On Conformal Mapping of Certain Classes of Jordan Domains

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Let C be a closed rectifiable Jordan curve and f a function which maps the interior of C conformally onto the unit disk. In this note we give bounds for

$$\int_C |f'(z)|^p \, |dz|$$

for three classes of curves C: (1) C has bounded arc length-chord length ratio, (2) C is of bounded rotation, and (3) C is sectionally smooth and has a finite number of corners. In all three cases the bounds obtained depend only on certain parameters of the curve C and are therefore useful when one desires bounds which hold uniformly for families of curves characterized by the same parameter values.

An application of the results for the classes (2) and (3) is found in [9]. It is also readily seen that in all three cases the existence of the integral

$$\int_C |f'(z)|^p \, |dz|$$

implies that $f' \in H_p(C)$ (*cf.* [9, p. 500]). Conditions which imply that higher derivatives $f^{(n)}$, $n \geqq 2$, are in $H_p(C)$ were given by SEIDEL [6] and SMIRNOFF [7].

1. Curves with Bounded Arc Length-Chord Length Ratio

A rectifiable Jordan curve C has a *bounded arc length-chord length ratio* if

$$\sigma_{PQ} \leqq b\, r_{PQ} \qquad (b \text{ constant}, \ 1 < b < \infty) \tag{1}$$

uniformly for all points $P, Q \in C$. Here σ_{PQ} is the shorter arc length along C between P and Q, and r_{PQ} is the Euclidean distance between P and Q.

Theorem 1. *Let $w = f(z)$ map R, the interior of a rectifiable Jordan curve C, conformally onto $|w| < 1$ such that $z_0 \in R$ corresponds to $w = 0$. Let d be the distance from z_0 to C, and assume that C satisfies condition* (1). *Then there exists a $\delta > 0$ which depends on b only such that for every h, $0 < h < \delta$,*

$$\int_C |f'(z)|^{1+h} \, |dz| \leqq 2\pi \left[1 + \left(\frac{2\pi}{d} \right)^{\delta} \frac{1}{e^{-h} - e^{-\delta}} \right]. \tag{2}$$

Proof. We make use of the following result of LAVRENTIEFF [1, Theorem 7]: Under the hypothesis of our Theorem 1 there exists a $\delta>0$, which depends on b only, such that under the mapping f a set $E\subset C$ of measure ε corresponds to a set $\mathscr{E}$ on $|w|=1$ with measure $m(\mathscr{E})\leqq 2\pi\left(\frac{\varepsilon}{d}\right)^{\delta}$.

Assume

$$\int_C |f'(z)|^p\,|dz|=A<\infty$$

for some $p\geqq 1$. (Since f is absolutely continuous on C, this is certainly valid for $p=1$ and $A=2\pi$.) Let $E_n=\{z\in C\colon e^n<|f'(z)|^p\}$ and $\mathscr{E}_n=f(E_n)$. Then the measure of the set E_n with respect to the arc length of C

$$m(E_n)=\int_{E_n}|dz|\leqq e^{-n}\int_{E_n}|f'(z)|^p\,|dz|\leqq A\,e^{-n},$$

and by LAVRENTIEFF'S result

$$m(\mathscr{E}_n)\leqq 2\pi\left(\frac{A\,e^{-n}}{d}\right)^{\delta}$$

where $\delta>0$ and depends only on b. Let $\varphi(w)=f^{-1}(w)$, and note that $\mathscr{F}_n=\{e^{i\theta}\colon e^n<|\varphi'(e^{i\theta})|^{-p}\leqq e^{n+1}\}\subset\mathscr{E}_n$. Then

$$\begin{aligned}
\int_C |f'(z)|^{1+ph}\,|dz| &= \int_0^{2\pi}|\varphi'(e^{i\theta})|^{-ph}\,d\theta\\
&\leqq 2\pi+\sum_{n=0}^{\infty}\int_{\mathscr{F}_n}|\varphi'(e^{i\theta})|^{-ph}\,d\theta\\
&\leqq 2\pi+\sum_{n=0}^{\infty}e^{(n+1)h}\,m(\mathscr{E}_n) \qquad (3)\\
&\leqq 2\pi+\sum_{n=0}^{\infty}e^{(n+1)h}\,2\pi\left(\frac{A\,e^{-n}}{d}\right)^{\delta}\\
&=2\pi\left[1+\left(\frac{A}{d}\right)^{\delta}\frac{1}{e^{-h}-e^{-\delta}}\right]
\end{aligned}$$

for $h<\delta$. Now (2) follows from (3) with $p=1$ and $A=2\pi$.

Corollary. *Under the hypothesis of Theorem 1*

$$\int_C |f'(z)|^{\sum_{\nu=0}^{n}h^{\nu}}\,|dz|\leqq 2\pi\sum_{\nu=0}^{n}\left(\frac{2\pi\,d^{-\delta}}{e^{-h}-e^{-\delta}}\right)^{\nu} \qquad (4)$$

for every non-negative $h<\delta$ where $\delta>0$ depends only on b. Consequently, for every non-negative $\eta<\frac{\delta}{1-\delta}$

$$\int_C |f'(z)|^{1+\eta}\,|dz|\leqq M(\eta,\delta(b),d). \qquad (5)$$

Proof. If

$$\int_C |f'(z)|^p |dz| \leqq A, \qquad A \geqq 1,$$

it follows from (3) that

$$\int_C |f'(z)|^{1+ph} |dz| \leqq 2\pi + \left(\frac{2\pi d^{-\delta}}{e^{-h} - e^{-\delta}}\right) A$$

for $h<\delta$ (note $\delta \leqq 1$). By substituting consecutively $p=1$ and $A=2\pi$, $p=1+h$ and

$$A = 2\pi\left[1 + \frac{2\pi d^{-\delta}}{e^{-h} - e^{-\delta}}\right], \ldots, p = \sum_{\nu=0}^{n-1} h^\nu \quad \text{and} \quad A = 2\pi \sum_{\nu=0}^{n-1} \left(\frac{2\pi d^{-\delta}}{e^{-h} - e^{-\delta}}\right)^\nu,$$

one obtains (4).

2. Curves of Bounded Rotation

A rectifiable Jordan curve C is called *of bounded rotation* [*3*; see also *4*, p. 225] if the forward half-tangent exists at every point and the tangent angle $\tau(s)$ which it makes with a fixed direction may be defined as a function of bounded variation of the arc length s, $0 \leqq s \leqq L$. Furthermore, $\tau(s)$ is so determined that its jumps do not exceed π in absolute value. We assume that the arc length parametrization corresponds to the positive orientation of C.

Theorem 2. *Suppose C is a rectifiable Jordan curve of bounded rotation whose interior R has area at most A and contains a disk of radius d about z_0. Suppose furthermore that condition* (1) *is satisfied. Let $v_+(s)$ and $v_-(s)$ be the positive and negative variation functions of $\tau(s)$, respectively, and*

$$a_\pm = \max_s \, [v_\pm(s+0) - v_\pm(s-0)]\ ^1.$$

Let $z=\varphi(w)$ map $|w|<1$ conformally onto R such that $\varphi(0)=z_0$. Then there exist constants M_p^+ and M_p^- depending only on p, $a_\pm$, b, d, A, and the function $v_\pm(s)$ such that uniformly for $0 \leqq r \leqq 1$

$$\int_0^{2\pi} |\varphi'(r e^{i\theta})|^p \, d\theta \leqq M_p^+ \qquad \text{for } 0 \leqq p < \frac{\pi}{a_+} \tag{6}$$

and

$$\int_0^{2\pi} \frac{d\theta}{|\varphi'(r e^{i\theta})|^p} \leqq M_p^- \qquad \text{for } 0 \leqq p < \frac{\pi}{a_-}. \tag{7}$$

Consequently, if $f(z) = \varphi^{-1}(z)$, then similar bounds $N_p^\pm$ exist, depending on the same parameters, such that

$$\int_C \frac{|dz|}{|f'(z)|^p} \leqq N_p^+ \qquad \text{for } 0 \leqq p < \frac{\pi}{a_+} - 1 \tag{8}$$

and

$$\int_C |f'(z)|^p \, |dz| \leqq N_p^- \qquad \text{for } 0 \leqq p < \frac{\pi}{a_-} + 1. \tag{9}$$

[1] The symbol $\pm$ denotes that the statement is interpreted throughout using either the $+$ sign or the $-$ sign.

If $a_+=0$, *then* (6) *and* (8) *are valid for all* $p\geqq 0$, *and if* $a_-=0$, *then* (7) *and* (9) *are valid for all* $p\geqq 0$ [2].

Remark 1. This theorem is sharp in the sense that the ranges of p given are the best possible. Indeed there exist curves (*e.g.*, polygons) satisfying the hypothesis but for which the integrals in (6)–(9) diverge if p is allowed to equal the least upper bound of its given range. This theorem contains statement B of [9, p. 499] as a special case.

Remark 2. *One obtains uniform bounds in* (6) *and* (8) *in the following manner. Let* V_+ *be a non-decreasing function defined on* $[0, S]$ *and*

$$A_+=\max_s\left[V_+(s+0)-V_+(s-0)\right].$$

Suppose that $L\leqq S$, *and that*

$$\int_I dv_+(s)\leqq \int_I dV_+(s)$$

over every interval I *of* $[0, L]$. *Then the bounds in* (6) *and* (8) *may be chosen to depend only on* p, A_+, b, d, A, *and the function* V_+. *The ranges of* p *then involve* A_+ *rather than* a_+. *A completely analogous statement is valid for the bounds in* (7) *and* (9).

Before proving Theorem 2 we first present a lemma.

Lemma 1. *Suppose* C *is a rectifiable Jordan curve satisfying condition* (1). *Let* $z=\varphi(w)$ *map* $|w|<1$ *conformally onto* R, *the interior of* C, *such that* $\varphi(0)=z_0$. *Assume that* R *has area at most* A *and contains a disk of radius* d *about* z_0.

Then there exist positive constants H *and* $\mathfrak{a}$ *depending only on* b, d, *and* A *such that*

$$|\varphi(w)-\varphi(\omega)|\leqq H|w-\omega|^\alpha \qquad \alpha=\frac{2}{(1+b)^2} \tag{10}$$

for $|w-\omega|\leqq\frac{1}{2}\mathfrak{a}$, $|w|\leqq 1$, $|\omega|=1$. *Furthermore, the arc length* $s(\theta)$ *of* C *as a function of* θ (*where* $z=\varphi(e^{i\theta})$) *satisfies*

$$|s(\theta)-s(\vartheta)|\leqq K|\theta-\vartheta|^\alpha \qquad (K=bH) \tag{11}$$

for $|\theta-\vartheta|\leqq\frac{1}{2}\mathfrak{a}$.

Proof. Lemma 1 in [*10*] establishes (10) for every ω with $|\omega|=1$ and $|w|\leqq 1$, provided $|w-\omega|\leqq\frac{1}{2}\mathfrak{a}$, and it is shown there that H depends only on b, A, and $\mathfrak{a}$. The constant $\mathfrak{a}$ is defined such that the image of the arc $\{w: |w|=1, |w-\omega|\leqq\mathfrak{a}\}$ under the mapping $z=\varphi(w)$ has a smaller arc length than the complementary arc of C. To prove our statement concerning H it is therefore sufficient to show that an $\mathfrak{a}$ may be chosen which depends only on b, d, and A.

Let $|\omega|=1$, and denote $k_r=\{w: |w-\omega|=r, |w|\leqq 1\}$. By a theorem of WOLFF [*11*, p. 217] there exists for every r, $0<r<1$, *a* ρ, $r\leqq\rho\leqq\sqrt{r}$, such that

[2] The same result is obtained if one uses the extended definition of PAATERO [*2*] for simply connected domains of bounded boundary rotation. In this case $\tau(s)$ is to be replaced by the function $\psi(\theta)$ used in the representation of log $\varphi'(w)$ by a Poisson-Stieltjes integral [*2*, p. 44] in $|w|<1$ and $v_+(s)$ and $v_-(s)$ by the positive and negative variations of $\psi(\theta)$.

$\gamma_\rho = \varphi(k_\rho)$ is a crosscut of R whose length

$$l_\rho \leqq \sqrt{\frac{2\pi A}{|\log r|}}.$$

For $r = \exp[-8\pi A\, b^2/d^2]$ the resulting crosscut γ_ρ has length

$$l_\rho \leqq \frac{1}{2}\,\frac{d}{b}.$$

Denote by Γ the arc of C which is the image of the arc $\{w: |w|=1, |w-\omega| \leqq \rho\}$. Since $\rho < 1$, the closed Jordan curve $(C-\Gamma) \cup \gamma_\rho$ contains $\varphi(0) = z_0$ in its interior R_0. Furthermore, since γ_ρ has length

$$l_\rho \leqq \frac{1}{2}\,\frac{d}{b} < \frac{1}{2}\, d,$$

the whole disk $|z - z_0| \leqq \frac{1}{2} d$ must be contained in R_0. If $C - \Gamma$ were to have an arc length less than or equal to that of Γ, then by (1) the arc length of $C - \Gamma$ would be at most $b\, l_\rho \leqq \frac{1}{2} d$. In that case the total length of $(C-\Gamma) \cup \gamma_\rho$ would not exceed d, but this is impossible since R_0 contains a disk of radius $\frac{1}{2} d$ which has circumference $\pi d > d$. Thus Γ has a shorter arc length than $C - \Gamma$, and $\mathfrak{a} = \exp[-8\pi A\, b^2/d^2]$ has the desired properties. Now the inequality (11) follows from (1) and (10) with $K = bH$.

Proof of Theorem 2. We express by means of the mapping $z = \varphi(e^{i\theta})$ the angle θ, $0 \leqq \theta \leqq 2\pi$, as a function of the arc length s, $0 \leqq s \leqq L$. Then for $|w| < 1$ (see [*2*, p. 44])

$$\begin{aligned} \log\left|\frac{\varphi'(w)}{\varphi'(0)}\right| &= -\frac{1}{\pi}\int_0^L \log|e^{i\theta(s)} - w|\, d\tau(s) \\ &= \frac{1}{\pi}\int_0^L \log\left|\frac{2}{e^{i\theta(s)} - w}\right| d\tau(s) - 2\log 2. \end{aligned} \tag{12}$$

Let $p > 0$ be restricted so that $p < \pi/a_\pm$. (If $a_\pm = 0$, p may be any positive number.) Then since the integrand is of one sign,

$$\begin{aligned} \log\left|\frac{\varphi'(w)}{\varphi'(0)}\right|^{\pm p} &= \frac{\pm p}{\pi}\int_0^L \log\left|\frac{2}{e^{i\theta(s)} - w}\right| d\tau(s) - (2\log 2)(\pm p) \\ &\leqq \frac{p}{\pi}\int_0^L \log\left|\frac{2}{e^{i\theta(s)} - w}\right| dv_\pm(s) + 2p\log 2. \end{aligned} \tag{13}$$

Let

$$h = \frac{1}{2\pi}(p\, a_\pm + \pi) < 1.$$

Since

$$\frac{\pi h}{p} > a_\pm,$$

there exists a $\delta > 0$ such that

$$v_\pm(s') - v_\pm(s'') \leqq \frac{\pi h}{p} \qquad \text{if } s' - s'' \leqq \delta \ (0 \leqq s'' < s' \leqq L).$$

Otherwise for every n there would exist points $s_n', s_n'' \in [0, L]$ such that

$$0 < s_n' - s_n'' < \frac{1}{n}$$

but

$$v_\pm(s_n') - v_\pm(s_n'') > \frac{\pi h}{p}. \tag{14}$$

We may assume that the sequences $\{s_n'\}$, $\{s_n''\}$ converge monotonically to a point ξ. If both sequences approach ξ from the same side, then the left hand side of (14) tends to zero; if they converge from opposite sides, the limit is

$$v_\pm(\xi+0) - v_\pm(\xi-0) \leqq a_\pm < \frac{\pi h}{p},$$

so that we obtain a contradiction in either case. Thus there exists a partition of $[0, L]$, $0 = s_1 < s_2 < \cdots < s_{n+1} = L$, such that

$$0 \leqq \lambda_k \equiv v_\pm(s_{k+1}) - v_\pm(s_k) \leqq \frac{\pi h}{p},$$

where each s_k may be chosen as a point where $v_\pm'(s_k)$ exists. We assume that the partition is sufficiently fine that

$$\theta(s_{k+1}) - \theta(s_k) \leqq \frac{1}{2}\,\mathfrak{a} < \frac{\pi}{2},$$

where $\mathfrak{a}$ is the constant of Lemma 1. By the result of LAVRENTIEFF [*1*, Theorem 7] quoted earlier this restriction on the norm of the partition of $[0, L]$ depends only on the parameters b, d, and A.

Now there exist constants A_k and B_k which depend only on b, d, A, and the function $v_\pm(s)$ such that

$$\int_{s_k}^{s_{k+1}} \log \frac{\pi}{\theta(s) - \theta(s_k)}\, dv_\pm(s) \leqq A_k \quad \text{and} \quad \int_{s_k}^{s_{k+1}} \log \frac{\pi}{\theta(s_{k+1}) - \theta(s)}\, dv_\pm(s) \leqq B_k.$$

To establish the bound A_k we first note that $v_\pm'(s_k)$ exists so that

$$G_k = \sup_{s_k < s < s_{k+1}} \frac{v_\pm(s) - v_\pm(s_k)}{s - s_k} < \infty.$$

Moreover, by Lemma 1 there exist positive constants K and α which depend only on b, d, and A such that $s - s_k \leqq K[\theta(s) - \theta(s_k)]^\alpha$ for $s_k \leqq s \leqq s_{k+1}$. Consequently, through integration by parts

$$\int_{s_k}^{s_{k+1}} \log \frac{\pi}{\theta(s) - \theta(s_k)}\, dv_\pm(s) \leqq \frac{1}{\alpha} \int_{s_k}^{s_{k+1}} \log \frac{K\pi^\alpha}{s - s_k}\, dv_\pm(s)$$

$$= \frac{1}{\alpha}[v_\pm(s_{k+1}) - v_\pm(s_k)] \log \frac{K\pi^\alpha}{s_{k+1} - s_k} + \frac{1}{\alpha} \int_{s_k}^{s_{k+1}} \frac{v_\pm(s) - v_\pm(s_k)}{s - s_k}\, ds$$

$$\leqq \frac{1}{\alpha}[v_\pm(s_{k+1}) - v_\pm(s_k)] \log \frac{K\pi^\alpha}{s_{k+1} - s_k} + \frac{1}{\alpha} G_k(s_{k+1} - s_k) \equiv A_k.$$

One develops the bound B_k in a similar manner.

Let $w=r\,e^{it}$, $0\leqq t\leqq 2\pi$, and let t be exterior to the interval $(\theta(s_k), \theta(s_{k+1}))$. Then for $s_k\leqq s\leqq s_{k+1}$

$$|e^{i\theta(s)}-w|\geqq \min\{\sin[\theta(s)-\theta(s_k)], \sin[\theta(s_{k+1})-\theta(s)]\}$$

$$\geqq \frac{2}{\pi}\min\{\theta(s)-\theta(s_k), \theta(s_{k+1})-\theta(s)\}.$$

As a result,

$$\int_{s_k}^{s_{k+1}} \log\left|\frac{2}{e^{i\theta(s)}-w}\right| dv_\pm(s) \leqq \int_{s_k}^{s_{k+1}} \log\frac{\pi}{\min\{\theta(s)-\theta(s_k), \theta(s_{k+1})-\theta(s)\}}\, dv_\pm(s) \leqq A_k+B_k. \tag{15}$$

Now if $\theta(s_j)\leqq t\leqq \theta(s_{j+1})$, we obtain from (13) and (15)

$$\log\left|\frac{\varphi'(w)}{\varphi'(0)}\right|^{\pm p} \leqq \frac{p}{\pi}\sum_{k=1}^{n}(A_k+B_k)+\frac{p}{\pi}\int_{s_j}^{s_{j+1}}\log\left|\frac{2}{e^{i\theta(s)}-w}\right| dv_\pm(s)+2p\log 2.$$

If

$$0<\lambda_j\leqq\frac{\pi h}{p},$$

then since $\log x$ is a convex function,

$$\frac{p}{\pi}\int_{s_j}^{s_{j+1}}\log\left|\frac{2}{e^{i\theta(s)}-w}\right| dv_\pm(s)\leqq\frac{1}{\lambda_j}\int_{s_j}^{s_{j+1}}\log\left|\frac{2}{e^{i\theta(s)}-w}\right|^h dv_\pm(s)$$

$$\leqq\log\left[\frac{1}{\lambda_j}\int_{s_j}^{s_{j+1}}\left|\frac{2}{e^{i\theta(s)}-w}\right|^h dv_\pm(s)\right],$$

and so

$$\left|\frac{\varphi'(w)}{\varphi'(0)}\right|^{\pm p}\leqq 4^p\, e^{\frac{p}{\pi}\sum_{k=1}^{n}(A_k+B_k)}\,\frac{1}{\lambda_j}\int_{s_j}^{s_{j+1}}\left|\frac{2}{e^{i\theta(s)}-w}\right|^h dv_\pm(s).$$

Thus if $0\leqq\lambda_j\leqq\frac{\pi h}{p}$,

$$\left|\frac{\varphi'(w)}{\varphi'(0)}\right|^{\pm p}\leqq H_\pm\left[1+\int_{s_j}^{s_{j+1}}\left|\frac{2}{e^{i\theta(s)}-w}\right|^h dv_\pm(s)\right]$$

where

$$H_\pm=4^p\, e^{\frac{p}{\pi}\sum_{k=1}^{n}(A_k+B_k)}\left[1+\max_{\lambda_k\neq 0}\frac{1}{\lambda_k}\right].$$

Therefore, since $h<1$,

$$\int_{\theta(s_j)}^{\theta(s_{j+1})}\left|\frac{\varphi'(r\,e^{it})}{\varphi'(0)}\right|^{\pm p} dt\leqq H_\pm\left[\int_{\theta(s_j)}^{\theta(s_{j+1})} dt+2\int_{s_j}^{s_{j+1}} dv_\pm(s)\int_{\theta(s_j)}^{\theta(s_{j+1})}\frac{dt}{|e^{i\theta(s)}-r\,e^{it}|^h}\right]$$

and

$$\int_0^{2\pi}\left|\frac{\varphi'(r\,e^{it})}{\varphi'(0)}\right|^{\pm p} dt\leqq H_\pm\left[2\pi+\frac{4\pi}{1-h}\int_0^L dv_\pm(s)\right]\equiv M_\pm \tag{16}$$

where $M_\pm$ depends only on p, $a_\pm$, b, d, A, and the function $v_\pm(s)$. Using the Cauchy estimate below and RENGEL's inequality above, we find $d \leqq |\varphi'(0)| \leqq \sqrt{A/\pi}$. The inequalities (6) and (7) follow for $0 \leqq r < 1$ with $M_p^+ = (A/\pi)^{\frac{1}{2}p} M_+$ and $M_p^- = d^{-p} M_-$.

By a well-known theorem of RIESZ the bounds M_p^+ and M_p^- may be extended to $r=1$. The inequalities (8) and (9) are immediate consequences of (6) and (7).

To verify the remark concerning uniform bounds, one replaces $a_\pm$ with $A_\pm$ throughout the proof and $v_\pm(s)$ with $V_\pm(s)$ beginning at (13). The partition satisfying (14) is then constructed with respect to the function $V_\pm$ and its whole interval $[0, S]$, $S \geqq L$, of definition. However, one restricts the range of integration of all integrals in the proof to $[0, L]$. Since the constants A_k and B_k can be constructed for all $k \leqq n$, $H_\pm$ is uniform with respect to $A_\pm$ and $V_\pm$. If one uses

$$\int_0^S dV_\pm(s)$$

in (16), then both $M_\pm$ and $M_p^\pm$ are also uniform with respect to $A_\pm$ and $V_\pm$.

3. Sectionally Smooth Curves

The following theorem considers curves with a finite number of corners.

Theorem 3. *Suppose C is a rectifiable Jordan curve with sectionally continuous tangent,* i.e., *the tangent angle $\tau(s)$ is a continuous function of the arc length s except for a finite number of points at which C has vertices with interior angles α_k, $0 < \alpha_k < 2\pi$, $k = 1, 2, \ldots, n$. Suppose furthermore that*

(i) *$\beta(t)$ is a modulus of continuity of $\tau(s)$ in each (closed) interval where $\tau(s)$ is continuous,* i.e., $|\tau(s \pm t) - \tau(s)| \leqq \beta(t)$, $t > 0$, *where $\beta(t)$ is non-decreasing and* $\lim_{t \to 0^+} \beta(t) = 0$;

(ii) *C satisfies condition* (1);

(iii) *the interior R of C has area A and contains a point z_0 at the distance d from C;*

(iv) $a_+ = \max_{1 \leqq k \leqq n} [\pi - \alpha_k, 0]$ *and* $a_- = \max_{1 \leqq k \leqq n} [\alpha_k - \pi, 0]$.

Let $z = \varphi(w)$ map $|w| < 1$ conformally onto R such that $\varphi(0) = z_0$. Then there exist constants M_p^+ and M_p^- depending only on p, $a_\pm$, b, d, A, n, and the function $\beta(t)$ such that uniformly for $0 \leqq r \leqq 1$

$$\int_0^{2\pi} |\varphi'(r e^{i\theta})|^p \, d\theta \leqq M_p^+ \qquad \textit{for } 0 \leqq p < \frac{\pi}{a_+} \tag{17}$$

and

$$\int_0^{2\pi} \frac{d\theta}{|\varphi'(r e^{i\theta})|^p} \leqq M_p^- \qquad \textit{for } 0 \leqq p < \frac{\pi}{a_-}. \tag{18}$$

Consequently, if $f(z) = \varphi^{-1}(z)$, then similar bounds $N_p^\pm$ exist, depending on the same parameters, such that

$$\int_C \frac{|dz|}{|f'(z)|^p} \leqq N_p^+ \qquad \textit{for } 0 \leqq p < \frac{\pi}{a_+} - 1 \tag{19}$$

and

$$\int_C |f'(z)|^p |dz| \leqq N_p^- \qquad \textit{for } 0 \leqq p < \frac{\pi}{a_-} + 1. \tag{20}$$

If $a_+ = 0$, *then* (17) *and* (19) *are valid for all* $p \geqq 0$, *and if* $a_- = 0$, *then* (18) *and* (20) *are valid for all* $p \geqq 0$.

Remarks. This theorem is an extension of the result in [*8*] for smooth curves and is proved in [*5*, Theorem 10]. Again the results are sharp in the sense that the ranges of p given are the best possible. This theorem contains statement A of [*9*, p. 499] as a special case.

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