pmatrix}B_1\\B_3\\B_2\end{pmatrix},$$
where
$$A_1=\sum x_1^2\sum x_1^2(y_2^2-y_3^2), \qquad A_2=\sum x_1y_1\sum x_1^2(x_2y_2-x_3y_3),$$
$$B_1=\sum x_1^4(y_2^2-y_3^2), \qquad B_2=\sum x_1^2x_2^2(y_1^2-y_2^2),$$
$$B_3=\sum x_1^3y_1(x_2y_2-x_3y_3),$$
and $\{C^*_{42}\}$ may be taken as $\sum x_1^2x_2^2(y_1^2-y_2^2)$, $\sum y_1^2y_2^2(x_1^2-x_2^2)$.

For $i_1 i_2 \ldots i_n=33$:
$$\begin{pmatrix}A_1\\A_2\\A_3+A_1-A_2\end{pmatrix}=\begin{pmatrix}1, & -1, & 1, & 0\\ & -1, & -1, & 1\\ & & 3, & 0\end{pmatrix}\begin{pmatrix}B_1\\B_2\\B_4\\B_3\end{pmatrix},$$

where

$$A_1 = \sum x_1^2(y_2^2 - y_3^2)\sum x_1 y_1, \qquad A_2 = \sum x_1^2(x_2 y_2 - x_3 y_3)\sum y_1^2,$$
$$A_3 = \sum y_1^2(x_2 y_2 - x_3 y_3)\sum x_1^2,$$
$$B_1 = \sum x_1^3 y_1(y_2^2 - y_3^2), \qquad B_2 = \sum x_1 y_1^3(x_2^2 - x_3^2),$$
$$B_3 = \sum x_1^2 y_1^2(x_2 y_2 - x_3 y_3), \qquad B_4 = \sum x_1 y_1(x_2^2 y_3^2 - x_3^2 y_2^2),$$

and $\{C_{33}^*\}$ may be taken as $\sum x_1^2 y_1^2(x_2 y_2 - x_3 y_3)$.

For $i_1 i_2 \dots i_n = 211: P_{211}^* = 2, \vartheta_{211}^* = 0$, hence $\{C_{211}^*\}$ may be taken as $\sum x_1^2(y_2 z_2 - y_3 z_3)$, $\sum y_1^2(x_2 z_2 - x_3 z_3)$, $\sum z_1^2(x_2 y_2 - x_3 y_3)$, $\sum x_1 y_1(x_2 z_2 - x_3 z_3)$, $\sum y_1 z_1(x_2 y_2 - x_3 y_3)$, $\sum x_1 z_1(y_2 z_2 - y_3 z_3)$.

For $i_1 i_2 \dots i_n = 411$:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 + A_1 - A_2 \end{pmatrix} = \begin{pmatrix} 1, & 0, & -1, & 1, & 0 \\ & 1, & -1, & -1, & 0 \\ & & -1, & 0, & 1 \\ & & & 3, & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_5 \\ B_4 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2(x_2 y_2 - x_3 y_3)\sum x_1 z_1, \qquad A_2 = \sum x_1^2(x_2 z_2 - x_3 z_3)\sum x_1 y_1,$$
$$A_3 = \sum x_1^2(y_2 z_2 - y_3 z_3)\sum x_1^2, \qquad A_4 = \sum x_1 y_1(x_2 z_2 - x_3 z_3)\sum x_1^2,$$
$$B_1 = \sum x_1^3 z_1(x_2 y_2 - x_3 y_3), \qquad B_2 = \sum x_1^3 y_1(x_2 z_2 - x_3 z_3),$$
$$B_3 = \sum x_1^2 y_1 z_1(x_2^2 - x_3^2), \qquad B_4 = \sum x_1^4(y_2 z_2 - y_3 z_3),$$
$$B_5 = x_1 x_2 x_3 \sum x_1(y_2 z_3 - y_3 z_2),$$

and we may take $\{C_{411}^*\} = \sum x_1^4(y_2 z_2 - y_3 z_3)$, $\sum y_1^4(x_2 z_2 - x_3 z_3)$, $\sum z_1^4(x_2 y_2 - x_3 y_3)$.

For $i_1 i_2 \dots i_n = 321$:

$$\begin{pmatrix} A_7 \\ A_6 \\ A_2 \\ A_1 \\ A_4 \\ A_3 + A_6 - A_1 \end{pmatrix} = \begin{pmatrix} 1, & 0, & -1, & -1, & 0, & 0, & 0 \\ & -1, & 0, & 0, & 0, & -1, & 1 \\ & & 1, & -1, & 1, & 0, & 0 \\ & & & -1, & 0, & 1, & 1 \\ & & & & 1, & 0, & -1 \\ & & & & & -3, & 0 \end{pmatrix} \begin{pmatrix} B_7 \\ B_6 \\ B_5 \\ B_2 \\ B_4 \\ B_3 \\ B_1 \end{pmatrix}$$

and

$$A_5 = 2A_1 - 2A_2 + A_3 + 3A_4 + A_6 - A_7,$$

where

$$A_1 = \sum x_1^2(x_2 y_2 - x_3 y_3)\sum y_1 z_1, \qquad A_2 = \sum x_1^2(x_2 z_2 - x_3 z_3)\sum y_1^2,$$
$$A_3 = \sum x_1^2(y_2 z_2 - y_3 z_3)\sum x_1 y_1, \qquad A_4 = \sum x_1 y_1(x_2 z_2 - x_3 z_3)\sum x_1 y_1,$$
$$A_5 = \sum y_1^2(x_2 z_2 - x_3 z_3)\sum x_1^2, \qquad A_6 = \sum y_1 z_1(x_2 y_2 - x_3 y_3)\sum x_1^2,$$
$$A_7 = \sum x_1^2(y_2^2 - y_3^2)\sum x_1 z_1,$$
$$B_1 = \sum x_1^2 y_1 z_1(x_2 y_2 - x_3 y_3), \qquad B_2 = \sum x_1 y_1^2 z_1(x_2^2 - x_3^2),$$
$$B_3 = \sum x_1^2 y_2 y_3(x_2 z_3 - x_3 z_2), \qquad B_4 = \sum x_1^2 y_1^2(x_2 z_2 - x_3 z_3),$$
$$B_5 = \sum x_1^2(x_2 y_3^2 z_2 - x_3 y_2^2 z_3), \qquad B_6 = \sum x_1^3 y_1(y_2 z_2 - y_3 z_3),$$
$$B_7 = \sum x_1^3 z_1(y_2^2 - y_3^2),$$

and we may take $\{C_{321}^*\} = \sum x_1^2 y_1 z_1(x_2 y_2 - x_3 y_3)$, $\sum x_1 y_1^2 z_1(y_2 z_2 - y_3 z_3)$, $\sum x_1 y_1 z_1^2(x_2 z_2 - x_3 z_3)$, $\sum x_1^2 y_1 z_1(x_2 z_2 - x_3 z_3)$, $\sum x_1 y_1^2 z_1(x_2 y_2 - x_3 y_3)$, $\sum x_1 y_1 z_1^2(y_2 z_2 - y_3 z_3)$.

For $i_1 i_2 \dots i_n = 222$:

$$\begin{pmatrix} A_2 \\ A_5 \\ A_6 \\ A_3 \\ A_7 \\ A_9 + A_8 + A_7 \\ A_4 - A_5 - A_6 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\ & 1, & -1, & 0, & 0, & 0, & 1, & 0 \\ & & 1, & -1, & 0, & 0, & 1, & 0 \\ & & & 1, & 1, & 0, & 0, & 0 \\ & & & & 1, & 1, & 0, & -1 \\ & & & & & 3, & 0, & 0 \\ & & & & & & -3, & 0 \end{pmatrix} \begin{pmatrix} B_2 \\ B_4 \\ B_5 \\ B_6 \\ B_3 \\ B_7 \\ B_8 \\ B_1 \end{pmatrix}$$

and

$$A_1 = B_1 + B_4 = -A_2 + 2A_3 + A_4 + A_6 - A_7 + A_9,$$
$$A_8 = B_1 - B_2 + B_7 = -3A_2 + 3A_3 + A_4 - A_5 + 2A_6 - A_7 + 2A_9,$$

where

$$A_1 = \sum x_1^2 (y_2 z_2 - y_3 z_3) \sum y_1 z_1, \qquad A_2 = \sum y_1^2 (x_2 z_2 - x_3 z_3) \sum x_1 z_1,$$
$$A_3 = \sum z_1^2 (x_2 y_2 - x_3 y_3) \sum x_1 y_1, \qquad A_4 = \sum x_1^2 (z_2^2 - z_3^2) \sum y_1^2,$$
$$A_5 = \sum x_1^2 (y_2^2 - y_3^2) \sum z_1^2, \qquad A_6 = \sum y_1^2 (z_2^2 - z_3^2) \sum x_1^2,$$
$$A_7 = \sum y_1 z_1 (x_2 y_2 - x_3 y_3) \sum x_1 z_1, \qquad A_8 = \sum x_1 z_1 (y_2 z_2 - y_3 z_3) \sum x_1 y_1,$$
$$A_9 = \sum x_1 y_1 (x_2 z_2 - x_3 z_3) \sum y_1 z_1,$$
$$B_1 = \sum x_1^2 y_1 z_1 (y_2 z_2 - y_3 z_3), \qquad B_2 = \sum x_1 y_1^2 z_1 (x_2 z_2 - x_3 z_3),$$
$$B_3 = \sum x_1 y_1 z_1^2 (x_2 y_2 - x_3 y_3), \qquad B_4 = \sum x_1^2 (y_2^2 z_2^2 - y_3^2 z_3^2),$$
$$B_5 = \sum y_1^2 (x_2^2 z_2^2 - x_3^2 z_3^2), \qquad B_6 = \sum z_1^2 (x_2^2 y_2^2 - x_3^2 y_3^2),$$
$$B_7 = \sum x_2 x_3 y_1 z_1 (y_2 z_3 - y_3 z_2), \qquad B_8 = \sum x_1^2 (y_2^2 z_3^2 - y_3^2 z_2^2).$$

Hence $\{C^*_{222}\}$ may be taken as $\sum x_1^2 y_1 z_1 (y_2 z_2 - y_3 z_3)$.

For $i_1 i_2 \dots i_n = 1111$: $P^*_{1111} = 3$, $\vartheta^*_{1111} = 0$, hence $\{C^*_{1111}\}$ may be taken as

$$\sum x_1 y_1 (z_2 u_2 - z_3 u_3), \quad \sum x_1 z_1 (y_2 u_2 - y_3 u_3), \quad \sum x_1 u_1 (y_2 z_2 - y_3 z_3).$$

For $i_1 i_2 \dots i_n = 3111$:

$$\begin{pmatrix} A_1 \\ A_3 \\ A_4 \\ A_6 \\ A_7 \\ A_9 \\ A_2 - A_1 - A_3 \\ A_5 - A_4 - A_6 \\ A_8 - A_7 - A_9 \\ A_{10} - A_{11} + A_{12} \end{pmatrix} = \begin{pmatrix} 1, & -1, & 0, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & 0 \\ & 1, & 0, & 0, & 0, & 0, & 1, & 0, & 0, & 0, & -1 \\ & & 1, & -1, & 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ & & & 1, & 0, & 0, & 0, & 1, & 0, & 0, & -1 \\ & & & & 1, & -1, & 0, & 0, & 1, & 0, & 0 \\ & & & & & 1, & 0, & 0, & 1, & 0, & -1 \\ & & & & & & -3, & 0, & 0, & 0, & 0 \\ & & & & & & & -3, & 0, & 0, & 0 \\ & & & & & & & & -3, & 0, & 0 \\ & & & & & & & & & 3, & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_4 \\ B_5 \\ B_7 \\ B_8 \\ B_3 \\ B_6 \\ B_9 \\ B_{11} \\ B_{10} \end{pmatrix}$$

and

$$3A_{11} = 2A_1 - 2A_2 - 4A_3 - A_4 + A_5 + 2A_6 - A_7 + A_8 + 2A_9 - 3A_{10},$$
$$3A_{12} = -A_1 + A_2 + 2A_3 + 2A_4 - 2A_5 - 4A_6 - A_7 + A_8 + 2A_9 + 3A_{10},$$

where

$$A_1 = \sum x_1 y_1 (z_2 u_2 - z_3 u_3) \sum x_1^2, \qquad A_2 = \sum x_1^2 (z_2 u_2 - z_3 u_3) \sum x_1 y_1,$$
$$A_3 = \sum x_1^2 (x_2 y_2 - x_3 y_3) \sum z_1 u_1, \qquad A_4 = \sum x_1 z_1 (y_2 u_2 - y_3 u_3) \sum x_1^2,$$
$$A_5 = \sum x_1^2 (y_2 u_2 - y_3 u_3) \sum x_1 z_1, \qquad A_6 = \sum x_1^2 (x_2 z_2 - x_3 z_3) \sum y_1 u_1,$$
$$A_7 = \sum x_1 u_1 (y_2 z_2 - y_3 z_3) \sum x_1^2, \qquad A_8 = \sum x_1^2 (y_2 z_2 - y_3 z_3) \sum x_1 u_1,$$

$$A_9 = \sum x_1^2 (x_2 u_2 - x_3 u_3) \sum y_1 z_1, \qquad A_{10} = \sum x_1 y_1 (x_2 z_2 - x_3 z_3) \sum x_1 u_1,$$
$$A_{11} = \sum x_1 y_1 (x_2 u_2 - x_3 u_3) \sum x_1 z_1, \qquad A_{12} = \sum x_1 z_1 (x_2 u_2 - x_3 u_3) \sum x_1 y_1,$$
$$B_1 = \sum x_1^3 y_1 (z_2 u_2 - z_3 u_3), \qquad B_2 = \sum x_1^2 z_1 u_1 (x_2 y_2 - x_3 y_3),$$
$$B_3 = \sum x_1^2 (x_2 y_2 z_3 u_3 - x_3 y_3 z_2 u_2), \qquad B_4 = \sum x_1^3 z_1 (y_2 u_2 - y_3 u_3),$$
$$B_5 = \sum x_1^2 y_1 u_1 (x_2 z_2 - x_3 z_3), \qquad B_6 = \sum x_1^2 (x_2 z_2 y_3 u_3 - x_3 z_3 y_2 u_2),$$
$$B_7 = \sum x_1^3 u_1 (y_2 z_2 - y_3 z_3), \qquad B_8 = \sum x_1^2 y_1 z_1 (x_2 u_2 - x_3 u_3),$$
$$B_9 = \sum x_1^2 (x_2 u_2 y_3 z_3 - x_3 u_3 y_2 z_2), \qquad B_{10} = \sum x_1 y_1 z_1 u_1 (x_2^2 - x_3^2),$$
$$B_{11} = x_1 x_2 x_3 \sum y_1 (z_2 u_3 - z_3 u_2).$$

Hence $\{C^*_{3111}\}$ may be taken as $\sum x_1 y_1 z_1 u_1 (x_2^2 - x_3^2)$, $\sum x_1 y_1 z_1 u_1 (y_2^2 - y_3^2)$,
$\sum x_1 y_1 z_1 u_1 (z_2^2 - z_3^2)$, $\sum x_1 y_1 z_1 u_1 (u_2^2 - u_3^2)$.

For $i_1 i_2 \ldots i_n = 2211$:

$$\begin{pmatrix} A_{10} \\ A_6 \\ A_5 \\ A_3 \\ A_{11} \\ A_{16} \\ A_1 \\ A_2 \\ A_{15} \\ A_4 \\ A_{14} \\ A_6 + A_7 - A_{11} \\ A_5 - A_{13} + A_{16} \\ A_8 - A_4 - A_{15} \\ A_{10} + A_{12} - A_3 \\ A_1 + A_9 - A_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ & & & & & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & 3 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 3 & 0 & 0 & 0 \\ & & & & & & & & & & 3 & 0 & 0 \\ & & & & & & & & & & & 3 & 0 \end{pmatrix} \begin{pmatrix} B_{13} \\ B_{11} \\ B_1 \\ B_2 \\ B_4 \\ B_7 \\ B_8 \\ B_{12} \\ B_{10} \\ B_9 \\ B_6 \\ B_3 \\ B_5 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 (y_2 z_2 - y_3 z_3) \sum y_1 u_1, \qquad A_2 = \sum x_1^2 (y_2 u_2 - y_3 u_3) \sum y_1 z_1,$$
$$A_3 = \sum x_1^2 (z_2 u_2 - z_3 u_3) \sum y_1^2, \qquad A_4 = \sum x_1 y_1 (x_2 z_2 - x_3 z_3) \sum y_1 u_1,$$
$$A_5 = \sum x_1 y_1 (x_2 u_2 - x_3 u_3) \sum y_1 z_1, \qquad A_6 = \sum x_1 z_1 (x_2 u_2 - x_3 u_3) \sum y_1^2,$$
$$A_7 = \sum y_1^2 (x_2 z_2 - x_3 z_3) \sum x_1 u_1, \qquad A_8 = \sum x_1 y_1 (y_2 u_2 - y_3 u_3) \sum x_1 z_1,$$
$$A_9 = \sum y_1 z_1 (y_2 u_2 - y_3 u_3) \sum x_1^2, \qquad A_{10} = \sum x_1^2 (y_2^2 - y_3^2) \sum z_1 u_1,$$
$$A_{11} = \sum y_1^2 (x_2 u_2 - x_3 u_3) \sum x_1 z_1, \qquad A_{12} = \sum y_1^2 (z_2 u_2 - z_3 u_3) \sum x_1^2,$$
$$A_{13} = \sum x_1 y_1 (y_2 z_2 - y_3 z_3) \sum x_1 u_1, \qquad A_{14} = \sum x_1 y_1 (z_2 u_2 - z_3 u_3) \sum x_1 y_1,$$
$$A_{15} = \sum x_1 z_1 (y_2 u_2 - y_3 u_3) \sum x_1 y_1, \qquad A_{16} = \sum x_1 u_1 (y_2 z_2 - y_3 z_3) \sum x_1 y_1,$$
$$B_1 = \sum x_1^2 y_1 u_1 (y_2 z_2 - y_3 z_3), \qquad B_2 = \sum y_1^2 z_1 u_1 (x_2^2 - x_3^2),$$
$$B_3 = \sum x_1^2 y_2 y_3 (z_2 u_3 - z_3 u_2), \qquad B_4 = \sum x_1^2 y_1 z_1 (y_2 u_2 - y_3 u_3),$$
$$B_5 = \sum x_1^2 y_1^2 (z_2 u_2 - z_3 u_3), \qquad B_6 = \sum x_1^2 (y_2^2 z_3 u_3 - y_3^2 z_2 u_2),$$
$$B_7 = \sum y_1^2 x_1 u_1 (x_2 z_2 - x_3 z_3), \qquad B_8 = \sum x_1 y_1 z_1 u_1 (x_2 y_2 - x_3 y_3),$$
$$B_9 = \sum x_1 z_1 y_2 y_3 (x_2 u_3 - x_3 u_2), \qquad B_{10} = \sum y_1 z_1 x_2 x_3 (y_2 u_3 - y_3 u_2),$$
$$B_{11} = \sum y_1^2 x_1 z_1 (x_2 u_2 - x_3 u_3), \qquad B_{12} = \sum y_1^2 x_2 x_3 (z_2 u_3 - z_3 u_2),$$
$$B_{13} = \sum x_1^2 z_1 u_1 (y_2^2 - y_3^2).$$

The 16×13 matrix above is clearly of at least rank 12. It is not of rank 13 since we obtain a column of zeros upon addition of columns $1, \ldots, 7$ and 13. Thus, the matrix is of rank 12 and we see that $\{C^*_{2211}\}$ may be taken as

$$\begin{array}{lll} \sum x_1^2 y_1^2 (z_2 u_2 - z_3 u_3), & \sum x_1^2 z_1^2 (y_2 u_2 - y_3 u_3), & \sum x_1^2 u_1^2 (y_2 z_2 - y_3 z_3), \\ \sum y_1^2 z_1^2 (x_2 u_2 - x_3 u_3), & \sum y_1^2 u_1^2 (x_2 z_2 - x_3 z_3), & \sum z_1^2 u_1^2 (x_2 y_2 - x_3 y_3). \end{array}$$

For $i_1 i_2 \ldots i_n = 21111$:

$$\begin{pmatrix} A_{14} \\ A_{11} \\ A_{10} \\ A_8 \\ A_7 \\ A_5 \\ A_4 \\ A_3 \\ A_2 \\ A_1 \\ A_{12} \\ A_{15} \\ A_{26} \\ A_{25} \\ A_{24} \\ A_{23} \\ A_{22} \\ A_{21} \\ A_{22}-A_{21}+A_{27} \\ A_{19}-A_{23}+A_{26} \\ A_{20}-A_{24}+A_{25} \\ A_7-A_{14}-A_{17} \\ A_5-A_8-A_9 \\ A_4-A_{15}-A_{18} \\ A_3-A_6-A_{12} \\ A_2-A_{11}-A_{13} \\ A_1-A_{10}-A_{16} \end{pmatrix} = \left(\begin{array}{cccccccccccccccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 & & & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 & & & & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & & & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & & & & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & & & & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & 3 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & 3 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & 3 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & & 3 & 0
\end{array}\right) \begin{pmatrix} B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \\ B \end{pmatrix}$$

where

$$\begin{array}{ll}
A_1 = \sum x_1 y_1 (z_2 u_2 - z_3 u_3) \sum x_1 v_1, & A_2 = \sum x_1 z_1 (y_2 u_2 - y_3 u_3) \sum x_1 v_1, \\
A_3 = \sum x_1 u_1 (y_2 z_2 - y_3 z_3) \sum x_1 v_1, & A_4 = \sum x_1 y_1 (z_2 v_2 - z_3 v_3) \sum x_1 u_1, \\
A_5 = \sum x_1 z_1 (y_2 v_2 - y_3 v_3) \sum x_1 u_1, & A_6 = \sum x_1 v_1 (y_2 z_2 - y_3 z_3) \sum x_1 u_1, \\
A_7 = \sum x_1 y_1 (u_2 v_2 - u_3 v_3) \sum x_1 z_1, & A_8 = \sum x_1 u_1 (y_2 v_2 - y_3 v_3) \sum x_1 z_1, \\
A_9 = \sum x_1 z_1 (x_2 u_2 - x_3 u_3) \sum y_1 v_1, & A_{10} = \sum x_1 y_1 (x_2 v_2 - x_3 v_3) \sum z_1 u_1, \\
A_{11} = \sum x_1 z_1 (x_2 v_2 - x_3 v_3) \sum y_1 u_1, & A_{12} = \sum x_1 u_1 (x_2 v_2 - x_3 v_3) \sum y_1 z_1, \\
A_{13} = \sum x_1 v_1 (y_2 u_2 - y_3 u_3) \sum x_1 z_1, & A_{14} = \sum x_1 z_1 (u_2 v_2 - u_3 v_3) \sum x_1 y_1, \\
A_{15} = \sum x_1 u_1 (z_2 v_2 - z_3 v_3) \sum x_1 y_1, & A_{16} = \sum x_1 v_1 (z_2 u_2 - z_3 u_3) \sum x_1 y_1, \\
A_{17} = \sum x_1 y_1 (x_2 z_2 - x_3 z_3) \sum u_1 v_1, & A_{18} = \sum x_1 y_1 (x_2 u_2 - x_3 u_3) \sum z_1 v_1, \\
A_{19} = \sum x_1^2 (y_2 u_2 - y_3 u_3) \sum z_1 v_1, & A_{20} = \sum x_1^2 (y_2 z_2 - y_3 z_3) \sum u_1 v_1, \\
A_{21} = \sum x_1^2 (z_2 u_2 - z_3 u_3) \sum y_1 v_1, & A_{22} = \sum x_1^2 (y_2 v_2 - y_3 v_3) \sum z_1 u_1, \\
A_{23} = \sum x_1^2 (z_2 v_2 - z_3 v_3) \sum y_1 u_1, & A_{24} = \sum x_1^2 (u_2 v_2 - u_3 v_3) \sum y_1 z_1,
\end{array}$$

$$A_{25} = \sum y_1 z_1 (u_2 v_2 - u_3 v_3) \sum x_1^2, \qquad A_{26} = \sum y_1 u_1 (z_2 v_2 - z_3 v_3) \sum x_1^2,$$
$$A_{27} = \sum y_1 v_1 (z_2 u_2 - z_3 u_3) \sum x_1^2, \qquad B_1 = \sum x_1^2 y_1 v_1 (z_2 u_2 - z_3 u_3),$$
$$B_2 = \sum x_1 z_1 u_1 v_1 (x_2 y_2 - x_3 y_3), \qquad B_3 = \sum x_1 x_2 z_3 u_3 (v_1 y_2 - v_2 y_1),$$
$$B_4 = \sum x_1^2 z_1 v_1 (y_2 u_2 - y_3 u_3), \qquad B_5 = \sum x_1 y_1 u_1 v_1 (x_2 z_2 - x_3 z_3),$$
$$B_6 = \sum x_1 x_2 y_3 u_3 (v_1 z_2 - v_2 z_1), \qquad B_7 = \sum x_1^2 u_1 v_1 (y_2 z_2 - y_3 z_3),$$
$$B_8 = \sum x_1 y_1 z_1 v_1 (x_2 u_2 - x_3 u_3), \qquad B_9 = \sum x_1 x_2 y_3 z_3 (v_1 u_2 - v_2 u_1),$$
$$B_{10} = \sum x_1^2 y_1 u_1 (z_2 v_2 - z_3 v_3), \qquad B_{11} = \sum x_1 x_2 z_3 v_3 (u_1 y_2 - u_2 y_1),$$
$$B_{12} = \sum x_1^2 z_1 u_1 (y_2 v_2 - y_3 v_3), \qquad B_{13} = \sum x_1 x_2 y_3 v_3 (u_1 z_2 - u_2 z_1),$$
$$B_{14} = \sum x_1 y_1 z_1 u_1 (x_2 v_2 - x_3 v_3), \qquad B_{15} = \sum x_1^2 y_1 z_1 (u_2 v_2 - u_3 v_3),$$
$$B_{16} = \sum x_1 x_2 u_3 v_3 (y_2 z_1 - y_1 z_2), \qquad B_{17} = \sum x_1^2 (y_2 z_2 u_3 v_3 - y_3 z_3 u_2 v_2),$$
$$B_{18} = \sum x_1^2 (y_2 u_2 z_3 v_3 - y_3 u_3 z_2 v_2), \qquad B_{19} = \sum x_1^2 (y_2 v_2 z_3 u_3 - y_3 v_3 z_2 u_2),$$
$$B_{20} = \sum y_1 z_1 u_1 v_1 (x_2^2 - x_3^2).$$

The 27×20 matrix above is clearly of at least rank 19. It is not of rank 20 since we obtain a column of zeros upon addition of columns 1, ..., 10 and 20. Thus, the matrix is of rank 19 and we see that $\{C^*_{21111}\}$ may be taken as

$$\sum y_1 z_1 u_1 v_1 (x_2^2 - x_3^2), \qquad \sum x_1 z_1 u_1 v_1 (y_2^2 - y_3^2), \qquad \sum x_1 y_1 u_1 v_1 (z_2^2 - z_3^2),$$
$$\sum x_1 y_1 z_1 v_1 (u_2^2 - u_3^2), \qquad \sum x_1 y_1 z_1 u_1 (v_2^2 - v_3^2).$$

For $i_1 i_2 \ldots i_n = 111111$:

A_{44}	0	0	0	−1	0	0	0	0	0	0	0	0	1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	B_2
A_{43}	1	0	0	−1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{41}	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	B_2
A_{40}	0	0	0	−1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{38}	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{37}	1	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{35}	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{34}	0	0	0	0	0	0	0	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	B_1
A_{32}	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	B_9
A_{31}	0	0	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	B_8
A_{29}	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	B_6
A_{28}	0	1	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	B_5
A_{23}	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	B_1
A_{22}	0	0	0	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	B_2
A_{16}	0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	B_3
A_{15}	0	0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	B_2
A_{27}	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	B_2
A_{26}		1	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	B_2
A_{20}			1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	B_2
A_{19}				−1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	B_2
A_{18}					−1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	B_2
A_{14}						1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	B_2
A_{11}							−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	B_1
A_{12}								1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	B_1
A_8									−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	1	B_1
A_9										−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	B_1
A_5											−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	1	B_1
A_6												−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	B_7
A_1													1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	B_4
A_2														−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	B_3
$A_{43}-A_{44}+A_{45}$															3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$A_{40}-A_{41}+A_{42}$																3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$A_{37}-A_{38}+A_{39}$																	3	0	0	0	0	0	0	0	0	0	0	0	0	0	
$A_{34}-A_{35}+A_{36}$																		3	0	0	0	0	0	0	0	0	0	0	0	0	
$A_{31}-A_{32}+A_{33}$																			3	0	0	0	0	0	0	0	0	0	0	0	
$A_{28}-A_{29}+A_{30}$																				3	0	0	0	0	0	0	0	0	0	0	
$A_{25}-A_{26}+A_{27}$																					3	0	0	0	0	0	0	0	0	0	
$A_{22}-A_{23}+A_{24}$																						3	0	0	0	0	0	0	0	0	
$A_{19}-A_{20}+A_{21}$																							3	0	0	0	0	0	0	0	
$A_{17}-A_{16}-A_{18}$																								3	0	0	0	0	0	0	
$A_{13}-A_{14}+A_{15}$																									3	0	0	0	0	0	
$A_{11}-A_{10}-A_{12}$																										3	0	0	0	0	
$A_7-A_8+A_9$																											3	0	0	0	
$A_4-A_5+A_6$																												3	0	0	
$A_1-A_2+A_3$																													3	0	

where

$$A_1 = \sum x_1 y_1 (z_2 u_2 - z_3 u_3) \sum v_1 w_1, \qquad A_2 = \sum x_1 y_1 (v_2 w_2 - v_3 w_3) \sum z_1 u_1,$$
$$A_3 = \sum z_1 u_1 (v_2 w_2 - v_3 w_3) \sum x_1 y_1, \qquad A_4 = \sum x_1 z_1 (y_2 u_2 - y_3 u_3) \sum v_1 w_1,$$
$$A_5 = \sum x_1 z_1 (v_2 w_2 - v_3 w_3) \sum y_1 u_1, \qquad A_6 = \sum y_1 u_1 (v_2 w_2 - v_3 w_3) \sum x_1 z_1,$$
$$A_7 = \sum x_1 u_1 (y_2 z_2 - y_3 z_3) \sum v_1 w_1, \qquad A_8 = \sum x_1 u_1 (v_2 w_2 - v_3 w_3) \sum y_1 z_1,$$
$$A_9 = \sum y_1 z_1 (v_2 w_2 - v_3 w_3) \sum x_1 u_1, \qquad A_{10} = \sum x_1 w_1 (z_2 u_2 - z_3 u_3) \sum y_1 v_1,$$
$$A_{11} = \sum x_1 w_1 (y_2 v_2 - y_3 v_3) \sum z_1 u_1, \qquad A_{12} = \sum z_1 u_1 (v_2 y_2 - v_3 y_3) \sum x_1 w_1,$$
$$A_{13} = \sum x_1 w_1 (y_2 z_2 - y_3 z_3) \sum u_1 v_1, \qquad A_{14} = \sum x_1 w_1 (v_2 u_2 - v_3 u_3) \sum y_1 z_1,$$
$$A_{15} = \sum y_1 z_1 (u_2 v_2 - u_3 v_3) \sum x_1 w_1, \qquad A_{16} = \sum x_1 u_1 (z_2 w_2 - z_3 w_3) \sum y_1 v_1,$$
$$A_{17} = \sum x_1 u_1 (v_2 y_2 - v_3 y_3) \sum z_1 w_1, \qquad A_{18} = \sum z_1 w_1 (v_2 y_2 - v_3 y_3) \sum x_1 u_1,$$
$$A_{19} = \sum x_1 v_1 (y_2 z_2 - y_3 z_3) \sum u_1 w_1, \qquad A_{20} = \sum x_1 v_1 (u_2 w_2 - u_3 w_3) \sum y_1 z_1,$$
$$A_{21} = \sum y_1 z_1 (u_2 w_2 - u_3 w_3) \sum x_1 v_1, \qquad A_{22} = \sum x_1 w_1 (y_2 u_2 - y_3 u_3) \sum z_1 v_1,$$
$$A_{23} = \sum x_1 w_1 (z_2 v_2 - z_3 v_3) \sum y_1 u_1, \qquad A_{24} = \sum y_1 u_1 (z_2 v_2 - z_3 v_3) \sum x_1 w_1,$$
$$A_{25} = \sum x_1 u_1 (y_2 w_2 - y_3 w_3) \sum z_1 v_1, \qquad A_{26} = \sum x_1 u_1 (z_2 v_2 - z_3 v_3) \sum y_1 w_1,$$
$$A_{27} = \sum y_1 w_1 (z_2 v_2 - z_3 v_3) \sum x_1 u_1, \qquad A_{28} = \sum x_1 y_1 (z_2 v_2 - z_3 v_3) \sum u_1 w_1,$$
$$A_{29} = \sum x_1 y_1 (u_2 w_2 - u_3 w_3) \sum z_1 v_1, \qquad A_{30} = \sum z_1 v_1 (u_2 w_2 - u_3 w_3) \sum x_1 y_1,$$
$$A_{31} = \sum x_1 y_1 (z_2 w_2 - z_3 w_3) \sum u_1 v_1, \qquad A_{32} = \sum x_1 y_1 (u_2 v_2 - u_3 v_3) \sum z_1 w_1,$$
$$A_{33} = \sum z_1 w_1 (u_2 v_2 - u_3 v_3) \sum x_1 y_1, \qquad A_{34} = \sum x_1 z_1 (y_2 v_2 - y_3 v_3) \sum u_1 w_1,$$
$$A_{35} = \sum x_1 z_1 (u_2 w_2 - u_3 w_3) \sum y_1 v_1, \qquad A_{36} = \sum y_1 v_1 (u_2 w_2 - u_3 w_3) \sum x_1 z_1,$$
$$A_{37} = \sum x_1 z_1 (y_2 w_2 - y_3 w_3) \sum u_1 v_1, \qquad A_{38} = \sum x_1 z_1 (u_2 v_2 - u_3 v_3) \sum y_1 w_1,$$
$$A_{39} = \sum y_1 w_1 (u_2 v_2 - u_3 v_3) \sum x_1 z_1, \qquad A_{40} = \sum x_1 v_1 (y_2 u_2 - y_3 u_3) \sum z_1 w_1,$$
$$A_{41} = \sum x_1 v_1 (z_2 w_2 - z_3 w_3) \sum y_1 u_1, \qquad A_{42} = \sum y_1 u_1 (z_2 w_2 - z_3 w_3) \sum x_1 v_1,$$
$$A_{43} = \sum x_1 v_1 (y_2 w_2 - y_3 w_3) \sum z_1 u_1, \qquad A_{44} = \sum x_1 v_1 (z_2 u_2 - z_3 u_3) \sum y_1 w_1,$$
$$A_{45} = \sum y_1 w_1 (z_2 u_2 - z_3 u_3) \sum x_1 v_1, \qquad B_1 = \sum x_1 y_1 v_1 w_1 (z_2 u_2 - z_3 u_3),$$
$$B_2 = \sum z_1 u_1 v_1 w_1 (x_2 y_2 - x_3 y_3), \qquad B_3 = \sum x_1 y_1 z_1 u_1 (v_2 w_2 - v_3 w_3),$$
$$B_4 = \sum x_1 y_1 (z_2 u_2 v_3 w_3 - z_3 u_3 v_2 w_2), \qquad B_5 = \sum x_1 z_1 v_1 w_1 (y_2 u_2 - y_3 u_3),$$
$$B_6 = \sum y_1 u_1 v_1 w_1 (x_2 z_2 - x_3 z_3), \qquad B_7 = \sum x_1 z_1 (y_2 u_2 v_3 w_3 - y_3 u_3 v_2 w_2),$$
$$B_8 = \sum x_1 u_1 v_1 w_1 (y_2 z_2 - y_3 z_3), \qquad B_9 = \sum y_1 z_1 v_1 w_1 (x_2 u_2 - x_3 u_3),$$
$$B_{10} = \sum x_1 u_1 (y_2 z_2 v_3 w_3 - y_3 z_3 v_2 w_2), \qquad B_{11} = \sum x_1 z_1 u_1 w_1 (v_2 y_2 - v_3 y_3),$$
$$B_{12} = \sum y_1 z_1 u_1 v_1 (x_2 w_2 - x_3 w_3), \qquad B_{13} = \sum x_1 w_1 (y_2 v_2 z_3 u_3 - y_3 v_3 z_2 u_2),$$
$$B_{14} = \sum x_1 y_1 z_1 w_1 (v_2 u_2 - v_3 u_3), \qquad B_{15} = \sum x_1 w_1 (y_2 z_2 u_3 v_3 - y_3 z_3 u_2 v_2),$$
$$B_{16} = \sum x_1 y_1 u_1 v_1 (z_2 w_2 - z_3 w_3), \qquad B_{17} = \sum x_1 u_1 (y_2 v_2 z_3 w_3 - y_3 v_3 z_2 w_2),$$
$$B_{18} = \sum y_1 z_1 u_1 w_1 (x_2 v_2 - x_3 v_3), \qquad B_{19} = \sum x_1 v_1 (y_2 z_2 u_3 w_3 - y_3 z_3 u_2 w_2),$$
$$B_{20} = \sum x_1 y_1 z_1 v_1 (u_2 w_2 - u_3 w_3), \qquad B_{21} = \sum x_1 y_1 u_1 w_1 (z_2 v_2 - z_3 v_3),$$
$$B_{22} = \sum x_1 w_1 (y_2 u_2 z_3 v_3 - y_3 u_3 z_2 v_2), \qquad B_{23} = \sum x_1 z_1 u_1 v_1 (y_2 w_2 - y_3 w_3),$$
$$B_{24} = \sum x_1 u_1 (y_2 w_2 z_3 v_3 - y_3 w_3 z_2 v_2), \qquad B_{25} = \sum x_1 y_1 (z_2 v_2 u_3 w_3 - z_3 v_3 u_2 w_2),$$
$$B_{26} = \sum x_1 y_1 (z_2 w_2 u_3 v_3 - z_3 w_3 u_2 v_2), \qquad B_{27} = \sum x_1 z_1 (y_2 v_2 u_3 w_3 - y_3 v_3 u_2 w_2),$$
$$B_{28} = \sum x_1 z_1 (y_2 w_2 u_3 v_3 - y_3 w_3 u_2 v_2), \qquad B_{29} = \sum x_1 v_1 (y_2 u_2 z_3 w_3 - y_3 u_3 z_2 w_2),$$
$$B_{30} = \sum x_1 v_1 (y_2 w_2 z_3 u_3 - y_3 w_3 z_2 u_2).$$

The 45×30 matrix above is clearly of at least rank 29. It is not of rank 30 since we obtain a column of zeros upon addition of columns $1, \ldots, 14$ and 30. Thus, the matrix is of rank 29 and it is clear that $\{C^*_{111\,111}\}$ may be taken as $\sum x_1 y_1 z_1 u_1 (v_2 w_2 - v_3 w_3)$.

Summarizing the conclusions reached above, we see that an irreducible integrity basis for the n vectors $\boldsymbol{A}^{(r)}$ $(r=1, \ldots, n)$ for the diploidal class is formed by the invariants (9.2), together with

$$\begin{array}{lll}
\sum x_1^2 x_2^2 (x_1^2 - x_2^2); & & \\
\sum x_1^2 (y_2^2 - y_3^2), & \sum x_1^2 y_1^2 (x_2 y_2 - x_3 y_3), & \\
\sum x_1^2 (x_2 y_2 - x_3 y_3), & \sum x_1^2 x_2^2 (y_1^2 - y_2^2), & \sum x_1^2 x_2^2 (x_1 y_1 - x_2 y_2), \\
\sum y_1^2 (x_2 y_2 - x_3 y_3), & \sum y_1^2 y_2^2 (x_1^2 - x_2^2), & \sum y_1^2 y_2^2 (x_1 y_1 - x_2 y_2); \\
\sum x_1^2 (y_2 z_2 - y_3 z_3), & \sum x_1 y_1 (x_2 z_2 - x_3 z_3), & \\
\sum x_1^2 y_1 z_1 (x_2 y_2 - x_3 y_3), & \sum x_1^2 y_1 z_1 (x_2 z_2 - x_3 z_3), & \sum x_1^4 (y_2 z_2 - y_3 z_3)
\end{array} \tag{9.3}$$

and the invariants obtained from these by cyclic permutation of x, y, z;

$$\begin{array}{lll}
\sum x_1^2 y_1 z_1 (y_2 z_2 - y_3 z_3); & & \\
\sum x_1 y_1 (z_2 u_2 - z_3 u_3), & \sum x_1 z_1 (y_2 u_2 - y_3 u_3), & \sum x_1 u_1 (y_2 z_2 - y_3 z_3), \\
\sum x_1 y_1 z_1 u_1 (x_2^2 - x_3^2), & \sum x_1 y_1 z_1 u_1 (y_2^2 - y_3^2), & \sum x_1 y_1 z_1 u_1 (z_2^2 - z_3^2), \\
\sum x_1 y_1 z_1 u_1 (u_2^2 - u_3^2), & \sum x_1^2 y_1^2 (z_2 u_2 - z_3 u_3), & \sum x_1^2 z_1^2 (y_2 u_2 - y_3 u_3), \\
\sum x_1^2 u_1^2 (y_2 z_2 - y_3 z_3), & \sum y_1^2 z_1^2 (x_2 u_2 - x_3 u_3), & \sum y_1^2 u_1^2 (x_2 z_2 - x_3 z_3), \\
\sum z_1^2 u_1^2 (x_2 y_2 - x_3 y_3); & & \\
\sum x_1 y_1 z_1 u_1 (v_2^2 - v_3^2) & &
\end{array}$$

and the invariants obtained from this by cyclic permutation of x, y, z, u and v;

$$\sum x_1 y_1 z_1 u_1 (v_2 w_2 - v_3 w_3).$$

(d) Hextetrahedral class $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)\cdot(\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$

We first generate an integrity basis for three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and for this purpose it is convenient to hold in abeyance the convention described in § 3. We see, from the discussion of the rhombic-disphenoidal class in § 5, that if a polynomial in the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ is invariant under the transformations $\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3$, it must be expressible as a polynomial in the quantities

$$\begin{array}{llllll}
x_1 x_2 x_3, & y_1 y_2 y_3, & z_1 z_2 z_3, & x_1 y_2 z_3, & x_2 y_3 z_1, & x_3 y_1 z_2, \\
x_1 y_3 z_2, & x_2 y_1 z_3, & x_3 y_2 z_1 & & &
\end{array} \tag{9.4}$$

and the quantities $X_i^{(\alpha)}$ $(i=1, 2, 3;\ \alpha=1, 2, \ldots, 12)$ defined by

$$\begin{array}{lll}
X_i^{(1)} = x_i^2, & X_i^{(2)} = y_i^2, & X_i^{(3)} = z_i^2, \\
X_i^{(4)} = y_i z_i, & X_i^{(5)} = z_i x_i, & X_i^{(6)} = x_i y_i
\end{array} \tag{9.5}$$

and

$$\begin{array}{lll}
X_1^{(7)} = x_2 x_3 y_1, & X_1^{(8)} = y_2 y_3 x_1, \ldots, & X_1^{(12)} = z_2 z_3 y_1, \\
X_2^{(7)} = x_3 x_1 y_2, & X_2^{(8)} = y_3 y_1 x_2, \ldots, & X_2^{(12)} = z_3 z_1 y_2, \\
X_3^{(7)} = x_1 x_2 y_3, & X_3^{(8)} = y_1 y_2 x_3, \ldots, & X_3^{(12)} = z_1 z_2 y_3.
\end{array} \tag{9.6}$$

We note that the product of any two quantities selected from (9.4) and (9.6), including the square of a single quantity, may be expressed as a monomial in the quantities (9.5). It follows that any polynomial in the quantities (9.4), (9.5) and (9.6) may be expressed as a polynomial in these quantities, of degree unity in (9.4) and (9.6). We note further that the quantities $x_1x_2x_3$, $y_1y_2y_3$, $z_1z_2z_3$ are invariant under $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$ and that any linear function of $x_1y_2z_3$, $x_2y_3z_1$, $x_3y_1z_2$, $x_1y_3z_2$, $x_2y_1z_3$, $x_3y_2z_1$ with coefficients which are polynomials in the quantities (9.5) is expressible in terms of (i) linear functions of $x_1y_2z_3, \ldots, x_3y_2z_1$ with coefficients which are polynomials in $\sum x_1^2$, $\sum y_1^2$, $\sum z_1^2$ and (ii) linear functions of the quantities (9.6) with coefficients which are polynomials in the quantities (9.5). This follows immediately from identities of the form

$$x_1y_2z_3 \cdot x_1y_1 = x_1^2 \cdot y_1y_2z_3, \qquad x_1y_2z_3 \cdot x_2y_2 = y_2^2 \cdot x_1x_2z_3,$$

$$x_1y_2z_3 \cdot x_3y_3 = y_3z_3 \cdot x_3x_1y_2, \qquad x_1y_2z_3 \cdot x_2^2 = x_2y_2 \cdot x_1x_2z_3,$$

$$x_1y_2z_3 \cdot x_3^2 = x_3z_3 \cdot x_3x_1y_2$$

and

$$x_1y_2z_3 \cdot x_1^2 = x_1y_2z_3 \cdot \sum x_1^2 - x_1y_2z_3 \cdot x_2^2 - x_1y_2z_3 \cdot x_3^2.$$

It follows immediately from theorem 4 and the further requirement of invariance under $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$ that an integrity basis for polynomials of type (i) is formed by $\sum x_1^2$, $\sum y_1^2$, $\sum z_1^2$, $\sum x_1(y_2z_3 + y_3z_2)$. The further requirement of invariance under these transformations implies that the polynomials in the quantities (9.5) and (9.6) must be invariant under all permutations of the subscripts on the X's. Consequently, from theorem 2 and bearing in mind the linearity of the polynomial in (9.4) and (9.6), it must be expressible as a polynomial in invariants of degree not greater than 7 in the vectors.

If we examine the invariants of total degree seven in the vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ generated by the application of theorem 2, we find in each case that they can be expressed as polynomials in invariants of lower total degree. We conclude, therefore, that the elements of an irreducible integrity basis for the vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ for the hextetrahedral class are of total degree not greater than six.

The elements of the integrity basis which involve one vector only are, from (9.4), (9.5) and theorem 2, $\sum x_1^2$, $\sum x_1^2x_2^2$ and $x_1x_2x_3$, together with the invariants obtained by substituting y and z for x. We shall determine, following the procedure described in § 9(a), elements of an irreducible integrity basis involving two and three vectors. It will be seen that the invariants in the irreducible integrity basis for three vectors so obtained are of degree not greater than four. It then follows from theorem 5 that the elements of an irreducible integrity basis for an arbitrary number of vectors have degree not greater than four.

Accordingly, we calculate $P_{i_1i_2}$ for all non-zero values of i_1 and i_2 such that $i_1 + i_2 \leqq 6$ and $P_{i_1i_2i_3}$ for all non-zero values of i_1, i_2 and i_3 such that $i_1 + i_2 + i_3 \leqq 6$ and, in addition, we calculate P_{1111}. These values are given in Table 11. Proceeding in the manner described in § 9(a) we obtain the following results, using again the notation of § 3.

Table 11

$i_1 i_2 \dots i_n$	11	21	31	22	41	32	51	42	33	111	211	311	221	411	321	222	1111
$P_{i_1 i_2 \dots i_n}$	1	1	2	3	2	3	4	6	6	1	3	4	5	7	9	12	4
$\vartheta_{i_1 i_2 \dots i_n}$	0	0	1	2	2	3	4	7	7	0	2	4	5	8	11	15	3
$R_{i_1 i_2 \dots i_n}$	0	0	1	2	2	3	4	6	6	0	2	4	5	7	9	12	3

For $i_1 i_2 \dots i_n = 11$: $P_{11}=1$, $\vartheta_{11}=0$, hence $\{C_{11}\}$ may be taken as $\sum x_1 y_1$.

For $i_1 i_2 \dots i_n = 21$: $P_{21}=1$, $\vartheta_{21}=0$, hence $\{C_{21}\}$ may be taken as $\sum x_1 x_2 y_3$, $\sum y_1 y_2 x_3$.

For $i_1 i_2 \dots i_n = 31$:

$$A_1 = (1,\ 1) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 \sum x_1 y_1,$$

$$B_1 = \sum x_1^3 y_1, \qquad B_2 = \sum x_1^2 (x_2 y_2 + x_3 y_3),$$

and hence $\{C_{31}\}$ may be taken as $\sum x_1^2 (x_2 y_2 + x_3 y_3)$, $\sum y_1^2 (x_2 y_2 + x_3 y_3)$.

For $i_1 i_2 \dots i_n = 22$:

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 0 \\ 0, & 1, & 2 \end{pmatrix} \begin{pmatrix} B_3 \\ B_1 \\ B_2 \end{pmatrix},$$

where

$$A_1 = (\sum x_1 y_1)^2, \qquad A_2 = \sum x_1^2 \sum y_1^2,$$

$$B_1 = \sum x_1^2 y_1^2, \qquad B_2 = \sum x_1 x_2 y_1 y_2, \qquad B_3 = \sum x_1^2 (y_2^2 + y_3^2),$$

and hence $\{C_{22}\}$ may be taken as $\sum x_1 x_2 y_1 y_2$.

For $i_1 i_2 \dots i_n = 41$:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1, & 1 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} B_2 \\ B_1 \end{pmatrix},$$

where

$$A_1 = \sum x_2 x_3 y_1 \sum x_1^2, \qquad A_2 = x_1 x_2 x_3 \sum x_1 y_1,$$

$$B_1 = x_1 x_2 x_3 \sum x_1 y_1, \qquad B_2 = \sum x_1^3 (x_2 y_3 + x_3 y_2),$$

and hence we see that $\{C_{41}\}$ contains no elements.

For $i_1 i_2 \dots i_n = 32$:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 0 \\ & 1, & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

where

$$A_1 = \sum x_1^2 \sum x_1 y_2 y_3, \qquad A_2 = \sum x_1 y_1 \sum x_1 x_2 y_3, \qquad A_3 = x_1 x_2 x_3 \sum y_1^2,$$

$$B_1 = \sum x_1^3 y_2 y_3, \qquad B_2 = \sum x_1^2 y_1 (x_2 y_3 + x_3 y_2), \qquad B_3 = x_1 x_2 x_3 \sum y_1^2,$$

and hence we see that $\{C_{32}\}$ contains no elements.

For $i_1 i_2 \dots i_n = 111$: $P_{111}=1$, $\vartheta_{111}=0$, hence $\{C_{111}\}$ may be taken as $\sum x_1 (y_2 z_3 + y_3 z_2)$.

For $i_1 i_2 \dots i_n = 211$:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 1 \\ & 1, & 1 \end{pmatrix} \begin{pmatrix} B_3 \\ B_2 \\ B_1 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 \sum y_1 z_1, \quad A_2 = \sum x_1 y_1 \sum x_1 z_1,$$
$$B_1 = \sum x_1^2 y_1 z_1, \quad B_2 = \sum x_1 y_1 (x_2 z_2 + x_3 z_3), \quad B_3 = \sum x_1^2 (y_2 z_2 + y_3 z_3),$$

and we may take $\{C_{211}\} = \sum x_1^2 y_1 z_1, \sum y_1^2 x_1 z_1, \sum z_1^2 x_1 y_1$.

For $i_1 i_2 \dots i_n = 311$:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 1, & 0 \\ & 1, & 0, & 1 \\ & & 1, & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix},$$

where

$$A_1 = \sum x_1^2 \sum x_1 (y_2 z_3 + y_3 z_2), \quad A_2 = \sum x_1 z_1 \sum x_2 x_3 y_1,$$
$$A_3 = \sum x_1 y_1 \sum x_2 x_3 z_1, \quad A_4 = x_1 x_2 x_3 \sum y_1 z_1,$$
$$B_1 = \sum x_1^3 (y_2 z_3 + y_3 z_2), \quad B_2 = \sum x_1^2 z_1 (x_2 y_3 + x_3 y_2),$$
$$B_3 = \sum x_1^2 y_1 (x_2 z_3 + x_3 z_2), \quad B_4 = x_1 x_2 x_3 \sum y_1 z_1,$$

and hence we see that $\{C_{311}\}$ contains no elements.

For $i_1 i_2 \dots i_n = 221$:

$$\begin{pmatrix} A_1 \\ A_4 \\ A_3 \\ A_2 \\ A_5 - A_1 + A_4 - A_3 + 2A_2 \end{pmatrix} = \begin{pmatrix} 1, & 1, & 0, & 1, & 0 \\ & 1 & 1, & 0, & 0 \\ & & 1, & 1, & 0 \\ & & & 1, & 1 \\ & & & & 3 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_5 \\ B_3 \\ B_4 \end{pmatrix},$$

where

$$A_1 = \sum x_1 (y_2 z_3 + y_3 z_2) \sum x_1 y_1, \quad A_2 = \sum x_1 y_2 y_3 \sum x_1 z_1,$$
$$A_3 = \sum x_1 x_2 y_3 \sum y_1 z_1, \quad A_4 = \sum y_1^2 \sum x_2 x_3 z_1, \quad A_5 = \sum x_1^2 \sum y_2 y_3 z_1,$$
$$B_1 = \sum x_1^2 y_1 (y_2 z_3 + y_3 z_2), \quad B_2 = \sum x_1 y_1^2 (x_2 z_3 + x_3 z_2),$$
$$B_3 = \sum x_1 y_1 z_1 (x_2 y_3 + x_3 y_2), \quad B_4 = \sum x_1^2 y_2 y_3 z_1, \quad B_5 = \sum y_1^2 x_2 x_3 z_1,$$

and hence we see that $\{C_{221}\}$ contains no elements.

For $i_1 i_2 \dots i_n = 1111$:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0, & 1 \\ & 1, & 0, & 1 \\ & & 1, & 1 \end{pmatrix} \begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_1 \end{pmatrix},$$

where

$$A_1 = \sum x_1 y_1 \sum z_1 u_1, \quad A_2 = \sum x_1 z_1 \sum y_1 u_1,$$
$$A_3 = \sum x_1 u_1 \sum y_1 z_1, \quad B_1 = \sum x_1 y_1 z_1 u_1,$$
$$B_2 = \sum x_1 y_1 (z_2 u_2 + z_3 u_3), \quad B_3 = \sum x_1 z_1 (y_2 u_2 + y_3 u_3),$$
$$B_4 = \sum x_1 u_1 (y_2 z_2 + y_3 z_3)$$

and we may take $\{C_{1111}\}$ as $\sum x_1 y_1 z_1 u_1$.

For $i_1 i_2 \dots i_n = 51, 42, 33, 411, 321, 222$, we have seen in the case of the hexoctahedral class that there are 4, 6, 6, 7, 9, 12 linearly independent invariants respectively which may be generated by the elements of the integrity basis for the vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ for the hexoctahedral class. From (9.2), (9.7) and the identity

$$\begin{aligned}\sum y_2 y_3 x_1 \sum z_2 z_3 x_1 + 2 \sum x_1^2 y_1 z_1 \sum y_1 z_1 + \sum y_1^2 x_1 z_1 \sum x_1 z_1 + \\ + \sum z_1^2 x_1 y_1 \sum x_1 y_1 - \sum x_1^2 \sum y_1 y_2 z_1 z_2 - \sum x_1 y_1 \sum x_1 z_1 \sum y_1 z_1 = 3 \sum x_1^2 y_1^2 z_1^2,\end{aligned}$$

it is readily seen that each of these basis elements for the hexoctahedral class is expressible in terms of basis elements of total degrees 2, 3 and 4 for the hextetrahedral class. Thus, $R_{i_1 i_2 \dots i_n}$ for $i_1 i_2 \dots i_n = 51, 42, 33, 411, 321, 222$ must be at least equal to 4, 6, 6, 7, 9, 12 respectively for the hextetrahedral class. Since, from Table 11, these coincide with $P_{i_1 i_2 \dots i_n}$ for $i_1 i_2 \dots i_n = 51, 42, \dots, 222$, it is clear that $\{C_{51}\}$, $\{C_{42}\}$, $\{C_{33}\}$, $\{C_{411}\}$, $\{C_{321}\}$ and $\{C_{222}\}$ contain no elements.

Summarizing the conclusions reached above, we see that an irreducible integrity basis for the n vectors $\boldsymbol{A}^{(r)}$ $(r = 1, \dots, n)$ for the hextetrahedral class is formed by

$$\begin{aligned}&\sum x_1^2, \quad \sum x_1^2 x_2^2, \quad x_1 x_2 x_3; \\ &\sum x_1 y_1, \quad \sum x_1 x_2 y_3, \quad \sum y_1 y_2 x_3, \quad \sum x_1 x_2 y_1 y_2, \\ &\sum x_1^2 (x_2 y_2 + x_3 y_3), \quad \sum y_1^2 (x_2 y_2 + x_3 y_3); \\ &\sum x_1 (y_2 z_3 + y_3 z_2), \quad \sum x_1^2 y_1 z_1, \quad \sum y_1^2 x_1 z_1, \quad \sum z_1^2 x_1 y_1; \\ &\sum x_1 y_1 z_1 u_1.\end{aligned} \tag{9.7}$$

(e) *Tetartoidal class* $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{I}, \boldsymbol{M}_1, \boldsymbol{M}_2)$

We shall first generate an integrity basis for the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and for this purpose it is convenient to hold in abeyance the convention described in § 3. As in our discussion of the hextetrahedral class, we see that if a polynomial in $\boldsymbol{x}$ and $\boldsymbol{y}$ is invariant under the transformations $\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3$, it must be expressible as a polynomial in $x_1 x_2 x_3$, $y_1 y_2 y_3$ and the quantities $X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(5)}$ $(i = 1, 2, 3)$ defined by

$$X_i^{(1)} = x_i^2, \qquad X_i^{(2)} = y_i^2, \qquad X_i^{(3)} = x_i y_i \tag{9.8}$$

and

$$\begin{aligned}X_1^{(4)} &= x_2 x_3 y_1, & X_2^{(4)} &= x_3 x_1 y_2, & X_3^{(4)} &= x_1 x_2 y_3, \\ X_1^{(5)} &= y_2 y_3 x_1, & X_2^{(5)} &= y_3 y_1 x_2, & X_3^{(5)} &= y_1 y_2 x_3.\end{aligned} \tag{9.9}$$

We note that the product of any two quantities selected from (9.9), including the square of a single quantity, may be expressed as a monomial in the quantities (9.8). It follows that any polynomial in the quantities (9.8) and (9.9) may be expressed as a polynomial of degree unity in (9.9). If further this polynomial is invariant under the transformations $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3) \cdot (\boldsymbol{M}_1, \boldsymbol{M}_2)$, it follows from Table 1 that it is invariant under cyclic permutation of the subscripts on the X's. Consequently, from theorem 3 and the linearity of the polynomial in (9.9), it follows that the elements of an irreducible integrity basis for the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, for the tetartoidal class, are of total degree not greater than seven.

If we examine the invariants of total degree seven in the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ generated by the application of theorem 3, we find in each case that they can be expressed as polynomials in invariants of lower total degree. We conclude, therefore, that the elements of an irreducible integrity basis for the vectors $\boldsymbol{x}$, $\boldsymbol{y}$ for the tetartoidal class are of total degree not greater than six.

The elements of the integrity basis which involve one vector only are, from theorem 3, $\sum x_1^2$, $\sum x_1^2 x_2^2$, $x_1 x_2 x_3$, $\sum x_1^2 x_2^2 (x_1^2 - x_2^2)$, together with the invariants obtained by substituting y for x.

We shall determine, following the procedure described in § 9(a), elements of an irreducible integrity basis involving two vectors. It will be seen that the invariants in the irreducible integrity basis for two vectors so obtained are of degree not greater than six. It then follows from theorem 6 that the elements of an irreducible integrity basis for an arbitrary number of vectors have degree not greater than six and therefore none of the elements can involve more than six vectors. In reaching this conclusion, we note that det$[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$, where $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ are three vectors, is invariant under the transformations of the tetartoidal class.

In constructing an irreducible integrity basis for an arbitrary number of vectors for the tetartoidal class, we note that any invariant of even total degree is an invariant for the diploidal class and can therefore be expressed as a polynomial in the elements of the integrity basis for that class. This integrity basis consists of the irreducible integrity basis for the hexoctahedral class (9.2) together with the invariants (9.3). We also note that any invariant for the hexoctahedral class is also an invariant for the hextetrahedral class and can therefore be expressed as a polynomial in the elements (9.7) of the irreducible integrity basis for that class. Consequently, any invariant of even total degree for the tetartoidal class can be expressed as a polynomial in the elements of (9.3) and (9.7). We note that all of the elements in (9.3) and (9.7) are invariants for the tetartoidal class. It is apparent, by inspection, that none of the elements in (9.3) and (9.7) is redundant in the sense that it can be expressed as a polynomial in the remaining elements.

Therefore, in order to construct an irreducible integrity basis for the tetartoidal class, we must construct those elements of odd total degree which together with (9.3) and (9.7) form such a basis. It is apparent from theorem 3 that there are no such additional invariants of degree 3, which involve two vectors only.

Analogously to the case of the diploidal class, we define $P^*_{i_1 i_2 \ldots i_n}$ as the number of linearly independent invariants of odd total degree $i_1 i_2 \ldots i_n$ for the tetartoidal class which are not invariants for the hextetrahedral class. $\vartheta^*_{i_1 i_2 \ldots i_n}$, $R^*_{i_1 i_2 \ldots i_n}$, $\{C^*_{i_1 i_2 \ldots i_n}\}$ are defined in a corresponding manner. Again, in constructing relations of the type (9.1), the A's and B's will, of course, refer only to invariants which are invariant for the tetartoidal class, but not for the hextetrahedral class. In

Table 12

$i_1 i_2 \ldots i_n$	41	32	111	311	221	2111	11111
$P^*_{i_1 i_2 \ldots i_n}$	1	1	1	3	3	6	10
$\vartheta^*_{i_1 i_2 \ldots i_n}$	0	0	0	1	1	4	10
$R^*_{i_1 i_2 \ldots i_n}$	0	0	0	1	1	3	6

Table 12 are given the values of $P^*_{i_1 i_2 \ldots i_n}$ for $i_1 + i_2 + \cdots + i_n = 5$ where at least two of the i's are non-zero and also for $i_1 i_2 \ldots i_n = 111$. We then proceed in the manner described in § 9(a) and obtain the following results.

For $i_1 i_2 \ldots i_n = 41$: $P^*_{41} = 1$, $\vartheta^*_{41} = 0$, hence

$\{C^*_{41}\}$ may be taken as $\sum x_1^3(x_2 y_3 - x_3 y_2)$, $\sum y_1^3(x_2 y_3 - x_3 y_2)$.

For $i_1 i_2 \ldots i_n = 32$: $P^*_{32} = 1$, $\vartheta^*_{32} = 0$, hence

$\{C^*_{32}\}$ may be taken as $\sum x_1^2 y_1(x_2 y_3 - x_3 y_2)$, $\sum y_1^2 x_1(x_2 y_3 - x_3 y_2)$.

For $i_1 i_2 \ldots i_n = 111$: $P^*_{111} = 1$, $\vartheta^*_{111} = 0$, hence

$\{C^*_{111}\} = \sum x_1(y_2 z_3 - y_3 z_2) = \det[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$.

For $i_1 i_2 \ldots i_n = 311$:

$$A_1 = (1,\ 1,\ \ -1)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

where

$$A_1 = \sum x_1(y_2 z_3 - y_3 z_2) \cdot \sum x_1^2, \qquad B_1 = \sum x_1^3(y_2 z_3 - y_3 z_2),$$
$$B_2 = \sum x_1^2 z_1(x_2 y_3 - x_3 y_2), \qquad B_3 = \sum x_1^2 y_1(x_2 z_3 - x_3 z_2),$$

and $\{C^*_{311}\}$ may be taken as

$$\sum x_1^2 z_1(x_2 y_3 - x_3 y_2), \quad \sum y_1^2 x_1(y_2 z_3 - y_3 z_2), \quad \sum z_1^2 y_1(x_2 z_3 - x_3 z_2),$$
$$\sum x_1^2 y_1(x_2 z_3 - x_3 z_2), \quad \sum y_1^2 z_1(x_2 y_3 - x_3 y_2), \quad \sum z_1^2 x_1(y_2 z_3 - y_3 z_2).$$

For $i_1 i_2 \ldots i_n = 221$:

$$A_1 = (1,\ 1,\ \ -1)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

where

$$A_1 = \sum x_1(y_2 z_3 - y_3 z_2) \cdot \sum x_1 y_1, \qquad B_1 = \sum x_1 y_1 z_1(x_2 y_3 - x_3 y_2),$$
$$B_2 = \sum x_1^2 y_1(y_2 z_3 - y_3 z_2), \qquad B_3 = \sum y_1^2 x_1(x_2 z_3 - x_3 z_2)$$

and $\{C^*_{221}\}$ may be taken as

$$\sum x_1^2 y_1(y_2 z_3 - y_3 z_2), \quad \sum y_1^2 z_1(x_2 z_3 - x_3 z_2), \quad \sum z_1^2 x_1(x_2 y_3 - x_3 y_2),$$
$$\sum y_1^2 x_1(x_2 z_3 - x_3 z_2), \quad \sum z_1^2 y_1(x_2 y_3 - x_3 y_2), \quad \sum x_1^2 z_1(y_2 z_3 - y_3 z_2).$$

For $i_1 i_2 \ldots i_n = 2111$:

$$\begin{pmatrix} A_4 \\ A_2 \\ A_3 \\ A_1 - A_2 + A_3 - A_4 \end{pmatrix} = \begin{pmatrix} 1, & -1, & 1, & 0, & 0, & 0 \\ & 1, & 0, & 0, & 1, & -1 \\ & & 1, & 1, & 0, & -1 \\ & & & 0, & 0, & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_4 \\ B_6 \\ B_2 \\ B_3 \\ B_5 \end{pmatrix},$$

where

$$A_1 = \sum x_1(y_2 z_3 - y_3 z_2) \cdot \sum x_1 u_1, \qquad A_2 = \sum x_1(y_2 u_3 - y_3 u_2) \cdot \sum x_1 z_1,$$
$$A_3 = \sum x_1(z_2 u_3 - z_3 u_2) \cdot \sum x_1 y_1, \qquad A_4 = \sum y_1(z_2 u_3 - z_3 u_2) \cdot \sum x_1^2,$$
$$B_1 = \sum x_1^2 u_1(y_2 z_3 - y_3 z_2), \qquad B_2 = \sum x_1 y_1 u_1(x_2 z_3 - x_3 z_2),$$
$$B_3 = \sum x_1 z_1 u_1(x_2 y_3 - x_3 y_2), \qquad B_4 = \sum x_1^2 z_1(y_2 u_3 - y_3 u_2),$$
$$B_5 = \sum x_1 y_1 z_1(x_2 u_3 - x_3 u_2), \qquad B_6 = \sum x_1^2 y_1(z_2 u_3 - z_3 u_2),$$

and $\{C^*_{2111}\}$ may be taken as $\sum x_1y_1u_1(x_2z_3-x_3z_2)$, $\sum x_1z_1u_1(x_2y_3-x_3y_2)$, $\sum x_1y_1z_1(x_2u_3-x_3u_2)$ and the invariants obtained from these by cyclic permutation of x, y, z and u.

For $i_1i_2\dots i_n=11111$:

$$\begin{Bmatrix} A_2 \\ A_1 \\ A_8 \\ A_7 \\ A_6 \\ A_5 \\ A_3-A_8+A_6-A_5 \\ A_4-A_1+A_7-A_5 \\ A_9-A_2+A_7-A_6 \\ A_{10}-A_2+A_1-A_8 \end{Bmatrix} = \begin{bmatrix} 1, & 0, & 0, & 0, & 0, & 0, & -1, & 1, & 0, & 0 \\ & 1, & 0, & 0, & 0, & 0, & -1, & 0, & 1, & 0 \\ & & 1, & 0, & 0, & 0, & 0, & -1, & 1, & 0 \\ & & & 1, & 0, & 0, & -1, & 0, & 0, & 1 \\ & & & & 1, & 0, & 0, & -1, & 0, & 1 \\ & & & & & 1, & 0, & 0, & -1, & 1 \\ & & & & & & 0, & 0, & 0, & 0 \\ & & & & & & & 0, & 0, & 0 \\ & & & & & & & & 0, & 0 \\ & & & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \end{Bmatrix},$$

where

$$\begin{aligned}
A_1 &= \sum x_1(z_2v_3-z_3v_2)\cdot\sum y_1u_1, & A_2 &= \sum x_1(u_2v_3-u_3v_2)\cdot\sum y_1z_1,\\
A_3 &= \sum y_1(z_2u_3-z_3u_2)\cdot\sum x_1v_1, & A_4 &= \sum y_1(z_2v_3-z_3v_2)\cdot\sum x_1u_1,\\
A_5 &= \sum x_1(y_2z_3-y_3z_2)\cdot\sum u_1v_1, & A_6 &= \sum x_1(y_2u_3-y_3u_2)\cdot\sum z_1v_1,\\
A_7 &= \sum x_1(y_2v_3-y_3v_2)\cdot\sum z_1u_1, & A_8 &= \sum x_1(z_2u_3-z_3u_2)\cdot\sum y_1v_1,\\
A_9 &= \sum y_1(u_2v_3-u_3v_2)\cdot\sum x_1z_1, & A_{10} &= \sum z_1(u_2v_3-u_3v_2)\cdot\sum x_1y_1,\\
B_1 &= \sum x_1y_1z_1(u_2v_3-u_3v_2), & B_2 &= \sum x_1y_1u_1(z_2v_3-z_3v_2),\\
B_3 &= \sum x_1y_1v_1(z_2u_3-z_3u_2), & B_4 &= \sum x_1z_1u_1(y_2v_3-y_3v_2),\\
B_5 &= \sum x_1z_1v_1(y_2u_3-y_3u_2), & B_6 &= \sum x_1u_1v_1(y_2z_3-y_3z_2),\\
B_7 &= \sum y_1z_1u_1(x_2v_3-x_3v_2), & B_8 &= \sum y_1z_1v_1(x_2u_3-x_3u_2),\\
B_9 &= \sum y_1u_1v_1(x_2z_3-x_3z_2), & B_{10} &= \sum z_1u_1v_1(x_2y_3-x_3y_2)
\end{aligned}$$

and $\{C^*_{11111}\}$ may be taken as $\sum y_1z_1u_1(x_2v_3-x_3v_2)$, $\sum y_1z_1v_1(x_2u_3-x_3u_2)$, $\sum y_1u_1v_1(x_2z_3-x_3z_2)$, $\sum z_1u_1v_1(x_2y_3-x_3y_2)$.

Thus, an integrity basis is formed by the quantities (9.3), (9.7) and the $\{C^*_{i_1i_2\dots i_n}\}$ given above. It remains to show that none of these quantities is redundant.

It is apparent, by inspection, that none of the elements of (9.3) and (9.7) can be expressed as a polynomial in the remaining elements of (9.3) and (9.7). It is also clear from the manner of derivation that none of the $\{C^*_{i_1i_2\dots i_n}\}$ is redundant. It is necessary however to verify that the products of elements of degree three from (9.7) and the $\{C^*_{i_1i_2\dots i_n}\}$ may not be used to eliminate elements of degree six from (9.3). We are assured of this by identities of the form

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}\cdot\begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix} = \begin{vmatrix} \sum x_1u_1, & \sum y_1u_1, & \sum z_1u_1 \\ \sum x_1v_1, & \sum y_1v_1, & \sum z_1v_1 \\ \sum x_1w_1, & \sum y_1w_1, & \sum z_1w_1 \end{vmatrix}$$

and

$$3\sum x_1(y_2z_3-y_3z_2)\cdot\sum u_1(v_2w_3+v_3w_2)=\sum x_1u_1(y_2v_2-y_3v_3)\cdot\sum z_1w_1$$
$$-\sum x_1u_1(z_2w_2-z_3w_3)\cdot\sum y_1v_1-\sum z_1w_1(v_2y_2-v_3y_3)\cdot\sum x_1u_1-$$
$$-\sum x_1w_1(z_2u_2-z_3u_3)\cdot\sum y_1v_1+\sum x_1w_1(y_2v_2-y_3v_3)\cdot\sum z_1u_1-$$
$$-\sum z_1u_1(v_2y_2-v_3y_3)\cdot\sum x_1w_1+\sum x_1w_1(y_2u_2-y_3u_3)\cdot\sum z_1v_1-$$
$$-\sum x_1w_1(z_2v_2-z_3v_3)\cdot\sum y_1u_1+\sum y_1u_1(z_2v_2-z_3v_3)\cdot\sum x_1w_1+$$
$$+\sum x_1u_1(y_2w_2-y_3w_3)\cdot\sum z_1v_1-\sum x_1u_1(z_2v_2-z_3v_3)\cdot\sum y_1w_1+$$
$$+\sum y_1w_1(z_2v_2-z_3v_3)\cdot\sum x_1u_1+\sum x_1v_1(y_2u_2-y_3u_3)\cdot\sum z_1w_1-$$
$$-\sum x_1v_1(z_2w_2-z_3w_3)\cdot\sum y_1u_1+\sum y_1u_1(z_2w_2-z_3w_3)\cdot\sum x_1v_1+$$
$$+\sum x_1v_1(y_2w_2-y_3w_3)\cdot\sum z_1u_1-\sum x_1v_1(z_2u_2-z_3u_3)\cdot\sum y_1w_1+$$
$$+\sum y_1w_1(z_2u_2-z_3u_3)\cdot\sum x_1v_1.$$

Thus, the invariants of the form $\sum x_1(y_2z_3-y_3z_2)\cdot\sum u_1(v_2w_3+v_3w_2)$ and $\sum x_1(y_2z_3-y_3z_2)\cdot\sum u_1(v_2w_3-v_3w_2)$ are seen to be expressible as products of invariants of degrees two and four from the sets (9.3) and (9.7) and hence may not be used to eliminate invariants of degree six from (9.3).

Summarizing the conclusions reached above, we see that an irreducible integrity basis for the vectors $\boldsymbol{A}^{(r)}$ $(r=1,\ldots,n)$ for the tetartoidal class is formed by the invariants (9.3), (9.7) and

$$\begin{aligned}
&\sum x_1^3(x_2y_3-x_3y_2), && \sum x_1^2y_1(x_2y_3-x_3y_2),\\
&\sum y_1^2x_1(x_2y_3-x_3y_2), && \sum y_1^3(x_2y_3-x_3y_2);\\
&\sum x_1(y_2z_3-y_3z_2)=\det[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]; &&\\
&\sum x_1^2y_1(x_2z_3-x_3z_2), && \sum x_1^2z_1(x_2y_3-x_3y_2),\\
&\sum x_1^2y_1(y_2z_3-y_3z_2), && \sum y_1^2x_1(x_2z_3-x_3z_2)
\end{aligned}\tag{9.10}$$

together with the invariants obtained from these latter invariants by cyclic permutation of x, y and z;

$$\sum x_1y_1z_1(x_2u_3-x_3u_2),\quad \sum x_1y_1u_1(x_2z_3-x_3z_2),\quad \sum x_1z_1u_1(x_2y_3-x_3y_2)$$

together with the invariants obtained from these by cyclic permutation of x, y, z and u;

$$\begin{aligned}
&\sum y_1z_1u_1(x_2v_3-x_3v_2), && \sum y_1z_1v_1(x_2u_3-x_3u_2),\\
&\sum y_1u_1v_1(x_2z_3-x_3z_2), && \sum z_1u_1v_1(x_2y_3-x_3y_2).
\end{aligned}$$

10. Integrity bases for absolute and polar vectors

For a proper orthogonal transformation, the transformation law for absolute and polar vectors are the same and consequently for the non-centrosymmetric groups considered (subgroups of the proper orthogonal group) the irreducible integrity bases obtained for absolute vectors are also irreducible integrity bases if some or all of the vectors are polar vectors.

Consider a centrosymmetric group $\{\mathscr{G}_2\}$, the transformations of which consist of those of some non-centrosymmetric group $\{\mathscr{G}_1\}$, together with the central inversion transformation and its products with the transformations of $\{\mathscr{G}_1\}$.

Let $\boldsymbol{A}_r$ $(r=1, 2, \ldots, n)$ and $\boldsymbol{B}_s$ $(s=1, 2, \ldots, m)$ be absolute and polar vectors respectively. Suppose that $I_1, I_2, \ldots, I_N$ and $J_1, J_2, \ldots, J_M$ are the elements of an irreducible integrity basis for these vectors under the group $\{\mathscr{G}_1\}$ of even and odd degrees respectively in the absolute vectors. Then $J_1, J_2, \ldots, J_M$ change sign under the central inversion transformation, while $I_1, I_2, \ldots, I_N$ are invariant under the central inversion transformation. Applying theorems 1 and 4, we see that an integrity basis for the vectors $\boldsymbol{A}_r$ $(r=1, 2, \ldots, n)$ and $\boldsymbol{B}_s$ $(s=1, 2, \ldots, m)$ under the group $\{\mathscr{G}_2\}$ is formed by $I_1, I_2, \ldots, I_N$ and $J_P J_Q$ $(P, Q=1, 2, \ldots, M;$ $P \leqq Q)$. From this integrity basis, an irreducible integrity basis can be derived by eliminating redundant elements by methods generally similar to those used in deriving irreducible integrity bases for absolute vectors. Also, the irreducibility of the integrity bases can be demonstrated by methods similar to those used earlier in this paper.

11. The anisotropic tensors

We define an anisotropic tensor with respect to a group $\{\mathscr{G}\}$ as a tensor the components of which are unaltered by each transformation of the group. It has been seen in a previous paper (Smith & Rivlin (1957)) that for any given group $\{\mathscr{G}\}$, there exists a finite number of anisotropic tensors in terms of which any anisotropic tensor may be expressed as a sum of outer products with scalar coefficients. This set of anisotropic tensors may be called a *tensor basis* for the group. If any element of the tensor basis may be expressed as a sum of outer products of other elements with scalar coefficients, it may be omitted from the basis and a tensor basis which is such that no element is expressible in this way, in terms of the remaining elements, is called an *irreducible tensor basis.*

It has been shown in the previous paper how a tensor basis for any group may be obtained if the integrity basis for an arbitrary number of vectors is known for that group. Let $I_1, I_2, \ldots, I_M$ be the elements of the integrity basis for the N vectors $A_i^{(R)}$ $(R=1, 2, \ldots, N)$ under the group $\{\mathscr{G}\}$, which are multilinear in their argument vectors. Then, a tensor basis is formed by the tensors

$$\frac{\partial^{\mu} I_R}{\partial A_i^{(R_1)}\, \partial A_j^{(R_2)} \ldots \partial A_k^{(R_\mu)}},$$

where I_R is multilinear in the μ vectors $A_i^{(R_1)}, A_i^{(R_2)}, \ldots, A_i^{(R_\mu)}$. If the set of invariants $I_1, I_2, \ldots, I_M$ is such that none of them is expressible as a polynomial in terms of the remainder, it is easily seen that the tensor basis generated is an irreducible one. For, suppose the tensor $\partial^{\mu} I_R / \partial A_i^{(R_1)}\, \partial A_j^{(R_2)} \ldots \partial A_k^{(R_\mu)}$ is expressible as a sum of outer products of other elements of the tensor basis, with scalar coefficients. Then, it is apparent that

$$\frac{\partial^{\mu} I_R}{\partial A_i^{(R_1)}\, \partial A_j^{(R_2)} \ldots \partial A_k^{(R_\mu)}} \quad A_i^{(R_1)} A_j^{(R_2)} \ldots A_k^{(R_\mu)} = I_R$$

is expressible as a polynomial in invariants other than I_R of the set $I_1, I_2, \ldots, I_M$.

The procedure described above has been used to obtain an irreducible tensor basis for each of the 31 transformation groups for which irreducible integrity bases for N vectors have been obtained. The tensor bases obtained are listed below.

Triclinic system

Pedial: $\delta_{\alpha i}$ $(\alpha = 1, 2, 3)$,

Pinacoidal: $\delta_{\alpha i}\delta_{\beta j}$ $(\alpha, \beta = 1, 2, 3)$.

Monoclinic system

Domatic: $\delta_{1i}\delta_{1j}$, δ_{2i}, δ_{3i},

Sphenoidal: δ_{1i}, $\delta_{\alpha i}\delta_{\beta j}$ $(\alpha, \beta = 2, 3)$,

Prismatic: $\delta_{1i}\delta_{1j}$, $\delta_{\alpha i}\delta_{\beta j}$ $(\alpha, \beta = 2, 3)$.

Rhombic system

Rhombic-pyramidal: δ_{1i}, $\delta_{2i}\delta_{2j}$, $\delta_{3i}\delta_{3j}$,

Rhombic-disphenoidal: $\delta_{\alpha i}\delta_{\alpha j}$, $\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k}$ $(\alpha, \beta, \gamma = 1, 2, 3;\ \alpha \neq \beta \neq \gamma \neq \alpha)$,

Rhombic-dipyramidal: $\delta_{\alpha i}\delta_{\alpha j}$ $(\alpha = 1, 2, 3)$.

Tetragonal system

Tetragonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{2j} - \delta_{2i}\delta_{1j}$, $\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{2l} - \delta_{2i}\delta_{2j}\delta_{2k}\delta_{1l}$,

Tetragonal-disphenoidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{2j} - \delta_{2i}\delta_{1j}$, $\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{2l} - \delta_{2i}\delta_{2j}\delta_{2k}\delta_{1l}$, $\delta_{3i}(\delta_{1j}\delta_{1k} - \delta_{2j}\delta_{2k})$,
$\delta_{3j}(\delta_{1k}\delta_{1i} - \delta_{2k}\delta_{2i})$, $\delta_{3k}(\delta_{1i}\delta_{1j} - \delta_{2i}\delta_{2j})$,
$\delta_{3i}(\delta_{1j}\delta_{2k} + \delta_{2j}\delta_{1k})$, $\delta_{3j}(\delta_{1k}\delta_{2i} + \delta_{2k}\delta_{1i})$,
$\delta_{3k}(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j})$,

Tetragonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{2j} - \delta_{2i}\delta_{1j}$, $\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{2l} - \delta_{2i}\delta_{2j}\delta_{2k}\delta_{1l}$,

Ditetragonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$,

Tetragonal-scalenohedral: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$, $\delta_{3i}(\delta_{1j}\delta_{2k} + \delta_{2j}\delta_{1k})$,
$\delta_{3j}(\delta_{1k}\delta_{2i} + \delta_{2k}\delta_{1i})$, $\delta_{3k}(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j})$,

Ditetragonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$,

Tetragonal-trapezohedral: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}$, $\delta_{3i}(\delta_{1j}\delta_{2k} - \delta_{2j}\delta_{1k})$
and tensors obtained by cyclic permutation of i, j, k,
$\delta_{3i}(\delta_{1j}\delta_{1k}\delta_{1l}\delta_{2m} - \delta_{2j}\delta_{2k}\delta_{2l}\delta_{1m})$
and tensors obtained by cyclic permutation of i, j, k, l, m.

Hexagonal system

Trigonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$, $\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j}$,
$\delta_{2i}\delta_{2j}\delta_{2k}-\delta_{2i}\delta_{1j}\delta_{1k}-\delta_{2j}\delta_{1k}\delta_{1i}-\delta_{2k}\delta_{1i}\delta_{1j}$,

Trigonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j}$, $\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j}$,
$\delta_{2i}\delta_{2j}\delta_{2k}-\delta_{2i}\delta_{1j}\delta_{1k}-\delta_{2j}\delta_{1k}\delta_{1i}-\delta_{2k}\delta_{1i}\delta_{1j}$,

Hexagonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$, $\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j}$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{1l}\delta_{1m}\delta_{1n}-\delta_{1l}\delta_{2m}\delta_{2n}-\delta_{1m}\delta_{2n}\delta_{2l}-\delta_{1n}\delta_{2l}\delta_{2m})$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{2l}\delta_{2m}\delta_{2n}-\delta_{2l}\delta_{1m}\delta_{1n}-\delta_{2m}\delta_{1n}\delta_{1l}-\delta_{2n}\delta_{1l}\delta_{1m})$,

Hexagonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$, $\delta_{1i}\delta_{2j}-\delta_{2i}\delta_{1j}$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{1l}\delta_{1m}\delta_{1n}-\delta_{1l}\delta_{2m}\delta_{2n}-\delta_{1m}\delta_{2n}\delta_{2l}-\delta_{1n}\delta_{2l}\delta_{2m})$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{2l}\delta_{2m}\delta_{2n}-\delta_{2l}\delta_{1m}\delta_{1n}-\delta_{2m}\delta_{1n}\delta_{1l}-\delta_{2n}\delta_{1l}\delta_{1m})$,

Rhombohedral: The tensors given for the hexagonal-dipyramidal class,
$\delta_{3i}(\delta_{1j}\delta_{1k}\delta_{1l}-\delta_{1j}\delta_{2k}\delta_{2l}-\delta_{1k}\delta_{2l}\delta_{2j}-\delta_{1l}\delta_{2j}\delta_{2k})$
and $\delta_{3i}(\delta_{2j}\delta_{2k}\delta_{2l}-\delta_{2j}\delta_{1k}\delta_{1l}-\delta_{2k}\delta_{1l}\delta_{1j}-\delta_{2l}\delta_{1j}\delta_{1k})$
and tensors obtained from these by cyclic permutation of i, j, k and l.

Ditrigonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$\delta_{2i}\delta_{2j}\delta_{2k}-\delta_{2i}\delta_{1j}\delta_{1k}-\delta_{2j}\delta_{1k}\delta_{1i}-\delta_{2k}\delta_{1i}\delta_{1j}$,

Ditrigonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$\delta_{2i}\delta_{2j}\delta_{2k}-\delta_{2i}\delta_{1j}\delta_{1k}-\delta_{2j}\delta_{1k}\delta_{1i}-\delta_{2k}\delta_{1i}\delta_{1j}$,

Trigonal-trapezohedral: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j}$, $\delta_{3i}(\delta_{1j}\delta_{2k}-\delta_{2j}\delta_{1k})$
and the tensors obtained from this by cyclic permutation of i, j, k,
$\delta_{3i}(\delta_{2j}\delta_{2k}\delta_{2l}-\delta_{2j}\delta_{1k}\delta_{1l}-\delta_{2k}\delta_{1l}\delta_{1j}-\delta_{2l}\delta_{1j}\delta_{1k})$
and the tensors obtained from this by cyclic permutation of i, j, k, l.

Dihexagonal-pyramidal: δ_{3i}, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{1l}\delta_{1m}\delta_{1n}-\delta_{1l}\delta_{2m}\delta_{2n}-\delta_{1m}\delta_{2n}\delta_{2l}-\delta_{1n}\delta_{2l}\delta_{2m})$,

Dihexagonal-dipyramidal: $\delta_{3i}\delta_{3j}$, $\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}$,
$(\delta_{1i}\delta_{1j}\delta_{1k}-\delta_{1i}\delta_{2j}\delta_{2k}-\delta_{1j}\delta_{2k}\delta_{2i}-\delta_{1k}\delta_{2i}\delta_{2j})\times$
$\times(\delta_{1l}\delta_{1m}\delta_{1n}-\delta_{1l}\delta_{2m}\delta_{2n}-\delta_{1m}\delta_{2n}\delta_{2l}-\delta_{1n}\delta_{2l}\delta_{2m})$,

Hexagonal-scalenohedral: The tensors given for the dihexagonal-dipyramidal class together with

$\delta_{3i}(\delta_{2j}\delta_{2k}\delta_{2l}-\delta_{2j}\delta_{1k}\delta_{1l}-\delta_{2k}\delta_{1l}\delta_{1j}-\delta_{2l}\delta_{1j}\delta_{1k})$

and tensors obtained from this by cyclic permutation of i, j, k, l.

Hexagonal-trapezohedral: The tensors given for the dihexagonal-dipyramidal class together with

$\delta_{3i}(\delta_{1j}\delta_{2k}-\delta_{2j}\delta_{1k})$, $\delta_{3i}(\delta_{1j}\delta_{1k}\delta_{1l}-\delta_{1j}\delta_{2k}\delta_{2l}-\delta_{1k}\delta_{2l}\delta_{2j}-$
$-\delta_{1l}\delta_{2j}\delta_{2k})(\delta_{2m}\delta_{2n}\delta_{2p}-\delta_{2m}\delta_{1n}\delta_{1p}-\delta_{2n}\delta_{1p}\delta_{1m}-\delta_{2p}\delta_{1m}\delta_{1n})$

and tensors obtained from these by cyclic permutation of i, j, k and i, j, k, l, m, n, p respectively.

Cubic system

Hexoctahedral class: δ_{ij}, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}$, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}\delta_{1m}\delta_{1n}$,

Diploidal class: δ_{ij}, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}$, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}\delta_{1m}\delta_{1n}$,
$\sum\delta_{1i}\delta_{1j}(\delta_{2k}\delta_{2l}-\delta_{3k}\delta_{3l})$, $\sum\delta_{1i}\delta_{1k}(\delta_{2j}\delta_{2l}-\delta_{3j}\delta_{3l})$,
$\sum\delta_{1i}\delta_{1l}(\delta_{2j}\delta_{2k}-\delta_{3j}\delta_{3k})$, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}(\delta_{2m}\delta_{2n}-\delta_{3m}\delta_{3n})$,

Hextetrahedral: δ_{ij}, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}$, $\sum\delta_{1i}(\delta_{2j}\delta_{3k}+\delta_{3j}\delta_{2k})$,

Tetartoidal: δ_{ij}, $\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}$, $\sum\delta_{1i}\delta_{1j}(\delta_{2k}\delta_{2l}-\delta_{3k}\delta_{3l})$,
$\sum\delta_{1i}\delta_{1k}(\delta_{2j}\delta_{2l}-\delta_{3j}\delta_{3l})$, $\sum\delta_{1i}\delta_{1l}(\delta_{2j}\delta_{2k}-\delta_{3j}\delta_{3k})$,
$\sum\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}(\delta_{2m}\delta_{2n}-\delta_{3m}\delta_{3n})$, $\sum\delta_{1i}(\delta_{2j}\delta_{3k}-\delta_{2k}\delta_{3j})$,
$\sum\delta_{1j}\delta_{1k}\delta_{1l}(\delta_{2i}\delta_{3m}-\delta_{3i}\delta_{2m})$, $\sum\delta_{1j}\delta_{1k}\delta_{1m}(\delta_{2i}\delta_{3l}-\delta_{3i}\delta_{2l})$,
$\sum\delta_{1j}\delta_{1l}\delta_{1m}(\delta_{2i}\delta_{3k}-\delta_{3i}\delta_{2k})$, $\sum\delta_{1k}\delta_{1l}\delta_{1m}(\delta_{2i}\delta_{3j}-\delta_{3i}\delta_{2j})$,
$\sum\delta_{1i}(\delta_{2j}\delta_{3k}+\delta_{3j}\delta_{2k})$.

12. The use of the tensor bases in the generation of integrity bases

Let $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ be an irreducible tensor basis for the group $\{\mathscr{G}\}$ which is the orthogonal group $\{\mathscr{O}\}$ or a subgroup of it. We shall prove that if I is a polynomial invariant of the tensors $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ and $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ under the orthogonal group $\{\mathscr{O}\}$, it is an invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ under the group $\{\mathscr{G}\}$.

Let $\boldsymbol{G}$ be a generic transformation of the group $\{\mathscr{G}\}$. Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ be transformed into $\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2, \ldots, \bar{\boldsymbol{A}}_R$ (say) by the transformation $\boldsymbol{G}$. Since $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ are anisotropic tensors with respect to the transformation $\boldsymbol{G}$, they are unchanged by the transformation. Since $\boldsymbol{G}$ is a transformation of the group $\{\mathscr{O}\}$, it follows from the fact that I is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ and $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ under the orthogonal group, that

$$I(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R; \boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N) = I(\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2, \ldots, \bar{\boldsymbol{A}}_R; \boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N).$$

Thus, I is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ under the transformation $\boldsymbol{G}$ and hence under the group $\{\mathscr{G}\}$.

We shall now prove the converse theorem: if I is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ under the group $\{\mathscr{G}\}$, it is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ and $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ under the group $\{\mathscr{O}\}$.

If I is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ under the group $\{\mathscr{G}\}$, it must be expressible as the sum of a number of terms of the form

$$\gamma \, \alpha_{i_1 i_2 \ldots i_P} \beta_{i_1 i_2 \ldots i_P},$$

where $\beta_{i_1 i_2 \ldots i_P}$ is an outer product formed from the tensors $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$, and $\alpha_{i_1 i_2 \ldots i_P}$ is an outer product formed from the tensors $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ and γ is a constant. It is easy to see that each such term and hence I is a polynomial invariant of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_R$ and $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_N$ under the group $\{\mathscr{O}\}$. Let

$$\bar{\alpha}_{i_1 i_2 \ldots i_P} = t_{i_1 j_1} t_{i_2 j_2} \ldots t_{i_P j_P} \alpha_{j_1 j_2 \ldots j_P}$$

$$\bar{\beta}_{i_1 i_2 \ldots i_P} = t_{i_1 j_1} t_{i_2 j_2} \ldots t_{i_P j_P} \beta_{j_1 j_2 \ldots j_P}$$

where $\|t_{ij}\|$ is a generic transformation of the group $\{\mathscr{O}\}$. Then, since $t_{ik} t_{jk} = t_{ki} t_{kj} = \delta_{ij}$, it follows immediately that

$$\bar{\alpha}_{i_1 i_2 \ldots i_P} \bar{\beta}_{i_1 i_2 \ldots i_P} = \alpha_{i_1 i_2 \ldots i_P} \beta_{i_1 i_2 \ldots i_P}.$$

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References

SMITH, G. F.: Arch. Rational Mech. Anal. **5**, 382 (1960).
SMITH, G. F., & R. S. RIVLIN: Q. Appl. Math. **15**, 308 (1957).
SMITH, G. F., M. M. SMITH & R. S. RIVLIN: Arch. Rational Mech. Anal. **12**, 93 (1963).

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