Acta Informatica 15, 365–384 (1981)

Acta Informatica

The Logic of Aliasing

Robert Cartwright[1] and Derek Oppen[2]

[1] Mathematical Sciences Department, Rice University, Houston, Texas 77001, USA
[2] Computer Science Department, Stanford University, Stanford, California 94305, USA

Summary. We present a new version of Hoare's logic that correctly handles programs with aliased variables. The central proof rules of the logic (procedure call and assignment) are proved sound and complete.

1. Introduction

One of the most discredited features common to many programming languages is *aliasing*, the practice of associating more than one name with a piece of storage in a program. Since explicitly changing the value of one variable may implicity change the values of other variables, it is widely argued that aliasing makes writing, debugging and understanding programs more difficult.

The major technical argument against aliasing is that it makes devising intelligible proof rules for reasoning about programs extremely difficult – that programming languages admitting aliasing cannot be satisfactorily axiomatized. The problem is most acute for assignment rules and procedure call rules. None of the assignment or procedure call rules published to date admit aliasing (see for example [9, 17, 3, 7, 11, 5, 12]).

Although the prohibition of aliasing is the most severe limitation imposed by existing proof rules, all place additional restrictions on procedures and procedure calls[1]. For instance, the most comprehensive procedure call rule proposed to date (for EUCLID by [12]) must:

An earlier version of this paper appeared in the Proceedings of the Fifth ACM Symposium on Principles of Programming Languages, 1978. This research has been partially supported by National Science Foundation grants MCS 76-14293 and MCS 76-000327

[1] Apt and de Bakker [1] have proposed procedure call and assignment rules which eliminate all of these restrictions, except 2. However, their proof rules nullify an important property of Hoare's logic: that proof rules not modify program text. Their procedure call rule rewrites the entire procedure body, destroying the direct relationship between asserted programs and the structure of proofs in Hoare's logic. Further, the Apt-de Bakker rules force the correctness of a procedure body to be separately proved for every possible aliasing relationship

0001-5903/81/0015/0365/$04.00

1. Prohibit aliasing in procedure calls.
2. Disallow passing procedures and functions as parameters.
3. Require that value parameters be read-only (that is, constant parameters).
4. Prohibit declareing a procedure within a procedure of the same name.
5. Require that (relatively) global variables accessed by a procedure be accessible at every point of call.

Our purpose in this paper is to develop a new version of Hoare's logic that handles unrestricted aliasing. We therefore concentrate on rules for assignment and for procedure calls. The proof rules we give are no more complex than existing rules of comparable scope that prohibit aliasing. The tradeoff is that proofs are more tedious when aliasing is actually used.

First we give a simple simultaneous assignment rule (similar to that given by Gries [8]) and then a simple procedure call rule (patterned after Hoare's rule [10] along lines similar to the EUCLID rule by London et al. [12]) for calls where no aliasing is present. Next, we propose generalized assignment and procedure call rules for contexts where aliasing is permitted. Both generalized rules collapse to the corresponding simple rules if no aliasing exists.

All the rules that we propose in this paper are proved sound and relatively complete (in the sense of Cook). Although this may seem a tedious and unnecessary exercise, we believe that it is essential to give formal justifications for proof rules. The semantics of procedure calls in "real" programming languages (such as PASCAL) are so complicated that none of the proposed axiomatizations for such languages in Hoare's logic [17, 12] is sound. We too made mistakes in our first attempts at axiomatizing aliasing, and we discovered our errors only when trying to formally justify our axiomatization.

The rules we give in this paper are more formally stated than is common in the literature. Since we wished to prove our rules sound, we had to state explicitly what assumptions our rules require. Consequently, our rules will appear longer and more complicated than comparable rules in the literature.

2. Mathematical Foundations

Before we can formulate and justify our proof rules, we must establish the mathematical foundations for our version of Hoare's logic. We introduce three sets of definitions.

2.1. State Vectors and Abstract Addresses

From an informal viewpoint, a *state vector* is a sequence of bindings of program variable names to data values, and procedure names to procedure bodies (as in a LISP association list). An *abstract address* is a sequence of data values (including variable names) serving as a canonical name for an entry in a state vector. For example, the abstract address for the variable x is $\langle' x\rangle$. Since x typically means the value of the variable x, we use the notation $'x$ to refer to the name of the variable. The abstract address for the array element $a[1]$ is $\langle' a, 1\rangle$.

More formally, we let $\mathscr{D}$ denote the set of data values that program variables may assume, and let $\mathscr{I}$ and $\mathscr{I}'$ denote the set of program identifiers $a, b, c, \ldots$, and quoted program identifiers $'a, 'b, 'c, \ldots$, respectively. We let $\mathscr{B}$ denote the set of procedure bodies. A *variable* is any legal left-hand side of an assignment statement. A *simple variable* is a variable consisting of a single identifier. For example, $a[x]$ and x are both variables; x is a simple variable, but $a[x]$ is not.

For the sake of simplicity, we limit our attention to a subset of PASCAL restricting the set of variables to simple variables and singly subscripted arrays. We assume the data value domain for our PASCAL dialect has the form $\bigcup_{j\in J}\mathscr{D}_j \cup \bigcup_{j,k\in J}(\mathscr{D}_j \to \mathscr{D}_k)$ where $\mathscr{D}_j, j\in J$, are disjoint sets of primitive data objects (for example, integers, characters, booleans) and $(\mathscr{D}_j \to \mathscr{D}_k)$ denotes the set of mappings (arrays) from $\mathscr{D}_j$ into $\mathscr{D}_k$. We call each set $\mathscr{D}_j$ and $(\mathscr{D}_j \to \mathscr{D}_k)$ a *type*. These restrictions are made only for explanatory purposes. All of our results generalize to arbitrary PASCAL data domains.

We define the *abstract address* corresponding to the simple variable v as the singleton sequence $\langle 'v\rangle$. For a variable of the form $a[e]$ (where a is an array and e is an expression), the abstract address is $\langle 'a, e_0\rangle$ where $e_0\in\mathscr{D}$ is the value of e. We define two abstract addresses to be *disjoint* if and only if neither is an initial segment of the other.

Let H be a finite set of variable declarations $v: T$ (where v is a program identifier and T is a type) and procedure declarations **procedure** $p(\alpha)$; **imports** β; **global** z; B (where p is a program identifier, α is a sequence of **var** and **value** parameter declarations, β is a list of the procedures imported by p, z is a list of the variables imported by p, and B is an optional procedure body). We call H a *declaration set*. H is *closed* if and only if every procedure or variable name imported by a procedure in H is declared in H. Henceforth, we will assume that all declaration sets are closed. A *state vector s consistent with H* is a mapping from $\mathscr{I}$ (identifiers) into $\mathscr{D}$ (data values)$\cup\mathscr{B}$ (procedure bodies) such that each identifier in H is bound to a data value or procedure body compatible with its declaration.

Note that closed declaration sets can contain multiple bindings to the same program identifier unless restriction 5 in Sect. 1 holds. For this reason we require restriction 5 in all our procedure call rules. As discussed in Sect. 7, this restriction may be eliminated if we allow rules to rewrite program text.

Typically, we are only interested in a finite restriction of the state vector s – the bindings of the variables and procedure names declared in H. In this case, we can think of s as a finite sequence of ordered pairs (x, d) where x is a program identifier declared in H and d is its binding.

We let $\mathscr{A}$ and $\mathscr{S}$ denote the set of abstract addresses and the set of state vectors respectively.

2.2. *Value and Update Functions*

We introduce two functions *Value* and *Update* to access and modify states, analogous to the array access and update functions defined by McCarthy [13]. *Value* maps a state vector s and an abstract address α into the binding of α in s.

Update maps a state vector s, an abstract address α, and a value d into the state vector s', where s' is identical to s except that the entry within s' specified by α has the new value d.

In more formal terms, *Value* is a mapping form $\mathscr{S} \times \mathscr{A}$ into $\mathscr{D} \cup \mathscr{B}$ and *Update* is a mapping from $\mathscr{S} \times \mathscr{A} \times (\mathscr{D} \cup \mathscr{B})$ into $\mathscr{S}$ satisfying the following axioms:

1. *Value* (*Update* $(s, \alpha, e), \alpha) = e$ for arbitrary state vector s, abstract address α, and value e, provided the entry specified by α exists in s.

2. *Value* (*Update* $(s, \alpha_1, e), \alpha_2) =$ *Value* (s, α_2) if α_1 and α_2 are disjoint abstract addresses and the entries specified by α_1 and α_2 exist in s.

3. Let *Select* be the standard array access function mapping $(\mathscr{D}_i \to \mathscr{D}_j) \times \mathscr{D}_i$ into $\mathscr{D}_j$ for all i, j. (See for instance [14].) Then *Value* (*Update* $(s, \langle 'v \rangle, d), \langle 'v, e \rangle)$ $=$ *Select* (d, e) for arbitrary state vector s, identifier v, array value d, and data value e, provided e is in the domain of d.

4. Let *Store* be the standard array update function mapping $(\mathscr{D}_i \to D_j) \times \mathscr{D}_i \times \mathscr{D}_j$ into $(\mathscr{D}_i \to \mathscr{D}_j)$ for all i, j. Then *Value* (*Update* $(s, \langle 'v, e \rangle, d), \langle 'v \rangle) =$ *Store* (*Value* $(s, \langle 'v \rangle), e, d)$ for arbitrary state vector s, identifier v, and data values d and e, provided e and d belong to the domain and range of *Value* $(s, \langle 'v \rangle)$ respectively.

We extend *Value* and *Update* to apply to sequences of disjoint abstract addresses as follows:

1. *Value** $(s, \langle \alpha_1, \ldots, \alpha_n \rangle) = \langle$ *Value* $(s, \alpha_1), \ldots,$ *Value* $(s, \alpha_n) \rangle$ for arbitrary state vector s and abstract addresses $\alpha_1, \ldots, \alpha_n$, provided the entries specified by $\alpha_1, \ldots \alpha_n$ exist in s.

2. *Update** $(s, \langle \alpha_1, \ldots, \alpha_n \rangle, \langle d_1, \ldots, d_n \rangle) =$ *Update* $(\ldots$*Update*$(s, \alpha_1, d_1) \ldots, \alpha_n, d_n)$ for arbitrary abstract addresses $\alpha_1, \ldots, \alpha_n$ and values $d_1, \ldots, d_n$ provided the specified updates are well-defined.

3. Let $\alpha_1, \ldots, \alpha_n$ be disjoint abstract addresses such that $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik}$ have the form $\langle 'v, e_p \rangle$, $p = 1, 2, \ldots, k$, where $'v$ is an identifier; and e_p is a data value. Let $\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jn-k}$ be the remaining abstract addresses, and let d denote *Value* $(s, \langle 'v \rangle)$. Then *Update** $(s, \langle \alpha_1, \ldots, \alpha_n \rangle, \langle d_1, \ldots, d_n \rangle) =$ *Update** $(s, \langle \alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jn-k} \rangle, \langle$*Store* $(\ldots$*Store* $(d, e_1, d_{i1}) \ldots, e_k, d_{ik}), d_{j1}, \ldots, d_{jn-k} \rangle)$ provided the specified updates are well-defined.

The final axiom above merely collects updates to various elements of the same array and combines them into a single update of the entire array. We can use this axiom to convert an arbitrary sequence of disjoint updates to an equivalent set of simple updates (that is, updates of simple variables rather than array elements). For example,

*Update** $(s, \langle \langle 'a, 1 \rangle, \langle 'b \rangle, \langle 'a, 4 \rangle, \langle 'c \rangle \rangle, \langle 1, 2, 3, 4 \rangle)$
$=$ *Update** $(s, \langle \langle 'a \rangle, \langle 'b \rangle, \langle 'c \rangle \rangle, \langle$*Store* $($*Store* $($*Value* $(s, \langle 'a \rangle), 1, 1), 4, 3), 2, 4 \rangle)$.

We denote the set of sequences of abstract addresses by $\mathscr{A}^*$.

2.3. *Definition of Truth*

In this section, we define the syntax and meaning of statements in our version of Hoare's logic.

2.3.1. The Base Logic. Assume we are given a base first order theory $(\mathscr{L}, \mathscr{M})$ (for the program data domain), consisting of a logical language $\mathscr{L}$ with equality and a model $\mathscr{M}$ for $\mathscr{L}$, with the following properties:

1. The domain of the model $\mathscr{M}$ includes $\mathscr{D}$ (data values), $\mathscr{I}'$ (quoted identifiers), $\mathscr{A}$ (abstract addresses), $\mathscr{A}^*$ (sequences over $\mathscr{A}$), and $\mathscr{B}$ (procedure bodies).

2. The variables of $\mathscr{L}$ include two disjoint sets: $\mathscr{I}$ (programming language identifiers) and $\mathscr{V}$, a set of logical variables which may not appear within programs.

3. The logic includes the binary function $\otimes$ and the unary function *Seq*. The $\otimes$ operator concatenates two sequences; that is, $\langle u_1, \ldots, u_m\rangle \otimes \langle v_1, \ldots, v_n\rangle = \langle u_1, \ldots, u_m, v_1, \ldots, v_n\rangle$. *Seq* maps a data object d (specifically a quoted identifier, a data value, or an abstract address) into the singleton sequence $\langle d\rangle$. With the functions $\otimes$ and *Seq*, we can construct arbitrary members of $\mathscr{A}$ and $\mathscr{A}^*$.

4. The logic includes all the primitive functions of the programming language including array access and update functions *Select* and *Store*. We let $a[e]$, where a is an identifier and e is a term, abbreviate the term *Select* (a, e).

5. The logic includes a characteristic predicate P_T for each data type T in $\mathscr{D}$. We will use the familiar Pascal notation $x: T$ to abbreviate $P_T(x)$.

6. The logic includes the predicates *Disjoint* and *Pair-Disjoint* with domains $\mathscr{A}^*$ and $\mathscr{A}^* \times \mathscr{A}^*$ respectively. *Disjoint* $(\langle \alpha_1, \ldots, \alpha_n\rangle)$ is true if and only if abstract addresses α_i and α_j are disjoint for all i, j such that $i \neq j$. *Pair-disjoint* $(\langle \alpha_1, \ldots, \alpha_m\rangle, \langle \beta_1, \ldots, \beta_n\rangle)$ is true if and only if every pair (α_i, β_j) is disjoint.

Given an arbitrary variable v, we can mechanically construct a term v^* in $\mathscr{L}$ such that the meaning of v^* is the abstract address for v. If v is a simple variable x, then v^* is simply *Seq* $('x)$. If v is an array element $a[e]$, then v^* is *Seq* $('a) \otimes$ *Seq* (e). We will frequently employ this construction in our proof rules.

2.3.2. Extended Terms and Formulas. For the sake of clarity, we prohibit formulas of $\mathscr{L}$ from using program identifiers as bound (quantified) variables. In addition, to conveniently handle updates to the state vector, we extend the logical language $\mathscr{L}$ to include updated formulas and terms. We define an *extended formula (term)* of $\mathscr{L}$ as follows. An extended formula (term) has a recursive definition identical to that of an ordinary formula (term) [6] except that there is an additional mechanism (called an update) for building new formulas and terms from existing ones. Given an extended formula (term) α, the form $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\alpha$ is also an extended formula (term), where $\mathbf{v}$ is a sequence of disjoint variables and $\mathbf{t}$ is a corresponding sequence of ordinary (not updated) terms in $\mathscr{L}$. We will call $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]$ a *simultaneous update*. Henceforth, we will simply use the term *formula (term)* to refer to an extended formula (extended term). We will abbreviate

$$[\![\mathbf{v}_1 \leftarrow \mathbf{t}_1]\!] \ldots [\![\mathbf{v}_n \leftarrow \mathbf{t}_n]\!] P \quad \text{by} \quad [\![\mathbf{v}_1 \leftarrow \mathbf{t}_1; \ldots; \mathbf{v}_n \leftarrow \mathbf{t}_n]\!] P.$$

Simultaneous updates are closely related to the modalities of dynamic logic [16] and the weakest precondition transformer in Dijkstra's program calculus [4]. When a program segment S is a sequence of assignments where all program operations are totally defined, the relation corresponding to S is the function

$\lambda e.$ *Eval* $[S, e]$ where *Eval* is the interpreter for the programming language. In this case, $\langle S\rangle P \equiv [S]P \equiv \mathsf{wp}(S, P) \equiv [\![S]\!]P$.

2.3.3. Hoare Assertions and Statements. Let Q be an arbitrary formula in $\mathscr{L}$ and let $x_1, \ldots, x_n$ be the program identifiers which occur in Q. Let H be a declaration set including declarations for $x_1, \ldots, x_n$. A *Hoare assertion* has the form

$$H \mid Q.$$

Let A be a program segment and P and Q be formulas in $\mathscr{L}$. Let H be a declaration set including declarations for all the free program variables and procedure names in A, P, and Q. A *Hoare statement* has the form

$$H \mid P\{A\}Q.$$

We define the meaning of Hoare assertions and statements as follows. Let $H \mid Q$ be an arbitrary Hoare assertion. The definition of truth for $H \mid Q$ is identical to the standard first-order definition of truth for Q [6] except:

1. $H \mid Q$ is vacuously true for states inconsistent with H.
2. The meaning of the updated formula (term) $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\alpha$ for state s is the meaning of the formula (term) α for state *Update** $(s, \mathbf{v}^*, \mathbf{t}_s)$ where $\mathbf{v}^*$ denotes the sequence of abstract addresses corresponding to $\mathbf{v}$, and $\mathbf{t}_s$ denotes the interpretation of $\mathbf{t}$ under state s.

Let $H \mid P\{A\}Q$ be an arbitrary Hoare statement and let *Eval* be an interpreter (a partial function) mapping states $\times$ program-segments into states. Then $H \mid P\{A\}Q$ is true if and only if for all states s either

1. $H \mid P$ is false for s.
2. *Eval* (s, a) is undefined.
3. Q is true for *Eval* (s, A).

2.3.4. Standard Proof Rules. The standard simple Hoare proof rules have obvious analogs in our version of the logic. The most fundamental rules – consequence, composition, and substitution – have the following form:

1. Consequence
$$\frac{H \mid P \supset Q,\ H \mid Q\{A\}R,\ H \mid R \supset S}{H \mid P\{A\}S}.$$
2. Composition
$$\frac{H \mid P\{A\}Q,\ H \mid Q\{B\}R}{H \mid P\{A; B\}R}.$$
3. Substitution
$$\frac{H \mid P\{A\}Q}{H \mid P(t/x)\{A\}Q(t/x)}$$

where x is a logical variable and $Q(t/x)$ denotes Q with every free occurrence of x replaced by t (renaming bound variables when necessary).

The other standard rule that we take as given is:

4. Declaration

$$\frac{H(\mathbf{x}'/\mathbf{x}, \mathbf{p}'/\mathbf{p}) \cup \{\mathbf{x}:\mathbf{T}, \mathbf{p}:\mathbf{B}\} \mid P(\mathbf{x}'/\mathbf{x})\{A\}\, Q(\mathbf{x}'/\mathbf{x})}{H \mid P\{\mathsf{begin}\ \mathbf{x}:\mathbf{T};\ \mathbf{p}:\mathbf{B};\ A\ \mathsf{end}\}\, Q}$$

where $\mathbf{x}:\mathbf{T}$ and $\mathbf{p}:\mathbf{B}$ are sequences of variable and procedure declarations, and $\mathbf{x}'$ and $\mathbf{p}'$ are sequences of fresh program variables and procedure names corresponding to $\mathbf{x}$ and $\mathbf{p}$. As Apt (personal communication) has observed, this rule is incomplete because it does not allow one to deduce the values of new variables on block entry. There are several possible solutions to this technical problem but they are beyond the scope of this paper.

2.3.5. Reasoning about Updated Formulas. In order to prove Hoare assertions involving updated formulas, we need special axioms about updates. For disjoint updates modifying entire formulas, the following axioms (derived from the corresponding axioms for *Update**) are sufficient:

1. $[\![\mathbf{x} \leftarrow \mathbf{t}]\!]\, Q \equiv Q(\mathbf{t}/\mathbf{x})$ where $\mathbf{x}$ is a sequence of distinct *simple* variables, and Q is a formula containing no updates.

2. Let $v_1, \ldots, v_n$ be disjoint variables where $v_{i1}, \ldots, v_{ik}$ have the form $a[e_l]$, $l=0, \ldots, k$, and a is a particular array identifier. Let $v_{j1}, \ldots, v_{jn-k}$ be the remaining variables. Let $\mathbf{v}'$ denote the sequence of variables $a, v_{j1}, \ldots, v_{jn-k}$ and let $\mathbf{t}'$ denote the sequence of terms $\langle Store(\ldots Store(a, e_1, t_{i1}) \ldots, e_k, t_{ik}), t_{j1}, \ldots, t_{jn-k} \rangle$. Then

$$[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q \equiv [\![\mathbf{v}' \leftarrow \mathbf{t}']\!]\, Q.$$

Given an arbitrary disjoint simultaneous update $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]$, we can eliminate it from a formula of the form $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q$ where Q is update free by using axiom 2 to eliminate all assignments to array elements and then applying axiom 1. We can similarly eliminate all updates from a formula of the form $[\![\ldots]\!]\, [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q$ where Q is update free by repeatedly applying the same simplification procedure.

3. Simple Simultaneous Assignment

Given the concept of simultaneous updates within formulas, it is easy to give a simple simultaneous assignment rule. Let $\mathbf{v} \leftarrow \mathbf{t}$ be a simultaneous assignment to disjoint variables $\mathbf{v}$, let $\mathbf{v}^*$ be the abstract address terms in $\mathscr{L}$ corresponding to $\mathbf{v}$, and let P be an arbitrary formula in $\mathscr{L}$. The rule is:

$$\frac{H \mid [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P \supset Disjoint(\mathbf{v}^*)}{H \mid [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P\{\mathbf{v} \leftarrow \mathbf{t}\}\, P}.$$

The soundness and relative completeness of this rule follows immediately from the definition of meaning of statements in the logic and the definition of simultaneous assignment.

4. Simple Procedure Call Rule

In this section we assume that our PASCAL subset:

1. Prohibits aliasing in procedure calls.
2. Disallows passing procedures and functions as parameters.
3. Requires that the global variables accessed by a procedure be explicitly declared at the head of the procedure and that these variables be accessible at the point of every call.

Assumption 3 guarantees that dynamic scoping and static scoping are equivalent.

Under these assumptions, it is straightforward to formulate a procedure call rule by treating procedure calls as simultaneous assignments to the variables passed to the procedure. The assigned values are any values consistent with the input-output assertions for the procedure.

Let p be declared as procedure $p(\text{var } \mathbf{x}:\mathbf{T}_x;\ \text{val } \mathbf{y}:\mathbf{T}_y);$ imports β; global $\mathbf{z}$; B in the declaration set H. B may not access any global variables other than $\mathbf{z}$. Let H' be H augmented by the declarations $\mathbf{x}:\mathbf{T}_x$ and $\mathbf{y}:\mathbf{T}_y$ (prior declarations of $\mathbf{x}$ and $\mathbf{y}$ are replaced). Let P and Q be formulas containing no free program variables other than $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{x}, \mathbf{z}$ respectively. Let $\mathbf{v}$ be the free logical variables of P and Q, and let $\mathbf{x}'$ and $\mathbf{z}'$ be fresh logical variables corresponding to $\mathbf{x}$ and $\mathbf{z}$. Then the (non-recursive) simple procedure call rule has the following form:

$$\frac{H \mid R \supset \mathit{Disjoint}\,(\mathbf{a}^* \otimes \mathbf{z}^*),\ H' \mid P\{B\}\,Q \qquad H \mid \forall \mathbf{v}[P(\mathbf{a}/\mathbf{x}, \mathbf{b}/\mathbf{y}) \supset Q(\mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})] \supset [R \supset [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, S]}{H \mid R\{p(\mathbf{a}; \mathbf{b})\}\, S}.$$

It is important to note that the free logical variables $\mathbf{x}'$ and $\mathbf{z}'$ in the third premise are implicitly universally quantified. The rule forces $R \supset [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, S$ to be true for arbitrary $\mathbf{x}'$ and $\mathbf{z}'$ consistent with $\forall \mathbf{v}[P(\mathbf{a}/\mathbf{x}, \mathbf{b}/\mathbf{y}) \supset Q(\mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})]$. In contrast, the EUCLID procedure call rule explicitly omits the corresponding quantifier - permitting false deductions. Like the EUCLID rule, our rule generalizes Hoare's original rule [10] to apply to a richer programming language. The main difference is between our rule and its predecessors (Hoare's original rule and the EUCLID rule) is that our rule precisely states the assumptions left implicit by the earlier rules.

4.1. Soundness

If *Eval* is properly defined, it is easy to prove the soundness of the simple procedure call rule. Let s be an arbitrary state, consistent with H such that $H \mid R$ is true for s and $\mathit{Eval}\,(s, p(\mathbf{a}; \mathbf{b}))$ is defined. We must show S is true for $\mathit{Eval}\,(s, p(\mathbf{a}; \mathbf{b}))$. Let s' be $[\![\mathbf{x}', \mathbf{z}' \leftarrow \mathbf{x}_0, \mathbf{z}_0]\!]\, s$ where $\mathbf{x}_0, \mathbf{z}_0$ are the output values of $\mathbf{x}$ and $\mathbf{z}$ in the call $p(\mathbf{a}; \mathbf{b})$ (that is, the values of $\mathbf{x}$ and $\mathbf{z}$ in the state $\mathit{Eval}\ ([\![\mathbf{x}, \mathbf{y} \leftarrow \mathbf{a}, \mathbf{b}]\!]\, s, B)$). By the second premise, s' satisfies $\forall \mathbf{v}[P(\mathbf{a}/\mathbf{x}, \mathbf{b}/\mathbf{y})$

$\supset Q(\mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})]$. Hence, by the final premise, s' must also satisfy $[\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!] S$. From the definition of *Eval*, the following states are equivalent

$$Eval\,(s, p(\mathbf{a}, \mathbf{b})) = [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}_0, \mathbf{z}_0]\!]\, s = [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, s'.$$

Consequently, $Eval\,(s, p(\mathbf{a}, \mathbf{b}))$ satisfies S. Q.E.D.

Although the soundness of the procedure call rule does not depend on the third assumption listed above (the accessibility of the procedure globals at the point of every call), the assumption is necessary to prove that *Eval* obeys static scoping. The natural definition of *Eval* (which we used in the soundness proof) employs dynamic scope rules. If the third assumption holds then static and dynamic scope rules are semantically equivalent.

4.2. Relative Completeness

It is also reasonably straightforward to prove that the simple procedure call rule is *relatively complete* for non-recursive programs in the sense of Cook [3]. Since our base logic includes such a rich collection of logical primitives for describing abstract addresses, the incompleteness results of Clarke [2] do not apply to our version of Hoare's logic. We assume that the assertion language $\mathscr{L}$ is *expressive*; that is, that given an arbitrary assertion P in $\mathscr{L}$ and a program segment A the strongest postcondition Q of A given precondition P is definable in $\mathscr{L}$. To show that the rule is complete relative to the completeness of the other proof rules and the axiomatization of the extended base logic, it suffices to show that for any program segment A and postcondition S the weakest liberal pre-condition R is provable, i.e. $H|R\{A\}\,S$ is provable. The proof proceeds by contradiction.

Assume $p'(\mathbf{a}'; \mathbf{b}')$ is a procedure call for which the rule is not complete. Let $p(\mathbf{a}; \mathbf{b})$ be the deepest procedure call in the calling tree of p' for which the simple procedure call rule is not complete. Let H be the declaration set at the point of the call, and let p be declared as procedure p(var $\mathbf{x}$:$\mathbf{T}_x$; val $\mathbf{y}$:$\mathbf{T}_y$); global $\mathbf{z}$; B in H. Let S be an arbitrary true postcondition for $p(\mathbf{a}; \mathbf{b})$. We define Q' as the strongest postcondition for B given the precondition $\mathbf{x}, \mathbf{y}, \mathbf{z} = \mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$. By assumption $H'|P\{B\}\,Q'$ is provable. We define Q to be $\exists \mathbf{y}'\, Q'(\mathbf{y}'/\mathbf{y})$. By the rule of consequence, $H|P\{B\}\,Q$ must be provable. Let R be $\forall \mathbf{x}', \mathbf{z}'[Q(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i, \mathbf{z}/\mathbf{z}_i, \mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z}) \supset [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, S]$. Then the last premise is provable by ordinary first order deduction, implying that $H|R\{p(\mathbf{a}; \mathbf{b})\}\,S$ is provable.

Assume R is not the weakest liberal precondition. Then there exists a state s consistent with H such that R is false and such that either $Eval\,(s, p(\mathbf{a}; \mathbf{b}))$ is undefined or S is true for $Eval\,(s, p(\mathbf{a}; \mathbf{b}))$. Let s' be $[\![\mathbf{x}, \mathbf{y} \leftarrow \mathbf{a}, \mathbf{b}]\!]\, s$. Either $Eval\,(s, B)$ is undefined or Q is true for $Eval\,(s', B)$. In the former case, $Q(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i, \mathbf{z}/\mathbf{z}_i, \mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})$ must be false for all $\mathbf{x}', \mathbf{z}'$ since $Q(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i, \mathbf{z}/\mathbf{z}_i)$ is false for all $\mathbf{x}, \mathbf{z}$. Hence R is true, generating a contradiction. In the other case, $Q(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i, \mathbf{z}/\mathbf{z}_i, \mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})$ is true only for states with $\mathbf{x}'$ and $\mathbf{z}'$ equal to the values of $\mathbf{x}$ and $\mathbf{z}$ in $Eval\,(s', B)$. But for such $\mathbf{x}'$ and $\mathbf{z}'$, $Eval\,(s, p(\mathbf{a}; \mathbf{b})) = [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, s$. Consequently, $[\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\, S$ is true for all states satisfying $Q(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i, \mathbf{z}/\mathbf{z}_i, \mathbf{x}'/\mathbf{x}, \mathbf{z}'/\mathbf{z})$ implying R is true. Again, we have a contradiction. Q.E.D.

4.3. *A Sample Proof*

Let's consider a simple example that most procedure call rules cannot handle. Let p be a standard integer variable swap procedure defined as follows:

```
procedure p(var x, y: integer);
begin
    pre x = x_i ∧ y = y_i;
    x, y ← y, x;
    post y = x_i ∧ x = y_i
end;
```

By the simultaneous assignment rule, we must show x, y: integer $\mid x = x_i \wedge y = y_i \supset [\![x, y \leftarrow y, x]\!] (y = x_i \wedge x = y_i)$ to establish the declared pre and postassertions for the swap. By the $[\![\ ]\!]$ substitution axiom (axiom 1 in 2.3.5).

$$[\![x, y \leftarrow y, x]\!] (y = x_i \wedge x = y_i) \equiv x = x_i \wedge y = y_i$$

which is precisely the preassertion. Q.E.D.

Now let us consider a sample application of the procedure call rule. Assume we want to prove:

a: array integer of integer, i: integer $\mid$

$$a[i] = a_0 \wedge i = i_0 \{p(a[i], i)\}\ a[i_0] = i_0 \wedge i = j_0.$$

Let H denote $\{a$: array integer of integer, i: integer$\}$; P' denote the substituted precondition $a[i] = x_i \wedge i = y_i$; Q' denote the substituted postcondition $y' = x_i \wedge x' = y_i$; R denote $a[i] = a_0 \wedge i = i_0$; and S denote $a[i_0] = i_0 \wedge i = a_0$. By the simple procedure call rule, we must show

1. $H \mid R \supset Disjoint\ (\langle' a, i\rangle, \langle' i\rangle)$.
2. The correctness of the input-output assertions for the procedure body.
3. $H \mid \forall x_0, y_0 [P' \supset Q'] \supset [R \supset [\![a[i], i \leftarrow x', y']\!] S]$.

Since 1 is trivial, and we have already proved 2, it suffices to prove 3. First we transform $[\![a[i], i \leftarrow x', y']\!] S$ into $[\![a, i \leftarrow Store\ (a, i, x'), y']\!] S \equiv Store\ (a, i, x')[i_0] = i_0 \wedge y' = a_0$. Since $i = i_0$ by hypothesis in R,

$$S' \equiv Store\ (a, i_0, x')[i_0] = i_0 \wedge y' = a_0 \equiv x' = i_0 \wedge y' = a_0.$$

By applying the equality hypothesis in R, we transform $x' = i_0 \wedge y' = a_0$ into $x' = i \wedge y' = a[i]$, which is an immediate consequence of $P' \supset Q'$ when x_i, y_i are instantiated as $a[i]$ and i respectively. Q.E.D.

4.4. *Handling Recursion*

Our simple rule can be extended to handle mutually recursive procedures by generalizing Hoare's original approach to the problem [10]. However, we must impose the following additional restriction on our PASCAL subset to ensure the soundness of the rule:

No procedure named p may be declared within the scope of another procedure named p.

Our rule is not unique in this respect. Every other proposed procedure call rule (with the exception of Apt and de Bakker [1]) requires an equivalent restriction. The restriction is necessary because the input-output specifications for a procedure p may be assumed for any procedure call within a procedure declared in the scope of p.

Let procedure p_i(var $\mathbf{x}_i$: $\mathbf{T}_{x_i}$; val $\mathbf{y}_i$: $\mathbf{T}_{y_i}$); imports β_i; global $\mathbf{z}_i$; B_i, $i=1, 2, \ldots, n$ be a sequence of procedure declarations at the head of some block. Let P_i and Q_i, $i=1, \ldots, n$ be assertions containing no free program variables other than $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ and $\mathbf{x}_i$, $\mathbf{z}_i$ respectively. Let $\mathbf{v}_i$ be the free logical variables in P_i and Q_i. Let H be a declaration set containing the declarations of $p_1, \ldots, p_n$ and let H' denote H with these declarations replaced by "forward" procedure declarations which only specify the procedures' formal parameters. Let H_i' denote H' augmented by the declarations $\mathbf{x}_i$: $\mathbf{T}_{x_i}$, $\mathbf{y}_i$: $\mathbf{T}_{y_i}$ (prior declarations of $\mathbf{x}_i$ and $\mathbf{y}_i$ are replaced). For $i=1, \ldots, n$ we define the recursion hypothesis I_i as the rule:

$$\frac{H \mid \Theta_1 \supset \mathit{Disjoint}\,(\mathbf{c}^* \otimes \mathbf{z}_i^*) \qquad H \mid \forall \mathbf{v}_i [P_i(\mathbf{c}/\mathbf{x}_i, \mathbf{d}/\mathbf{y}_i) \supset Q_i(\mathbf{x}'/\mathbf{x}_i, \mathbf{z}'/\mathbf{z}_i)] \supset [\Theta_1 \supset [\![\mathbf{c}, \mathbf{z}_i \leftarrow \mathbf{x}_i', \mathbf{z}_i']\!]\, \Theta_2]}{H \mid \Theta_1 \{p_i(\mathbf{c}; \mathbf{d})\}\, \Theta_2}$$

where Θ_1, Θ_2, $\mathbf{c}, \mathbf{d}$ and H are arbitrary. Then the recursive version of the rule has the form:

$$\frac{\begin{array}{l} H' \mid R \supset \mathit{Disjoint}\,(\mathbf{a}^* \otimes \mathbf{z}_i^*) \\ I_1, \ldots, I_n \vdash H_j' \mid P_j \{B_j\}\, Q_i, j=1, \ldots, n \\ H' \mid \forall \mathbf{v}_i [P_i(\mathbf{a}/\mathbf{x}_i, \mathbf{b}/\mathbf{y}_i) \supset Q_i(\mathbf{x}_i'/\mathbf{x}_i, \mathbf{z}_i'/\mathbf{z}_i)] \supset [R \supset [\![\mathbf{a}, \mathbf{z}_i \leftarrow \mathbf{x}_i', \mathbf{z}_i']\!]\, S] \end{array}}{H \mid R \{p_i(\mathbf{a}; \mathbf{b})\}\, S}$$

where $I_1, I_2, \ldots, I_n \vdash H_j' \mid P_j \{B_j\}\, Q_j$ means we may use the rules I_i to prove $H_j' \mid P_j \{B_j\}\, Q_j$.

Unlike Hoare's original rule and the EUCLID rule, our recursive rule is relatively complete, even for programs utilizing mutual recursion. Of the rules previously proposed in the literature, our rule most closely resembles that of [7]. Gorelick uses a more complex set of potentially mutually recursive procedures instead of $p_1, \ldots, p_n$ and divides the procedure call rule into two parts: a rule of modification and a rule of invariance. We originally formulated our procedure call rules in two-part form, but abandoned the approach after we failed to devise a complete two-part rule. Gorelick achieves relative completeness by restricting actual var parameters to simple variables.

We can prove that the recursive version of the simple procedure call rule is sound by generalizing the argument we used for the non-recursive rule. First, we construct the sequences of procedures $p_{0i}, p_{1i}, \ldots, p_{ki}, \ldots$; $i=1, \ldots, n$ as follows. We let p_{0i} be a non-terminating procedure with parameters identical to p_i. For $k=1, 2, \ldots$, we let p_{ki} be defined by the procedure p_{ki} (var $\mathbf{x}_i$: $\mathbf{T}_{x_i}$; val $\mathbf{y}_i$: $\mathbf{T}_{y_i}$); global $\mathbf{z}_i$; $B_i(p_{jk-1}/p_j, j=0, \ldots, n)$, that is, by the same declaration as p_i except each call $p_j(\mathbf{c}; \mathbf{d})$ within the body of p_i is replaced by the call $p_{k-1j}(\mathbf{c}; \mathbf{d})$. Clearly,

if the evaluation of an arbitrary call $p_i(\mathbf{a};\mathbf{b})$ requires less than k levels of nested calls on $p_1, p_2, \ldots, p_n$, then the call $p_{ki}(\mathbf{a};\mathbf{b})$ is equivalent to $p_i(\mathbf{a};\mathbf{b})$. (Note that this statement does not hold if the restriction on procedure names is violated.) By the soundness of the non-recursive rule and simple induction on k, we know that the recursive rule is sound if we interpret p_j in the premises by p_{k-1j}, $j=1, \ldots, n$ and p_i in the conclusion by p_{ki}. Without loss of generality we may assume $p_i(\mathbf{a};\mathbf{b})$ terminates; otherwise, the rule is vacuously true. Let k be an integer greater than the maximum recursion calling depth on $p_1, \ldots, p_n$ in the evaluation of $p_i(\mathbf{a};\mathbf{b})$. By assumption, the premises are true for any interpretation of p_j, $j=1, \ldots, n$ consistent with H'. Hence they must hold for p_j interpreted as p_{k-1j}, implying the conclusion of the rule holds for $p_{ki}(\mathbf{a};\mathbf{b})$. Since $p_{ki}(\mathbf{a};\mathbf{b})$ is equivalent to $p_i(\mathbf{a};\mathbf{b})$, the conclusion of the rule must be true. Q.E.D.

The relative completeness of the recursive rule can be established by a similar inductive generalization of the proof for the non-recursive rule. We assume $\mathscr{L}$ is expressive. The proof proceeds by induction on the structure of a program. For every procedure p(var $\mathbf{x}$; val $\mathbf{y}$) global $\mathbf{z}$; B in the program, we let the pre and post assertions be $\mathbf{x}, \mathbf{y}, \mathbf{z}=\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ and $\exists \mathbf{y}' Q'(\mathbf{y}'/\mathbf{y})$ respectively, where Q' is the strongest postcondition for the program segment B given the precondition $\mathbf{x}, \mathbf{y}, \mathbf{z}=\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$. Let $p_1, \ldots, p_n$ be a sequence of procedures declared at the head of a block B such that the pre and post assertions for every procedure declared within $p_1, \ldots, p_n$ are provable. We must show

1. The pre/post assertion pair for the body of each procedure p_i is provable, and
2. The weakest precondition for any procedure call in the body of B is provable.

For each procedure p_i, we let P_i denote the pre assertion $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i = \mathbf{x}_{0i}, \mathbf{y}_{0i}, \mathbf{z}_{0i}$ and let Q_i denote the post assertion $\exists \mathbf{y}' Q'(\mathbf{y}'/\mathbf{y})$, where Q' is the strongest post condition of B_i given pre-condition P_i.

Let $q(\mathbf{c};\mathbf{d})$ be an arbitrary call in the body B_i of p_i. If q is internal to p_i, then the pre and post assertion of q are provable by assumption. If q is not internal to p_i, then the recursion hypothesis for q is available. In either case, by the same argument we used in the non-recursive case, the weakest precondition of $q(\mathbf{c};\mathbf{d})$, given an arbitrary postcondition S, is provable. Hence, since the remaining rules of the logic are complete by assumption, $P_i\{B_i\}Q_i$, $i=1, \ldots, n$ is provable. By applying the same argument again, we conclude that the weakest liberal precondition of any procedure call in the body of B is provable.

By induction on the structure of a program, we can repeatedly apply the previous argument to derive that the procedure call rule is complete for all calls appearing in the program. Q.E.D.

5. Rules for Programs with Aliasing

We now extend our version of Hoare's logic to handle aliasing. The modifications required are surprisingly minor.

Hoare's original assignment axiom has the form:

$$P(e/x)\{x \leftarrow e\}\, P$$

where x is a simple variable, e is an expression (term in the logical language $\mathscr{L}$) and P is a formula. This axiom is invalid if x is a reference parameter or an array reference, since there may be syntactically distinct variables in P with abstract addresses identical to x. While Hoare's substitution style axiom can be patched to handle array assignment (by viewing the assignment $a[e_1] \leftarrow e_2$ as an abbreviation for the simple assignment $a \leftarrow \mathit{Store}\,(a, e_1, e_2)$), it breaks down in the case of aliasing.

In contrast, our assignment call rule does not rely on the concept of substitution (although it collapses to that form in trivial cases). As a result, our rule is able to handle array assignment and aliasing without any modification.

5.1. Reference Parameters

In a programming language with unrestricted reference parameters like PASCAL, we interpret procedure calls as passing the abstract addresses of the actual reference parameters to the procedure. In other words, the interpreter (*Eval*) binds a formal reference parameter to the abstract address of the corresponding actual parameter. For example, if p is a procedure with the single reference parameter x, then the procedure call $p(\alpha)$, where α is a variable, binds x to the abstract address for α and evaluates the procedure body. In a language like PASCAL, every reference to a formal reference parameter is automatically dereferenced.

If x is a formal reference parameter bound to an actual parameter α, an assignment to x in the procedure body changes the binding of α (the variable to which x is bound); it does not change the binding of x. The binding of the formal reference parameter x is unchanged for the duration of the call.

Consequently, we consider PASCAL's notation for referring to formal reference parameters misleading. To remedy the situation in our PASCAL dialect, we require that every reference to a formal reference parameter x in the body of the procedure have the form $x\uparrow$ instead of x. (We have taken the $\uparrow$ operator from Pascal, where it serves as a "dereferencing" operator for pointers.) For instance, if x is a reference parameter, then the standard Pascal statement $x \leftarrow x+1$ is (implicitly) written as $x\uparrow \leftarrow x\uparrow +1$ in our dialect. We also require formal reference parameter declarations to have the form x: ref T instead of $x:T$.

To accommodate aliasing within our logic, we must extend the set of Hoare assertions to include terms of the form $x\uparrow$ where x is declared in the declaration set H as x: ref T for some type T. We prohibit the dereferencing operator from appearing in other contexts. The meaning of $x\uparrow$, given state s consistent with H, is $\mathit{Value}\,(s, \mathit{Value}\,(s, \langle' x\rangle))$. The abstract address for $x\uparrow$ is the *Value* of x. Consequently, the abstract address term for $x\uparrow$ is simply x.

Our proof rule for assignments to dereferenced formal reference parameters is identical to our ordinary assignment rule:

$$[\![x\uparrow \leftarrow e]\!]\, P\{x\uparrow \leftarrow e\}\, P$$

where we extend the definition of the simultaneous update $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha$ as follows. Let α be a term or formula in $\mathscr{L}$; let $\mathbf{v}$ be a sequence of variables, possibly including dereferenced formal reference parameters; and let $\mathbf{t}$ be a corresponding sequence of terms (not containing updates). The meaning of $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha$ for s is the meaning of α for *Update** $(s, \mathbf{v}^*, \mathbf{t})$ where *Update** is extended to overlapping abstract addresses. *Update** is defined by exactly the same axioms as before, except that axiom 2 (Sect. 2.2.) no longer requires the abstract addresses $\langle \alpha_1, \ldots, \alpha_n \rangle$ to be disjoint. Informally, a simultaneous update $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]$ with overlapping variables is performed in left-to-right order.

The soundness and relative completeness of the assignment rule stated above are an immediate consequence of the fact that

$$Eval\,(s, x\uparrow \leftarrow e) = Update\,(s, x, e_s)$$

where e_s is the interpretation of e under state s.

In order to reason about updated formulas containing updates to dereferenced variables, we need the following axioms about updates. Let P and Q be arbitrary formulas, $u_1, \ldots, u_k$ be arbitrary terms, and $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]$ be an arbitrary simultaneous update. Then:

1. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,(P \wedge Q) \equiv [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P \wedge [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q$.
2. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,(P \vee Q) \equiv [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P \vee [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q$.
3. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,(P \supset Q) \equiv [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P \supset [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, Q$.
4. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, \neg P \equiv \neg [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P$.
5. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, \forall \mathbf{x} P \equiv \forall \mathbf{x} [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P$ where $\mathbf{x}$ not free in $\mathbf{t}$.
6. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, \exists \mathbf{x} P \equiv \exists \mathbf{x} [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P$ where $\mathbf{x}$ not free in $\mathbf{t}$.
7. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, P_i(u_1, \ldots, u_k) \equiv P_i([\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, u_1, \ldots, [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, u_k)$ for every predicate symbol P_i (including equality).
8. $[\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, f_i(u_1, \ldots, u_k) = f_i([\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, u_1, \ldots, [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\, u_k)$ for every function symbol f_i.

These axioms enable us to move updates inside a formula to the point where they apply only to program and logical variables. We also need axioms for updates to program and logical variables. Let $v_1, \ldots, v_n$ be variables and $t_1, \ldots, t_n$ be corresponding terms. Let $[\![\ldots]\!][\![v_1, \ldots, v_n \leftarrow t_1, \ldots, t_n]\!]\,\alpha$ be an arbitrary updated variable. Then:

1. $[\![\ldots]\!]\,(v_n^* = \alpha^*) \supset [\![\ldots]\!][\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha = [\![\ldots]\!]\, t_n$.
2. $[\![\ldots]\!]\,(v_n^* \otimes Seq\,(d) = \alpha^*) \supset [\![\ldots]\!][\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha = [\![\ldots]\!]\, Select\,(t_1, d)$.
3. $[\![\ldots]\!]\,(v_n^* = \alpha^* \otimes Seq\,(d)) \supset [\![\ldots]\!][\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha = [\![\ldots]\!][\![v_1, \ldots, v_{n-1} \leftarrow t_1, \ldots, t_{n-1}]\!]$ $Store\,(\alpha, d, t_n)$.
4. $[\![\ldots]\!]\, Disjoint\,(Seq\,(v_n^*) \otimes Seq\,(\alpha^*)) \supset [\![\ldots]\!][\![\mathbf{v} \leftarrow \mathbf{t}]\!]\,\alpha = [\![\ldots]\!][\![v_1, \ldots, v_{n-1} \leftarrow t_1, \ldots, t_{n-1}]\!]\,\alpha$.

Since updates do not affect logical variables, the following axiom holds for arbitrary updated logical variable $[\![\ldots]\!]\, x'$:

5. $[\![\ldots]\!]\, x' = x'$.

The soundness of all the axioms for updates is an immediate consequence of the definition of truth for updated formulas.

We can use the axioms for updates to convert an arbitrary formula to update-free form. To accomplish this transformation, we repeatedly apply the following procedure. First, we push all updates inside the formula so that they apply only to variables and logical variables. We eliminate all updates to logical variables by applying axiom 5 above. Then for each updated variable $[\![\ldots]\!]\ [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\ \alpha$, we perform a case split on the relationship between $[\![\ldots]\!]\ v_n^*$ and α^* and apply the appropriate reduction (axioms 1, 2, 3, or 4 above) to each case, reducing the complexity of the updates involved.

While the update elimination procedure is of dubious practical value (since it can exponentially increase the size of a formula), it demonstrates that our axioms for updates are complete relative to the unextended base theory.

5.2. *Generalized Simultaneous Assignment Rule*

Given the generalized concept of update described in the previous section, we can generalize the simultaneous assignment axiom to permit overlapping variables on the left-hand side of the statement. The new simultaneous assignment axiom is identical to the old one except that the disjointness premise is omitted. Let $\mathbf{v} \leftarrow \mathbf{t}$ be a simultaneous assignment statement; P be a formula; and H be a declaration set declaring all the program variables appearing in P, $\mathbf{v}$, or $\mathbf{t}$. Then the generalized assignment rule states

$$H \mid [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\ P\{\mathbf{v} \leftarrow \mathbf{t}\}\ P.$$

The soundness and completeness of the rule are an immediate consequence of the fact that $Eval(s, \mathbf{v} \leftarrow \mathbf{t}) = [\![\mathbf{v} \leftarrow \mathbf{t}]\!]\ s$ and the definition of truth for statements in the logic.

5.3. *Generalized Procedure Call Rule*

Assume our PASCAL subset satisfies the restrictions listed in Sect. 3. Our generalized procedure call rule is nearly identical to the simple rule. Let p be declared as procedure p(var $\mathbf{x}$: ref $\mathbf{T}_x$; val $\mathbf{y}$: $\mathbf{T}_y$); global $\mathbf{z}$; B in the declaration set H; let P and Q be formulas containing no quoted identifiers and no program variables other than $\mathbf{x}, \mathbf{x}\uparrow, \mathbf{y}, \mathbf{z}$ and $\mathbf{x}, \mathbf{x}\uparrow, \mathbf{z}$ respectively; let $\mathbf{v}$ be the free logical variables in P and Q; let $\mathbf{x}'$ and $\mathbf{y}'$ be fresh logical variables corresponding to $\mathbf{x}$ and $\mathbf{y}$; let R and S be formulas; and let H' denote H augmented by $\mathbf{x}$:ref $\mathbf{T}_x$, $\mathbf{y}$:$\mathbf{T}_y$, and *Pair-Disjoint* $(\mathbf{x}, \mathbf{x}^* \otimes \mathbf{y}^*)$ (where prior declarations of $\mathbf{x}$ and $\mathbf{y}$ are replaced). Then:

$$\frac{\begin{array}{c} H' \mid P\{B\}\ Q \\ H \mid \forall\, \mathbf{v}[P(\mathbf{a}^*/\mathbf{x}, \mathbf{a}/\mathbf{x}\uparrow, \mathbf{b}/\mathbf{y}) \supset Q(\mathbf{x}'/\mathbf{x}\uparrow, \mathbf{z}'/\mathbf{z})] \supset [R \supset [\![\mathbf{a}, \mathbf{z} \leftarrow \mathbf{x}', \mathbf{z}']\!]\ S] \end{array}}{H \mid R\{p(\mathbf{a}; \mathbf{b})\}\ S}.$$

The disjointness hypothesis in H' asserts that the abstract addresses for the formal parameters are disjoint from the passed actual reference parameter abstract addresses. From this hypothesis we can deduce that the dereferenced formal reference parameters do not have any of the formal parameters as aliases. We must add an analogous hypothesis to the declaration rule given in Sect. 2.3.4.

5.3.1. Soundness and Relative Completeness. The soundness and relative completeness proofs for the generalized procedure call rule differ only in minor details from the corresponding proofs for the simple rule. The only complication concerns the definition of *Eval*. We must not let *Eval* be confused by formal parameter names. The simplest solution is to force *Eval* to rename the actual parameters conflicting with formal parameter names before evaluating the procedure body. After evaluating the procedure body, *Eval* performs the appropriate simultaneous assignment.

5.3.2. A Sample Proof Involving Aliasing. Let swap be the standard integer swap procedure defined by

```
procedure swap (var x, y ref integer):
   begin
      pre x↑ = x_i ∧ y↑ = y_i;
      x↑, y↑ ← y↑, x↑;
      post y↑ = x_i ∧ x↑ = y_i
   end;.
```

First we prove the correctness of the pre and post assertions. Let H be a declaration set including the declaration of swap. Let H_i be H augmented by the formal parameter declarations of swap and the disjointness hypothesis. By the simultaneous assignment rule, proving the pre and post assertions for swap reduces to proving the verification condition:

$$H' \mid x\uparrow = x_i \wedge y\uparrow = y_i \supset [\![x\uparrow, y\uparrow \leftarrow y\uparrow, x\uparrow]\!](y\uparrow = x_i \wedge x\uparrow = y_i).$$

Moving the update inside generates the equivalent assertion:

$$H' \mid x\uparrow = x_i \wedge y\uparrow = y_i \supset [\![x\uparrow, y\uparrow \leftarrow y\uparrow, x\uparrow]\!]\, y\uparrow = x_i \wedge [\![x\uparrow, y\uparrow \leftarrow y\uparrow, x\uparrow]\!]\, x\uparrow = y_i$$

which immediately reduces to:

$$H' \mid x\uparrow = x_i \wedge y\uparrow = y_i \supset x\uparrow = x_i \wedge [\![x\uparrow, y\uparrow \leftarrow y\uparrow, x\uparrow]\!]\, x\uparrow = y_i.$$

Since x and y are both ref integers we know that $H' \mid x = y \vee \mathit{Disjoint}(\mathit{Seq}(x) \otimes \mathit{Seq}(y))$. In the former case ($x = y$), $[\![x\uparrow, y\uparrow \leftarrow y\uparrow.\ x\uparrow]\!]\, x\uparrow$ equals $x\uparrow$ reducing the verification condition to

$$H' \mid x\uparrow = x_i \wedge y\uparrow = y_i \supset x\uparrow = x_i \wedge x\uparrow = y_i$$

which is true since $x = y$. In the other case (x and y disjoint), $[\![x\uparrow, y\uparrow \leftarrow y\uparrow, x\uparrow]\!]\, x\uparrow$

equals $y\uparrow$, reducing the verification condition to

$$H' \mid x\uparrow = x_i \wedge y\uparrow = y_i \supset x\uparrow = x_i \wedge y\uparrow = y_i$$

which is an obvious tautology. Q.E.D.

Now let us examine a sample application of the generalized procedure call rule involving aliasing. Let H include the declarations a: array integer of integer, i: integer, j: integer. Assume we want to prove:

$$H \mid a[i] = a_1 \wedge a[j] = a_2 \{swap(a[i], a[j])\}\ a[j] = a_1 \wedge a[i] = a_2.$$

By the generalized procedure call rule, we must show

$$H \mid \forall x_0\, y_0 [a[i] = x_0 \wedge a[j] = y_0 \supset y' = x_0 \wedge x' = y_0]$$
$$\supset [a[i] = a_1 \wedge a[j] = a_2 \supset [\![a[i], a[j] \leftarrow x', y']\!](a[j] = a_1 \wedge a[i] = a_2)].$$

Let S' denote the consequent of the final implication. Moving the updates within S' further inside yields

$$[\![a[i], a[j] \leftarrow x', y']\!](a[j]) = a_1 \wedge [\![a[i], a[j] \leftarrow x', y']\!](a[i]) = a_2$$

which reduces to

$$y' = a_1 \wedge [\![a[i], a[j] \leftarrow x', y']\!](a[i]) = a_2.$$

We instantiate the logical variables x_0, y_0 in the major hypothesis as a_1 and a_2 respectively, giving us the hypothesis

$$a[i] = a_1 \wedge a[j] = a_2 \supset y' = a_1 \wedge x' = a_2.$$

Since the premise of this hypothesis is identical to the minor hypothesis, we deduce the new hypothesis

$$y' = a_1 \wedge x' = a_2.$$

If $i \neq j$ then S' reduces precisely to this formula. On the other hand, if $i = j$ then S' reduces to

$$y' = a_1 \wedge y' = a_2$$

which is a simple consequence of the hypotheses $i = j$, $a[i] = a_1 \wedge a[j] = a_2$, and $y' = a_1 \wedge x' = a_2$. Q.E.D.

5.3.3. Handling Recursion. The recursive form of the generalized procedure call rule is completely analogous to the recursive generalization of the simple procedure call rule. The soundness and relative completeness proofs are also nearly identical to those for the simple rule.

6. Reducing the Complexity of Proofs Involving Aliasing

Although our rules for procedures with aliasing are no more complicated than comparable rules prohibiting aliasing, they are more cumbersome to use in

practice, because they force all variable parameters to be passed by reference. Most procedures exploiting aliasing are designed to work only for a small subset of the possible aliasing configurations. If all variable parameters are passed by reference, the pre and post assertions for such a procedure must include a long list of disjointness assumptions.

We believe that a procedural programming language should provide two distinct classes of formal variable parameters: those that can have aliases and those that cannot. The explicit syntactic differentiation between these two classes greatly reduces the number of possible aliasing configurations, simplifying reasoning about updates.

To incorporate this modification into our PASCAL dialect, we establish the following new syntax for procedures:

```
procedure p(var w:ref T_w, x:T_x; val y:T_y);
    aliased global z_1;
        global z_2;
        B
```

where $\mathbf{w}$ are reference parameters (as described in Sect. 4.1.), $\mathbf{x}$ are variable parameters that have no aliases within the procedure, $\mathbf{y}$ are standard val parameters, $\mathbf{z}_1$ are global variables that may have aliases in the procedure and $\mathbf{z}_2$ are global variables that may not.

Within the procedure code block B, an assignment to any parameter v other than a reference parameter has the standard form:

$$v \leftarrow e.$$

In contrast, all references to a reference parameter must be explicitly dereferenced. Hence, an assignment to a reference parameter w has the form:

$$w\uparrow \leftarrow e.$$

The generalized procedure call rule (without recursion) for this extension of PASCAL has the following form. Let p be declared as shown above in a declaration set H; let P and Q be formulas in $\mathscr{L}$ containing no quoted identifiers and no program variables other than $\mathbf{w}$, $\mathbf{w}\uparrow$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{w}$, $\mathbf{w}\uparrow$, $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{z}_2$ respectively; let $\mathbf{v}$ be the free logical variables in P and Q; let $\mathbf{w}'$, $\mathbf{x}'$, $\mathbf{z}_1'$, $\mathbf{z}_2'$ be logical variables corresponding to $\mathbf{w}\uparrow$, $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{z}_2$ respectively; let R and S be arbitrary formulas; and let H' be H augmented by $\mathbf{w}$:ref T_w, $\mathbf{x}$:$\mathbf{T}_x$, $\mathbf{y}$:$\mathbf{T}_y$, and *Pair-Disjoint* $(\mathbf{w}, \mathbf{w}^* \otimes \mathbf{x}^* \otimes \mathbf{y}^* \otimes \mathbf{z}_2^*)$ (with prior declarations of $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ deleted). Then

$$\frac{\begin{array}{c} H \mid R \supset [\mathit{Disjoint}(\mathbf{b}^* \otimes \mathbf{z}_1^* \otimes \mathbf{z}_2^*) \wedge \mathit{Pair\text{-}Disjoint}(\mathbf{a}^*, \mathbf{b}^* \otimes \mathbf{z}_2^*)] \\ H' \mid P\{B\}\,Q \\ H \mid \forall \mathbf{v}[P(\mathbf{a}^*/\mathbf{w}, \mathbf{a}/\mathbf{w}\uparrow, \mathbf{b}/\mathbf{x}, \mathbf{c}/\mathbf{y}) \supset Q(\mathbf{w}'/\mathbf{w}\uparrow, \mathbf{x}'/\mathbf{x}, \mathbf{z}_1'/\mathbf{z}_1, \mathbf{z}_2'/\mathbf{z}_2)] \supset [R[\![\mathbf{a}, \mathbf{b}, \mathbf{z}_1, \mathbf{z}_2 \leftarrow \mathbf{w}', \mathbf{x}', \mathbf{z}_1', \mathbf{z}_2'} \end{array}}{H \mid R\{p(\mathbf{a}, \mathbf{b}, \mathbf{c})\}\,Q}$$

The soundness and relative completeness proofs for the modified rule are essentially unchanged from before.

7. Eliminating the Remaining Restrictions

Our most general procedure call rules still require the following restrictions:

1. No parameters or functions may be passed as parameters.
2. Every global variable accessed in a procedure must be accessible at the point of every call.
3. No procedure named p may be declared within the scope of a procedure p.

As Donahue [5] has pointed out, restriction 2 can be eliminated by making the declaration rule rename new variables within program text. A similar strategy can be used to eliminate restriction 3. In essence, this approach makes the rules rename program identifiers so that restrictions 2 and 3 hold after the renaming. We dislike the idea, however, because it modifies the text of a program (and any embedded assertions) in the course of a proof.

Fortunately, neither of these restrictions handicaps the programmer in any way. They simply force him to unabiguously name his variables and procedures. For this reason, we believe these two restrictions are a reasonable part of a practical programming language definition.

In contrast, the remaining restriction – the prohibition of procedures and functions as parameters – prevents the programmer from using an important language construct. In some application areas (such as numerical analysis), procedures and functions as parameters are nearly indispensable. We intend to extend Hoare's logic to handle this language construct in a subsequent paper.

Acknowledgements. We are grateful to Hans Boehm and David Gries for helpful discussions.

References

1. Apt, K.R., DeBakker, J.W.: Semantics and proof theory of PASCAL procedures. Tech. Rept., Stichting Mathematsch Centrum, Amsterdam, 1977
2. Clarke, E.M.: Programming language constructs for which it is impossible to obtain good Hoare-like axiom systems. Proceedings of Fourth ACM Symposium on Principles of Programming Languages, Los Angeles, 1976
3. Cook, S.: Axiomatic and Interpretive Semantics for an Algol Fragment. Tech. Rpt. 79, Dept. of Comp. Sci., Univ. of Toronto, 1975
4. Dijkstra, E.: A discipline of programming. Englewood Cliffs, N.J.: Prentice-Hall 1976
5. Donahue, J.E.: Complementary definitions of programming language semantics. Berlin Heidelberg New York: Springer 1976
6. Enderton, H.: A mathematical introduction to logic. New York: Academic Press 1972
7. Gorelick, G.A.: A complete axiomatic system for proving assertions about recursive and non-recursive programs. Tech. Rpt. 75, Dept. of Comp. Sci., Univ. of Toronto, 1975
8. Gries, D.: The multiple assignment statement. IEEE Trans. Software Engrg. **SE-4**, 89–93 (1978)
9. Hoare, C.A.R.: An axiomatic approach to computer programming. CACM **12**, 332–329 (1969)
10. Hoare, C.A.R.: Procedures and parameters: An axiomatic approach. In: Symposium on semantics of algorithmic languages (E. Engeler, ed.), Lecture Notes in Mathematics, Vol. 188, pp. 102–116. Berlin Heidelberg New York: Springer 1971
11. Igarashi, S., London, R., Luckham, D.: Automatic program verification I: a logical basis and its implementation. Acta Informat. **4**, 145–182 (1975)
12. London, R.L., Guttag, J.V., Horning, J.J., Lampson, B.W., Mitchell, J.G., Popek, G.J.: Proof rules for the programming language EUCLID. Acta Informat. **10**, 1–26 (1976)

13. McCarthy, J.: A basis for a mathematical theory of computation. In: Computing programming and formal systems (P. Braffort, D. Hirshberg, eds.). Amsterdam: North-Holland 1963
14. Nelson, C.G., Oppen, D.C.: Simplification by cooperating decision procedures. ACM Trans. Programming Languages Systems, **2**, 245–257 (1979)
15. Oppen, D.C.: On logic and program verification. Tech. Rpt. 82, Dept. of Comp. Sci., Univ. of Toronto, 1975
16. Pratt, V.R.: Semantical considerations on Floyd-Hoare logic, 17th Symposium on Foundations of Computer Science, IEEE, October 1976
17. Wirth, N.: The programming language PASCAL. Acta Informat. **1**, 35–63 (1971)

Received December 18, 1978/November 29, 1980