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A SEMANTICAL THEORY OF ANALYTIC IMPLICATION

The system of Analytic Implication of Parry [2], [3] is based on the idea that for A to imply B every concept occurring in B must also occur in A. Dunn [1] proves an algebraic completeness result for an extension of Parry's system. In this paper we extend this completeness result to a modal extension of Dunn's system, for which we also prove decidability.

1. Axiomatic formulations

The systems to be discussed are formulated with denumerably many propositional variables $P_0, \ldots, P_k \ldots$ with the connectives $\rightarrow$, &, $\sim$ and $\Box$ primitive (the $\Box$ operator is present only in the system AIN described below); the set of well-formed formulas is as usual. $A \leftrightarrow B$, $A \vee B$ and $A \supset B$ are defined as $(A \rightarrow B)\ \&\ (B \rightarrow A)$, $\sim(\sim A\ \&\ \sim B)$ and $\sim(A\ \&\ \sim B)$ respectively.

The axioms of Parry [2] are the instances of the following schemata.

A1. $A\ \&\ B \rightarrow B\ \&\ A$
A2. $A \rightarrow A\ \&\ A$
A3. $A \rightarrow \sim\sim A$
A4. $\sim\sim A \rightarrow A$
A5. $A\ \&\ (B \vee C) \rightarrow (A\ \&\ B) \vee (A\ \&\ C)$
A6. $A \vee (B\ \&\ \sim B) \rightarrow A$
A7. $(A \rightarrow B)\ \&\ (B \rightarrow C) \rightarrow .A \rightarrow C$
A8. $A \rightarrow B\ \&\ C \rightarrow .A \rightarrow C$
A9. $(A \rightarrow B)\ \&\ (C \rightarrow D) \rightarrow .A\ \&\ C \rightarrow B\ \&\ D$
A10. $(A \rightarrow B)\ \&\ (C \rightarrow D) \rightarrow .A \vee C \rightarrow B \vee D$
A11. $(A \rightarrow B) \rightarrow (A \supset B)$
A12. $(A \leftrightarrow B)\ \&\ f(A) \rightarrow f(B)$
A13. $f(A) \rightarrow .A \rightarrow A.$

The sole rule of inference is *detachment* (from A and $A \rightarrow B$ to infer B). In A12 and A13 $f(B)$ is the result of replacing one or more occurrences

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of A in $f(A)$ by B. Parry [3] adds the rule of adjunction (from A and B to infer $A \,\&\, B$) and the following axiom schemata.

A14. $(\sim M \sim A \,\&\, (A \rightarrow B)) \rightarrow \sim M \sim B$
A15. $\sim (A \supset B) \rightarrow \sim (A \rightarrow B)$

where MA is defined to be $\sim (A \rightarrow \sim A)$.

Dunn [1] adds the schema

A16. $A \rightarrow . \sim A \rightarrow A.$

We shall refer to the system of Parry [3] as Analytic Strict Implication or ASI; ASI with the addition of A16 we shall refer to as Analytic Implication or AI. It should be noted that this formulation of AI does not coincide with that of Dunn [1]. Dunn's formulation is as above, save that A14 is omitted. In fact A14 is redundant in AI (as pointed out by a referee); we shall prove this as T14 below.

The system which forms the main object of our investigations is formed by adding to AI an explicit necessity operator $\Box$, for which we have the additional axiom schemata

A17. $\Box (A \rightarrow B) \rightarrow . \Box A \rightarrow \Box B$
A18. $\Box (A \,\&\, B) \rightarrow \Box A \,\&\, \Box B$
A19. $\Box A \rightarrow A$
A20. $\Box A \rightarrow \Box \Box A$
A21. $(A \rightarrow A) \rightarrow \Box (A \rightarrow A)$,

and the rule of necessitation (from A to infer $\Box A$). To A12 we add the restriction that in $f(A)$ there be no occurrence of A within the scope of a necessity operator. This system we shall refer to as AIN, or Analytic Implication with Necessity.

We now list some theorem schemata of AIN, referring the reader in most cases to Dunn [1] for proofs.

T1. $B \rightarrow .A \rightarrow A$, where B is a formula in which occur all variables in A.

Proof. As in [1] T4, using A17, A21.

T2. A, where A is a tautology of classical logic in & and $\sim$.

Proof. See [1], §2.

Let us write $S \vdash A$ if A is derivable from the set of formulas S and the axioms of AIN by the rules of detachment and adjunction.

T3. If $S \cup \{A\} \vdash B$, then $S \vdash A \supset B$

Proof. By T2.

T4. If $S \cup \{A\} \vdash B$ and every variable which occurs in B also occurs in A then $S \vdash A \rightarrow B$.

Proof. By T1 and T3, as in [1], §2.

T5. $A \mathbin{\&} B \rightarrow .A \leftrightarrow B$, provided A and B contain exactly the same propositional variables.

Proof. By T4.

T6. $\Box A_1 \mathbin{\&} \ldots \mathbin{\&} \Box A_n \rightarrow \Box (A_1 \mathbin{\&} \ldots \mathbin{\&} A_n)$

Proof. By A18.

T7. $\Box (A \supset B) \supset . \Box A \supset \Box B$

Proof. By T3, A18.

T8. $\sim A \mathbin{\&} \sim B \mathbin{\&} (\sim A \rightarrow \sim B) \rightarrow .A \rightarrow B$

Proof. See [1], T24.

T9. $A \mathbin{\&} B \mathbin{\&} (\sim A \rightarrow \sim B) \rightarrow .A \rightarrow B$

Proof. Assume $A \mathbin{\&} B \mathbin{\&} (\sim A \rightarrow \sim B)$. Then B, and $(\sim A \rightarrow \sim B)$ by the conjunction axioms. Now $\sim A \rightarrow \sim A$ by A3, A4, A7, hence $\sim A \rightarrow \sim A \mathbin{\&} \sim B$ by A2, A7, A9. $\sim A \leftrightarrow \sim A \mathbin{\&} \sim B$ follows by adjunction. Now we have $A \vee B \rightarrow B$ by T4, hence $A \rightarrow B$ by A3, A4, T7 and A12. Thus T9 follows by T4.

We take this opportunity to remark that T9 was inadvertently omitted from [1], §4. The author is indebted to Professor Dunn for the proof given above.

T10. $\sim A \mathbin{\&} B \mathbin{\&} (A \rightarrow B) \rightarrow . \sim A \rightarrow B.$

Proof. By [1] T26, and T9 above.

T11. $\sim A \mathbin{\&} \sim B \mathbin{\&} (A \rightarrow B) \rightarrow . \sim A \rightarrow \sim B.$

Proof. By T9, A3, A4.

A set S of sentences is *consistent* if for no B is it the case that $S \vdash B \,\&\, \sim B$, *maximally consistent* if S is consistent but is not properly contained in any consistent set.

T12. Every consistent set of sentences is contained in a maximally consistent set.

Proof. By the methods of the corresponding theorem for classical logic.

T13. If M is a maximally consistent set, then for any A, B, (i) A is in M iff $M \vdash A$, (ii) $\sim A$ is in M iff A is not in M, (iii) $A \,\&\, B$ is in M iff A is in M and B is in M.

Proof. By T2.

T14. $(\sim M \sim A \,\&\, (A \rightarrow B)) \rightarrow \sim M \sim B.$

Proof. First note that in primitive notation T14 reads:

$$(\sim\sim(\sim A \rightarrow \sim\sim A) \,\&\, (A \rightarrow B)) \rightarrow \sim\sim(\sim B \rightarrow \sim\sim B).$$

By A3, A4, adjunction and A12 this schema is equivalent to

$$((\sim A \rightarrow A) \,\&\, (A \rightarrow B)) \rightarrow (\sim B \rightarrow B).$$

Now assume $(\sim A \rightarrow A) \,\&\, (A \rightarrow B)$; assuming further $\sim B$, we have $\sim A \rightarrow A$ and $A \rightarrow B$ by A8, hence $\sim A \vee B$ by A11. Hence $((\sim A \vee B) \,\&\, \sim B)$, so $\sim A$ by A1, A5, A6. By detachment, we conclude that A, and so B. Hence the above schema follows by two applications of T4. Note that in the above proof A14 is nowhere used; hence A14 is redundant in our formulation of AI and AIN.

2. Semantics

$Q = (I, W, w, \leqslant, R, F)$ is an *aw*-model if (i) I is a non-empty set, (ii) $w \in W$, (iii) $\leqslant$ is a transitive reflexive relation on W, (iv) $R \subseteq W \times I$, (v) F is a function defined on the non-negative integers taking subsets Fk of $I \cup W$ as values. Relative to Q we define two valuation relations, one having I as domain, the other W.

For x in I,

$x \models_Q P_k$ iff $x \in F_k$,
$x \models_Q A \,\&\, B$ iff $x \models_Q A$ or $x \models_Q B$,
$x \models_Q \sim A$ iff $x \models_Q A$,
$x \models_Q A \rightarrow B$ iff $x \models_Q A$ or $x \models_Q B$,
$x \models_Q \Box A$ iff $x \models_Q A$.

For u in W,

$u \Vdash_Q P_k$ iff $u \in F_k$,
$u \Vdash_Q A \,\&\, B$ iff $u \Vdash_Q A$ and $u \Vdash_Q B$,
$u \Vdash_Q \sim A$ iff not $u \Vdash_Q A$,
$u \Vdash_Q A \rightarrow B$ iff for all x in I such that uRx, if $x \models_Q B$ then $x \models_Q A$, and if $u \Vdash_Q A$ then $u \Vdash_Q B$.
$u \Vdash_Q \Box A$ iff for all v such that $u \leqslant v$, $v \Vdash_Q A$.

$Q=(I, w, F)$ is an *a*-model if I and F are as in an *aw*-model. The consequence relation relative to I is defined as above, save that the clauses for $\Box$ are omitted, and the clause for $\rightarrow$ is simplified to read:

$w \Vdash_Q A \rightarrow B$ iff for all x in I, if $x \models_Q B$ then $x \models_Q A$, and if $w \Vdash_Q A$ then $w \Vdash_Q B$.

A formula A is *true* in a model Q if $w \Vdash_Q A$ and *aw-valid* (*a-valid*) if true in all *aw*-models (*a*-models).

An informal interpretation may be attached to these semantics. I may be taken to be a set of *concepts*, while W is a set of *possible worlds*, w being the real world. The relation $\leqslant$ is one of relative possibility, while for $u \in W$, $x \in I$, 'uRx' is to be read as 'x is a concept conceivable in world u'. For x in I, '$x \models_Q A$' may be read 'concept x occurs in sentence A', while '$u \Vdash_Q A$' is to be read 'A is true in possible world u'. Under this interpretation most of the valuation rules are self-explanatory. The only rule worthy of comment is that for $\rightarrow$ relative to W. According to this rule, $A \rightarrow B$ is true in world u iff (i) B follows from A in the sense of classical logic and (ii) every conceivable concept (that is, every concept conceivable in world u) which occurs in B also occurs in A.

Gödel in the discussion reported in [2] raised the question of completeness for Parry's system:

> '**p** impliziert **q** analytisch', kann man vielleicht so interpretieren: '**q** ist aus **p** und den logischen Axiomen ableitbar und

> enthält keine anderen Begriffe als **p**' und es wäre, nachdem man diese Definition genauer präzisiert hat, ein Vollständigkeitsbeweis für die *Parry*schen Axiome zu erstreben, in dem Sinn, dass alle Sätze, welche für die obige Interpretation von $\rightarrow$ gelten, ableitbar sind.

The definitions in conjunction with the informal interpretation given above can be seen as an attempt at making precise Gödel's definition.

3. Completeness of AI and AIN

THEOREM 1. A formula of AIN is provable in AIN iff it is *aw*-valid.

THEOREM 2. A formula of AI is provable in AI iff it is *a*-valid.

It is left to the reader to prove semantic consistency. We now proceed to prove the converse. Let A be unprovable in AIN. Then $\{\sim A\}$ is consistent, so by T12, $\sim A$ is in some maximally consistent set M. We shall show A *aw*-invalid by constructing an *aw*-model Q in which w is M and for any B, $w \Vdash_Q B$ iff $B \in M$.

Let W be the set of all maximally consistent sets of sentences. For u, v in W let $u \leqslant v$ hold iff for any A, if $\Box A$ is in u then A is in v. For an arbitrary set of sentences x let uIx hold if (i) $x \subseteq u$, (ii) if A, $B \in x$ then $A \& B \in x$, (iii) if $A \rightarrow B \in u$ and $A \in x$ then $B \in x$. Let $uR(v, x)$ hold if $u = v$ and vIx. Let $u(A)$ be A or $\sim A$ according to whether A is or is not in u (note that for u in W, $u(A) \in u$ by T13). Let (u, x) be in F_k if $u(A)$ is not in x, and let $u \in F_k$ if $P_k \in u$, for u in W, (u, x) in I.

By A19, $\leqslant$ is reflexive and by A20 $\leqslant$ is transitive, so $Q = (I, W, M, \leqslant, R, F)$ is an *aw*-model. Completeness follows from the following

LEMMA. For any A, (a) $(u, x) \vDash_Q A$ iff $u(A) \notin x$, (b) $u \Vdash_Q A$ iff $A \in u$.

Proof of part (a). Part (a) holds for propositional variables by definition. The inductive cases all follow from T5. We illustrate by treating the case of $\rightarrow$. Firstly, note that $u(A) \,\&\, u(B) \,\&\, u(A \rightarrow B) \rightarrow .u(A) \,\&\, u(B) \leftrightarrow \leftrightarrow u(A \rightarrow B)$ is an instance of T5. Since the antecedent is in u, so is

$$u(A) \,\&\, u(B) \leftrightarrow u(A \rightarrow B).$$

Hence:

$$(u, x) \vDash_Q A \rightarrow B \quad \text{iff} \quad (u, x) \vDash_Q A \quad \text{or} \quad (u, x) \vDash_Q B$$

iff $u(A)\notin x$ or $u(B)\notin x$
iff $u(A\rightarrow B)\notin x$.

Proof of part (b). The basis case holds by definition. The inductive steps for & and $\sim$ follow by T13. It only remains to prove (b) for the cases $\Box$ and $\rightarrow$.

If $\Box A\in u$ then $u \Vdash_Q \Box A$ by the definition of $\leqslant$. Conversely suppose that $\Box A\notin u$. Consider $S=\{B: \Box B\in u\}$. If $S\cup\{\sim A\}$ were inconsistent, then $S\vdash A$, so that

$$\vdash B_1 \& \cdots \& B_n \supset A, \qquad B_i\in S$$
$$\vdash \Box(B_1 \& \cdots \& B_n \supset A) \quad \text{(necessitation rule)}$$
$$\vdash \Box(B_1 \& \cdots \& B_n) \supset \Box A \quad \text{by T7}$$
$$\vdash \Box B_1 \& \cdots \& \Box B_n \supset \Box A \quad \text{by T6.}$$

But then $S\cup\{\sim A\}$ is consistent and by T12 has a maximally consistent extension v. Hence, for some v, $u\leqslant v$ but not $v\Vdash_Q A$, so that $u \Vdash_Q \Box A$ is false, as was to be proved.

Now for the case of $\rightarrow$. Firstly, let $A\rightarrow B\in u$. Then by A11, T13, either $A\notin u$ or $B\in u$, so by inductive assumption either not $u \Vdash_Q A$ or $u \Vdash_Q B$. Let uIx. We observe that for any u,

$u(A) \& u(B) \& (A\rightarrow B)\rightarrow. u(A)\rightarrow u(B)$ is provable by T9, T10. Hence if $u(A)\in x$, $u(B)\in x$, so that if $(u, x) \vDash_Q B$ then $(u, x) \vDash_Q A$ by part (a). It follows that $u\Vdash_Q A\rightarrow B$. Secondly, let $A\rightarrow B\notin u$, but either not $u \Vdash_Q A$ or $u\Vdash_Q B$. We aim to show that for some (u, x) such that uIx, $(u, x) \vDash_Q B$ but not $(u, x) \vDash_Q A$. Let a be the set $\{C: u(A)\rightarrow C\in u\}$. By A7, A9, uIa, so $uR(u, a)$. Evidently, $u(A)\in a$ so $(u, a) \vDash_Q A$ is false, by part (a). Now by T8, A16,

$u(A) \& u(B) \& (u(A)\rightarrow u(B))\rightarrow. A\rightarrow B$ is provable, since either $u(A)$ is $\sim A$ or $u(B)$ is B. Hence, if $u(B)\in a$ then $u(A)\rightarrow u(B)\in u$, so $A\rightarrow B\in u$, contrary to assumption. It follows that $u(B)$ is not in a, so $(u, a) \vDash_Q B$. This concludes the proof of the Lemma, and so of Theorem 1.

The proof of Theorem 2 proceeds just as for Theorem 1, but omitting any references to W, $\leqslant$ or R.

It must be emphasized that theorem 2, unlike Theorem 1, is not a new result. In [1] Dunn shows completeness of AI relative to a certain class of algebraic structures he calls Parry matrices. In §6 Dunn gives a representation of Parry matrices and places an intuitive interpretation upon this representation which he credits to Robert Meyer. The net effect of all this is that an assignment to a formula of an element in a Parry matrix can be looked at as an assignment of an ordered pair whose first element is the *content* of a formula, the second element being its truth value. The computation rules for these pairs correspond exactly to our rules in §2. Thus there is no significant difference either mathematically or philosophically between Dunn's semantics and ours, and Theorem 2 is an easy consequence of Dunn's work.

Decidability can be proved for AIN and AI by the method of *filtrations* due to Lemmon and Scott, and applied in Segerberg [4]. As it does not involve any new ideas in principle, the proof is omitted here. Decidability of AI is already shown in [1] by algebraic matrix methods.

We conclude with a conjecture concerning the completeness question for ASI. For A a formula of ASI, we define $A^{\Box}$, the translation of A into the language of AIN, as follows. $P_k^{\Box}$ is P_k, $(A \,\&\, B)^{\Box}$ is $(A^{\Box} \,\&\, B^{\Box})$, $({\sim}A)^{\Box}$ is ${\sim}(A^{\Box})$, $(A \rightarrow B)^{\Box}$ is $\Box(A^{\Box} \rightarrow B^{\Box})$. We conjecture that A is provable in ASI iff $A^{\Box}$ is provable in AIN.

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