J. Math. Biology (1986) 23: 341-359

Journal of
Mathematical
Biology

The asymptotic speed of propagation of the deterministic non-reducible *n*-type epidemic

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Abstract. A model has been formulated in [7] to describe the spatial spread of an epidemic involving n types of individuals, when triggered by the introduction of infectives from outside. Wave solutions for such a model have been investigated in [5] and [8] and have been shown only to exist at certain speeds. This paper establishes that the asymptotic speed of propagation, as defined in Aronson and Weinberger [1, 2], of such an epidemic is in fact c_0, the minimum speed at which wave solutions exist. This extends the known result for the one-type and host-vector epidemics.

Key words: Non-reducible n-type epidemic — Measles — Host-vector — Carrier-borne — rabies — Spatial spread — Asymptotic speed of propagation — Pandemic theorem

1. Introduction

In our recent papers [5, 7, 8, and 9], deterministic models were formulated to describe the spatial spread of an epidemic involving n types of individual. Special cases of such models include epidemics such as measles, host-vector and carrier-borne epidemics and rabies involving several species of animal.

An epidemic was considered in [7] which was triggered by the introduction of an initial infection from outside. A condition for a major epidemic was obtained, and the final size and pandemic theorems established. Wave solutions were investigated in [5] and [8]; the condition for a major epidemic being established as a necessary condition for the existence of such wave solutions. A further condition was established on the contact distributions. For radial contact distributions this condition was exponential domination in the tail. Under such conditions the existence of a critical speed c_0 was established. Wave solutions were shown to exist and be unique modulo translation at each positive speed $c \geqslant c_0$. No wave solutions were possible at speeds below c_0.

The main purpose of this paper is to show that c_0 is the asymptotic speed of propagation of the epidemic; a result indicated by an approximation obtained in our paper [9]. This extends the results obtained by Diekmann [3] and Thieme

[13] for the one-type epidemic, and by the present authors [10] for the host-vector epidemic. We use the methods of Diekmann [3].

The behaviour when the contact distributions are radial, but not all exponentially dominated in the tail, is briefly considered. The epidemic, if it is major, will then spread too fast to eventually propagate at a finite speed. In addition the pandemic theorem is established for all dimensions, in contrast to [6, 7] where the result was only obtained for one and two dimensions.

2. The model

Consider n populations, the ith of uniform density σ_i in R^N, each population consisting solely of susceptible and infected individuals. The rate of infection of susceptible individuals in population i from infected individuals in population j who were infected time τ ago is $\lambda_{ij}(\tau)$, and the contact distribution representing the distance $\mathbf{r}$ over which infection occurs has density $p_{ij}(\mathbf{r})$. In population i, let $x_i(\mathbf{s}, t)$ and $I_i(\mathbf{s}, t, \tau)\, d\tau$ be the proportions of individuals at position $\mathbf{s}$ and time t who were respectively susceptible, and infected in the time interval $(t-\tau-d\tau, t-\tau)$. Note that $I_i(\mathbf{s}, t, \tau) = I_i(\mathbf{s}, t-\tau, 0)$.

Infected individuals of k types are introduced from outside at time $t=0$, and the spread of infection through the n populations caused by these infected individuals is studied. Let $\varepsilon_j(\mathbf{s}, \tau)\, d\mathbf{s}\, d\tau$ be the number of such individuals of type j in the region $(\mathbf{s}, \mathbf{s}+d\mathbf{s})$ who were infected in the time interval $(-\tau-d\tau, -\tau)$. The rate of infection from such individuals, of susceptibles from population i, is $\lambda^*_{ij}(\tau)$ and the contact distribution has density $p^*_{ij}(\mathbf{r})$. Let $\varepsilon_i(\mathbf{s}) = \int_0^\infty \varepsilon_i(\mathbf{s}, \tau)\, d\tau$ and $\varepsilon_i = \int_{R^N} \varepsilon_i(\mathbf{s})\, d\mathbf{s}$ exist.

The model is then described by the equations:

$$\left.\begin{aligned} \frac{\partial x_i(\mathbf{s}, t)}{\partial t} &= -x_i(\mathbf{s}, t)\left\{\sum_{j=1}^{n} \sigma_j \int_{R^N} \int_0^t I_j(\mathbf{s}-\mathbf{r}, t, \tau) p_{ij}(\mathbf{r}) \lambda_{ij}(\tau)\, d\tau\, d\mathbf{r} + h_i(\mathbf{s}, t)\right\} \\ &\text{and} \\ I_i(\mathbf{s}, t, 0) &= -\frac{\partial x_i(\mathbf{s}, t)}{\partial t}, \quad \text{for } i = 1, \ldots, n, \\ &\text{where} \\ h_i(\mathbf{s}, t) &= \sum_{j=1}^{k} \int_{R^N} \int_0^\infty \varepsilon_j(\mathbf{s}-\mathbf{r}, \tau) p^*_{ij}(\mathbf{r}) \lambda^*_{ij}(t+\tau)\, d\tau\, d\mathbf{r}. \end{aligned}\right\} \quad (1)$$

The initial conditions are $x_i(\mathbf{s}, 0) \equiv 1$, for $i = 1, \ldots, n$.

Any solution to (1) is equivalent to a solution of the following equations:

$$\frac{\partial x_i(\mathbf{s}, t)}{\partial t} = x_i(\mathbf{s}, t)\left\{\sum_{j=1}^{k} \int_{R^N} \int_0^t \frac{\partial x_j(\mathbf{s}-\mathbf{r}, t-\tau)}{\partial t} p_{ij}(\mathbf{r}) \gamma_{ij}(\tau)\, d\tau\, d\mathbf{r} - h_i(\mathbf{s}, t)\right\}, \quad i = 1, \ldots, n, \quad (2)$$

where $\gamma_{ij}(\tau) = \sigma_j \lambda_{ij}(\tau)$ and $x_i(\mathbf{s}, t)$ is monotone decreasing in t with $x_i(\mathbf{s}, 0) \equiv 1$.

An n-type model with constant infectivities λ_{ij} and λ^*_{ij} and removal rates μ_j and μ^*_j can be postulated. This is an obvious extension of model 1 for the host-vector epidemic given in our papers [6] and [11]. This leads to a special case of the model considered in this paper with $\lambda_{ij}(\tau)=\lambda_{ij}\exp(-\mu_j t)$ and $\lambda^*_{ij}(\tau)=\lambda^*_{ij}\exp(-\mu^*_j t)$.

Conditions are now imposed on the rates of infection and the contact distributions. The contact distributions $p_{ij}(\mathbf{r})$ and $p^*_{ij}(\mathbf{r})$ are restricted to be bounded, continuous radial functions in R^N. The rates of infection $\lambda_{ij}(\tau)$ and $\lambda^*_{ij}(\tau)$ are taken to be bounded with bounded derivatives. In addition $\gamma^*_{ij}=\int_0^\infty \lambda^*_{ij}(\tau)\,d\tau$ is taken to be finite, as we are interested in the development of an epidemic within the n populations which is only triggered by the introduction of infectives from outside.

Define $w_i(\mathbf{s},t)=-\log x_i(\mathbf{s},t)$. In our paper [9] the solutions $x_i(\mathbf{s},t)$ of (2) were related to the solutions $w_i(\mathbf{s},t)$ of the following equations:

$$w_i(\mathbf{s},t)=\sum_{j=1}^{n}\int_{R^N}\int_0^t (1-\exp\{-w_j(\mathbf{s}-\mathbf{r},t-\tau)\})\gamma_{ij}(\tau)p_{ij}(\mathbf{r})\,d\tau\,d\mathbf{r}+H_i(\mathbf{s},t) \quad (3)$$

where $H_i(\mathbf{s},t)=\int_0^t h_i(\mathbf{s},w)\,dw$, for $i=1,\ldots,n$. The lemma relating these solutions is stated below.

Lemma 1. *Every non-negative solution $x_i(\mathbf{s},t)$, for $i=1,\ldots,n$, of* (2) *which is monotone decreasing in t and has $x_i(\mathbf{s},0)\equiv 1$, is continuous in t uniformly with respect to $\mathbf{s}$ for $t\in[0,\infty)$. There is a one to one correspondence between such solutions $x_i(\mathbf{s},t)$ of* (2) *and the solutions $w_i(\mathbf{s},t)$ of* (3), *which are non-negative monotone increasing in t with $w_i(\mathbf{s},0)\equiv 0$, given by the relation $w_i(\mathbf{s},t)=-\log x_i(\mathbf{s},t)$.*

The following lemma may also easily be established. We omit the proof.

Lemma 2. *Every non-negative solution $w_i(\mathbf{s},t)$ of* (3), *which is monotone increasing in t with $w_i(\mathbf{s},0)\equiv 0$, is continuous in $\mathbf{s}$ for each $t\in[0,\infty)$.*

We use the same matrix and vector notation as in our paper [5], and define $\rho(\mathbf{A})$, for a non-negative, square matrix $\mathbf{A}$. If $\mathbf{A}$ is finite then $\rho(\mathbf{A})$ is the maximum of the moduli of the eigenvalues of $\mathbf{A}$. When $\mathbf{A}$ is non-reducible, with at least one infinite element, $\rho(\mathbf{A})=\infty$. Let $\mathbf{\Gamma}=(\gamma_{ij})$, where $\gamma_{ij}=\int_0^\infty \gamma_{ij}(\tau)\,d\tau$, which may be infinite. The matrix $\mathbf{\Gamma}$ is taken to be non-reducible.

In our papers [5] and [7] it was shown that a major epidemic is only possible if $\rho(\mathbf{\Gamma})>1$. If a major epidemic occurs, then wave solutions are only possible if the $p_{ij}(\mathbf{r})$ are exponentially dominated in the tail; i.e. for some positive real λ, $\int_{R^N} p_{ij}(\mathbf{r})\exp(\lambda\{\mathbf{r}\}_1)\,dr$ is finite, where $\{\mathbf{r}\}_1$ is the first component of $\mathbf{r}$. In Sect. 3 we show that, with these restrictions on $\mathbf{\Gamma}$ and the $p_{ij}(\mathbf{r})$, and with $\varepsilon_j(\mathbf{r})$ and $p^*_{ij}(\mathbf{r})$ restricted so that the continuing effect of the infectives from outside does not dominate the ultimate behaviour of the epidemic, there is a finite speed of propagation. This speed is the minimal speed at which a wave solution exists. The case $\rho(\mathbf{\Gamma})\leq 1$, when there is no major epidemic, corresponds to zero speed of propagation.

In Sect. 4 the case is briefly considered when the $p_{ij}(\mathbf{r})$ are not all exponentially dominated in the tail. Essentially if $\rho(\mathbf{\Gamma})>1$, the epidemic will spread too fast for wave solutions to exist and for the epidemic eventually to propagate at a finite speed. This may be considered as corresponding to an infinite speed of propagation. The case $\rho(\mathbf{\Gamma})\leq 1$ again corresponds to zero speed of propagation.

Finally, in Sect. 5 the pandemic theorem is established for all dimensions N. Note that in our paper [7], when $\mathbf{\Gamma}$ is finite with $\rho(\mathbf{\Gamma})>1$, the pandemic theorem was only established for dimensions $N=1$ and 2.

3. The asymptotic speed of propagation when the contact distributions are exponentially dominated in the tail

We restrict each $p_{ij}(\mathbf{r})$ so that $P_{ij}(\lambda)=\int_{R^N}\exp(\lambda\{\mathbf{r}\}_1)p_{ij}(\mathbf{r})\,d\mathbf{r}$ exists for some positive real λ. Let Δ_v be the minimum of the abscissae of convergence of the $P_{ij}(\lambda)$. Define $V_{ij}(\lambda)=P_{ij}(\lambda)\Lambda_{ij}(c\lambda)$ and $\{\mathbf{V}(\lambda)\}_{ij}=V_{ij}(\lambda)$, where $\Lambda_{ij}(\lambda)=\int_0^\infty e^{-\lambda\tau}\gamma_{ij}(\tau)\,d\tau$. Then $K_c(\lambda)=\rho(\mathbf{V}(\lambda))$ exists for all λ real with $0<\lambda<\Delta_v$ and $K_c(0)=\rho(\mathbf{\Gamma})$, which may be infinite. In our paper [5] it was shown that no wave solution exists at any speed if $\rho(\mathbf{\Gamma})\leq 1$. When $\rho(\mathbf{\Gamma})>1$, a critical speed c_0 was defined by $c_0=\inf\{c\in R_+\colon K_c(\lambda)<1$ for some $\lambda\in(0,\Delta_v)\}$. It was shown that, provided $c_0\neq 0$, c_0 is the minimum speed at which wave solutions exist. Lemma 3 shows that the conditions restricting the $p_{ij}(\mathbf{r})$ to be radial functions ensures that $c_0>0$.

Lemma 3. *If* $\rho(\mathbf{\Gamma})>1$, *then* $c_0>0$.

Proof. If Δ_v is infinite, since $\mathbf{\Gamma}$ is non-reducible, there exists a distinct sequence $i_1,\ldots,i_{m-1}$, for some $1\leq m-1\leq n$, such that $\gamma_{i_j i_{j+1}}\neq 0$ for $j=1,\ldots,(m-1)$, where $i_m\equiv i_1$. Hence, as in the proof of Lemma 3 part (i) of our paper [5], $K_c(\lambda)\geq\{\prod_{s=1}^{m-1}V_{i_s,i_{s+1}}(\lambda)\}^{1/(m-1)}$. Let $\gamma(\tau)$ be the convolution of $\gamma_{i_j,i_{j+1}}(\tau)$ for $j=1,(m-1)$ and $p(x)$ be a similar convolution of $\tilde{p}_{i_j,i_{j+1}}(x)$, where $\tilde{p}_{i_j,i_{j+1}}(\{\mathbf{r}\}_1)=\int_{R^{N-1}}p_{ij}(\mathbf{r})\,d\{\mathbf{r}\}_2\,d\{\mathbf{r}\}_3\cdots d\{\mathbf{r}\}_N$. As the $p_{ij}(\mathbf{r})$ are radial functions, each $\tilde{p}_{ij}(x)$ and hence $p(x)$ is symmetric about $x=0$. Also $\int_0^\infty\gamma(\tau)\,d\tau=\prod_{j=1}^{m-1}\gamma_{i_j,i_{j+1}}\neq 0$. Hence there exist reals a, b, A, B, γ^* and p^* with $0\leq a<b$, $0<A<B$, $\gamma^*>0$ and $p^*>0$ such that $\gamma(\tau)\geq\gamma^*$ for $\tau\in[a,b]$ and $p(x)\geq p^*$ for $x\in[A,B]$. Hence, for λ real and positive,

$$\begin{aligned}\{K_c(\lambda)\}^m&\geq p^*\gamma^*(\{e^{\lambda B}-e^{\lambda A}\}/\lambda)(\{e^{-c\lambda a}-e^{-c\lambda b}\}/\{c\lambda\})\\&\geq p^*\gamma^*(b-a)(B-A)\,e^{\lambda(A-cb)}.\end{aligned}$$

Thus for $0<c^*<A/b$, $\lim_{\lambda\to\infty}K_{c^*}(\lambda)=\infty$. Take such a c^*. Then there exists a finite positive λ^* such that $K_{c^*}(\lambda)>1$ for $\lambda>\lambda^*$. From the properties of $K_c(\lambda)$ given in [5], it then immediately follows that $K_c(\lambda)>1$ for $\lambda>\lambda^*$ and $0<c\leq c^*$.

If Δ_v is finite and $P_{ij}(\Delta_v)$ is infinite for some i,j (with $\gamma_{ij}\neq 0$), then $\lim_{\lambda\uparrow\Delta_v}K_c(\lambda)$ is infinite for all $c>0$. If Δ_v is finite and $P_{ij}(\Delta_v)$ is finite for all i, j such that $\gamma_{ij}\neq 0$, then for each such i, j, $P_{ij}(\Delta_v)>P_{ij}(0)$ since $p_{ij}(\mathbf{r})$ is a radial function. Hence $\lim_{c\downarrow 0}K_c(\Delta_v)\geq\rho(\mathbf{\Gamma})>1$. In both cases these exists a $c>0$ and $\lambda^*>0$ such that $K_c(\lambda)>1$ for $\lambda^*\leq\lambda\leq\Delta_v$ and $0<c\leq c^*$.

Now suppose that for each $c>0$ there exists a $\lambda\in[0,\Delta_v]$ such that $K_c(\lambda)<1$. Then there exists a monotone decreasing sequence $\{c_n\}$, with $c_1\leq c^*$, and a sequence $\{\lambda_n\}$ such that $K_{c_n}(\lambda_n)<1$ for $n=1,2.\ldots,$ and $\lim_{n\to\infty}c_n=0$. Since $c_n\leq c^*$, necessarily $\lambda_n\leq\lambda^*$. Hence there exists a subsequence $n_1,n_2,\ldots$ with $c_{n_j}>0$ and $\lambda_{n_j}\to\lambda$ as $j\to\infty$, where $\lambda\in[0,\lambda^*]$. Hence $K_0(\lambda)\leq 1$. But $P_{ij}(\lambda)\Lambda_{ij}(0)\geq\gamma_{1j}$. Thus $K_0(\lambda)\geq\rho(\mathbf{\Gamma})>1$ and we have obtained a contradiction. Therefore $c_0>0$.

Consider $K_c(\lambda)$. If $\rho(\mathbf{\Gamma})\leq 1$, then $K_c(0)=\rho(\mathbf{\Gamma})\leq 1$. The contact distributions are radial functions and, from the proof of Lemma 4 of our paper [9],

$$K_c'(\lambda)=\sum_i\sum_j V_{ij}'(\lambda)\{\mathrm{Adj}(\rho(\mathbf{V}(\lambda))\mathbf{I}-\mathbf{V}(\lambda))\}_{ij}/\mathrm{trace}\,\mathrm{Adj}(\rho(\mathbf{V}(\lambda))\mathbf{I}-\mathbf{V}(\lambda)).$$

It is then easily established that $K_c'(0)<0$ for all $c>0$. Hence for each $c>0$ and $\lambda^*\in(0,\Delta_v)$ there exists a $\lambda\in(0,\lambda^*)$ such that $K_c(\lambda)<1$.

When $\rho(\mathbf{\Gamma})>1$, from [8] and Lemma 3, there is a unique $\lambda_0\in(0,\Delta_v]$ such that $K_{c_0}(\lambda_0)=1$. For each $c>c_0$ there exists a $\lambda\in(0,\lambda_0)$ such that $K_c(\lambda)<1$.

This suggests the formulation of part (ii) of Theorem 1, from which two corollaries, relating to the cases $\rho(\mathbf{\Gamma})\leq 1$ and $\rho(\mathbf{\Gamma})>1$ respectively, follow immediately. It also suggests conditions to be placed on $\varepsilon_j(\mathbf{r})$ and $p_{ij}^*(\mathbf{r})$. Note that Theorem 1 part (i) has been proved in a slightly different way in our paper [7]. The adapted method of proof is briefly indicated here as it is necessary for the proof of Theorem 1 part (ii).

Let $P_{ij}^*(\lambda)=\int_{R^N}p_{ij}^*(\mathbf{r})\exp(\lambda\{\mathbf{r}\}_1)\,d\mathbf{r}$ and $E_j(\lambda)=\sup_{|\mathbf{r}|\in R^N}\{e^{\lambda|\mathbf{r}|}\varepsilon_j(\mathbf{r})\}$. Then $p_{ij}^*(\mathbf{r})$ and $\varepsilon_j(\mathbf{r})$ are restricted so that there exists a positive real λ^* with $P_{ij}^*(\lambda)$ and $E_j(\lambda)$ finite for all i, j and $\lambda\in(0,\lambda^*)$. If $\rho(\mathbf{\Gamma})>1$, then they are restricted to be finite for $\lambda\in(0,\lambda_0)$. This ensures that the infection from outside does not spread the epidemic faster than it is spread by infection within the n populations of susceptibles. This is consistent with the infection from outside triggering the epidemic but not dominating its behaviour. These conditions are clearly met if $\varepsilon_i(\mathbf{r})$ has finite support and the individuals from outside are of the same types as those within the n populations of susceptibles so that for each i, j, $p_{ij}^*(\mathbf{r})=p_{it}(\mathbf{r})$ for some t.

Theorem 1. (i) *There exists a non-negative, monotone increasing (in t) solution $w_i(\mathbf{s},t)$ to equations* (3) *with $w_i(\mathbf{s},0)\equiv 0$, $(i=1,\ldots,n)$, which is unique.*

(ii) *For any $c^*>0$ such that $K_{c^*}(\lambda)<1$ for some $\lambda\in(0,\Delta_v)$, $\lim_{t\to\infty}\sup\{w_i(\mathbf{s},t): |\mathbf{s}|\geq c^*t\}=0$ for $i=1,\ldots,n$.*

Proof. (i) For any $\rho(\mathbf{\Gamma})$, there exists a $c>0$ and $\lambda\in(0,\Delta_v)$ such that $K_c(\lambda)<1$ and $P_{ij}^*(\lambda)$ and $E_j(\lambda)$ are finite for all i, j. Take such a c and λ and define

$$y_i(\mathbf{s},t)=w_i(\mathbf{s},t)\exp(\lambda(\{\mathbf{s}\}_1-ct)).$$

Let $y_i^{(0)}(\mathbf{s},t)=H_i(\mathbf{s},t)\exp(\lambda(\{\mathbf{s}\}_1-ct))$. Then

$$|y_i^{(0)}(\mathbf{s},t)|\leq\sum_{j=1}^k\gamma_{ij}^*\int_{R^N}[\varepsilon_j(\mathbf{s}-\mathbf{r})\exp(\lambda(\{\mathbf{s}\}_1-\{\mathbf{r}\}_1))][p_{ij}^*(\mathbf{r})\exp(\lambda\{\mathbf{r}\}_1)]\,d\mathbf{r}$$

$$\leq\sum_{j=1}^k\gamma_{ij}^*E_j(\lambda)P_{ij}^*(\lambda).$$

Hence $y_i^{(0)}(\mathbf{s},t)$ is uniformly bounded for $\mathbf{s}\in R^N$ and $t\geq 0$.

Define $y_i^{(m+1)}(\mathbf{s},t)$ recursively for $m=0,1,\ldots$ by

$$\begin{aligned}y_i^{(m+1)}(\mathbf{s},t)=\sum_{j=1}^n\int_0^t\int_{R^N}&[1-\exp\{-y_j^{(m)}(\mathbf{s}-\mathbf{r},t-\tau)\exp[-\lambda(\{\mathbf{s}-\mathbf{r}\}_1-c(t-\tau))]\}]\\&\times[\gamma_{ij}(\tau)\,e^{-\lambda c\tau}][p_{ij}(\mathbf{r})\exp(\lambda\{\mathbf{r}\}_1)]\\&\times[\exp\{\lambda(\{\mathbf{s}-\mathbf{r}\}_1-c(t-\tau))\}]\,d\mathbf{r}\,d\tau+H_i(\mathbf{s},t)\exp(\lambda(\{\mathbf{s}\}_1-ct)).\end{aligned}$$

Let $u_i^{(m)} = \sup|y_i^{(m+1)}(\mathbf{s}, t) - y_i^{(m)}(\mathbf{s}, t)|$, where the sup is taken over $\mathbf{s} \in R^N$ and $t \geq 0$, and define $\{\mathbf{u}^{(m)}\}_i = u_i^{(m)}$. Then

$$\mathbf{u}^{(m+1)} \leq \mathbf{V}(\lambda)\mathbf{u}^{(m)}.$$

Take $\mathbf{a}' > \mathbf{0}'$ to be the left eigenvector corresponding to $\rho(\mathbf{V}(\lambda))$. Then

$$\mathbf{a}'\mathbf{u}^{(m+1)} \leq K_c(\lambda)\mathbf{a}'\mathbf{u}^{(m)}.$$

Hence

$$u_i^{(m+1)} \leq (K_c(\lambda))^{m+1}\mathbf{a}'\mathbf{u}^{(0)}/\{\mathbf{a}\}_i.$$

But $y_i^{(0)}(\mathbf{s}, t)$ is uniformly bounded Hence there exists a positive vector $\mathbf{D}$ such that $y_i^{(0)}(\mathbf{s}, t) \leq \{\mathbf{D}\}_i$. Therefore $u^{(0)} \leq V(\lambda)D$ and

$$u_i^{(m+1)} \leq (K_c(\lambda))^m \mathbf{a}'\mathbf{D}/\{\mathbf{a}\}_i.$$

Since $K_c(\lambda) < 1$ it immediately follows that $y_i^{(m)}(\mathbf{s}, t)$ converges uniformly for $t \geq 0$ and $\mathbf{s} \in R^N$ to a limit $y_i(\mathbf{s}, t)$ which satisfies the following equation:

$$\begin{aligned} y_i(\mathbf{s}, t) = \sum_{j=1}^n \int_0^t \int_{R^N} & [1 - \exp\{-y_j(\mathbf{s}-\mathbf{r}, t-\tau) \exp[-\lambda(\{\mathbf{s}-\mathbf{r}\}_1 - c(t-\tau))]\}] \\ & \times [\gamma_{ij}(\tau)\, e^{-\lambda c\tau}][p_{ij}(\mathbf{r}) \exp(\lambda\{\mathbf{r}\}_1)] \\ & \times [\exp\{\lambda(\{\mathbf{s}-\mathbf{r}\}_1 - c(t-\tau))\}]\, d\mathbf{r}\, d\tau + H_i(\mathbf{s}, t) \exp(\lambda(\{\mathbf{s}\}_1 - ct)). \end{aligned}$$

The uniqueness of $y_i(\mathbf{s}, t)$ follows from a similar contraction argument. This then establishes the existence and uniqueness of $y_i(\mathbf{s}, t)$ and hence of $w_i(\mathbf{s}, t)$. The mononicity of $w_i(\mathbf{s}, t)$ in t is easily verified since each $y_i(\mathbf{s}, t) \exp\{-(\lambda\{\mathbf{s}\}_1 - ct)\}$ can be shown to be monotonic increasing in t.

(ii) Consider any $c^* > 0$ and $\lambda \in (0, \Delta_v)$ such that $K_{c^*}(\lambda) < 1$ and $P_{ij}^*(\lambda)$ and $E_j(\lambda)$ are finite. Since $K_c(\lambda)$ is a continuous function of c for each fixed λ, there exists a positive $c < c^*$ such that $K_c(\lambda) < 1$. Note that if $\rho(\boldsymbol{\Gamma}) > 1$ then necessarily $c > c_0$.

Define $y_i^{(m)}(\mathbf{s}, t)$ as in the proof of part (i) for these values of c and λ. Let $\{\mathbf{y}^{(m)}\}_i = y_i^{(m)} = \sup\{y_i^{(m)}(\mathbf{s}, t)\}$, where the sup is taken over $\mathbf{s} \in R^N$ and $t \geq 0$. From the proof of part (i), $y_i^{(0)} \leq \{\mathbf{D}\}_i$. Note that the constant does not alter if we rotate the co-ordinate axes in R^N. Then

$$\mathbf{y}^{(m+1)} \leq \mathbf{V}(\lambda)\mathbf{y}^{(m)} + \mathbf{y}^{(0)}.$$

Taking $\mathbf{a}'$ as in the proof of part (i), we obtain

$$\mathbf{a}'\mathbf{y}^{(m+1)} \leq K_c(\lambda)\mathbf{a}'\mathbf{y}^{(m)} + \mathbf{a}'\mathbf{y}^{(0)} \leq \mathbf{a}'\mathbf{y}^{(0)}/(1 - K_c(\lambda)) \leq \mathbf{a}'\mathbf{D}/(1 - K_c(\lambda)).$$

Therefore $y_i^{(m+1)} \leq D_i^*$, where $D_i^* = \mathbf{a}'\mathbf{D}/((1 - K_c(\lambda))\{\mathbf{a}\}_i)$. Hence $w_i(\mathbf{s}, t) \leq D_i^* \exp(\lambda(ct - |\mathbf{s}|))$ for $\mathbf{s} \in R^N$, $t \geq 0$ and $i = 1, \ldots, n$. Now $\sup\{w_i(\mathbf{s}, t) : |\mathbf{s}| \geq c^* t\} \leq D_i^* e^{\lambda(ct - c^* t)}$. The result then follows immediately since $c^* > c$.

Corollary 1. *If* $\rho(\boldsymbol{\Gamma}) \leq 1$, *then for any* $c > 0$, $\lim_{t\to\infty} \sup\{w_i(\mathbf{s}, t) : |\mathbf{s}| \geq ct\} = 0$ *for* $i = 1, \ldots, n$.

Corollary 2. *If* $\rho(\mathbf{\Gamma})>1$, *then for any speed* $c>c_0$, $\lim_{t\to\infty}\sup\{w_i(\mathbf{s},t): |\mathbf{s}|\geq ct\}=0$ *for* $i=1,\ldots,n$.

The asymptotic speed of propagation, as defined in Aronson and Weinberger [1, 2], is c if for any c_1 and c_2 with $0<c_1<c<c_2$,

(i) the solution $w_i(\mathbf{s},t)$ tends uniformly to zero in the region $|\mathbf{s}|\geq c_2t$;

(ii) the solution $w_i(\mathbf{s},t)$ is bounded away from zero uniformly in the region $|\mathbf{s}|\leq c_1t$ for t sufficiently large.

Corollary 1 shows that if $\rho(\mathbf{\Gamma})\leq 1$, no matter how slowly you travel, the proportion of infectives ahead of you will tend to zero. This corresponds to an epidemic which is not severe. The speed of propagation may be considered to be zero.

When $\rho(\mathbf{\Gamma})>1$, Corollary 2 establishes that c_0 satisfies part (i) of the definition of the asymptotic speed of propagation. We now state Theorem 2, which says that c_0 satisfies part (ii) of the definition, and hence proves that c_0 is the asymptotic speed of propagation of the epidemic. Before proceeding to the proof of Theorem 2, it is necessary to prove certain lemmas and state a comparison principle. The proof of Theorem 2 is then given at the end of this section.

Theorem 2. *For* $\rho(\mathbf{\Gamma})>1$ *and any* $c_1\in[0,c_0)$, *there exist positive constants* b_i *and constants* T_i *sufficiently large such that* $\min\{w_i(\mathbf{s},t): |\mathbf{s}|\leq c_1t\}\geq b_i$ *for all* $t\geq T_i$ *and* $i=1,\ldots,n$.

We assume for the rest of this section that $\rho(\mathbf{\Gamma})>1$. Define B_R to be the closed ball of radius R, centred on the origin, in R^N. Observe that if $c_1<c<c_0$, then $K_c(\lambda)>1$ for all $\lambda\in[0,\Delta_v]$. We need to construct a function E operating on $w_i(\mathbf{s},t)$ so that $w_i(\mathbf{s},t)\geq E[w_i(\mathbf{s},t)]$ with an associated function $k(y)$ with Laplace transform which is everywhere greater than one. We can then use a subsolution and comparison lemma as in Diekmann [3].

Using eqs. (3) we have

$$w_i(\mathbf{s},t)\geq\sum_{j=1}^{n}\int_{R^N}\int_0^t(1-\exp\{-w_j(\mathbf{s}-\mathbf{r},t-\tau)\})\gamma_{ij}(\tau)p_{ij}(\mathbf{r})\,d\tau\,d\mathbf{r}.$$

We may reuse this inequality for $w_j(\mathbf{s}-\mathbf{r},t-\tau)$ in the integrand on the right hand side to obtain a further inequality. This may be repeated any number of times and in the final replacement of the inequality, the summation may be truncated to include the term $j=i$ only. The associated $k(y)$ then has Laplace transform $\{\mathbf{V}^m(\lambda)\}_{ii}$. For c such that $0<c<c_0$ we can choose m such that $\{\mathbf{V}^m(\lambda)\}_{ii}>1$ for all $\lambda\in[0,\Delta_v]$. In fact E is defined so that $k(y)$ has compact support with its Laplace transform arbitrarily close to $\{\mathbf{V}^m(\lambda)\}_{ii}$, and hence still able to be made greater than one for all λ. It is therefore appropriate to prove Lemma 4 before defining the function E.

We first make the following definitions. Let $\mathbf{V}(\lambda;R,T)$ be the matrix with ijth element

$$\{V(\lambda;R,T)\}_{ij}=\int_0^T\int_{B_R}\exp(\lambda[\{\mathbf{x}\}_1-ct])p_{ij}(\mathbf{x})\gamma_{ij}(\tau)\,d\mathbf{x}\,d\tau.$$

Define $\{\mathbf{V}^s(\lambda)\}_{ij}=\lim_{R,T\to\infty}\{\mathbf{V}^s(\lambda;R,T)\}_{ij}$ for any positive integer s. Then $\mathbf{V}^s(\lambda)$

has the usual interpretation when $\mathbf{V}(\lambda)$ is finite; in addition it is defined when some of the elements of $\mathbf{V}(\lambda)$ are infinite. Note that we may choose R and T sufficiently large so that $\{\mathbf{V}(\lambda;\ R,\ T)\}_{ij}=0$ if and only if $\gamma_{ij}=0$. Hence the zero elements of $\mathbf{V}^s(\lambda)$ are in the same positions for all real finite λ.

For any positive c, R and T, define

$$L_c(\lambda;\ R,\ T,\ m)=\int_{-\infty}^{\infty}\int_0^T e^{\lambda(v-cu)}\sum_{j_1=1}^n\sum_{j_2=1}^n\cdots$$
$$\sum_{j_{m-1}=1}^n \tilde{p}_{i,j_1,\dots,j_{m-1},i}(v)\gamma_{i,j_1,\dots,j_{m-1},i}(u)\,du\,dv$$

where $\hat{p}_{ij}(x)$ is $p_{ij}(x)$ truncated outside the region $x\in B_{R/m}$, and

$$\tilde{p}_{i,j_1,\dots,j_{m-1},i}(\{\mathbf{x}\}_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{p}_{ij_1}*\cdots*\hat{p}_{j_{m-1}i}(\mathbf{x})\,d\{\mathbf{x}\}_2\cdots d\{\mathbf{x}\}_N$$

and

$$\gamma_{i,j_1,\dots,j_{m-1},i}(u)=\gamma_{ij_1}*\gamma_{j_1j_2}*\cdots*\gamma_{j_{m-1}i}(u).$$

Lemma 4. *For any positive $c<c_0$, there exists a positive integer m, a positive real $h<1$ and positive reals R_0 and T_0 sufficiently large such that $hL_c(\lambda;\ R,\ T,\ m)>1$ for $\lambda\in R$ and $R\geqslant R_0$ and $T\geqslant T_0$.*

Proof. We first show that, for any suffix i, there exists a positive integer m such that $\{\mathbf{V}^m(\lambda)\}_{ii}>1$ for all $\lambda\geqslant 0$; with $\{\mathbf{V}^m(\lambda)\}_{ii}$ infinite for $\lambda>\Delta_v$ when Δ_v is finite. Define $\rho(\lambda)\equiv\rho(\mathbf{V}(\lambda))$.

We use the results in Schaefer, ([12], Chap. 1, Prop. 7.3 and 7.4). Let $\mathbf{A}$ be a finite non-reducible, non-negative square matrix, which is not the 1×1 zero matrix. There exist positive integers l and s so that, by choosing a suitable permutation to apply to both rows and columns,

$$\mathbf{A}^{ls}=\begin{pmatrix}\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2 & \cdots & \mathbf{0}\\ \cdots & \cdots & \cdots & \\ \mathbf{0} & \cdots & \cdots & \mathbf{B}_l\end{pmatrix}$$

where $\mathbf{B}_j$ is a positive square matrix with $\rho(\mathbf{B}_j)=(\rho(\mathbf{A}))^{ls}$ for $j=1,\dots,l$. The positions of the zero entries of $\mathbf{A}$ determine the appropriate permutation, the integer l, the smallest possible s and the sizes of $\mathbf{B}_1,\dots,\mathbf{B}_l$. Take $\mathbf{A}$ with $\{\mathbf{A}\}_{ij}=0$ if and only if $\gamma_{ij}=0$. Find l and s. Then for R and T sufficiently large so that $\{\mathbf{V}(\lambda,\ R,\ T)\}_{ij}=0$ if and only if $\gamma_{ij}=0$, by relabelling the populations $1,\dots,n$,

$$\mathbf{V}^{ls}(\lambda;\ R,\ T)=\begin{pmatrix}\mathbf{B}_1(\lambda;\ R,\ T) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2(\lambda;\ R,\ T) & \cdots & \mathbf{0}\\ \cdots & \cdots & \cdots & \\ \mathbf{0} & \mathbf{0}\cdots & \cdots & \mathbf{B}_l(\lambda;\ R,\ T)\end{pmatrix}$$

where $\mathbf{B}_j(\lambda;\ R,\ T)>0$ and $\rho(\mathbf{B}_j(\lambda;\ R,\ T))=\rho(\mathbf{V}(\lambda;\ R,\ T))^{ls}$.

Now if $\mathbf{V}(\lambda^*)$ has an infinite element, then each $\mathbf{B}_j(\lambda^*; R, T)$ must have an element which tends to infinity as R, $T \to \infty$. Hence if $\mathbf{B}_j^r(\lambda) = \lim_{R,T\to\infty} \mathbf{B}_j^r(\lambda; R, T)$, then

$$\mathbf{V}^{rls}(\lambda) = \begin{pmatrix} \mathbf{B}_1^r(\lambda) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^r(\lambda) & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \\ \mathbf{0} & \cdots & \cdots & \mathbf{B}_l^r(\lambda) \end{pmatrix}$$

where $\mathbf{B}_j^r(\lambda)$ has all infinite elements for $r \geq 3$, $\lambda = \lambda^*$ and $j = 1, \ldots, l$.

Note that if Δ_v is finite this implies that $\{\mathbf{V}^{rls}(\lambda)\}_{ii} = \infty$ for $r \geq 3$ and $\lambda > \Delta_v$. If Δ_v is finite with $\rho(\Delta_v)$ infinite the result also holds at $\lambda = \Delta_v$. When $\rho(0) = \infty$ the result is valid at $\lambda = 0$. When Δ_v is finite take $m^* = 3ls$.

If Δ_v is infinite, since $\rho(\lambda) > 1$ for all λ, necessarily $\lim_{\lambda\to\infty} \rho(\lambda) = \infty$. Hence there exists a sequence $i_1, \ldots, i_t$ with $i_t = i_1$, such that $\lim_{\lambda\to\infty} \prod_{j=1}^{t-1} \{\mathbf{B}_j(\lambda)\}_{i_j i_{j+1}} = \infty$, (see [5] Lemma 3). Hence there exists a positive integer $w \geq 3$ such that $\lim_{\lambda\to\infty} \{\mathbf{B}_j^w(\lambda)\}_{vt} = \infty$ for all v, t, j. Take $m^* = lsw$. Then $\lim_{\lambda\to\infty} \{\mathbf{V}^{m^*}(\lambda)\}_{ii} = \infty$.

When $\mathbf{V}(0)$ is finite, take $\lambda_0 = 0$. If $\mathbf{V}(0)$ has an infinite element then $\{\mathbf{V}^{m^*}(0)\}_{ii} = \infty$. Hence we may find a $\lambda_0 > 0$ such that $\{\mathbf{V}^{m^*}(\lambda)\}_{ii} > 1$ for $0 \leq \lambda < \lambda_0$. When Δ_v is finite with $\mathbf{V}(\Delta_v)$ finite, take $\lambda_0^* = \Delta_v$. In all other cases $\lim_{\lambda\to\Delta_v} \{\mathbf{V}^{m^*}(\lambda)\}_{ii} = \infty$, and we may find a $\lambda_0^* < \Delta_v$ such that $\{\mathbf{V}^{m^*}(\lambda)\}_{ii} > 1$ for $\lambda > \lambda_0^*$.

Thus in all cases $\{\mathbf{V}^{m^*}(\lambda)\}_{ii} > 1$ for $\lambda < \lambda_0$ or $\lambda > \lambda_0^*$. Also if Δ_v is finite $\{\mathbf{V}^{m^*}(\lambda)\}_{ii} = \infty$ for $\lambda > \Delta_v$. Note that $\mathbf{V}^{m^*}(\lambda)$ is finite for $\lambda \in [\lambda_0, \lambda_0^*]$.

Now suppose that there is no positive integer q so that $\{\mathbf{V}^{qm^*}(\lambda)\}_{ii} > 1$ for all $\lambda \geq 0$. Then there exist sequences $\{q_j\}$ amd $\{\lambda_j\}$, with $\lim_{j\to\infty} q_j = \infty$, such that $\{\mathbf{V}^{q_j m^*}(\lambda_j)\}_{ii} \leq 1$. Note that necessarily $\lambda_j \in [\lambda_0, \lambda_0^*]$. Hence there exists a convergent subsequence; the subsequence of $\{\lambda_j\}$ tending to $\bar{\lambda} \in [\lambda_0, \lambda_0^*]$. Then $\lim_{q\to\infty} \{\mathbf{V}^{qm^*}(\bar{\lambda})\}_{ii} \leq 1$. But m^* is a multiple of ls and $\lim_{r\to\infty} \mathbf{B}_j^r(\bar{\lambda})/(\rho(\bar{\lambda}))^{rls} = \mathbf{E}_j(\bar{\lambda}) > 0$, where $\mathbf{E}_j(\bar{\lambda})$ is the idempotent of $\mathbf{B}_j(\bar{\lambda})$ corresponding to $(\rho(\bar{\lambda}))^{ls}$. Since $\rho(\bar{\lambda}) > 1$, $\lim_{q\to\infty} \{\mathbf{V}^{qm^*}(\bar{\lambda})\}_{ii} = \infty$, which gives a contradiction.

Hence there exists a positive integer q so that $\{\mathbf{V}^{qm^*}(\lambda)\}_{ii} > 1$ for all $\lambda \geq 0$. Take $m = qm^*$. Note that $\{\mathbf{V}^m(\lambda)\}_{ii} = \infty$ for $\lambda > \Delta_v$ if Δ_v is finite.

Now $L_c(\lambda; R, T, m)$ may be written as the Laplace transform of a non-negative function. In addition it is an increasing function of R and T, which tends to a limit $\{\mathbf{V}^m(\lambda)\}_{ii} > 1$ as R and T tend to infinity. We can clearly choose R and T sufficiently large so that $L_c(\lambda; R, T, m) \to \infty$ as $\lambda \to \infty$. Take R_1, T_1 and λ^* so that $L_c(\lambda; R, T, m) > 1$ for $\lambda \geq \lambda^*$, $R \geq R_1$ and $T \geq T_1$. Suppose no R_0 and T_0 exist such that $L_c(\lambda; R_0 \cdot T_0, m) > 1$ for $\lambda \in [0, \infty)$. Then there exist sequences $\{R_j\}$, $\{T_j\}$ and $\{\lambda_j\}$ such that $L_c(\lambda_j; R_j, T_j) \leq 1$ with $\lim_{j\to\infty} R_j = \infty$, $\lim_{j\to\infty} T_j = \infty$ and necessarily $\lambda_j \in [0, \lambda^*)$. Hence there exists a convergent subsequence. Then there is a $\bar{\lambda} \in [0, \lambda^*]$ with $\lim_{R\to\infty, T\to\infty} L_c(\bar{\lambda}; R, T, m) \leq 1$, which gives a contradiction. Thus there exists an m, R_0 and T_0 such that $L_c(\lambda; R_0, T_0, m) > 1$ for $\lambda \in [0, \infty)$. Note that $L_c(\lambda; R, T, m) \geq L_c(|\lambda|; R, T, m)$ for λ real. It then follows that $L_c(\lambda; R, T, m) > 1$ for all real λ and $R \geq R_0$ and $T \geq T_0$.

Since $\lim L_c(\lambda; R_0, T_0, m) = \infty$, both as $\lambda \to \infty$ and as $\lambda \to -\infty$, we obtain the result that $\inf_{\lambda\in R} L_c(\lambda; R_0, T_0, m) > 1$. Take h such that $\{\inf_{\lambda\in R} L_c(\lambda; R_0, T_0, m)\}^{-1} \leq h < 1$. The lemma then follows immediately.

Choose an m, R_0, T_0 and h as in Lemma 4. Take R and T so that $R \geq R_0$ and $T \geq T_0$. We then define an operator E, operating on $\psi(\mathbf{s}, t)$. Let $g(x) = 1 - e^{-x}$. Define

$$\left\{E_T^{(0)}\left(\psi\left(\mathbf{x} - \sum_{r=1}^{m-1} \mathbf{s}_r, t - \sum_{r=1}^{m-1} \tau_r\right)\right)\right\}_j$$

$$= \int_0^{T - \sum_{r=1}^{m-1} \tau_r} \int_{R^N} \gamma_{ji}(\tau_0) \hat{p}_{ji}(\mathbf{s}_0) g\left(\psi\left(x - \sum_{r=1}^{m-1} \mathbf{s}_r - \mathbf{s}_0, t - \sum_{r=1}^{m-1} \tau_r - \tau_0\right)\right) d\mathbf{s}_0 \, d\tau_0.$$

Then successively for $1 \leq k \leq (m-2)$ we define

$$\left\{E_T^{(k)}\left(\psi\left(\mathbf{x} - \sum_{r=k+1}^{m-1} \mathbf{s}_r, t - \sum_{r=k+1}^{m-1} \tau_r\right)\right)\right\}_j$$

$$= \sum_{l=1}^{n} \int_0^{T - \sum_{r=k+1}^{m-1} \tau_r} \int_{R^N} \gamma_{jl}(\tau_k) \hat{p}_{jl}(\mathbf{s}_k)$$

$$\times g\left(\left\{E_T^{(k-1)}\left(\psi\left(\mathbf{x} - \sum_{r=k}^{m-1} \mathbf{s}_r, t - \sum_{r=k}^{m-1} \tau_r\right)\right)\right\}_j\right) d\mathbf{s}_k \, d\tau_k.$$

Finally we define

$$E_T(\psi(\mathbf{x}, t)) = \sum_{j=1}^{n} \int_0^T \int_{R^N} \gamma_{ij}(\tau_{m-1}) \hat{p}_{ij}(\mathbf{s}_{m-1})$$

$$\times g(\{E_T^{(m-2)}(\psi(\mathbf{x} - \mathbf{s}_{m-1}, t - \tau_{m-1})\}_j) \, d\mathbf{s}_{m-1} \, d\tau_{m-1}.$$

Observe that m, R_0, T_0 and h, and hence E_T, depend on i. Note also that $w_i(\mathbf{x}, t) \geq E_T(w_i(\mathbf{x}, t))$ for $t \geq T$.

The definition of the operator E_T enables us to state a comparison principle which is essentially that of Diekmann ([3], Lemma 1). The proof is identical so is omitted. Define $\phi > \psi$ if ϕ and ψ are continuous functions defined in R^N with $\phi(\mathbf{x}) \geq \psi(\mathbf{x})$, the inequality being strict for $\mathbf{x} \in \operatorname{supp} \psi$.

Lemma 5 (Comparison Principle). *Suppose that* $E_T[\psi](\cdot, t) > \psi(\cdot, t)$ *for all* $t \geq T$, *where* $\psi: R^N \times R_+ \to R$ *is a non-negative continuous function such that*

(i) *for any* $t_1 > 0$ *there exists an* $S = S(t_1) < \infty$ *such that for any* $t \in [0, t_1]$, $\operatorname{supp} \psi(\cdot, t) \subset B_S$;

(ii) *if* $\{(\mathbf{s}_n, t_n)\}_{n=1}^{\infty} \subset R^N \times R_+$ *is a sequence for which* $\mathbf{s}_n \in \operatorname{supp} \psi(\cdot, t_n)$ *and* $\lim_{n\to\infty} (\mathbf{s}_n, t_n) = (\mathbf{s}, t)$, *then necessarily* $\mathbf{s} \in \operatorname{supp} \psi(\cdot, t)$.

If there exists a $t_0 \geq 0$ *such that* $w_i(\cdot, t_0 + t) > \psi(\cdot, t)$ *for all* $0 \leq t \leq T$, *then the relation holds for all* $t \geq 0$.

Consider $L(\lambda) = hL_c(\lambda; R, T, m) = \int_R e^{\lambda y} k(y) \, dy$, where

$$k(y) = h \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \tilde{p}_{i,j_1,\ldots,j_{m-1},i}(y + c\tau) \gamma_{i,j_1,\ldots,j_{m-1},i}(\tau) \, d\tau.$$

Note that $k(y)$ has compact support and $L(\lambda) > 1$ for all real λ. We now define a subsolution $\psi(\mathbf{s}, t)$, using the definitions and a lemma from Diekmann

[3], which are stated below. The subsolution ψ is chosen to satisfy the condition $E_T[\psi](\cdot, t) > \psi(\cdot, t)$ for $t \geq T$.

Define

$$q(y; \alpha, \beta) = \begin{cases} e^{-\alpha y} \sin(\beta y) & \text{for } 0 \leq y \leq \pi/\beta \\ 0 & \text{for } y \in R \backslash [0, \pi/\beta]. \end{cases}$$

Lemma 6. *Let $k \in L_1(R)$ be a non-negative function with compact support such that $L(\lambda) = \int_{-\infty}^{\infty} e^{\lambda y} k(y)\, dy > 1$ for all $\lambda \in R$. Then there exists a positive number β_0, a continuous function $\tilde{\alpha} = \tilde{\alpha}(\beta)$ and a positive function $\Delta = \Delta(\beta)$ defined on $[0, \beta_0]$ such that, for any $\beta \in [0, \beta_0]$ and any $\delta \in [0, \Delta(\beta)]$,*

$$\phi * k > \phi_\delta,$$

where $\phi(y) = q(y; \tilde{\alpha}(\beta), \beta)$ and $\phi_\delta(y) = \phi(y - \delta)$.

Starting from q, a three-parameter family of non-decreasing functions r is formed as follows,

$$r(y; \alpha, \beta\ \gamma) = \max_{\eta \geq -\gamma} q(y + \eta; \alpha, \beta),$$

or equivalently

$$r(y; \alpha, \beta, \gamma) = \begin{cases} M & \text{for } y \leq \gamma + \rho \\ q(y - \gamma; \alpha, \beta) & \text{for } \gamma + \rho \leq y \leq \gamma + (\pi/\beta) \\ 0 & \text{for } y \geq \gamma + (\pi/\beta), \end{cases}$$

where $M = M(\alpha, \beta) = \max\{q(y; \alpha, \beta) : 0 \leq y \leq (\pi/\beta)\}$ and $\rho = \rho(, \beta)$ is the value of y for which the maximum is achieved.

Define $\psi(\mathbf{s}, t) \equiv \sigma r(|\mathbf{s}|; \alpha, \beta, D + ct)$ for any $\sigma > 0$.

Lemma 7. *There exists a $\sigma^* > 0$ and $D > 0$ such that, for any $0 < \sigma < \sigma^*$, $E_T[\psi](\cdot, t) > \psi(\cdot, t)$ for $t \geq T$.*

Proof. Note that $\psi(s, t) \leq \sigma M$ for all $\mathbf{s} \in R^N$ and $t \geq 0$. We can choose σ^* sufficiently small so that $(1 - e^{-x}) \geq h^{1/m} x$ for $0 \leq x \leq \sigma^* M \max\{1, \{n \max_{j,k} \int_0^T \gamma_{jk}(\tau)\, d\tau\}^{m-1}\}$.

Take σ such that $0 < \sigma < \sigma^*$. Then

$$E_T[\psi](\mathbf{s}, t) \geq h \int_0^T \int_{R^N} \int_0^{T - \tau_{m-1}} \int_{R^N} \cdots \int_0^{T - \sum_{j=1}^{m-1} \tau_j}$$
$$\times \int_{R^N} \psi\left(\mathbf{s} - \sum_{j=1}^{m-1} \mathbf{s}_j, t - \sum_{j=1}^{m-1} \tau_j\right) \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \gamma_{ij_1}(\tau_{m-1})$$
$$\times \gamma_{j_1 j_2}(\tau_{m-2}) \cdots \gamma_{j_{m-1} i}(\tau_0) \hat{p}_{ij_1}(\mathbf{s}_{m-1}) \cdots \hat{p}_{j_{m-1} i}(\mathbf{s}_0)\, d\mathbf{s}_0\, d\tau_0 \cdots d\mathbf{s}_{m-1}\, d\tau_{m-1}$$
$$= h \int_0^T \int_{R^N} \psi(\mathbf{r}, t - \phi) \sum_{j_1=1}^{n} \sum_{j_{m-1}=1}^{n} \hat{p}_{ij_1} * \cdots * \hat{p}_{j_{m-1} i}(\mathbf{s} - \mathbf{r}) \gamma_{i, j_1, \ldots, j_{m-1}, i}(\phi)\, d\mathbf{r}\, d\phi$$
$$= h \int_0^T \int_{B_R} \sigma r(|\mathbf{s} - \mathbf{x}|; \alpha, \beta, D + c(t - \phi)) \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \hat{p}_{ij_1} * \cdots * \hat{p}_{j_{m-1} 1}(\mathbf{x})$$
$$\times \gamma_{i, j_1, \ldots, j_{m-1}, i}(\phi)\, d\mathbf{x}\, d\phi.$$

We first observe that if $\psi(s, t)=0$, i.e. if $|\mathbf{s}| \geq D+ct+(\pi/\beta)$, then trivially $E_T[\psi](\mathbf{s}, t) > \psi(\mathbf{s}, t)$. We need to show this result holds for all $\mathbf{s}$ and t.

Case (*i*). If $|\mathbf{s}| \leq D+c(t-\tau)+\rho-R$, since $|\mathbf{s}-\mathbf{x}| \leq |\mathbf{s}|+|\mathbf{x}| \leq |\mathbf{s}|+R$ for $\mathbf{x} \in B_R$, then $|\mathbf{s}-\mathbf{x}| \leq D+c(t-\phi)+\rho$ for $\mathbf{x} \in B_R$ and $0 \leq \phi \leq T$. Therefore

$$E_T[\psi](\mathbf{s}, t) \geq h \int_0^T \int_{B_R} \sigma M \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \hat{p}_{ij_1} * \cdots * \hat{p}_{j_{m-1}i}(\mathbf{x}) \gamma_{i,j_1,\ldots,j_{m-1},i}(\phi)\, d\mathbf{x}\, d\phi$$

$$= \sigma M h L_c(0; R, T, m) > \sigma M \geq \psi(\mathbf{s}, t).$$

Case (*ii*). If $D+c(t-\tau)+\rho-R < |\mathbf{s}| \leq D+ct+(\pi/\beta)$, then if we choose $D \geq (R^2/\{2\delta\})-\rho+R$, $|\mathbf{s}-\mathbf{x}| = (\mathbf{s}'\mathbf{s}+\mathbf{x}'\mathbf{x}-2\mathbf{s}'\mathbf{x})^{1/2} \leq |\mathbf{s}|-(\mathbf{s}'\mathbf{x}/|\mathbf{s}|)+\delta$. Note that $0<\delta<\Delta(\beta)$ as in Lemma 6.

Since $r(y; \alpha, \beta, \gamma)$ is a decreasing function of y and $\hat{p}_{ij_1} * \cdots * \hat{p}_{j_{m-1}i}(\mathbf{x})$ is a radial function, then

$$E_T[\psi](\mathbf{s}, t) \geq h\sigma \int_0^T \int_{B_R} r(|\mathbf{s}|-\{\mathbf{x}\}_1+\delta; \alpha, \beta, D+c(t-\phi))$$

$$\times \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \hat{p}_{ij_1} * \cdots * \hat{p}_{j_{m-1}i}(\mathbf{x})$$

$$\times \gamma_{i,j_1,\ldots,j_{m-1},i}(\phi)\, d\mathbf{x}\, d\phi$$

$$= h\sigma \int_0^T \int_{-R}^{R} \max_{\eta \geq -D-c(t-\phi)} q(|\mathbf{s}|-\{\mathbf{x}\}_1+\delta+\eta; \alpha, \beta)$$

$$\times \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \tilde{p}_{i,j_1,\ldots,j_{m-1},i}(\{\mathbf{x}\}_1) \gamma_{i,j_1,\ldots,j_{m-1},i}(\phi)\, d\{\mathbf{x}\}_1\, d\phi.$$

Let $u=\{\mathbf{x}\}_1-c\phi$ and $\eta^*=\eta-c\phi$. Then

$$E_T[\psi](\mathbf{s}, t) \geq h\sigma \int_{-\infty}^{\infty} \max_{\eta^* \geq -D-ct} q(|\mathbf{s}|-u+\delta+\eta^*; \alpha, \beta)$$

$$\times \sum_{j_1=1}^{n} \cdots \sum_{j_{m-1}=1}^{n} \gamma_{i,j_1,\ldots,j_{m-1},i}(\phi) \tilde{p}_{i,j_1,\ldots,j_{m-1},i}(u+c\phi)\, d\phi\, du$$

$$= h\sigma \int_{-\infty}^{\infty} \max_{\eta^* \geq -D-ct} q(|\mathbf{s}|-u+\delta+\eta^*; \alpha, \beta) k(u)\, du$$

Hence from Lemma 6, $E_T[\psi](\mathbf{s}, t) > h\sigma \max_{\eta^* \geq -D-ct} q(|\mathbf{s}|+\eta^*) = \psi(\mathbf{s}, t)$. This completes the proof.

We now show that $w_i(\mathbf{s}, t) > 0$ for $(\mathbf{s}, t) \in B_R \times [t_0, t_0+T]$. Since $w_i(\mathbf{s}, t)$ is monotone increasing in t, the inf of $w_i(\mathbf{s}, t)$ in this range is identical to inf $w_i(\mathbf{s}, t_0)$ over $\mathbf{s} \in B_R$. From Lemma 2, $w_i(\mathbf{s}, t_0)$ is continuous in $\mathbf{s}$. This then implies that $\inf w_i(\mathbf{s}, t) > 0$ for $(\mathbf{s}, t) \in B_R \times [t_0, t_0+T]$. This enables us to complete the proof of Theorem 2.

Lemma 8. *For any $R>0$ there exists a $t_0=t_0(R)$ such that $w_i(\mathbf{s}, t)>0$ for $(\mathbf{s}, t) \in B_R \times [t_0, \infty)$.*

Proof. The conditions imposed on $\varepsilon_j(\mathbf{s}, \tau)$, $\lambda_{ij}^*(\tau)$ and $p_{ij}^*(\mathbf{r})$ ensure that there exists an open set F in R^N and a positive constant T, such that for some j, $H_j(\mathbf{s}, t) > 0$ for $t \geq T$ and $\mathbf{s} \in F$. We may shift the origin so that, without loss of generality, $F \supset B_A$ for some positive A. Hence $w_j(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_A$ and $t \geq T$.

Since $\boldsymbol{\Gamma}$ is non-reducible, there exists a sequence $j_1, \ldots, j_l$ with $j_1 = j$ and $j_l = j$ and $\gamma_{j_s, j_{s+1}} \neq 0$ for $s = 1, \ldots, (l-1)$. Now $w_{j_{s+1}} > 0$ if

$$\int_0^T \int_{R^N} w_{j_s}(\mathbf{r}, \tau)\gamma_{j_{s+1}j_s}(t-\tau)p_{j_{s+1}j_s}(\mathbf{s}-\mathbf{r})\, d\mathbf{r}\, d\tau > 0.$$

It is easily seen therefore that $w_j(\mathbf{s}, t) > 0$ if

$$\int_0^T \int_{R^N} w_j(\mathbf{r}, \tau)\gamma_{j_1 j_2} * \cdots * \gamma_{j_{l-1}j_l}(t-\tau)p_{j_1 j_2} * \cdots * p_{j_{l-1}j_l}(\mathbf{s}-\mathbf{r})\, d\mathbf{r}\, d\tau > 0.$$

Now for some B, C, T_1, T_2, $p_{j_1 j_2} * \cdots * p_{j_{l-1}j_l}(\mathbf{x}) > 0$ for $B \leq |\mathbf{x}| \leq C$ and $\gamma_{j_1 j_2} * \cdots * \gamma_{j_{l-1}j_l}(\tau) > 0$ for $\tau \in [T_1, T_2]$, with $B < C$ and $T_1 < T_2$. Hence $w_j(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_A$ and/or $B - A \leq |\mathbf{s}| \leq A + C$ and $t \geq T + T_1$. Repeating this procedure, in two steps we obtain the result that $w_j(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_{A+C-B}$ and $t \geq T + 2T_1$.

If $j = i$, choose a non-negative integer r such that $R \leq A + r(C-B)$; then $w_i(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_R$ and $t \geq t_0$ where $t_0 = T + 2rT_1$.

If $j \neq i$, there exists a sequence $i_1, \ldots, i_k$, with $i_k = j$, $i_1 = i$ and $\gamma_{i_s i_{s+1}} \neq 0$ for $s = 1, \ldots, (k-1)$. Now $w_i(\mathbf{s}, t) > 0$ if $\int_0^t \int_{R^N} w_j(\mathbf{r}, \tau)\gamma_{i_1 i_2} * \cdots * \gamma_{i_{k-1}i_k}(t-\tau) \times p_{i_1 i_2} * \cdots * p_{i_{k-1}i_k}(\mathbf{s}-\mathbf{r})\, d\mathbf{r}\, d\tau > 0$. There exist non-negative reals S_1, S_2, D and E such that $\gamma_{i_1 i_2} * \cdots * \gamma_{i_{k-1}i_k}(t) > 0$ for $t \in [S_1, S_2]$ and $p_{i_1 i_2} * \cdots * p_{i_{k-1}i_k}(\mathbf{x}) > 0$ for $D \leq |\mathbf{x}| \leq E$. Choose r such that $R + D \leq A + r(C-B)$. Then $w_j(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_{R+D}$ and $t \geq T + 2rT_1$. Hence $w_i(\mathbf{s}, t) > 0$ for $\mathbf{s} \in B_R$ and $t \geq T + 2rT_1 + S_1$. In this case we take $t_0 = T + 2rT_1 + S_1$. This completes the proof.

Proof of Theorem 2. From Lemma 8, for any finite positive T,-inf $w_i(\mathbf{s}, t) > 0$ for $(\mathbf{s}, t) \in B_R \times [t_0, t_0 + T]$. Choose σ such that $0 < \sigma < \sigma^*$ and $\sigma M < \inf w_i(\mathbf{s}, t)$ where the inf is taken over $(\mathbf{s}, t) \in B_R \times [t_0, t_0 + T]$. Hence $w_i(\mathbf{s}, t_0 + t) > \psi(\mathbf{s}, t)$ for $0 \leq t \leq T$. Using the comparison lemma, (Lemma 5), we then obtain the result that $w_i(\mathbf{s}, t_0 + t) > \psi(\mathbf{s}, t)$ for all $t \geq 0$. Hence $w_i(\mathbf{s}, t_0 + t) \geq \sigma M$ for $|\mathbf{s}| \leq \rho + D + ct$ and $t \geq 0$. Thus $w_i(\mathbf{s}, t) \geq \sigma M$ for $|\mathbf{s}| \leq \rho + D + c(t - t_0)$ and $t \geq t_0$. Therefore $w_i(\mathbf{s}, t) \geq \sigma M$ for $|\mathbf{s}| \leq c_1 t$ if $c_1 t \leq \rho + D + c(t - t_0)$ and $t \geq t_0$, i.e. if $t \geq \max\{t_0, (ct_0 - \rho - D)/(c - c_1)\}$. Then if we take $b_i = \sigma M > 0$ and $T_i = \max\{t_0, (ct_0 - \rho - D)/(c - c_1)\}$, we obtain the result $\min\{w_i(\mathbf{s}, t): |\mathbf{s}| \leq c_1 t\} \geq b_i$ for all $t \geq T_i$.

Note that we can define the appropriate operator E_T for each value of i, and hence obtain the corresponding b_i and T_i, so that Theorem 2 holds for all $i = 1, \ldots, n$.

Corollary 2 and Theorem 2 together show that if $\rho(\boldsymbol{\Gamma}) > 1$ then the asymptotic speed of propagation is c_0, the minimum speed at which wave solutions exist. Corollary 1 shows that if $\rho(\boldsymbol{\Gamma}) \leq 1$ the speed of propagation is zero. In order to prove these results we require each $p_{ij}(\mathbf{r})$ to be exponentially dominated in the tail and suitable restrictions to be placed on the $\varepsilon_i(\mathbf{r})$ and $p_{ij}^*(\mathbf{r})$ so that the effect of the infectives from outside does not dominate the ultimate behaviour of the epidemic. The restrictions on the $\varepsilon_i(\mathbf{r})$ and the $p_{ij}^*(\mathbf{r})$ are only required to prove theorem 1, so that, if ($\rho(\mathbf{\Gamma})>1$), the speed of propagation is at least c_0 even if these restrictions do not hold. If $P^*_{ij}(\lambda)$ and $E_i(\lambda)$ are finite for all i, j and $0\leqslant\lambda<\alpha<\lambda_0$ where α is such that $K_{c^*}(\alpha)=1$, then it may easily be shown that the speed of propagation c, (when $\rho(\mathbf{\Gamma})>1$), satisfies the inequality $c_0\leqslant c\leqslant c^*$. In this case the infection from outside is feeding the epidemic ahead of the epidemic generated within the n populations of susceptibles.

4. The behaviour when at least one contact distribution is not exponentially dominated in the tail

In this section the behaviour is considered when some $P_{ij}(\lambda)$ is infinite for λ real and non-zero. The existence and uniqueness of $w_i(\mathbf{s}, t)$ may then be established as in our paper [7, theorem 4]. We first prove the analogue of Lemma 4.

Lemma 9. *When $\rho(\mathbf{\Gamma})>1$, for any $c>0$ there exists a positive integer m, a positive real $h<1$ and positive reals R_0 and T_0 sufficiently large so that $hL_c(\lambda; R, T, m)>1$ for $\lambda\in R$, $R\geqslant R_0$ and $T\geqslant T_0$.*

Proof. As in Lemma 4, there exist positive integers l and s so that, by relabelling the populations $1,\dots,n$,

$$\mathbf{V}^{ls}(\lambda)=\begin{pmatrix}\mathbf{B}_1(\lambda) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2(\lambda) & \cdots & \mathbf{0}\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_l(\lambda)\end{pmatrix}$$

where $\mathbf{B}_j(\lambda)>0$. Also $\{\mathbf{V}^{rls}(\lambda)\}_{ii}=\infty$ for $\lambda>0$, $r\geqslant3$ and all i. If $\mathbf{V}(0)$ has an infinite element, then $\{V^{rls}(0)\}_{ii}=\infty$ for $r\geqslant3$ and all i. Take $m=3ls$.

When $\mathbf{V}(0)$ is finite, since $\rho(0)>1$ and $\mathbf{B}_j(0)>0$ for all j, $\lim_{r\to\infty}\mathbf{B}^r_j(0)/(\rho(0))^{rls}=\mathbf{E}_j(0)>\mathbf{0}$; where $\mathbf{E}_j(0)$ is the idempotent of $\mathbf{B}_j(0)$ corresponding to the eigenvalue $(\rho(0))^{ls}$. Then there exists an $r\geqslant3$ with $\{\mathbf{V}^{rls}(0)\}_{ii}>1$ for all i. Take $m=rls$.

Then $\{\mathbf{V}^m(\lambda)\}_{ii}>1$ for all $\lambda\geqslant0$ and all i, and $\{\mathbf{V}^m(\lambda)\}_{ii}=\infty$ for $\lambda>0$.

We may then proceed exactly as in the proof of Lemma 4, noting that $\lim_{R,T\to\infty}L_c(\lambda; R, T, m)=\{\mathbf{V}^m(\lambda)\}_{ii}>1$ for $\lambda\geqslant0$.

In an identical manner to Sect. 3 we may then use this lemma to define the operator E_T and hence to prove the following theorem.

Theorem 3. *When $\rho(\mathbf{\Gamma})>1$, for any $c>0$ there exist positive constants b_i and T_i, with T_i sufficiently large so that* $\min\{w_i(\mathbf{s}, t)\colon |\mathbf{s}|\leqslant ct\}\geqslant b_i$ *for $t\geqslant T_i$, $(i=1,\dots,n)$.*

When $\rho(\mathbf{\Gamma})>1$ and at least one $p_{ij}(\mathbf{r})$ is not exponentially dominated in the tail the asymptotic speed of propagation may then be considered to be infinite.

In order to establish the result that the speed of propagation is zero if $\rho(\mathbf{\Gamma})\leqslant1$, we impose the further condition that each $p_{ij}(\mathbf{r})$ is monotone decreasing in $|\mathbf{r}|$, differentiable with bounded derivatives and convex in the tails. Since $w_i(\mathbf{s}, t)$ is monotone increasing in t and uniformly bounded for $\mathbf{\Gamma}$ with all finite elements, $w_i(\mathbf{s})=\lim_{t\to\infty}w_i(\mathbf{s}, t)$ exists and is finite.

Theorem 4. *If* $\rho(\Gamma)\leq 1$, *then* $\lim_{t\to\infty}\sup\{w_i(\mathbf{s},t): |\mathbf{s}|\geq ct\}=0$ *for any* $c>0$ *and* $i=1,\ldots,n$.

Proof. Take $H_i^*(\mathbf{s},t)=\sup\{H_i(\mathbf{x},t): |\mathbf{x}|\geq|\mathbf{s}|\}$. Then $H_i^*(\mathbf{s},t)$ is a radial function of s and is monotone decreasing in $|\mathbf{s}|$. It is also continuous in s and uniformly bounded. Let $w_i^*(\mathbf{s},t)$ be the unique solution to

$$w_i^*(\mathbf{s},t)=\sum_{j=1}^{n}\int_{R^N}\int_0^t \gamma_{ij}(\tau)p_{ij}(\mathbf{r})g(w_j^*(\mathbf{s}-\mathbf{r},t-\tau))\,d\tau\,d\mathbf{r}+H_i^*(\mathbf{s},t), \qquad (4)$$

with $w_i^*(\mathbf{s},t)$ monotone increasing in t and $w_i^*(\mathbf{s},0)\equiv 0$ for $i=1,\ldots,n$. Note that $g(x)=1-\exp(-x)$.

Using the construction of the solution $w_i^*(\mathbf{s},t)$, and that of $w_i(\mathbf{s},t)$, given in [7] Theorem 4, it is easily seen that $w_i(\mathbf{s},t)\leq w_i^*(\mathbf{s},t)$ and $w_i^*(\mathbf{s},t)$ is a radial function of s which is monotone decreasing in $|\mathbf{s}|$. Let $w_i^*(\mathbf{s})=\lim_{t\to\infty}w_i^*(\mathbf{s},t)$ and $a_i^*(\mathbf{s})=\lim_{t\to\infty}H_i^*(\mathbf{s},t)$. If monotone convergence is now applied to (4), we obtain for each $i=1,\ldots,n$,

$$w_i^*(\mathbf{s})=\sum_{j=1}^{n}\gamma_{ij}\int_{R^N}p_{ij}(\mathbf{r})g(w_j^*(\mathbf{s}-\mathbf{r}))\,d\mathbf{r}+a_i^*(\mathbf{s}).$$

Since $w_i^*(\mathbf{s})$ is a radial function of s and is montone decreasing in $|\mathbf{s}|$, $w_i^*=\lim_{|\mathbf{s}|\to\infty}w_i^*(\mathbf{s})$ exists. Note that $w_i^*\geq 0$. Let $a_i(\mathbf{s})=\lim_{t\to\infty}H_i(\mathbf{s},t)$. The conditions imposed on γ_{ij}^* and $\varepsilon_i(\mathbf{s},\tau)$ imply that $a_i(\mathbf{s})$ is uniformly continuous with $\int_{R^N}a_i(\mathbf{s})\,d\mathbf{s}$ finite; and hence that $\lim_{|\mathbf{s}|\to\infty}a_i^*(\mathbf{s})=0$. Therefore $w_i^*=\sum_{j=1}^n\gamma_{ij}g(w_j^*)$ for $i=1,\ldots,n$. From our paper, [7], since $\rho(\Gamma)\leq 1$, we obtain the result that $w_i^*=0$.

Now for any $c>0$, $\sup\{w_i(\mathbf{s},t): |\mathbf{s}|\geq ct\}\leq\sup\{w_i^*(\mathbf{s},t): |\mathbf{s}|\geq ct\}=w_i^*(ct,t)$. For $t\geq T$, we have $w_i^*(ct,t)\leq w_i^*(cT,t)\leq w_i^*(cT)$. Hence, since $w^*=0$, we obtain the result that for any $c>0$ and $i=1,\ldots,n$, $\lim_{t\to\infty}\sup\{w_i(\mathbf{s},t): |\mathbf{s}|\geq ct\}=0$.

5. The pandemic theorem

Since $x_i(\mathbf{s},t)$ is monotone decreasing in t and bounded, $v_i(\mathbf{s})=1-\lim_{t\to\infty}x_i(\mathbf{s},t)$ exists, and measures the proportion of the population at position s who eventually suffer the epidemic. Note that $a_i(\mathbf{s})=\lim_{t\to\infty}H_i(\mathbf{s},t)$ is continuous and integrable, so that $a_i=\inf_s a_i(\mathbf{s})$ is necessarily zero for all i.

In our paper, [7], we did not restrict the a_i to be all zero, and proved the pandemic theorem giving a lower bound for $v_i(\mathbf{s})$, which depends on the a_i. In that paper the theorem was established for all dimensions N if at least one $a_i>0$ and/or Γ has at least one infinite element. Essén's result [4], which restricted the validity of the proof to $N=1$ and $N=2$ only, was only necessary when considering the case where $a_i=0$ all i and Γ is finite.

In this section we assume that $\rho(\Gamma)>1$, so that a major epidemic occurs. We can improve the lower bounds b_i in Theorem 2 for $w_i(\mathbf{s},t)$ and interpret them in terms of $v_i(\mathbf{s},t)=1-x_i(\mathbf{s},t)$. We show that for $c>0$, with $c<c_0$ if all the $p_{ij}(\mathbf{r})$ are exponentially dominated in the tail, $\lim_{t\to\infty}\inf[\min\{v_i(\mathbf{s},t): |\mathbf{s}|\leq ct\}]\geq\eta_i$, for $i=1,\ldots,n$, where η_i is defined as follows:

(i) If $\mathbf{\Gamma}$ has at least one infinite element in each row, then $\eta_i = 1$ for $i = 1, \ldots, n$.

(ii) If $\mathbf{\Gamma}$ is partitioned into

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{21} & \mathbf{\Gamma}_{22} \end{pmatrix},$$

where $\mathbf{\Gamma}_{11}$ is $m \times m$ and $(\mathbf{\Gamma}_{21}\mathbf{\Gamma}_{22})$ has at least one infinite element in each row, but $(\mathbf{\Gamma}_{11}\mathbf{\Gamma}_{12})$ has no infinite elements, then $\eta_i = 1$ for $i = m+1, \ldots, n$, and η_i for $i = 1, \ldots, m$ is the unique positive solution $y_i = \eta_i$ to

$$-\log(1-y_i) = \sum_{j=1}^{m} \gamma_{ij} y_j + \sum_{j=m+1}^{n} \gamma_{ij} \quad \text{for } i = 1, \ldots, m.$$

(iii) If $\mathbf{\Gamma}$ is finite, then η_i for $i = 1, \ldots, n$ is the unique positive solution $y_i = \eta_i$ to

$$-\log(1-y_i) = \sum_{j=1}^{n} \gamma_{ij} y_j \quad \text{for } i = 1, \ldots, n.$$

This is a stronger form of the pandemic theorem. The pandemic theorem given in [7], for the case $a_i = 0$ all i, then follows immediately as a corollary, and is valid for all dimensions N.

Note that this section may easily be rewritten without restricting a_i to be zero for all i. Only minor modifications are needed to lemmas 10 and 11. This would then establish the stronger form of pandemic theorem for all a_i, and also give an alternative proof in all cases of the pandemic theorem of [7]. We omit the details.

Let $\gamma_{ij}(R, T) = \int_0^T \gamma_{ij}(\tau)\, d\tau \int_{B_R} p_{ij}(r)\, dr$ and $\mathbf{\Gamma}(R, T) = (\gamma_{ij}(R, T))$. Note that $\mathbf{\Gamma}(R, T)$ is non-reducible for R and T sufficiently large, and $\rho(\mathbf{\Gamma}(R, T))$ is a continuous, increasing function of R and T. Hence there exists an R_0 and T_0 sufficiently large so that $\mathbf{\Gamma}(R, T)$ is non-reducible with $\rho(\mathbf{\Gamma}(R, T)) > 1$ for $R \geq R_0$ and $T \geq T_0$. For $R \geq R_0$ and $T \geq T_0$, define $\eta_i(R, T)$ to be the unique positive solution $y_i = \eta_i(R, T)$ to

$$-\log(1-y_i) = \sum_{j=1}^{n} \gamma_{ij}(R, T) y_j \quad \text{for } i = 1, \ldots, n.$$

Take any positive speed c such that either (i) $p_{ij}(\mathbf{r})$ is exponentially dominated in the tail for all i, j, and $c < c_0$, or (ii) at least one $p_{ij}(\mathbf{r})$ is not exponentially dominated in the tail. For any such c, any $\varepsilon > 0$, $R \geq R_0$ and $T \geq T_0$, we show that there exists a $t^* > T$ such that $\min\{w_i(\mathbf{s}, t) : |\mathbf{s}| \leq ct\} \geq -\log(1 - \eta_i(R, T)) - \varepsilon$, for $t \geq t^*$.

To establish the stronger form of pandemic theorem from this, it is necessary to show that $\eta_i(R, T) \uparrow \eta_i$ as R and T tend to infinity. The result is established in the following lemma.

Lemma 10. *For each $i = 1, \ldots, n$, $\eta_i(R, T) \uparrow \eta_i$ as R and T tend to infinity.*

Proof. Take any $R \geq R^* \geq R_0$ and $T \geq T^* \geq T_0$, where $\mathbf{\Gamma}(R_0, T_0)$ is non-reducible with $\rho(\mathbf{\Gamma}(R_0, T_0)) > 1$. Now, for each i,

$$-\log(1 - \eta_i(R, T)) = \sum_{j=1}^{n} \gamma_{ij}(R, T)\eta_j(R, T) = \sum_{j=1}^{n} \gamma_{ij}(R^*, T^*)\eta_j(R, T) + d_i,$$

where $d_i = \sum_{j=1}^{n} (\gamma_{ij}(R, T) - \gamma_{ij}(R^*, T^*))\eta_j(R, T) \geq 0$. Hence from [7], Theorem 1, $\eta_i(R^*, T^*) \leq \eta_i(R, T)$.

We first consider the case where $\mathbf{\Gamma}$ is finite. We may proceed as above using $\mathbf{\Gamma}$ in place of $\mathbf{\Gamma}(R, T)$, to obtain the result $\eta_i(R^*, T^*) \leq \eta_i$ for $R^* \geq R_0$ and $T^* \geq T_0$. Hence $\eta_i(R, T)$ is an increasing function of R and T, for $R \geq R_0$ and $T \geq T_0$, which is bounded above by η_i and below by $\eta_i(R_0, T_0)$. Then $\eta_i(R, T)$ tends to a positive limit y_i as $R \to \infty$ and $T \to \infty$, satisfying

$$-\log(1 - y_i) = \sum_{j=1}^{n} \gamma_{ij} y_j \quad \text{for } i = 1, \ldots, n.$$

Then from [7], Theorem 1, necessarily $y_i = \eta_i$ for $i = 1, \ldots, n$. Hence $\lim_{R,T\to\infty} \eta_i(R, T) = \eta_i$.

Now suppose that for some $0 \leq m < n$, $\mathbf{\Gamma}$ is partitioned into

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{21} & \mathbf{\Gamma}_{22} \end{pmatrix},$$

where $\mathbf{\Gamma}_{11}$ is $m \times m$, $(\mathbf{\Gamma}_{11}\mathbf{\Gamma}_{12})$ has all finite elements and $(\mathbf{\Gamma}_{21}\mathbf{\Gamma}_{22})$ has at least one infinite element in every row. Take $R \geq R_0$ and $T \geq T_0$ so that $\rho(\mathbf{\Gamma}(R, T)) > 1$. For each $i > m$ there exists a j such that $\gamma_{ij} = \infty$. Then, for such an i and j,

$$-\log(1 - \eta_i(R, T)) \geq \gamma_{ij}(R, T)\eta_j(R, T) \geq \gamma_{ij}(R, T)\eta_j(R_0, T_0).$$

Hence $\lim_{R,T\to\infty} \eta_i(R, T) = 1$ for $i = m+1, \ldots, n$. If $m = 0$, i.e. $\mathbf{\Gamma}$ has an infinity in every row, this completes the proof.

If $m > 0$, then for $i = 1, \ldots, m$ and $R \geq R_0$ and $T \geq T_0$,

$$-\log(1 - \eta_i(R, T)) = \sum_{j=1}^{m} \gamma_{ij}(R, T)\eta_j(R, T) + d_i(R, T),$$

where $d_i(R, T) = \sum_{j=m+1}^{n} \gamma_{ij}(R, T)\eta_j(R, T)$. Since $\eta_j(R, T)$ is monotone increasing in R and T and bounded above by η_j, it tends to a limit y_j as R and T tend to infinity satisfying the equations

$$-\log(1 - y_i) = \sum_{j=1}^{m} \gamma_{ij} y_j + \sum_{j=m+1}^{n} \gamma_{ij} \quad \text{for } i = 1, \ldots, m.$$

We can then use [7], Theorem 1 to obtain the result that $y_i = \eta_i$ and hence $\lim_{R,T\to\infty} \eta_i(R, T) = \eta_i$ for $i = 1, \ldots, m$.

Lemma 11. *Let $\mathbf{B} = (\beta_{ij})$ be a non-negative, non-reducible matrix of finite elements, with $\rho(\mathbf{B}) > 1$, and let $y_i = \phi_i$ be the unique positive solution to*

$$-\log(1 - y_i) = \sum_{j=1}^{n} \beta_{ij} y_j \quad \textit{for } i = 1, \ldots, n. \tag{5}$$

Take $\mathbf{u} > \mathbf{0}$ to be the right eigenvector corresponding to $\rho(\mathbf{B})$, scaled so that if $\{\mathbf{a}\}_i = -\log(1 - \{\mathbf{u}\}_i)$, then $\rho(\mathbf{B})g(\{\mathbf{a}\}_i) > \{\mathbf{a}\}_i$ for all i, where $g(x) = 1 - e^{-x}$. Define $\mathbf{N}_0 = \mathbf{a}$ and successively define for $m = 1, 2, \ldots,$

$$\{\mathbf{N}_{m+1}\}_i = \sum_{j=1}^{n} \beta_{ij} g(\{\mathbf{N}_m\}_j) \quad \textit{for } i = 1, \ldots, n$$

Then for every $\boldsymbol{\varepsilon} > \mathbf{0}$ *there exists an* M *such that* $\mathbf{N}_m > \boldsymbol{\phi} - \boldsymbol{\varepsilon}$ *for* $m \geq M$, *where* $\{\boldsymbol{\phi}\}_i = \phi_i$.

Proof. Now $\{\mathbf{N}_1\}_i = \sum_{j=1}^n \beta_{ij} g(\{\mathbf{N}_0\}_j) = \sum_{j=1}^n \beta_{ij}\{\mathbf{u}\}_j$. Hence $\mathbf{N}_1 = \mathbf{B}\mathbf{u} = \rho(\mathbf{B})\mathbf{u} > \mathbf{a} = \mathbf{N}_0$. Clearly from the definition of $\mathbf{N}_m$ and the non-reducibility of $\mathbf{B}$, $\{\mathbf{N}_m\}$ is a strictly increasing sequence. It is bounded above since $\{\mathbf{N}_m\}_i \leq \sum_{j=1}^n \beta_{ij}$ for $i = 1, \ldots, n$. Hence $\mathbf{N}_m$ tends to a limit $\mathbf{N}$ as $m \to \infty$, where $\mathbf{N}$ satisfies the equations

$$\{\mathbf{N}\}_i = \sum_{j=1}^n \beta_{ij} g(\{\mathbf{N}\}_j) \quad \text{for } i = 1, \ldots, n.$$

As $\mathbf{N} > \mathbf{N}_0 > \mathbf{0}$, it follows that $\mathbf{N} = \boldsymbol{\phi}$. Therefore there exists an M sufficiently large so that $\mathbf{N}_m > \boldsymbol{\phi} - \boldsymbol{\varepsilon}$ for $m \geq M$.

Theorem 5. *Consider any positive speed c such that either* (i) $p_{ij}(\mathbf{r})$ *is exponentially dominated in the tail for all i, j, and* $c < c_0$; *or* (ii) $p_{ij}(\mathbf{r})$ *is not exponentially dominated in the tail for some i, j. For any* $d_0 > c$ *still satisfying these conditions, take R and T sufficiently large, and* $h < 1$ *such that* $h\rho(\boldsymbol{\Gamma}(R, T)) > 1$. *Then for every* $\varepsilon_i > 0$ $(i = 1, \ldots, n)$ *there exists a* $t^* > T$ *such that* $\min\{w_i(\mathbf{s}, t) \colon |\mathbf{s}| \leq ct\} \geq -\log(1 - \eta_i(R, T)) - \varepsilon_i$ *for* $i = 1, \ldots, n$ *and* $t \geq t^*$, *where* $y_i = \eta_i(R, T)$ *is the unique positive solution to* (5) *when* $\mathbf{B} = \boldsymbol{\Gamma}(R, T)$.

Proof. From Theorem 2, there exist positive constants b_i and T_0 such that $\min\{w_i(\mathbf{s}, t) \colon |\mathbf{s}| \leq d_0 t\} \geq b_i$ for all $t \geq T_0$ and $i = 1, \ldots, n$. In Lemma 11 take $\mathbf{B} = \boldsymbol{\Gamma}(R, T)$ and the corresponding $a > 0$ so that $\{\mathbf{a}\}_i \leq \mathbf{b}_i$ for $i = 1, \ldots, n$. Then $w_i(\mathbf{s}, t) \geq \{\mathbf{N}_0\}_i$ for $|\mathbf{s}| \leq d_0 t$ and $t \geq T_0$, $(i = 1, \ldots, n)$.

Now for $t \geq T$,

$$w_i(\mathbf{s}, t) \geq \sum_{j=1}^n \int_0^T \int_{B_R} g(w_j(\mathbf{s} - \mathbf{r}, t - \tau))\gamma_{ij}(\tau) p_{ij}(\mathbf{r})\, d\mathbf{r}\, d\tau$$

$$\geq \sum_{j=1}^n \gamma_{ij}(R, T) g(\{\mathbf{N}_0\}_j) = \{\mathbf{N}_1\}_i,$$

if $|\mathbf{s} - \mathbf{r}| \leq d_0(t - \tau)$ for $\mathbf{r} \in B_R$ and $(t - \tau) \geq T_0$ when $\tau \leq T$. These conditions will certainly hold if $|\mathbf{s}| \leq -R - d_0 T + d_0 t$ and $t \geq T_0 + T$.

Next take $d_1 \in (c, d_0)$. Then $w_i(\mathbf{s}, t) \geq \{N_1\}_i$ for $|\mathbf{s}| \leq d_1 t$ and $t \geq T_1$, where $T_1 = \max\{T_0 + T, (R + Td_0)/(d_0 - d_1)\}$. By successively choosing $d_{j+1} \in (c, d_j)$ and defining $T_{j+1} = \max\{T_j + T, (R + Td_j)/(d_j - d_{j+1})\}$ for $j = 1, 2, \ldots$, we obtain the result $w_i(\mathbf{s}, t) \geq \{\mathbf{N}_j\}_i$ for $|\mathbf{s}| \leq d_j t$ and $t \geq T_j$ for $i = 1, \ldots, n$ and $j = 1, 2, \ldots$.

Now for any $\varepsilon_i > 0$ $(i = 1, \ldots, n)$ choose M as in Lemma 11 so that $\{\mathbf{N}_M\}_i \geq -\log(1 - \eta_i(R, T)) - \varepsilon_i$. Then $w_i(\mathbf{s}, t) \geq -\log(1 - \eta_i(R, T)) - \varepsilon_i$ for $|\mathbf{s}| \leq d_M t$ and $t \geq T_M$. Hence if we take $t^* = T_M$, we obtain the result

$$\min\{w_i(\mathbf{s}, t) \colon |\mathbf{s}| \leq ct\} \geq -\log(1 - \eta_i(R, T)) - \varepsilon_i \text{ for } i = 1, \ldots, n \text{ and } t \geq t^*.$$

The result for $w_i(\mathbf{s}, t)$ may easily be converted into an equivalent result for $v_i(\mathbf{s}, t) = 1 - x_i(\mathbf{s}t)$, namely

$$\min\{v_i(\mathbf{s}, t) \colon |\mathbf{s}| \leq ct\} \geq 1 - \exp(\log(1 - \eta_i(R, T)) + \varepsilon_i) = 1 - (1 - \eta_i(R, T))e^{\varepsilon_i}$$

for $i=1,\dots,n$ and $t \geq t^*$. From this result, and Lemma 10, the following two corollaries are immediate.

Corollary 3. *Under the conditions on* c *specified in Theorem* 5, $\lim_{t\to\infty} \inf \min\{v_i(\mathbf{s}, t): |\mathbf{s}| \leq ct\} \geq \eta_i$ *for* $i=1,\dots,n$.

Corollary 4. (The pandemic theorem). *For all* $\mathbf{s}$ *and* $i=1,\dots,n$, $v_i(\mathbf{s}) \geq \eta_i$.

Note that Corollary 4 gives a lower bound for the proportion of individuals at position $\mathbf{s}$, in population i, who eventually suffer the epidemic; this bound holding when $\rho(\mathbf{\Gamma}) > 1$ so that a major epidemic occurs.

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Received February 22, revised June 10, 1985