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TRUTH, BELIEF, AND VAGUENESS

When Jones believes that Horatio planted petunias in the garden yesterday, and Smith shares this belief, then there is something which both Smith and Jones believe. The most convenient name I can think of for the thing which both believe is 'proposition'. And since I think all of us have many beliefs, I also think there are many propositions, at least some of which are believed by at least one of us. Of course, it is nothing philosophically new to cast propositions in the role of objects of belief, disbelief, and various other so-called 'propositional attitudes'. Moreover, if Jones, or anyone else for that matter, believes that Horatio planted petunias in the garden yesterday, it would be unreasonable for Jones to deny that Horatio planted petunias. It would be unreasonable because obviously Jones' belief logically implies that Horatio planted petunias, even though logicians may dispute about the formalities of this implication. This illustrates the point that the propositions which can serve as the objects of belief (or of propositional attitudes in general) have logical implications, and also are implied by propositions which in turn can be the objects of propositional attitudes. Thus, the very same propositions which serve as the objects for propositional attitudes *also* serve as subject matter for logic, although the preceding considerations leave open the possibility that the class of propositions dealt with in logic is wider than the class of possible objects for propositional attitudes of people. (Some propositions included in the subject matter of logic, for example, may be too complex to be believed by any person.)

Logic, then, has at least the task of providing an account of the implications of and the various other logical relations between our beliefs, our disbeliefs, and the objects of our other propositional attitudes. (In addition, if there are propositions not subject to being objects of such attitudes, logic will also deal with them.) Because the set of propositions which logic must handle includes those propositions which real people actually believe in their day-to-day lives, logic must be prepared to deal with vague propositions, for people often have vague beliefs. This is not to say that in order to fulfill its mission to handle vague propositions logic itself must become vague. (The

Journal of Philosophical Logic 5 (1976) 47–78.

study of dead civilizations need not itself be dead.) But in order to deal with vague propositions, traditional logic does have to be generalized, since traditional logic is designed to handle only precise propositions; it thus cuts itself off from telling us about the logical relations between the propositions we actually believe most of the time. The purpose of this paper is to provide a philosophical discussion of the issues involved in the generalization of traditional logic. The paper is accessible to those with a background in elementary model theory.

When I say that ordinary beliefs are often vague, I do not mean the linguistic expression which may be given the belief by the believer, or an onlooker, is ambiguous – i.e., possessed of multiple senses, or characterized by indeterminacy of reference due to the existence of competing candidates to serve in the role of referents for some of the referring expressions in the linguistic expression. If I say Jones believes Horatio planted petunias in the garden yesterday, my utterance may very well be ambiguous, in that there may be more than one Horatio, or there may be several gardens of which I might be speaking. But such ambiguity does not cause any peculiar difficulties for classical theory of inference so far as I can see, although it is both interesting and difficult to handle in doing formal semantics. In any case, ambiguity is not what I mean by vagueness.

Jones' belief about Horatio might not be very specific, either. He may not have any particular belief about the tools used by Horatio, or the time of day at which the planting was done. But this lack of specific detail in Jones' belief is not vagueness, in the sense in which I mean to use the term. Again, no peculiar problems for classical logic seem to arise from the generality or lack of specificity characterizing some beliefs. So long as there is a determinate range of facts which would make the belief true or false, were any of the facts in this range to obtain, the belief is precise, for my purposes. The vagueness of belief which requires that classical logic be generalized is the vagueness which results in indeterminacy with respect to the truth conditions for the belief; so that for each vague belief (or any other vague proposition) the range of possible facts which would make that belief (or proposition) true were these facts to obtain is at least somewhat indeterminate. This means that for each vague proposition there are possible worlds in which the proposition's truth value is in some way peculiar, or indeterminate, or lacking entirely. It is this sort of vagueness which characterizes most of our beliefs, and which requires changes be made in logic in order to allow logic to deal with the properties and relations of vague propositions.

For example, classical logic insists on the principle of bivalence: Any given proposition is either simply true or simply false, but not both. However, Jones' belief that Horatio planted petunias in the garden yesterday fails to satisfy the principle of bivalence, because there are many conceivable circumstances in which the belief is neither simply true nor simply false. Suppose Horatio's idea of planting petunias is to soak the intended flower bed and then throw the baby petunia plants onto the top of the soil where they are allowed to lie untouched and unimplanted. Some of them may even survive and grow, if the next few days are cloudy and Horatio pours on the water. I for one, though, do not think it simply true under these circumstances that Horatio planted petunias in the garden yesterday. Nor simply false. The number of such peculiar but possible cases seems limited only by the power of one's imagination. Surely there does not exist a simple 'yes' or 'no' answer in each such case to the question, 'True?', asked of Jones' belief.

The classical logician, wishing to preserve hallowed traditions, may quite naturally try to come to the rescue of bivalence, perhaps with a speech proclaiming that whether Horatio succeeded in planting petunias in our imagined case above depends on what concept of *planting* is relevant. The difficulty with this speech is that it is *Jones'* concept of planting that is relevant, and Jones' concept of planting is vague – provided Jones is like the rest of us – with the result that one probably cannot settle the question whether Horatio planted by looking at Jones' concept of planting. (Of course, the question *could* be settled that way if Jones' concept of planting definitely does or definitely does not include Horatio's actions, but then there will be other cases for which Jones' concept of planting provides no ready answer.)

In short and in sum, there is no reason to suppose that once one has gotten clear about Jones' concepts, and the context of Jones' coming to believe that Horatio would do his bit to beautify the world, and anything or everything else about Jones that could possibly be relevant, then one will always have information which will determine a unique, old-fashioned truth value for the proposition Jones believes. That proposition does not relate to the detailed facts of the world in a neat two-valued way. It cannot, therefore, be formally represented as a function from possible worlds to old-fashioned truth values. Nor can it be thought of as a fully detailed, fleshed-out 'state of affairs', such as, e.g., Horatio Van Alstanwine III planted_5 petunias_2 fully within the outermost boundaries of a plot of ground located . . . , between 12 midnight, April 23, 1975, and 12 midnight, April 24, 1975.[1] Jones does not believe the things which can be represented in these ways, because Jones' belief is not nearly so precise. So long as logic is thought to deal only with such precise propositions, logic will not be able to treat of many of the objects of propositional attitudes.

Since many of the propositions people believe are not bivalent, it is important to discover what becomes of such propositions when they are neither simply true nor simply false. The most elegant view would seem to be that in such a case they are lacking a truth value entirely. I say this view is elegant, because if one adopts it, one need not try making sense of the idea that there are more than two truth values.

Thomason, Fine, and Van Fraassen at least have indicated sympathy for handling vague propositions in this elegant way.[2] Their move is made formally by using supervaluations.[3] Roughly, the idea is that a vague proposition's truth value is its supervaluation, which is a function of the proposition's tentative classical valuations. Each different tentative classical valuation is the ordinary classical truth value the proposition would have if it were made precise in some particular way, so as to rule out all borderline cases. For each way of making the proposition precise, we get a new tentative classical valuation for that proposition, indicating whether the proposition as newly interpreted is true or is false. If *every* way of making the proposition precise makes the proposition classically true, all the tentative classical valuations will be true, for that proposition. If every precise version of the proposition is false, all the tentative classical valuations are false. Otherwise, we get a mixture of tentative valuations. The supervaluation of the original vague proposition is then said to be truth just in case all the tentative classical valuations are true; false if and only if all the tentative valuations are false; and undefined otherwise. Thus, on this view a vague proposition is true just in case all ways of making it precise are true propositions, false just in case all precise versions of it are false, and neither true nor false otherwise. When the proposition is neither true nor false it has no truth value at all. On this view, Jones' belief about Horatio's petunia planting would presumably be neither true not false under the circumstances described earlier, since some ways of making Jones' belief precise would render the belief false while others would render the belief true.

One presumed advantage of the supervaluation approach is that the theorems of classical logic remain logically true when interpreted as outlined above, since these theorems will always all be true on every tentative classical valuation. Whether this feature of the supervaluation approach is in fact an advantage or a disadvantage I suppose might be questioned. My own intuitions happen to run counter to claiming that the law of noncontradiction, for example, is always completely true. But I will not press that line here, for a battle of raw intuitions is liable to bore onlookers.

Instead, I think it more fruitful to point out that Tarski's convention (*T*) would appear to favor a more radical approach to vague propositions. Applying (*T*) to our gardening example yields what seems to be an obvious truth:

(1) "Horatio planted petunias in the garden yesterday" is true if and only if Horatio planted petunias in the garden yesterday.

(1) can be abbreviated as

(2) $T(Q)$ if and only if P

where 'Q' is a name of the sentence whose quotation-name appears in (1), and 'P' is an abbreviation for that sentence. If Q lacks a truth value entirely, as it may on the supervaluation approach, or on any other approach which denies truth values to vague propositions,

(3) '$T(Q)$' is (out-and-out) false

because

(4) Q has no truth value.

Treating 'if and only if' in (2) and (1) as material equivalence will allow us to infer

(5) Q is false

from (2) and (3). But surely (4) and (5) are incompatible. That is, on this interpretation of (2), we cannot allow (4). It looks as if we must give up Tarski's (*T*) as applied in (1) if we are to allow vague propositions to lack truth value.

However, in such matters it pays to be cautious. One might read the 'if and only if' in (2) and (1) to mean that the truth of '$T(Q)$' in some sense necessitates the truth of Q (i.e., the truth of 'P'), and the truth of 'P' (i.e., Q) in the same way necessitates the truth of '$T(Q)$'.[4] Then, from (2) and (3) it would follow that Q is not true (for if it were, '$T(Q)$' would be true as well, contrary to (3)); but proving that Q is not true does not amount to proving that Q is false (i.e., does not amount to proving (5)).

Nevertheless, there will still be trouble, even on the new reading of (2) and (1). Consider what happens if we assume (2), (3), and (4) are completely and unproblematically true, as they ought to be on the supervaluation approach. In addition, we perversely assume that Horatio planted petunias in the garden yesterday. I.e., we assume

(6) P.

(2) under the present interpretation tells us, among other things, that if 'P' is true, '$T(Q)$' is bound to be true as well. Hence, (2) warrants the inference from (6) to

(7) $T(Q)$

and, similarly, from (7) to

(8) $T(T(Q))$

which clearly contradicts (3). Thus, we cannot allow both (3) and (6) if we want to keep (2). This much might seem obvious, and unproblematic, since (3) and (6) are intuitively incompatible.

Operating classically, we could go on to conclude that from (2) and (3) we would obtain

(9) $\sim P$

by indirect proof. This lands us in hot water. Suppose we take the plunge: From (9) we get

(10) $T(R)$

where 'R' is a name of '$\sim P$'; but (10) is presumably definitionally equivalent to (5), which is still incompatible with (4). This would show the supervaluation approach to be incompatible with Tarski's convention (T), if we were allowed to infer (9) by indirect proof. But the supervaluation approach cannot allow the validity of indirect proof in general. On that approach, a valid argument leading to a contradiction does lead to a clearly false conclusion, but that merely shows not all the premises in the argument are true. It may be the untrue premises aren't false, either, for they may lack a value entirely. If they do lack a truth value, so do their negations. Hence, one ought not infer (9) as we did.

But is this rejection of indirect proof rationally justified when dealing with vagueness? The rejection does show the supervaluation approach to be less classical than it might at first appear, for even though it preserves all the classical tautologies, it does not preserve all the classical rules of inference. I would prefer a logic of vagueness which would allow the inference to (9) once (8) and (3) are seen to be completely incompatible. I.e., it seems very desirable to adopt a view of logical consequence which has it that whenever a set of premises, S, together with some additional proposition, Φ, have as logical consequence two completely incompatible propositions, Ψ and Δ, then S has $\sim\Phi$ as a logical consequence. I can see that in a somewhat different context, when Ψ and Δ are vague propositions whose truth values are in some doubt, the inference to $\sim\Phi$ may not be warranted, since in such a context Ψ and Δ may not be completely incompatible. But in the case at hand, involving the inference from the incompatibility of (8) and (3) to the assertion (9), we have a case of *complete* incompatibility, on the supervaluation account.[5] And thus in the case at hand, the supervaluation approach (as well as any other approach which denies truth values to vague propositions) runs afoul of what seem to be reasonable requirements on what can count as valid inferences. Accordingly, I urge we search for another approach which will allow us to interpret the connective in (2) as material equivalence and which will preserve the limited version of indirect proof described above as being reasonable.

There may at this point seem to be nowhere left to go. We abandoned bivalence earlier. Now we are prohibited from denying truth values to vague propositions. There is, however, one alternative left to the stout-hearted: Vague propositions must take on unusual truth values when they fail to have the usual truth or falsehood as values.

Such unusual truth values might conveniently be thought of as degrees of truth and falsehood, so that when Horatio throws the petunias on the soggy ground and floods them with water it is more true to say Horatio planted petunias than it would have been were Horatio to dump the petunias in the garbage can and go off to sun himself at the beach. This approach has been advocated by several authors – notably, the mathematicians Zadeh and

Goguen – and is the one for which I am attempting to provide adequate motivation.[6]

Perhaps if one is firmly tied to a correspondence theory of truth, the notion of degrees of truth will not ring any more strangely than the notion of degrees of correspondence – the latter notion surely being at least moderately intelligible. On such a view, propositions may be formally represented as functions from possible worlds to the new, improved truth set containing as many truth values as one needs to deal with the logical phenomena. And we can represent Jones' believings in the most straightforward way as instances of the believing relation, holding between Jones and various propositions most of which are vague.

In outline, then, I want to allow for inversely variant degrees of truth and falsehood in vague propositions, with the classical truth values representing complete truth and complete falsehood. Although my inclinations in this matter are at least verbally in agreement with the common sense view that some beliefs are "truer" than others, that agreement cannot be taken at face value as an indication that the common man thinks of degrees of truth in the same way I do. The everyday assertion that P is truer than Q can in fact mean many different things. It might mean that P is more epistemically certain than Q. Or, that although P and Q are both really false, P is somewhat more accurate than Q. This probably means that the state of affairs which would make P true, were that state to obtain, is more similar to the actual state of affairs in the world than is the state of affairs which would make Q true were that state to obtain. It is important to distinguish these ordinary ways of talking about degrees of truth from the way of talking about degrees of truth which I wish to adopt. The conception of degrees of truth which is relevant in dealing with vagueness, and which could serve as a useful notion in logic when characterizing validity of argument forms, tautologous sentence forms, and the like, is *not* an *epistemic* notion like degrees of certainty; nor is it to be used to award consolation prizes to statements that really are false. E.g., since some crows are not in fact black, "All crows are black" will be considered just plain false on my view, even though *most* crows are black, and people might commonly say the statement is 'nearly' true.

But if a proposition is true *only* to a degree (i.e., not fully), then it seems that it must also be *false* to at least some degree, since to whatever extent the proposition fails to correspond fully with the way the world is that proposition is false. In application to our petunia planting example, it would seem there are many actions which Horatio could perform which are not full-fledged, clear examples of petunia planting, but yet are not full-fledged, clear examples of failure to plant petunias, either. So when we say it is only to some degree true that Horatio planted petunias, we shall not mean that Horatio simply *failed* to plant petunias, but came close; we shall mean instead that to some degree he succeeded in planting petunias, and to some degree he failed. And if we say that it is true only to some degree that this color chip is red, we shall *not* mean the color chip is simply not red, but comes close to being red; we shall mean instead that the chip really is somewhat red.

The task now before us then is to discuss the foundations of a many-valued logic in which the various values are to be construed as degrees of truth or falsity as outlined above. Of course, a large number of many-valued formal logical systems already exist. But the bare many-valuedness of a system provides no guarantee the system is suitable for the purposes at hand. We need to consider what conditions ought to be met by a logical system suitable for our purposes.

One limitation on the system which may be laid down at the outset is almost dictated by the fact that vague propositions sometimes take the classical truth values (now thought of as 'complete truth' and 'complete falsehood'), and when they do, the usual classical treatment will be just as acceptable for them as it is for precise propositions. I have no desire to quarrel here with the classical treatment of the usual truth-functional connectives, even though there are some cases in which such connectives clearly fail to provide adequate representations of connectives in a natural language such as English. (I have in mind, e.g., the failure of classical logic to provide an adequate treatment of the non-truth-functional connective, 'because'.) There is no reason to suppose a generalization of classical logic adequate to deal with vagueness should also thereby become adequate to deal with connectives in natural language which are non-truth-functional even when flanked by precise propositions. Hence, I will require for the sake of simplicity that the logic of vagueness be *normal*, in the sense that the sentential connectives shall be defined in such a way that when operating on propositions with classical values they yield propositions with the usual classical values. The normality requirement rules out, for example, a negation operator which when applied to a (completely) true proposition yields a proposition which is something besides (completely) false. To this extent, anyway, the logic we want is truth-functional.

We may now go on to ask whether our desired logical system is *entirely* truth-functional with regard to its sentential connectives. I find this a very difficult question to settle decisively, although I shall argue for the truth-functional approach. In this regard, too, the system I favor differs from that advocated in the supervaluation method. From the point of view of supervaluation theory, the truth value of a complex formula in the sentential calculus is a function of the tentative classical valuations of the complex formula taken as a whole. Each tentative classical valuation is, of course, truth-functional. But the resulting supervaluations are not. E.g., '$p \vee \sim p$' is true on each tentative classical valuation; hence any proposition of this form has truth for its supervalue as well – all of this despite the possibility that the proposition substituted for 'p' may itself lack a (super) truth value entirely, being vague. In point of truth-functionality, then, the approach I advocate is more classical than the theory generated by supervaluations, but this similarity to the classical position does not by itself constitute an argument for the appropriateness of treating the sentential connectives truth-functionally.[7]

It does not seem entirely clear at the outset whether, say, a formula of the form 'p & q' should always find its value from the values of 'p' and 'q', even when 'p' or 'q' have intermediate values. For, what do we say about an expression of the form 'p & $\sim p$' when 'p' is 'half' true? After all, both conjuncts are true to a degree. Does that mean that contradictions can be 'half' true, as wholes? If we take the truth-functional approach, it seems we shall be committed to the partial truth of some contradictions. On the other hand, if we were willing to give up truth-functionality, we could insist that in the case of a contradiction, even though both conjuncts are true to a degree, nevertheless the whole formula is completely false, because the conjuncts are not logically independent of one another. One might be reminded here of probability theory, in which the conjunction of two events, A and B, each having probability greater than 0 and less than 1, may nevertheless have probability 0 when A and B are mutually exclusive events.

Similar questions may be raised with respect to the law of the excluded middle, '$p \vee \sim p$', when neither 'p' nor '$\sim p$' is completely true or completely false. In such a case, the truth-functional approach demands that '$p \vee \sim p$' be treated just like any other formula of the form '$p \vee q$'. So it would seem that the most natural sort of truth-functional definitions of the connectives '$\vee$' and '&' in our multi-valued logic are likely to result in the loss of both the law of noncontradiction and the law of the excluded middle.

Indeed, this is exactly what happens in the logic for which I shall argue. I take it that the loss of these laws may seem initially to be a sufficient ground for rejecting my approach; my strategy will be to try to make the loss seem appropriate and welcome. It happens that one can give up these laws without destroying logic; in fact in a sense it even turns out that these laws are preserved in the system to be described below – although they will not be always completely true, they will always be at least half true.

Of course, in designing a formal multi-valued logic of vague propositions there is nothing to prevent us from defining some truth-functional connectives which we may choose capriciously to call 'and', 'or', etc. But what we really want is to define these connectives in such a way that they work as good symbolizations of logical functions actually employed in our vague speaking. So the issue is whether a truth-functional 'and', or 'or', or whatever, will be useful as a tool in adequately symbolizing complex vague propositions. And this issue is to be settled here in the same way it is settled in classical logic, by appeal to our understanding of the truth conditions for various sorts of complex propositions. But it is well-known that the classical sentential logic connectives do not all fit their English counterparts well. There are all sorts of difficulties about causal 'if . . . then' propositions, for example. I do not intend to solve any of these problems here. Rather, I have the more modest aim of providing the foundations for a logic of vague propositions which will handle them as well as classical first-order predicate logic handles precise propositions. My contention is that a truth-functional approach will do this job, even though in a sense we have to give up noncontradiction and the excluded middle in the process.

Consider negation first, because it seems clearest. I do not know of any reason why one would want negation to operate in anything but a truth-functional way in a logic of vagueness. There are, however, several different truth-functions which one might allow to play the role of negation. Of these the most natural have it that as 'p' gets truer, '$\sim p$' gets falser, and vice versa, with the values of 'p' and '$\sim p$' more or less equal when 'p' is about half true. It is just such a notion of negation which caused the trouble over noncontradiction and the excluded middle above, and it is just such a notion of negation which I wish to adopt.

Consider conjunction next, taking Jones' belief about Horatio once again for an example. We want to say Jones' vague belief is not completely true, but neither is it completely false. For the same reasons I should think it natural to say the negation of Jones' belief is neither out-and-out true nor out-and-out false. If you will, Jones' belief is true to some extent and also false to some extent. Similarly, for the negation of Jones' belief. It should by now be clear that in so characterizing Jones' belief, or its negation, I am not thereby trying to say part of the proposition he believes is true and part of it is false. Nor that in some respect, or in some sense, or at some time it is true, and in some other respect, sense, or time it is false. I.e., I am carefully avoiding those common misunderstandings of the law of noncontradiction which lead people to suppose the law false for bad reasons. I am trying to indicate, instead, that Jones' belief and its negation are both neither completely true nor completely false in the same respect, in the same sense, at the same time. Ontologically, I suppose this means there is not a relation of planting, not even a determinable as opposed to a determinate one, which Jones believes Horatio to stand in with respect to the petunias. For, if there were such a relation, I should think that Horatio would either stand in it with respect to the petunias, or not, as the case might be. Rather, what is going on here is that there are *many* properties and relations having to do with Horatio's actions vis a vis the petunias – all of which are relevant to the truth value of Jones' belief. And that belief on this occasion is constructed in such a way that these properties and relations do not all add up to something definitely falling within, or without, Jones' concept of petunia planting. This can happen because that concept does not pick out a precisely bounded set of such properties and relations which would then constitute petunia planting. Instead, a fairly indefinite range of such properties and relations is included under the concept, so that troublesome cases like the one under consideration can arise.

Given this understanding of the truth conditions for Jones' belief, it seems quite reasonable to say that Horatio to some extent planted petunias in the garden yesterday and that to some extent he did not plant petunias in the garden yesterday, in the same respect, in the same sense, and at the same time. The most reasonable thing to say in this case, then, seems to me to be that Horatio to some extent both planted and did not plant petunias in the garden yesterday. Given our understanding of degrees of truth it then becomes reasonable to say the proposition that Horatio planted and did not plant petunias in the garden yesterday is at least partially true. To deny Horatio to some extent did both seems unreasonable. In other words, a truth-functional definition of '&' in our logic will give us just the sort of result we want, since it now seems to be an essential characteristic of a vague proposition that a contradiction consisting of the proposition and its negation can be partially true.

This argument for allowing contradictions to be at least partially true does not work at all if one supposes that 'degrees of truth' merely represent degrees of epistemic certainty. If the *only* issue here were lack of certainty, one might well insist that even though no one *knows* whether Horatio planted those petunias, nevertheless he either did or did not succeed in doing the requisite planting, but not both. If that were the case the proposition that he both planted and did not plant would be plainly false, although no one would know which conjunct was true and which false. One would not want truth-functionality in a 'logic' whose 'truth values' represented degrees of epistemic certainty.

Similarly, one would not want truth-functionality if the 'truth values' represented the truth values a proposition could have if it were made precise, as we noted earlier in the discussion of supervaluations. It should be clear by now that the truth values which will appear in the system advocated here represent neither degrees of epistemic certainty nor the results of evaluating propositions after having made them more precise in various ways.

We turn to disjunction next. I wish to treat disjunction in the same way as conjunction – i.e., truth-functionally – for reasons quite parallel to those offered in favor of the truth-functionality of conjunction. As noted earlier, this approach immediately calls into question the law of the excluded middle, since a partially true '*p*' yields a partially true '$\sim p$', so that '$p \vee \sim p$' has neither disjunct completely true in such a case, with the result that it is hard to see how '$p \vee \sim p$' could be completely true on a truth-functional interpretation of 'v'. In fact, I do want to claim that in such a case, the value of '$p \vee \sim p$' is an intermediate, or nonclassical one. So there will be propositions of the form '$p \vee \sim p$' which are not completely true. But as it turns out, there will not be any that are more than half-way false. So we will have a law of the more or less excluded middle.

Given the truth-functionality of negation, conjunction, and disjunction it seems pointless to argue whether we ought to include a truth-functional analog of classical material implication in our logic. We already know that material implication does not correspond well to the English 'if . . . then' precisely because the English is non-truth-functional whereas material implication is truth-functional. Material implication in classical logic merely represents the best truth-functional way to handle many 'if . . . then' propositions (ignoring the effects of vagueness). We shall try for something similar in our logic, leaving the questions of better non-truth-functional representations of 'if . . . then' to another time, since these questions seem to have nothing special to do with vagueness. The details will be developed below along with the precise definitions of '~', '&', and 'v'.

I hope the preceding discussion has showed that those logical notions which are truth-functional for precise propositions as in classical logic are also truth-functional in the logic of vagueness, and that we might expect vague propositions to be just about as adequately symbolized in our logic of vagueness with its truth-functional connectives as precise propositions are in classical first-order logic.

We now have formulated two restrictions which any formal logic will have to meet in order to be suitable for our purposes: 1) normality, and 2) truth-functionality. There are of course many different systems which fulfill both 1) and 2). I wish now to consider further restrictions which ought to be placed on the system we want.

It would be nice to know something more precise about the set of truth values which our propositions can have. So far we have merely talked vaguely about 'more true', 'less true', and so on, and have assumed that the truth set has cardinality greater than or equal to three. Now, if we were to choose a truth set with only three elements, there would not be any chance to accurately represent situations in which a number of borderline cases arrange themselves in a natural ordering with respect to the degree to which they are *F*'s. We would simply have to say that all the borderline cases of an *F* are in the same truth-boat – i.e., to say of any one of them that it is an *F* is to say something with the one and only nonclassical truth value. In fact it is conceivable that we shall at times have a continuum of borderline cases with respect to some predicate *F*, and that to identify the degree to which any one of them is an *F* with the degree to which any other one is an *F* would seem completely arbitrary. This suggests that we really want a continuum of truth values, with an ordering relation defined on it. I will use the unit interval, with 0 representing complete falsehood and 1 representing complete truth, as has become fairly common practice.[8]

This means that we may on occasion assign $1/\pi$ to some proposition, to serve as its truth value; one may well wonder what sense can be made of such assignments. I am myself unable to see how one might arrive at such an assignment with any confidence that it, and it alone out of the whole continuum of possible assignments, correctly gives the truth value of a given proposition under a given set of circumstances. However, the assignments need not be completely arbitrary. For example, if we are dealing with the classification of bald men, an empirical investigation could reveal at approximately what point people begin to feel unsure whether sample baldish men are really bald; conceivably a great many variables could be involved, such as the age of the sample man, the color of his hair, and of course its density and distribution. After lengthy investigation, however, some patterns in the common man's classification of people as bald or not bald should emerge. One could then use these patterns to assign truth values to the propositions asserting that various intermediately baldish men are bald. The result would not be completely determined by the empirical data, but neither would it be completely arbitrary. It would have something of the character of a scientific hypothesis in empirical semantics. Fortunately, the assignment of exact values usually doesn't matter much for deciding on logical relations between vague propositions; what is of importance instead is the ordering relation between the values of various propositions.

We turn now to the precise definition of the connectives. As it happens, here again, as above, the problem is not to devise a new system but rather to lay down well-motivated constraints on the system we are looking for. It turns out that the Łukasiewicz system known as $L_{\aleph}$ satisfies them all. Thus, I am urging that Łukasiewicz' calculus is well-suited to serve as a logic of vagueness – an interpretation of the system quite different from that which Łukasiewicz himself placed on it.[9]

First of all, negation seems quite naturally defined in the usual way as follows: $/{\sim}p/ = 1 - /p/$. (Here the slashes around a formula are read: 'the value of'.) This definition gives us normality, and the inverse relation between $/p/$ and $/{\sim}p/$ which we want. There are, of course, other definitions which would do the same thing, but all of them seem arbitrary by comparison.

Given our decision to have truth-functional '&' and 'v', these definitions cause no trouble: if $/p/ = m$ and $/q/ = n$, then $/p \,\&\, q/$ clearly ought to be no truer than the maximum of m and n. Moreover, we ought also to have $/p \,\&\, q/ \geqslant \min(m, n)$. The only possible question here is whether we could have strict inequality in the last formula. I am inclined to think of conjunction in a fairly classical way here: When one conjunct is false, that makes the whole conjunction false, no matter how true the other conjunct might be. Accordingly, I will require $/p \& q/ = \min(/p/, /q/)$.[10] After all, if one allows $/p \& q/ > \min(/p/, /q/)$, then the conjunction of the premises in a given argument could be truer than the falsest premise in that argument, so that the argument from $\{p, q\}$ to r might be truth-preserving while the argument from $\{p \& q\}$ to r might not be. Such a result seems peculiar to me. Similarly, I will think of 'v' in a somewhat classical way, requiring $/p \vee q/ = \max(/p/, /q/)$.

I have proceeded here by working out definitions of '~', '&', and 'v' independently of one another, rather than, say, defining $/p \& q/ = /{\sim}({\sim}p \vee {\sim}q)/$, simply because it is not always clear which classical tautology, if any, ought to continue to link these connectives in a logic of vagueness. I prefer to derive the definitions of the connectives from a list of conditions which seem reasonable for our purposes. It does happen, though, that $/p \& q/ = /{\sim}({\sim}p \vee {\sim}q)/$, and $/p \vee q/ = /{\sim}({\sim}p \& {\sim}q)/$, all of which is very nice.

We do need to consider very carefully how to generalize material implication for our logic. Intuitions are no clear guide here. For example, it is tempting to try to take a shortcut by simply defining '⊃' by means of the relation $/p \supset q/ = /{\sim}p \vee q/$. But this would be a mistake, I think, which illustrates the need for developing our definitions to satisfy reasonable conditions rather than defining the connectives directly in terms of one another by means of classically valid formulas. To see that the proposed definition is questionable, consider the formula '$p \supset p$'. It seems that even in logic which admits vague instantiations for 'p', all instantiations of this formula ought to be completely true, since the truth values of the antecedent and consequent are of necessity always equal. Thinking of English sentences which would be symbolized this way, we would be inclined to say they are analytically true (if we don't mind the concept of analyticity). But if we define '⊃' as proposed, and let $/p/ = 1/2$, then $/p \supset p/ = 1/2$, rather than 1.

There are further considerations which will help settle this issue. It would be nice to preserve the very important classical connection between '⊃' and logical inference; i.e., we want a proposition of the form '$p \supset q$' to take value 1 in all models just in case the inference from p to q is valid in the sentential calculus. Although we have not yet given a formal definition of argument validity for the multi-valued system being generated here, it is clear that validity will have to amount to something like truth-preservation in virtue of logical form. I.e., when an argument instantiates an argument form possessing the property that its conclusion must always be at least as true as the falsest premise, the argument will be fully valid. Translation of this notion of argument validity into a condition on the definition of '⊃' yields the result that '⊃' must be defined in such a way that whenever the consequent is at least as true as the antecedent, the whole conditional statement is completely true. From these considerations, we obtain the following restriction on '⊃':

$$\text{If } /p/ < /q/ \text{ and } /r \supset p/ = 1, \text{ then } /r \supset q/ = 1. \tag{1}$$

We can obtain additional restrictions on '⊃' if we decide that when $/q/ < /p/$ and $/q/ \neq /r/$, then $/p \supset q/ \neq /p \supset r/$. The only reasonable alternative to such a decision would be to rule that $/p \supset q/ = 0$ uniformly, for all values of $/q/$ such that $/q/ < /p/$. A choice of the latter alternative, as opposed to the former, would break down the classical connection between '⊃' and logical inference, because in developing the theory of logical inference in a multi-valued calculus we shall surely want to draw some distinction between those forms of argument which are *nearly* truth-preserving and those which are not at all truth-preserving. If there is a form of argument such that the truth value of its conclusion must always be at least 80% of the truth value of its falsest premise, that form of argument deserves higher logical honors than a form of argument like

$$\begin{array}{l} p \\ \underline{q} \\ r \end{array}$$

in which there is no guarantee at all that the value of the conclusion is greater than 0 even when the premises are valued at 1. Accordingly, I believe the first alternative for defining '⊃' is more appropriate than the second. A choice of the first alternative suggests the following conditions:

$$\text{If } /p/ < /q/ \leqslant /r/, \text{ then } /r \supset p/ < /r \supset q/. \tag{2}$$

$$\text{If } /r/ \leqslant /p/ < /q/, \text{ then } /q \supset r/ < /p \supset r/. \tag{3}$$

Conditions (1), (2), and (3) jointly uniquely determine a definition for '⊃' when the set of truth values is *finite*. We can generalize the definition of '⊃' obtained from these conditions to the case of the infinite truth set.[11] To see what definition of '⊃' is implied by (1), (2), and (3) when the set of truth values is finite, suppose there are n such truth values: 0, $1/n-1$, $2/n-1, \ldots, 1$. First, we use (2) to find the values of '$r \supset p$' when $/r/ = 1$ and $/p/$ varies through n possibilities. (2) requires each such value of '$r \supset p$' be different and ordered so that as $/p/$ becomes smaller, so does $/r \supset p/$. Since there are only n different values to work with, $/r \supset p/ = /p/$ must hold. A similar argument shows that (3) requires $/p \supset r/ = 1 - /p/$ when $/r/ = 0$, since in this case as we run through the various values for 'p' we need to have $/p \supset r/$ increase as $/p/$ decreases. We now know the values of the conditional when the antecedent has value 1 or the consequent has value 0. By exactly similar reasoning we can fill in the rest of the values for the conditional when the antecedent is at least as true as the consequent. In each case we can use (2) or (3) to establish an ordering between the various values of '$r \supset p$' (or '$p \supset r$') keeping $/r/$ fixed at some intermediate value. A rigorous proof would show by mathematical induction on **n** that for any finite cardinality **n** of the set of truth values, $/p \supset q/ = 1 - /p/ + /q/$ when $/p/ \geqslant /q/$. We then use (1) to establish $/p \supset q/$ when $/p/ < /q/$. This turns out to be trivial, since we already know from our earlier work that $/r \supset p/ = 1$ when $/r/ = /p/$, so that if we let $/q/ > /p/$, $/r \supset q/ = 1$, by (1). Hence $/p \supset q/ = 1$ when $/q/ > /p/$. Summarizing these results we obtain

$$/p \supset q/ \;=\; \begin{cases} 1, & /q/ > /p/ \\ 1 - /p/ + /q/, & /q/ \leqslant /p/. \end{cases}$$

The degree to which the conditional is true thus amounts to an inverse measure of the degree to which the consequent fails to be as true as the antecedent. The amount by which $/p \supset q/$ falls short of complete truth is the amount by which $/q/$ fails to be as true as $/p/$. The above definition of $/p \supset q/$ can of course be used also when the set of truth values is infinite. We shall do just that.

We shall also adopt for convenience the definition

$$/p \equiv q/ \;=\; /(p \supset q) \;\&\; (q \supset p)/.$$

As I noted earlier, the system proposed here is due to Łukasiewicz. I know of no place in which Łukasiewicz himself indicated any interest in thinking of his logic as a logic of vagueness. In fact he seems to have had a quite different interpretation in mind – an interpretation of the values as probabilities. In retrospect his interpretation seems doomed to failure, since probability logic ought not be truth-functional. In fact, I know of no one who has proposed that the system just described be used for capturing the logical relations between vague propositions – a use for which the logic seems eminently well-suited.

In order to provide for quantification theory in our logic and at the same time to provide a set-theoretic semantics which will accord with the definitions of the connectives already given, we can employ a generalized set theory described by Zadeh and developed by Goguen. Basically, the generalized set theory differs from ordinary set theory by dint of allowing the set membership relation to admit of degrees. Formally, this is achieved by mapping an ordinary set into an (ordinary) index set, so that the mapping function can be thought of as a so-called 'fuzzy' set. An element of the domain of the function 'belongs to' the fuzzy set to the degree indicated by the element of the index set which is its image under the mapping. We shall use the unit interval for our index set, in order to provide a natural connection between the present semantics and our earlier decision to use the unit interval as our truth set; however, this move is not absolutely necessary, and one should not confuse the two roles of the unit interval as index set and as truth set.

Roughly, we will follow the strategy of assigning a fuzzy set to each predicate letter in the calculus which we wish to interpret. If a given predicate letter is not vague, or if it is vague but happens not to have any problematic instances in the domain of discourse at issue, the fuzzy set which serves as its extension will presumably map the domain into $\{0, 1\}$ – i.e., the fuzzy set won't have any fuzziness and will behave like an ordinary set.

However, this picture is to be modified a bit in order to provide for a formal display of various types of vagueness. It seems that there are at least three different sorts of vagueness which arise relative to predicates within a natural language, and my intention is to provide a formal mechanism for capturing these varieties of vagueness. We can name these types of vagueness with suitably descriptive phrases: (a) Conflict Vagueness, (b) Gap Vagueness, and (c) Weighting Vagueness. Briefly, Conflict Vagueness occurs when a single predicate is used in such a way that the semantical rules governing its application on the occasion in question conflict with one another. Gap Vagueness occurs when the semantical rules for a predicate fail to say anything at all about whether certain sorts of possible objects are to be included in the extension of the predicate. And Weighting Vagueness occurs when the natural semantics governing the use of the predicate provides that some one property or some combination of properties of a given object count to only a certain limited extent toward placing the object into the extension of the predicate, even though these properties are the only ones which are at all relevant in deciding the applicability of the predicate to the object. All the well-worked examples of vagueness occurring in the literature are of this latter sort.

I do not wish to argue here that all three types of vagueness actually exist in, say, English. I have given examples of these types of vagueness elsewhere.[12] The three types of vagueness mentioned above do not correspond in any neat way with the more standard classification scheme introduced by Alston in which vagueness is divided into two types: (a) that which stems from a 'lack of a precise cutoff point along some dimension', and (b) that which derives from the interaction of 'a number of independent conditions of application' for a predicate.[13] I classify vagueness in the unique way described above because the formal representation of each type is distinct in what follows.

In order to represent Conflict Vagueness, I allow a given predicate letter to be assigned more than one (fuzzy) partial extension in the model. Each such partial extension is intended to represent the extension determined by one *nonconflicting* set of criteria for application of the predicate expression being abbreviated by that predicate letter. The degree to which a given element of the domain belongs to one such partial extension need not equal the degree to which that same element belongs to another such partial extension of the same predicate letter: A given object **b** in the domain may be clearly an *F* when judged by one set of criteria for *F*-ness, but only partially an *F* when judged by another, equally appropriate, set of criteria. The valuation of '*Fb*' will then be settled by taking some appropriately weighted average of the values one would obtain for '*Fb*' under each set of criteria taken separately. In order to represent Gap Vagueness, I allow the functions which set the assignment of extensions to predicate letters to remain silent as to whether a given element, say, **b**, of the domain of discourse is in the fuzzy extension of a given predicate letter, say, '*F*'. In such a case, there is a question about what should be done about the truth value of '*Fb*'. Two possibilities suggest themselves: (a) '*Fb*' lacks truth value. (b) '*Fb*' is just as true as it is false – i.e., it has value 1/2. In order to avoid the diffficulties with truth-valueless propositions, I opt for the latter alternative, which has the advantage of making '$Fb \supset Fb$' a tautology. Finally, in order to represent

Weighting Vagueness I allow predicate letters to have fuzzy extensions such that some members of the domain are 'in' the extension of a given predicate letter only to a limited extent. In so far as mere computation of truth values is concerned, most of this complexity in assigning extensions to predicate letters could be avoided – we could have a model theory for vague predicates simply by using fuzzy extensions, without making distinctions between the three types of vagueness just described. But if our model theory is intended to reveal important semantic relationships, and is intended to be useful in some of the more formal aspects of doing semantics of a natural language, then the addition of some such detail as I have outlined may be justifiable.

Let the language to be interpreted be a first-order predicate calculus (without identity, definite descriptions, or operation constants), with a denumberable set of predicate letters for each finite number of places, and a denumberable set of individual constants. (The introduction of operation constants poses no special problems, but identity and definite descriptions are another matter.[14]) An interpretation, **M**, for such a language consists of the following:

(1) A non-empty set **D**, called the *domain* of the interpretation **M**.

(2) The unit interval, **I**, called the *index* of the interpretation **M**.

(3) The set **E**, called the *set of possible extensions*, consisting of all the ordered pairs whose first members are n-place predicate letters ($n \geqslant 1$) and whose second members are n-tuples of elements of **D** where the number of places in the predicate letter equals the number of places in the n-tuple in each case.

(4) A finite set **F**, the set of *predicate interpretation functions*, each member of which is a function having a subset of **E** as domain and a (perhaps improper) subset of **I** as range. For each predicate letter we require that at least one member of **F** have in its domain an element of **E** having that predicate letter as first member. (We do not require that all the various elements of **F** map a given element of **E** onto the same element of **I**, nor that all the elements of **E** be in the domain of some element or other of **F**. This allows for gaps, conflicts, and weighting.)

(5) A function **d** called the *denotation function*, which assigns to each individual constant an element of **D**. (However, for simplicity of notation we shall merely write, e.g., 'a' below in those places where "d('a')" would strictly be required.)

(6) *A valuation function*, **v**, such that:

(a) Each sentence letter is assigned a value in [0, 1] by **v**, and

(b) **v** assigns to each n-place predicate letter, ϕ, followed by n individual constants, $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$, a value in [0, 1] satisfying the following conditions:

(b.1) If only one element, **f**, of **F** interprets ϕ at $\langle a_1, a_2, \ldots, a_n \rangle$, then $\mathbf{v}(\phi(\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n)) = \mathbf{f}(\langle \phi, \langle a_1, a_2, \ldots, a_n \rangle\rangle)$.

(b.2) If no elements of **F** interpret ϕ at $\langle a_1, a_2, \ldots, a_n \rangle$, then $\mathbf{v}(\phi(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)) = .5$.

(b.3) If more than one element of **F** interprets ϕ at $\langle a_1, a_2, \ldots, a_n \rangle$, then $\mathbf{v}(\phi(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n))$ shall be chosen so as to lie somewhere within the range of values given to $\langle \phi, \langle a_1, a_2, \ldots, a_n \rangle\rangle$ by these elements of **F**.

(c) If a predicate letter has variables in any of its argument places, an assignment of values (in **D**) to these variables is made in the usual way and then **v** of the whole is determined, relative to this assignment, in a manner analogous to that outlined in (b) above.

(d) For any wffs, A and B, and a given assignment of values to variables, $\mathbf{v}(\sim A) = 1 - \mathbf{v}(A)$; $\mathbf{v}(A \,\&\, B) = \min(\mathbf{v}(A), \mathbf{v}(B))$; $\mathbf{v}(A \vee B) = \max(\mathbf{v}(A), \mathbf{v}(B))$; $\mathbf{v}(A \supset B) = 1 - \mathbf{v}(A) + \mathbf{v}(B)$ when $\mathbf{v}(B) < \mathbf{v}(A)$ and $\mathbf{v}(A \supset B) = 1$ otherwise.

(e) For any wff, A, $\mathbf{v}((\forall x)A)$, relative to an assignment of values to variables, is the greatest lower bound of the various values of $\mathbf{v}(A)$ relative to all possible assignments which differ from one another at most with respect to the value assigned to x. (If there are no free variables in A other than x, $\mathbf{v}((\forall x)A)$ will no longer be relative to an assignment, but will instead be uniquely determined for all assignments.) Similarly, $\mathbf{v}((\exists x)A)$, relative to an assignment of values to variables, is the least upper bound of the values of $\mathbf{v}(A)$ relative to all possible assignments which differ from one another at most with respect to the value assigned to x. Let '$\text{glb}_x(\mathbf{v}(A))$' denote the greatest lower bound described above; let '$\text{lub}_x(\mathbf{v}(A))$' denote the least upper upper bound described above.

If A and B are fuzzy sets, we define $A \cup B$ to be the fuzzy set such that (1) x belongs to $A \cup B$ iff x belongs to A to some degree or x belongs to B to some degree, and (2) x belongs to $A \cup B$ to the higher of the degrees to which x belongs to A or x belongs to B. Similarly, $A \cap B$ is the fuzzy set such that x belongs to $A \cap B$ to the lower of the degrees to which it belongs to A or to B.[15] These definitions preserve the usual connection between conjunction and set intersection and between disjunction and set union.

The value of $\mathrm{v}(A)$ for any *sentence* A is to be thought of as A's truth value. (If A is not a sentence, and A contains at least one free variable, then the valuation assigned to A by v will be the truth value of A, but only relative to an assignment of values to variables.)

It is not hard to show that $1 - \mathrm{glb}_x(1 - \mathrm{v}(A)) = \mathrm{lub}_x(\mathrm{v}(A))$ for any wff. A, and hence that the classical relation between the universal and existential quantifiers is maintained – i.e., $/{\sim}(\forall x) \sim A/ = /(\exists x)A/$. Similarly, $/(\forall x)A/ = /{\sim}(\exists x) \sim A/$. Moreover, the classical relations between conjunction, disjunction, and quantification in a finite D continue to hold: E.g., $/(\forall x)Fx/ = /Fa_1 \,\&\, Fa_2 \,\&\, \ldots \,\&\, Fa_n/$ where the a_i exhaust the domain, and $/(\exists x)Fx/ = /Fa_1 \vee Fa_2 \vee \ldots \vee Fa_n/$ similarly, since when the domain is finite, the glb = min and the lub = max.

Given the preceding understandings regarding quantification, sentential connectives, predication, and the truth set, there are of course an enormous number of interesting questions to explore. One might ask about axiomatizations, about natural deduction systems, and then about completeness and consistency, and about key theorems. Before doing these investigations, however, some additional concepts would need definition: the concept of a valid argument, and of a tautology or valid formula, at least. We have said nothing about these important matters yet. A brief discussion of these latter concepts will complete our sketch of the foundations of a logic of vagueness.

The most commonly used technique in multi-valued logic for defining 'valid argument' and 'valid formula' employs the notion of *designation*: Informally speaking, to designate an element or a set of elements of the truth set is to pick out these elements as being true-like, in some sense; so that when a formula is assigned a designated value, that formula is to be thought of as somehow true, or true-like. It has never been clear to me what understanding of truth lies behind such talk. However, designation does provide a convenient crutch on which one may lean in order to construct a notion of a valid formula or valid argument: A logically valid formula (or tautology) is one which can take on only designated truth values, no matter what consistent assignments are made to its various parts. And an argument form is valid if and only if the assignment of any designated value to its premises guarantees its conclusion will have a designated value as well.

Given our understanding of the truth set as representing degrees of truth, is there any sense in designating some of these truth values? Yes, I suppose some attention should be paid to the task of isolating those formulas which uniformly take on the value 1, and those argument forms which are constructed in such a way that the complete truth of all the premises guarantees the complete truth of the conclusion. It might also be interesting to explore these same questions with regard to the values greater than or equal to .5.[16] However, the more interesting question is the more general one of analysing argument forms with an eye to the *degree* to which they are truth-preserving, and analysing formulas with the goal of discovering the range of possible truth values they can take on. We really want to know for a given form of argument what constraints, if any, are placed by that form on the possible truth value of the conclusion if the premises have given truth values. For example, if all the premises have value greater than **n** does it not follow that the conclusion has value greater than n? Greater than $\frac{1}{2}$n? And so on. Similarly, we may ask of individual formulas whether there is some minimum or maximum value which it is possible for them to receive. We need not restrict our attention to the formulas which uniformly take on value 1.

Hence, rather than a notion of tautology, I propose we use a notion of a *minimally* **n**-*valued formula*: A formula is minimally **n**-valued iff. it can never have a value less than **n**. We can also use the parallel notion of a *maximally* **n**-*valued formula*. Then one can ask, with respect to a given formula, for the maximal **n** such that the formula is minimally **n**-valued. And instead of the notion of a designation-preserving form of argument, we want the notion of a *truth-preserving argument form*: A form of argument is truth-preserving iff its conclusion must be at least as true as its falsest premise. (We could call such argument forms 'valid'.) Finally, this notion can be generalized to the notion of the *degree of truth-preservation* possessed by an argument form: Here what is wanted is the function which determines the minimal truth value possible for a given argument form's conclusion given the various possible values of the premises.

In this connection, we can confirm that there is an interesting and useful relationship between our definition of '$\supset$' and the business of truth-preservation in argument forms: (This relationship is the analog of the relation between the classical horseshoe and classically valid argument forms.) Let "Φ implies Ψ to degree **n**" mean roughly that $/\Psi/$ can not dip below $/\Phi/$ by more than $1 - \mathbf{n}$, so that when **n** is near 1, the difference between $/\Phi/$ and $/\Psi/$ must be near 0. More precisely, let "$\Phi \vdash_{\overline{n}} \Psi$" mean that **n** is the least upper bound of all the numbers **m** such that $(/\Phi/ - /\Psi/) \leqslant (1 - \mathbf{m})$ for all possible assignments of values to Φ and Ψ, provided that at least some of those assignments result in a positive value for $(/\Phi/ - /\Psi/)$; if $(/\Phi/ - /\Psi/)$ is always negative or 0, then we stipulate that $\mathbf{n} = 1$. Given this definition of "$\Phi \vdash_{\overline{n}} \Psi$", it follows that **n** is the greatest lower bound of the set of possible values of the formula "$\Phi \supset \Psi$". That is, if Ψ is always at least as true as Φ, then $\Phi \vdash_{\overline{1}} \Psi$ and $/\Phi \supset \Psi/$ is always 1, but if Ψ can get a bit falser than Φ, then $\Phi \vdash_{\overline{n}} \Psi$ where n is some number near 1 and $/\Phi \supset \Psi/$ will sometimes dip a bit below 1 – in fact, it will dip just as far as **n**. And if it is possible for $/\Psi/$ to be 0 even when $/\Phi/ = 1$, then $\Phi \vdash_{\overline{0}} \Psi$ and $/\Phi \supset \Psi/$ goes as low as 0 – which is to say that Φ in this case does not imply Ψ at all.

The logical system outlined above can be used to solve the ancient *sorites* paradox. I shall consider only one of the most straightforward members of the *sorites* family of paradox-generating arguments, but I believe it fair to assume other members of the family would yield to similar analysis.[17] In English, the argument goes like this:

Horatio the would-be petunia planter has no hair on his head. (1)

Anyone who has no hair is bald. (2)

Anyone who has just one more hair on his head than any bald man is also bald. (3)

Anyone who has 10^7 hairs on his head is bald. (4)

From (1) and (2) we are to conclude Horatio is bald; from this conclusion and (3) we are to conclude that anyone who has just one more hair than Horatio is bald. From the latter conclusion and (3) we obtain the further result that anyone having just one more hair than anyone having just one more hair than Horatio is bald. And so on, until (4) is obtained.

Adopting the following scheme of abbreviation will yield a formalization of the argument sufficient for our purposes:

Nx: x has no hair on his head

Mxy: x has just one more hair on his head than y

Bx: x is bald

The first steps of the argument then might be symbolized as follows:

$$Nh \quad (5)$$
$$(\forall x)(Nx \supset Bx) \quad (6)$$
$$\overline{Bh} \quad (7)$$
$$(\forall x)(\forall y)(Mxy \,\&\, By \supset Bx) \quad (8)$$
$$\overline{(\forall x)(Mxh \supset Bx)} \quad (9)$$
$$(\forall x)(\forall y)(Mxy \,\&\, By \supset Bx) \quad (8)$$
$$\overline{(\forall x)(\forall y)(Mxy \,\&\, Myh \supset Bx)} \quad (10)$$
$$(\forall x)(\forall y)(Mxy \,\&\, By \supset Bx) \quad (8)$$
$$\overline{(\forall x)(\forall y)(\forall z)(Mxy \,\&\, Myz \,\&\, Mzh \supset Bx)} \quad (11)$$

Here (5), (6), and (8) are symbolizations of the original English premises, and (8) is used repeatedly to obtain further conclusions. Sufficient repetition of the pattern exhibited in the steps from (9)–(11) would ultimately yield a very long sentence equivalent to (4). In classical logic the argument (5)–(11) is valid; hence the paradox if (5) happens to be true, since (6) and (8) seem true a priori.

In contrast to the classical analysis of (5)–(11), our nonclassical interpretation of the logical constants in the argument shows the argument to be in trouble from the outset, for the argument from (5) and (6) to (7) is not fully valid. To be sure, when (5) and (6) have value 1, $/Bh/ = 1$ also, as can be seen from clauses (d) and (e) in the definition of **v**. (One of the assignments of values to variables which one considers in evaluating (6) is the assignment of h to 'x'.) Even when (5) has value 1 and (6) is less than completely true, things still look fine: Suppose (6) has value $m < 1$. Then $/Bh/$ can differ from $/Nh/$ by at most $(1 - m)$; i.e., $|/Nh/ - /Bh/| \leqslant (1 - m)$. But we are assuming for the moment that $/Nh/ = 1$. Hence $(1 - /Bh/) \leqslant (1 - m)$, or $/Bh/ \geqslant m$; i.e., the value of (7), the conclusion, is at least as great as the value of the falsest premise, (6). Nevertheless, when we consider the case in which (5) is *not* fully true, the form of argument exhibited in (5)–(7) is revealed not to be fully valid. E.g., let $/Nh/ = .6$ and $/(\forall x)(Nx \supset Bx)/ = .4$. Then we know only that $(/Nx/ - /Bx/) \leqslant .6$ for every assignment of values to 'x'. It could happen that $/Nh/ - /Bh/ = .6$; in this case, $/Bh/ = 0$ since $/Nh/ = .6$, resulting in a completely false conclusion from premises not completely false. Hence the argument form is not completely truth-preserving.

In general, if $/(5)/ = n$ and $/(6)/ = m$, then the best guarantee on the truth of (7) comes from the inequality $(/Nh/ - /Bh/) \leqslant (1 - m)$ which we get from the value of (6). Putting $/Nh/ = n$ in this inequality yields the guarantee that $/Bh/ \geqslant (m + n - 1)$. It is only to this degree that the argument form is valid in general. When $(/(5)/ + /(6)/) > 1$, $/Bh/ > 0$. When $(/(5)/ + /(6)/)$ is close to 2, $/Bh/$ is close to 1. Thus it would be unfair to say simply that the argument form of the *sorites* is invalid. That would not explain its deceptiveness. In fact, given the assumption that (5) and (6) are in reality completely true, we know from our analysis that (7) is then completely true as well, so that in a sense the *sorites* argument is quite all right down through (7). *That* is part of its deceptiveness.

The argument begins to get into more difficulty with the introduction of premise (8). The value of (8) is a function of the truth values of

$$Mxy \mathbin{\&} By \supset Bx \tag{12}$$

obtained from various assignments of denotation to 'x' and 'y'. For simplicity, assume 'M' is not vague. When 'x' and 'y' are assigned denotations not in relation M, or when 'y' is assigned a completely nonbald individual $/(12)/ = 1$. But we want to know the glb of the values of (12). $/(12)/$ dips below 1 when $/Mxy/ = 1$ and $/Bx/ < /By/$, for then $/Mxy \mathbin{\&} By/ = /By/$ and $/Mxy \mathbin{\&} By \supset Bx/ = 1 - /By/ + /Bx/$. Presumably, on the intended interpretation of 'B', any two individuals standing in relation M will be nearly equally B's. But at least within a certain range of such individuals, we shall want to say that a man with just one more hair is just a very, very small amount less bald. In this range, neither 'Bx' nor 'By' will be valued at 1, and the difference $/By/ - /Bx/$ will be some very small positive number, ϵ, perhaps on the order of 10^{-5}. The net result is that $/(8)/ = 1 - \epsilon$, rather than 1. Here is another factor in the deceptiveness of the *sorites* argument. It is easy to suppose (8) is completely true since its instances in many cases are completely true and in all cases are at least almost completely true.

Essentially the same analysis done earlier on the argument (5)–(7) will reveal with respect to (7)–(9) that so long as $/(7)/ = 1$, $/(9)/ \geqslant /(8)/$. I.e., given that $/(5)/$ and $/(6)/ = 1$, and $/(8)/ = (1 - \epsilon)$, we know $/(9)/ \geqslant (1 - \epsilon)$. Now, however, for the argument (9)–(10), it may be that *neither* premise (9) nor (8) is completely true. $/(8)/$ still $= (1 - \epsilon)$ and $/(9)/$ might be the same. What guarantee do we get for /(10)/? Assume 'x' and 'y' are assigned values such that 'Mxy' and 'Myh' are simultaneously completely true. From $/(9)/ \geqslant (1-\epsilon)$ we know $/By/ \geqslant (1-\epsilon)$. Moreover, since $/(8)/ = (1-\epsilon)$, we know $/By/ - /Bx/ \geqslant \epsilon$ for x such that $/Mxy/ = 1$. If $/Bx/$ did fall as much as ϵ below $/By/$, and $/By/$ took its lowest possible value, $(1-\epsilon)$, then $/Bx/ = (1-2\epsilon)$ and $/(10)/ = (1-2\epsilon)$. This is the worst that can happen to (10). I.e., we now know $/(10)/ \geqslant (1-2\epsilon)$.

As the pattern (9)–(10) is repeated in (10)–(11) and beyond, the guarantee of truth of each successive conclusion diminishes by ϵ for each repetition. Thus, the guarantee for (11) is down to $1-3\epsilon$, and the guarantee for the next conclusion would be $1-4\epsilon$. Clearly, if $\epsilon = 10^{-5}$, in approximately 10^5 steps, the guarantee of truth will be $1-10^5(10^{-5})$; i.e., there will be no guarantee left. Each step of the argument is slightly invalid, so the truth guarantee slowly leaks away as we try to carry it along the chain. The *sorites* is thus handled in what seems to me to be a very natural way when formalized in $L_\aleph$.

Note that if we use

$$(\forall x)(\forall y) \sim (Mxy \,\&\, By \,\&\, {\sim}Bx) \tag{12}$$

instead of (8) to symbolize (3), we get a different result (because ${\sim}(P \,\&\, {\sim}Q)$ is not equivalent in $L_\aleph$ to $(P \supset Q)$). Since there will be elements of the domain assigned to 'x' and 'y' for which $/By/ \approx .5$ and $/{\sim}Bx/ \approx .5$ even though $/Mxy/ = 1$, the value of (12) will be no higher than the neighborhood of .5; it cannot go lower either, since all values of 'x' and 'y' for which $/Mxy/ = 1$ will make $/By/$ very close to $/Bx/$, with the result that as $/By/$ decreases below .5 $/{\sim}Bx/$ increases above .5, and values of 'x' and 'y' which do not make $/Mxy/ = 1$ make $/Mxy/ = 0$. On this reading of premise (3) the *sorites* argument is not nearly so plausible. This, I think, is as it should be, for (12) essentially says that it never happens that of two persons differing by just a hair it can be said that one is bald and the other isn't. Since it can be somewhat true as well as somewhat false that *one* individual is bald, when his hair is very sparse, it can naturally also be quite true that *two* individuals, roughly alike, are both bald to some extent and not bald to some extent. Hence, the low truth value of (12) which denies this can happen.

The $L_\aleph$ approach to the *sorites* argument taken earlier allows us to have just about everything we want and yet we escape the paradox: The inductive premise of the argument (premise (3)) is interpreted as being quite true, so it is no wonder it seems plausible. The argument form has some validity, and in fact preserves truth quite well for many steps when the initial premises are quite true, so it is not surprising the logic of the argument should appear acceptable. There is no one point in the argument chain at which we can say that we have completely lost the guarantee of truth all in one big jump. Our result is what the common man wants: He's convinced that such 'slippery slope' arguments are fine if they're not carried too far. We can agree, because on the present view the truth guarantee for the conclusions in the chain of argument goes down rather slowly.

Let us check now to see how Tarski's convention (T) and *reductio ad absurdum* fare in $L_\aleph$. I take it that I have been saying that the predicate 'is true' applies to various propositions in varying degrees, so that $/$'p' is true$/$ $=$ $/p/$ (semantic paradoxes aside). Symbolizing (T) as follows

$$T(P) \equiv P$$

yields the result then that Tarksi's convention (T) is *completely* true, for $/p \equiv q/$ in general $= 1$ when $/p/ = /q/$ on our definition of '$\equiv$'. Of course, in order to obtain this result I have interpreted the connective in (T) as '$\equiv$' in $L_\aleph$. But if $L_\aleph$ is the proper logical system for handling vagueness, that is surely appropriate, since 'is true' is vague, as are many of the substitutions on 'p' in (T).

A form of reductio can also be maintained in $L_\aleph$. In $L_\aleph$ a proposition of the form $p \mathbin{\&} {\sim}p$ always has a value $\leqslant .5$. Hence, if the argument from the set of premises $S \cup \{Q\}$ to the conclusion $P \mathbin{\&} {\sim}P$ is completely valid, on every assignment at least one proposition in $S \cup \{Q\}$ has value $\leqslant .5$. There are only two cases to consider, then: (1) $\min(/S_i/) \leqslant .5$, where the S_i are the members of S, and (2) $\min(/S_i/) > .5$. In the first case, we know nothing about $/Q/$, except that $0 \leqslant /Q/ \leqslant 1$. But this is enough to ensure that $/{\sim}Q/$ can never fall more than .5 below $\min(/S_i/)$. In the second case, since $\min(/S_i/) > .5$, but $\min(/S_i/, /Q/) \leqslant .5$, we know $/Q/ \leqslant .5$. Thus, in this case, $/{\sim}Q/ \geqslant .5$, and once again $/{\sim}Q/$ can't fall more than .5 below $\min(/S_i/)$. Hence, consideration of the two cases shows that reductio is somewhat truth preserving. More precisely, if $S \cup \{Q\} \vdash_{1} P \mathbin{\&} {\sim}P$, then $S \vdash_{.5} {\sim}Q$, using an obvious extension of the notion of degreed implication discussed earlier. I.e., reductio is a 'half-way valid' mode of inference in $L_\aleph$.

An additional result regarding reductio in $L_\aleph$ shows $L_\aleph$ to be consistent with the remarks made earlier in the discussion of Tarski's convention (T).

If $S \cup \{Q\} \vdash_T P \& \sim P$, and if $/P \& \sim P/ = 0$, then we know $\min(/S_i/, /Q/) = 0$, since $\min(/S_i/, /Q/) \leqslant /P \& \sim P/$. Then, if $\min(/S_i/) > 0$, we know $/Q/ = 0$ and thus that $/\sim Q/ = 1$. On the other hand if $\min(/S_i/) = 0$, we know only that $0 \leqslant /Q/ \leqslant 1$. But in either case, we know $\min(/S_i/) \leqslant /\sim Q/$. That is, if $S \cup \{Q\} \vdash_T P \& \sim P$ and $/P \& \sim P/ = 0$, we know the inference from S to $\sim Q$ is *completely* truth-preserving. When a valid argument from $S \cup \{Q\}$ leads to an out-and-out contradiction, one may legitimately infer $\sim Q$ from S, which is just what was done in our earlier discussion of convention (T).

Finally, with respect to the philosophy of mathematics, a few remarks may be in order. Since the logic presented here is normal and since I assume precise propositions take on only classical truth-values, nothing in the classical analysis of precise mathematical propositions is affected by my insistence on a peculiar logic. I have assumed classical logic is fine as far as it goes; if one is convinced otherwise, as are the intuitionists, my proposals in this paper will be seen as generalizations of the wrong logic. But in any case that is a separate issue – neither intuitionistic nor classical logic is designed to handle vague propositions. And if classical logic is indeed fine for precise propositions we may cheerfully go on using it in doing philosophy of mathematics so long as we restrict attention to precise mathematical propositions. The claims made in this paper become relevant only when the propositions of mathematics are subject to vagueness. However, there are important cases of potentially vague mathematical propositions worth mentioning here, I think. The principle of mathematical induction and the least number principle, to name two, can have vague instances if these principles are taken to apply for ordinary predicates. The former principle can be understood to claim that if any given predicate meets certain conditions, then it follows that predicate truthfully applies to all the natural numbers; the latter principle can mean that there is a least number that has a given predicate meeting certain conditions. If I am right, these principles, understood in the way just described, are completely true only when limited to precise predicates. If one allows the substitution of vague predicates into these hallowed principles, it turns out that versions of the sorites paradox will be generated, as Cargile and Black have shown.[18] In addition to difficulties with mathematical induction and the least number principle, vagueness may cause trouble in the application of mathematics to physics or everyday life. Investigation of such problems would be a worthy subject for further efforts in the study of vagueness.[19]

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NOTES

[1] This is intended as a parody of sorts on the kind of account of propositions given by Chisholm. Cf., e.g., 'Problems of Identity' in *Identity and Individuation*, ed. by Milton K. Munitz, New York Univ. Press, 1971, pp. 24–26.
[2] Thomason and van Fraassen in conversation and unpublished essays. Kit Fine in 'Vagueness, Truth and Logic', *Synthese* **30** (1975), 265–300. Cf. D. Lewis, 'General Semantics', *Synthese* 22 (1970), 18–67. Fine argues that any approach such as the one I take will fail to capture the 'penumbral connections' between vague predicates. By a 'penumbral connection' he means a logical relation, such as contrariety. I think Fine is right to say even vague predicates have logical relations to one another, and that it is important to insist, as he does, that an adequate logic of vagueness capture such relations, or at least not rule them out. But I believe it is not at all obvious that the penumbral connections on which he builds his case against the sort of theory I hold really exist. E.g., he claims that 'red' and 'pink', even though vague and admitting of borderline cases of applicability, are nevertheless logically connected so that to say of some color shade that it is both red and pink is obviously to say something false. I must confess being completely insensitive to that intuition of a penumbral connection.
[3] Cf. B.C. van Fraassen, 'Presupposition, Implication, and Self-Reference', *J. Phil.* **65** (1968), 136–152, for an account of supervaluations in another context.
[4] This reading is due to van Fraassen, *ibid.*
[5] It seems to me that even if one does not approve of the unrestricted use of reductio to draw 'irrelevant' conclusions from a set of premises, one could still approve its use in the present instance where relevance is not a problem.
[6] L.A. Zadeh, 'Fuzzy Sets', *Information and Control* 8 (1965), 338–353, and J.A. Goguen, 'The Logic of Inexact Concepts', *Synthese* 19 (1968–69), 325–373. Cf. also my 'Vague Predicates', *Am. Phil. Quart.* 9 (1972), 225–233, for a fuller argument that there are vague propositions and for a modification of the Goguen semantics for a first-order logic of vagueness.
[7] One need not use supervaluations to ground a non-truth-functional logic of vagueness. David Sanford adopts a unique approach to the development of such a system in 'Borderline Logic', *Am. Phil. Quart.* **12** (1975), 29–40.
[8] There are other, more unfamiliar, constructs one might try out as truth sets, but the present paper is an exploration into what can be done with the unit interval. However, use of the unit interval does raise questions, especially since the interval's total ordering makes the truth values of all vague propositions pairwise comparable. Some would consider this result counterintuitive and would prefer a merely partially ordered truth set. It seems to me, however, that the difficulties about comparability are really just difficulties about how to assign degrees of truth to propositions, and that in general the unit interval can serve quite well as our truth set.
[9] Cf. N. Rescher, *Many-valued Logic*, New York, 1969, pp. 36 ff., and bibliography; and Jan Łukasiewicz and Alfred Tarski, 'Untersuchungen über den Aussagenkalkül,' *Comptes rendus . . . Varsovie*, Classe III **23** (1930), 51–77, tr. by J.H. Woodger in Alfred Tarski, *Logic, Semantics, Metamathematics*, Oxford, 1956, pp. 38–59.
[10] J.A. Goguen, in 'The Logic of Inexact Concepts', *Synthese* **19** (1968–69), 347, suggests that multiplication instead of min yields the more adequate symbolization of the English 'and'. I fail to see this, especially in view of Goguen's own earlier definition of fuzzy set intersection (p. 338) which has it that an element belongs to the intersection

of two fuzzy sets to the minimal degree to which it belongs to either of the fuzzy sets taken singly. Later in his paper, Goguen proves that various formulas are tautologies, assuming that the truth set has certain properties. As an example of such a truth set, Goguen mentions [0, 1], with the operation of multiplication serving as the analog of conjunction. (p. 355; cf. pp. 361 ff.) However, the operator min will serve just as well on [0, 1], with 1 as the identity (i.e. min $(a, 1) = a = \min(1, a)$ for any $a \in [0, 1]$), so that Goguen's theorems also go through using min instead of multiplication.

[11] Arto Salomaa in 'On Many-valued Systems of Logic', *Ajatus* 22 (1959), 138–145, discusses various conditions which might be placed on many-valued connectives. My set (1)–(3) is equivalent to his set C_8. Most of Salomaa's sets of conditions employ the device of *designating* some subset of the truth value set and treating this designated subset as if it were the value true. I find the use of designation in the semantics for many-valued logic philosophically very puzzling, despite its technical usefulness.

[12] In 'Vague Predicates'.

[13] William P. Alston, *Philosophy of Language*, Prentice-Hall, 1964, p. 87.

[14] The interpretations satisfying the conditions given below make reference precise. In a full treatment of vagueness, I believe vague reference would have to be taken into account as well, so that one would say "John existed on April 24" has an intermediate truth value if John was, if anything, only a partially developed fetus on April 24. In such cases the problems of vagueness are apparently connected with vague reference. I hope to consider such problems in a future paper.

[15] For a fairly full development of this sort of set theory see Goguen's work.

[16] Richard C.T. Lee and Chin-Liang Chang have proved that in the fragment of the present logic which employs only negation, conjunction, and disjunction a formula uniformly takes on values greater than or equal to .5 if and only if that formula is a classical tautology, and that a formula uniformly takes on values less than or equal to .5 if and only if that formula is a classical contradiction. Cf. 'Some Properties of Fuzzy Logic', *Information and Control* 19 (1971), 417–431. However, the Lee-Chang result cannot be generalized to the whole logic, for $/\sim (P \,\&\, (P \supset \sim P))/ = .4$ when $/P/ = .6$. (I owe this example to my colleague, Lawrence Eggan.)

[17] For some other versions of the paradox, see James Cargile, 'The Sorites Paradox', *Brit. J. Phil. Sci.* 20 (1969), 193–202, and Max Black, 'Reasoning with Loose Concepts', in *Margins of Precision*, Ithaca, N.Y., 1970, pp. 1–13. The solution to the paradox given below is the same in spirit as that given by Goguen.

[18] James Cargile, *loc. cit.*; Max Black, *loc. cit.*

[19] I am grateful for the careful comments of the referee – comments which provided useful suggestions, and which saved me from at least one major blunder.