GRAHAM PRIEST

SENSE, ENTAILMENT AND *MODUS PONENS* *

0. INTRODUCTION

The purpose of this paper is to provide a natural account of the semantics of entailment. I will approach the question via the notion of containment of sense. After an informal analysis in §§1–3 I will give in §§4–6 a formal semantics together with proof theory and completeness proof. §7 shows how the semantics can take account of enthymematic implication and §8 discusses some consequences of the analysis for *modus ponens* and naive set theory. An appendix relates these semantics to those of Priest [1979a].

1. MAKING SENSE OF SENSE

As all philosophy undergraduate students know, Frege held that every indicative sentence had two semantic components, sense and reference. (In fact he held that other linguistic units had these dual semantic features too but this need not concern us here.) The reference of a sentence he took to be a truth value, true or false, while the sense was the thought expressed by, or the objective content of, the sentence. The nature of sentence denotation, truth, has been widely investigated and now finds expression in truth theories of the kind Tarski instigated. The nature of sense, on the other hand, has remained largely in the state that Frege left it. One reason for this neglect is doubtlessly the fact that the notion of sense is more difficult to get to grips with than the notion of truth. Frege himself was able to do little more than provide a few hints and suggestions about sense. A more important reason however, is that modern logic with its extensionalist ideology has had no use for the notion of sense. In recent years the defects of extensionalism – in particular its inability to give a decent account of entailment – have been widely aired (or at least they have ceased to be widely ignored). The time is therefore ripe for an analysis of sense. The analysis provided, its use will not be far behind. Indeed in §3 it will be used to give a decent account of entailment.

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Kant defined an analytic truth as a (subject/predicate) judgement in which the subject is contained in the predicate, where he takes both subject and predicate to be concepts or, as we would put it now, senses. As a definition of analyticity this is clearly inadequate. Yet Kant's idea contains an important insight. The insight is that the notion of containment as a relation between senses, can have an explanatory function. Kant gives as an example of an analytic truth 'All bodies are extended'. (*Critique of Pure Reason* A7, B11) and explains its peculiar logical status by saying that the concept *body* does not go beyond, or is contained in, that of *extended*. Kant has often been criticized on the grounds that the use of 'contains' here is only a metaphor. But this is far too easy a dismissal. Many good explanatory theories are based in the first instance on a metaphor. The only way we have of assessing such an explanatory metaphor is by developing it into a fully grown theory and assessing the theory so produced. Kant failed to push the metaphor of containment any further but as we shall see, it can be pushed and used to produce a good theory.

Working on the metaphor, we can see that the following are properties we can reasonably suppose the containment relation to have: if the content of A contains the content of B and vice versa then the contents of A and B are identical; if the content of A is within the content of B and that of B is within the content of C then the content of A is within the content of C; the content of A is contained within itself.

Suppose we let $a, b, c \ldots$ be the objective contents (i.e., senses) of sentences and read '$a \leqslant b$' as 'a contains b'. What we have seen is that:

(i) $a \leqslant a$,

(ii) if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$,

(iii) if $a \leqslant b$ and $b \leqslant a$ then $b = a$.

In other words, the containment relation is a partial ordering on the domain of sentence senses. There is no reason to suppose that given any two senses one must be contained in the other (in other words that $\leqslant$ is a total ordering). Indeed it seems quite clear that the senses of 'the sun is shining' and 'familiarity breeds contempt' are incomparable.

So much for the containment relation itself. What of its relation to the senses of compound sentences? According to the Fregean principle, which seems a reasonable one, the sense of a compound sentence is a function of the senses of its components. How does the content of a conjunction relate to the contents of its components? It seems reasonable to suppose that a conjunction has at least as much content as each of the conjuncts and moreover that anything which has at least as much content as A and as B has at least as much content as $\ulcorner A \wedge B \urcorner$. In other words, the content of $\ulcorner A \wedge B \urcorner$ is the greatest lower bound of the contents of A and of B under the containment ordering. Dually, the content of a disjunction is the least upper bound of the contents of its disjuncts. Some authors, such as Parry (see Anderson and Belnap [1975] §29.6.1) have denied that the content of $\ulcorner A \vee B \urcorner$ is even an upper bound of the contents of A and of B, i.e., that the sense of A (and B) contains that of $A \vee B$. They do this on the grounds that $\ulcorner A \vee B \urcorner$ may refer to things or contain concepts not contained in A, say, alone. However, this is to confuse the content of a sentence with something like the aggregate of concepts mentioned. The content of an assertion is something like the amount of information the assertion gives you about the world. And clearly A may give you more information than B even though B refers to things not referred to by A, e.g., 'All men are mortal' tells you more than 'All men, except possibly Socrates, are mortal'. Similarly, the content of A really is greater than that of $\ulcorner A \vee B \urcorner$: it tells you more about the world. What we have seen is that the set of sentence contents forms a lattice with the operation corresponding to conjunction, $\cap$, being the lattice meet and that corresponding to disjunction, $\cup$, the lattice join. Moreover, the usual considerations concerning conjunction and disjunction suggest that it is a distributive lattice, i.e., that the content of $\ulcorner (A \wedge B) \vee (A \wedge C) \urcorner$ is included in that of $\ulcorner A \wedge (B \vee C) \urcorner$. (The converse inclusion already follows from the other principles.)

What of negation? Again it seems plausible to suppose that the content of $\ulcorner \neg A \urcorner$ is (in some sense) the opposite of the content of A. Suppose we let a^* be the opposite of the content a. The opposite of an opposite is the thing you started with, i.e., $a = a^{**}$.

Moreover, if the sense of B is contained in the sense of A then presumably the opposite of the sense of A is contained in the opposite of the sense of B, i.e., if $a \leqslant b$, then $b^* \leqslant a^*$. The inverse operation * makes the lattice a De Morgan lattice, or, as we will call it, a *sense lattice.* There is no reason in general to suppose that the lattice has a minimal element, though if we are concerned with the senses of sentences of a language with a truth predicate, the sense of 'Everything is true' seems a good candidate for a minimal element. But even then there seems no good reason to suppose that the sense of an arbitrary contradiction contains it. The content of 'Everything is true' is much wider than the content of 'it is raining and it is not raining'. Thus if we let 0 be the minimal element of the lattice (if it has one) we do not have in general

$$a \cap a^* \leqslant 0$$

i.e., the lattice is not a Boolean algebra.

To summarize this section, it is reasonable to suppose that the structure of senses under the containment relation and operations corresponding to the truth functions $\wedge, \vee$ $\neg$ is a De Morgan lattice, though not a Boolean algebra. Of course, the arguments used in no sense prove this but they do make it highly plausible. This is, quite generally, how arguments obtained by developing a metaphor work. The ultimate test of a theory based on a metaphor is, of course, its fruitfulness, simplicity, adequacy to the data and all the other criteria much beloved of philosophers of science.

Before we leave the question of sense, a few remarks concerning sense and possible worlds would not go amiss. There is currently abroad a certain suggestion to identify the sense of a sentence with a function from possible worlds to truth values. However, this will not work. For if the possible worlds are those of the standard, Kripke semantics for model logics all logically necessary truths, have the same sense. Thus it follows that, 'If it is raining it is raining' has the same sense as 'All bachelors are unmarried' – an obvious absurdity. Routely has proposed a somewhat more general notion of possible world and indeed advocated the analysis of content with respect to this. (See Routley [1977] §11.) This avoids the former problem but is in turn, thrown into others. For any sentence can fail in some world. Hence the distinction between logical truths and others collapses. We may still characterize logical truths as those provable in a certain axiom system but without a semantical underpinning this characterization is vacuous. Virtually *any* set of assertions can be characterized in this way. Some might be happy to see this distinction collapse. However, even the attacks on analyticity do not normally impugn the notion of logical truth.

In virtue of the fact that any sentence may not hold in some world, the reader who is unfamiliar with Routley/Meyer semantics may wonder how they ever manage to show a logic to be sound. The answer is that they require the base world of a model structure to be regular – in effect requiring that all logical truths hold at the base world. Leaving aside the obvious *ad hoc*ness of this proposal, we should pass on to ask what it would be like for the world to be irregular. What would a world be like in which a logical truth fails? We may be told that a world is really just a set of sentences closed under certain conditions and that is easy enough to see how a logical truth could fail to be a member of that. But if this is all there is to worlds, we lapse into semantic instrumentalism. To avoid this, the set of sentences must be a possible description of reality, or better, a description of a possible reality. And what would a reality be like where a law of logic failed? Even to ask the question causes the mind to boggle. What this shows is the bankruptcy of the possible world semantics. Possible world semantics are based on a metaphor (and may be none the worse for that). However, the metaphor breaks at its most crucial point, viz, when we need to give an account of logical truths. At this point the metaphor ceases to guide our semantic theory. Instead we have to use other intuitions about logical truth and by impositing arbitrary constraints on the possible worlds ensure that they are satisfied. Similar remarks can be made concerning the properties of the ternary relation in Routley/Meyer semantics. These are specified *ad hoc* to suit intuitions and arguments drawn from elsewhere. For this reason I consider the "possible worlds" metaphor to be a poor one on which to base a semantics of entailment: the flow of information has to go into the semantics to make it work. In a good semantics the flow of information goes the other way: the semantics should be genuinely informative for the notions concerned. This is precisely what the content metaphor does and it is for this reason that I think content semantics are preferable to world semantics.

2. FILTERING OUT THE TRUTH

So much for sense. Let us now return to the other arm of Fregean semantics, reference – in our case, truth. If different sentences have the same content they can hardly have different truth values. Coincidence of intension guarantees coincidence of extension though not, of course, vice versa. So let M be the set of senses of sentences of a certain language and let T be the subset of M containing the senses of true sentences. What can we say about T?

First, if a is the sense of a true sentence and $a \leqslant b$ then b is the sense of a true sentence. For if all the content of A is true and contains all that of B, then all the content of B is true.

What of compound propositions? Fairly obviously

$$a \cap b \in T \text{ iff } a \in T \text{ and } b \in T,$$

and

$$a \cup b \in T \text{ iff } a \in T \text{ or } b \in T.$$

These are the standard truth conditions for conjunction and disjunction.

What of negation? To the classical mind the obvious condition for negation is

$$a \in T \text{ iff } a^* \notin T.$$

However, this is incorrect. For it ignores the possibility that there are sentences such that both they and their negations are true. That there are such sentences I have argued elsewhere (Priest [1979a]) and I will not pursue the matter here. Thus T need not be consistent, in the sense that there may be an a such that $a \in T$ and $a^* \in T$. By contrast, it does seem reasonable to suppose that T is complete in the sense that for any a either $a \in T$ or $a^* \in T$. This might seem to rule out *a priori* the possibility that there are sentences that are neither true nor false. This may be no loss. I have argued elsewhere that at least one important kind of sentence often taken to be truth valueless – sentences containing denotationless singular terms – have truth values. (See Priest [1979b]). However, even if there are truth valueless sentences, we may still insist that the negation of a sentence A (or at least one of its negations) is true whenever A fails to be true (i.e., when A is either false or truth valueless). Thus it does seem reasonable to impose the condition

$$a \in T \text{ or } a^* \in T$$

on T.

The conditions laid down so far ensure, algebraically, that the set T is a prime ultrafilter on the set of senses. We will call such a filter a *truth filter.*

3. MAKING SENSE OF ENTAILMENT AND ENTAILMENT OUT OF SENSE

I now turn, as promised, to an analysis of entailment. In [1920] Moore defined entailment as the converse of deducibility and this is the way it is now generally understood. Moore's definition, though it gives us a start, does not get us very far. *A* entails *B* iff the inference from *A* to *B* is a valid deduction. But what is a valid deduction?

The currently orthodox answer to this question is that an inference is (deductively) valid if it is truth preserving, i.e., there is no valuation which makes the premise(s) true and the conclusion false. Although orthodox, this extensionalist reduction of validity is sadly inadequate. Its prime defect is that according to it any logical truth (i.e., a sentence true under all evaluations) is a valid consequence of anything and dually any logical falsity has anything as a consequence. The absurdity of this has been elaborated sufficiently for it to require little comment. (See Anderson and Belnap [1975].) I have discussed the matter in my [1979c] §5, and will not pursue it here. The important point is that though we certainly wish valid inferences to be truth preserving, truth preservation is not a sufficient guarantee of validity. The so-called paradoxes of implication arise exactly because truth preservation is taken to be both a necessary and a sufficient condition for validity.

The prevailing account of deductive validity in the 19th Century was somewhat different to the current one. It was based on the content metaphor and was to the effect that an inference is valid if the content of the conclusion is contained in that of the premises. Thus, for example, we find Mill saying:

> It is universally allowed that a syllogism is vicious if there be anything more in the conclusion than was assumed in the premise. *A System of Logic* Ch. III, § 1.

Although this view has lapsed now, there are still, fortunately, writers around to remind us of it. An interesting example is Salmon [1966]. He gives the orthodox characterization of a demonstrative (deductive) inference as one which is necessarily truth preserving. He then says (p. 8)

> Since demonstrative inferences have been characterised in terms of their basic property of necessary truth preservation, it is natural to ask how they achieve this very desirable trait. For a large group of demonstrative inferences, including those discussed under 'valid deduction' in most logic texts, the answer is rather easy. Inferences of this type purchase necessary truth preservation by sacrificing any extension of content. The conclusion of such an inference says no more than do the premises – often less. The conclusion can not be false if the premises are true *because* the conclusion says nothing that was not already in the premises. The conclusion is a mere reformulation of all or part of the content of the premises. In some cases the reformulation is unanticipated and therefore psychologically

> surprising, but the conclusion cannot augment the content of the premises. Such inferences are ***non-ampliative***; an ampliative inference, then, has a conclusion with content not present either explicitly or implicitly in the premises.

Salmon then goes on to argue that in all arguments which are truth preserving, the content of the conclusion is contained in that of the premises. This we know to be false. In §1 we saw that the lattice of senses was not a Boolean algebra. In particular, the fact that a sentence is logically true (i.e., that its sense is a member of every truth filter) does not guarantee that it is greater than every element of the lattice. However, Salmon has already swallowed the paradoxes of implication whole in his aside on logic texts; so perhaps this is not surprising. Anyone who swallows that will swallow anything!

Salmon's identification of truth preservation with validity aside, the paragraph provides a clear statement of what it is for an inference to be content non-increasing. He calls such an inference non-ampliative. I propose that we call such an inference (deductively) valid. In effect this resuscitates the 19th Century account of validity. In sum then an inference is valid if the sense of the conclusion is contained in the sense of the premise(s). And entailment being the converse of deducibility, an entailment is true if the sense of the consequent is contained in the sense of the antecedent. We can now spell out the semantics of entailment.

Let us write $\ulcorner A \rightarrow B \urcorner$ for $\ulcorner$that A entails that $B\urcorner$. According to the Fregean principle, the sense of $\ulcorner A \rightarrow B \urcorner$ must be a function of the senses of A and B. But what else can we say about it? The answer is 'In general, nothing'. There seems to be no connection, in general, between the content of $\ulcorner A \rightarrow B \urcorner$ and the contents of A and of B. Certainly we cannot identify the sense of $\ulcorner A \rightarrow B \urcorner$ with that of $\ulcorner \neg A \vee B \urcorner$. For whenever B is true, so is $\ulcorner \neg A \vee B \urcorner$; whereas it is ridiculous to claim that $\ulcorner A \rightarrow B \urcorner$ is, i.e., that the content of an arbitrary true proposition is, contained in that of any proposition. Thus $\ulcorner \neg A \vee B \urcorner$ and $\ulcorner A \rightarrow B \urcorner$ have, in general, different truth values and therefore different senses.

What of the relation between entailments and the truth filter? The answer is implicit in our discussion of entailment. $\ulcorner A \rightarrow B \urcorner$ is true iff the sense of A contains that of B. Thus if $\leqslant$ is the ordering of a sense lattice, T is a truth filter on it, and $\Rightarrow$ the function corresponding to '$\rightarrow$'), we require that

$$a \Rightarrow b \in T \quad \text{iff} \quad a \leqslant b.$$

We will call a function ➡ satisfying this requirement an *entailment function.*

It is true that the truth conditions of entailments violate the Fregean principle that the reference of a compound must be a function of the references (rather than the senses) of its parts. However, this latitude is no greater than Frege allowed himself for dealing with intensional contexts. (Though he hides his deviation from the principle, nominalistically, by calling sense 'indirect reference'.)

This provides all the information we require on the semantics of entailment and we can now pass to a formal presentation.

4. FORMAL SEMANTICS

4.0 Formal semantics of the kind I shall now give are not new. De Morgan lattices are a well articulated part of the theory of first degree entailment. See Anderson and Belnap [1975] §18. Moreover, they have been extended to provide an algebraic semantics for E and R, see, e.g., Meyer and Routley [1972]. However, it is clear that they have to specify a number of their algebraic postulates *ad hoc*, exemplifying exactly the fault that I commented on in connection with world semantics at the end of §1. By contrast the following semantics based on the heuristic argument of the previous sections is entirely natural. However, in virtue of the fact that the algebra is well known, I will give the accound fairly baldly.

4.1 A relation $\leqslant$ is a *partial ordering* on a set M iff for all $a, b, c \in M$:

(i) $a \leqslant a$,

(ii) if $a \leqslant b$ and $b \leqslant c$, $a \leqslant c$,

(iii) if $a \leqslant b$ and $b \leqslant a$, $a = b$.

4.2 A structure $\langle M \cap \cup * \leqslant \rangle$ is a sense (De Morgan) lattice iff $\leqslant$ is a partial ordering on M, $\cup$ and $\cap$ are functions from M^2 to M and $*$ is a function from M to M such that:

(i) $a \cap b \leqslant a, \quad a \cap b \leqslant b$,

(ii) if $a \leqslant b$ and $a \leqslant c$, $a \leqslant b \cap c$,

(iii) $a \leqslant a \cup b,\ b \leqslant a \cup b,$

(iv) if $a \leqslant b$ and $c \leqslant b,\ a \cup c \leqslant b,$

(v) $a \cap (b \cup c) \leqslant (a \cap b) \cup (a \cap c),$

(vi) $a^{**} = a$

(vii) if $a \leqslant b,\ b^* \leqslant a^*.$

4.3 If $L = \langle M \cup \cap * \leqslant \rangle$ is a sense lattice a *filter* on M is a set T such that

(i) $T \subseteq M,$

(ii) if $a \in T$ and $a \leqslant b,\ b \in T$

(iii) if $a \in T$ and $b \in T,\ a \cap b \in T$

T is a *prime filter* if in addition

(iv) if $a \cup b \in T,\ a \in T$ or $b \in T$

and a filter is an *ultrafilter* if

(v) $a \in T$ or $a^* \in T$

T is a *truth filter* if it is a prime ultrafilter.

4.4 Let $L = \langle M \cap \cup * \leqslant \rangle$ be a sense lattice and T a truth filter.
If $\Rightarrow$ is a function from M^2 to M, $\Rightarrow$ is an *entailment function* on L and T iff

$$a \Rightarrow b \in T \text{ iff } a \leqslant b.$$

4.5 *A model structure* (m.s.), is a triple $\langle L\ T \Rightarrow \rangle$ where L is a sense lattice, T is a truth filter and $\Rightarrow$ is an entailment function.

4.6 Let P be a set of propositional parameters and let F be the closure of P under the connectives $\wedge, \vee, \neg, \rightarrow$. Given a m.s. $\mathfrak{A}$ an *evaluation* ν on $\mathfrak{A}$ is a map from P to M. An evaluation is extended to an assignment (which we will also write as ν) of lattice values to all members of F as follows:

(i) $\nu(A \wedge B) = \nu(A) \cap \nu(B),$

(ii) $\nu(A \vee B) = \nu(A) \cup \nu(B)$,

(iii) $\nu(\neg A) = \nu(A)^*$

(iv) $\nu(A \rightarrow B) = \nu(A) \Rightarrow \nu(b)$.

If $\mathfrak{A}$ is a m.s. and ν an assignment on $\mathfrak{A}$ we will call the pair $\langle \mathfrak{A}\nu \rangle$ an *interpretation.*

4.7 If Γ is a subset of F and $A \in F$ then $\Gamma \models A$ iff there is no interpretation $I = \langle\langle LT \Rightarrow \rangle \nu\rangle$ such that $\nu(A) \notin T$ but

$$\nu(B) \in T \text{ for all } B \in \Gamma.$$

'$\models$' is the sign of truth preservation. It would perhaps be better to use it as the sign for validity. However, because of the orthodox identification of validity with truth preservation, its use to indicate truth preservation has become entrenched. Hence I will give it up as lost and use a new sign for validity. As usual '$\models A$' means $\phi \models A$, i.e., A is a logical truth.

4.8 If $I = \langle\langle LT \Rightarrow \rangle \nu\rangle$ is an interpretation and $\Sigma = \{B_1 \ldots B_n\} \subseteq F$ we will write

$$\Sigma \int_I A \text{ iff } \nu(B_1 \wedge \ldots \wedge B_n) \leqslant \nu(A).$$

Thus $\Sigma \int_I A$ indicates that on the interpretation I, the sense of $B_1 \wedge \ldots \wedge B_n$ contains that of A, i.e., that the inference from $B_1 \ldots B_n$ to A is valid. We have immediately that

$$\Sigma \int_I A \text{ iff } \nu(B_1 \wedge \ldots \wedge B_n \rightarrow A) \in T.$$

This is the symbolic representation of Moore's definition of entailment: B entails A iff A is validly deducible from B. Some inferences depend for their validity on the content of the non-logical symbols involved (e.g., Henry is a bachelor. Hence Henry is unmarried.) However, others are independent of the content of the non-logical symbols. These are the formally valid inferences much beloved by logicians. We can define formal validity simply. If Σ is a finite subject of F

$$\Sigma \int A \text{ iff for all } I, \ \Sigma \int_I A.$$

$\int$ is the sign of formal validity. It is immediate from the definition that

$$\{B_1 \ldots B_n\} \int A \text{ iff } \models B_1 \wedge \ldots B_n \rightarrow A.$$

4.9. THEOREM. The first degree logical truths are just the first degree entailments. However, there are logical truths which are not substitution instances of first degree entailments.

Proof. If A and B contain no occurrences of $\rightarrow$ then the conditions for the logical truth of $A \rightarrow B$ are exactly the same as those given for the validity of a first degree entailment in Anderson and Belnap [1975] §18. Hence $\models A \rightarrow B$ iff $A \rightarrow B$ is a valid first degree entailment. However, for any interpretation $\langle \mathfrak{A}\nu \rangle$, $\nu(A \wedge A) = \nu(A)$. Hence $\models (A \rightarrow B) \rightarrow (A \wedge A \rightarrow B)$ which is not a substitution instance of a first degree entailment.

5. PROOF THEORY

5.0 I will now give a proof theory for the semantics. I will give this in the form of a sequent calculus. A sequent is an item of the form $\Sigma : A$ where Σ is a finite subset of F and $A \in F$. The notions of proof, etc., are the standard ones

5.1 The basic (initial) sequents are:

(1) $A : A$,

(2) $\phi : A \wedge B \rightarrow A(B)$,

(3) $\phi : A(B) \rightarrow A \vee B$,

(4) $\phi : A \rightarrow \neg\neg A$,

(5) $\phi : \neg\neg A \rightarrow A$,

(6) $\phi\, A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge B)$,

(7) $\phi : A \vee \neg A$.

5.2 The rules governing the nodes of the proof trees are

(1) $$\frac{\Sigma, A : B \quad \Pi, C : B}{\Pi, \Sigma, A \vee C : B}$$

(2) $$\frac{\Sigma : A \to B \quad \Pi : A \to C}{\Pi, \Sigma : A \to B \wedge C}$$

(3) $$\frac{\Sigma : A \to B \quad \Pi : C \to B}{\Pi, \Sigma : A \vee C \to B}$$

(4) $$\frac{\Sigma : A \to B}{\Sigma : \neg B \to \neg A}.$$

(5) $$\frac{\Sigma : A \quad \Pi : A \to B}{\Pi, \Sigma : B}$$

(6) $$\frac{\Sigma : A \to B \quad \Pi : B \to C}{\Sigma, \Pi : A \to C}$$

(7) $$\frac{\Sigma : A \quad \Pi : B}{\Sigma, \Pi : A \wedge B}$$

(8) $$\frac{\Sigma : A \leftrightarrow B}{\Sigma : C \leftrightarrow C(A/B)}$$

where $\ulcorner A \leftrightarrow B \urcorner$ is $\ulcorner (A \to B) \wedge (B \to A) \urcorner$ and $C(A/B)$ is C with all occurrences of A replaced by B.

(9) $$\frac{\Pi, A : B \quad \Sigma : A}{\Sigma, \Pi : B}$$

5.3 If Γ is a subset of F and $A \in F$,
$\Gamma \vdash A$ iff there is a finite subset Γ_0 of Γ
and a proof of the sequent $\Gamma_0 : A$
$\vdash A$ iff $\phi \vdash A$

5.4 THEOREM. If $\Gamma \vdash A$, $\Gamma \models A$.
Proof. Suppose $\Gamma \vdash A$ then for some finite $\Gamma_0 \subseteq \Gamma$ there is a proof of $\Gamma_0 : A$. By a simple induction over proof trees it can be shown that if there is a proof of $\Sigma : A$, then $\Sigma \models A$.
Hence the result.

5.5 COROLLARY. If $\vdash A$, $\models A$.

5.6 COROLLARY. If $\vdash B_1 \wedge \ldots \wedge B_n \to A$, $\{B_1 \ldots B_n\} \int A$.

6. COMPLETENESS

6.0 We can now prove completeness for the various semantical notions involved. We will do this in a series of steps leading up to theorems at 6.4–6.6

6.1 Let $\Gamma \subseteq F$. Define $A \sim B$ as $\Gamma \vdash A \leftrightarrow B$.
$\sim$ is easily seen to be an equivalence relation.
Let $F/{\sim}$ be the set of equivalence classes and let $[A]$ be the equivalence class of A. Define the relation $\leqslant$ thus:

$$[A] \leqslant [B] \text{ iff } \Gamma \vdash A \rightarrow B,$$

and functions $\cap, \cup, *$, thus

$$[A] \cap [B] = [A \wedge B],$$

$$[A] \cup [B] = [A \vee B],$$

$$[A]^* = [\neg A].$$

These functions and $\leqslant$ are well defined and $L_\Gamma = \langle F/{\sim} \cap \cup * \leqslant\rangle$ is a sense lattice. The details to be checked here are routine and omitted.

6.2 If $\Gamma \subseteq F$ and $A \in F$ such that $\Gamma \nvdash A$, then there is a Γ^+ such that:

(i) $\Gamma^+ \supseteq \Gamma$,

(ii) $\Gamma^+ \nvdash A$,

(iii) $B \in \Gamma^+$ iff $\Gamma^+ \vdash B$,

(iv) $C \vee B \in \Gamma^+$ iff $C \in \Gamma^+$ or $B \in \Gamma^+$.

Proof. Enumerate the formulas of F, $A_1, A_2, \ldots$
Define a sequence:

$$\Gamma_0 = \Gamma,$$

$$\Gamma_{n+1} = \Gamma_n \cup \{A_n\} \text{ if } \Gamma_n \cup \{A_n\} \nvdash A,$$

$$\Gamma_{n+1} = \Gamma_n \text{ otherwise},$$

$$\Gamma^+ = \bigcup_{n<\omega} \Gamma_n,$$

Clearly Γ^+ satisfies (i) and (ii). From left to right (iii) is trivial.

From right to left, suppose that $B \notin \Gamma^+$. Then for some n $\Gamma_n, B \vdash A$. Now if $\Gamma^+ \vdash B$ then $\Gamma^+ \vdash A$ (by rule 9) contradicting (ii). Thus $\Gamma^+ \nvdash B$. From right to left (iv) is trivial. From left to right, suppose that $B \notin \Gamma^+$ and $C \notin \Gamma^+$. Then for some n, m, $\Gamma_n, C \vdash A$ and Γ_m, $B \vdash A$.
Thus $\Gamma^+, B \vee C \vdash A$ (by rule 1) and

$$\Gamma^+ \nvdash B \vee C \text{ (by rule 9 and (ii))}$$

6.3 Suppose $\Gamma \vdash A$. Let Γ^+ be defined as in 6.2 and L_{Γ^+}, as defined in 6.1. The $\Gamma^+/\!\sim\ \subseteq F/\!\sim$ and is a truth filter on $F/\!\sim$.
Define $[A] \Rightarrow [B]$ as $[A \to B]$
$\Rightarrow$ is well defined and an entailment function on $L_{\Gamma^+}, \Gamma^+/\!\sim$.
Moreover, $[A] \notin \Gamma^+/\!\sim$. There are a number of details here but they are all straightforward applications of the properties of Γ^+ established in 6.2 and so are omitted.

6.4 THEOREM. If $\Gamma \models A, \Gamma \vdash A$. (The converse of 5.4.)
Proof. Suppose that $\Gamma \nvdash A$. Then by 6.1 and 6.3 = $\langle L_{\Gamma^+}\Gamma^+/\!\sim\ \Rightarrow\rangle$ is a model structure and $[A] \notin \Gamma^+/\!\sim$. Let $\nu(p) = [p]$.
It is simple to check that for all $A \in F$ $\nu(A) = [A]$.
Hence $\nu(A) \notin \Gamma^+/\!\sim$. But if $B \in \Gamma$, $\Gamma^+ \vdash B$; so $\nu(B) \in \Gamma^+/\!\sim$. Thus $\Gamma \not\models A$.

6.5 COROLLARY. If $\models A, \vdash A$. (The converse of 5.5.)

6.6 COROLLARY. If $\{B_1 \dots B_n\} \int A, \vdash B_1 \wedge \dots \wedge B_n \to A$. (The converse of 5.6.)

7. ENTHYMEMATIC IMPLICATION

The theory of entailment of the previous sections is a fairly weak one. It is stronger (though not much) than first degree entailment closed under substitution of formulas of arbitrary degree (see 4.9) but it is much weaker than E and R. Some may be dismayed at this weakness (though dismay is not an argument). However, for such people it is worth pointing out that given any theory of entailment it is possible to provide an account of enthymematic implication (which is more like 'if . . . then . . . ') which is stronger. The idea belongs to Anderson and Belnap [1961]). Let t be the conjunction of all truths. A enthymematically implies B iff $A \wedge t \rightarrow B$. The formal theory of t is easy enough to give. Details are as before except that we now insist that a truth filter be *principle*, i.e., that there is an a such that $T = \{b | a \leqslant b\}$. Let us write the distinguished member of the filter as 1. Clearly 1 is the greatest lower bound of T, and is therefore essentially the conjunction of all truths. We suppose our language to be furnished with a sentential constant t, and demand that for any evaluation ν, $\nu(t) = 1$. If we now add the basic sequents

$$A : t \rightarrow A \qquad \phi : t$$

to the others §5.1 it is easy enough to check that these are truth preserving and are just what are required to show that $[t]$ is the minimal element of the equivalence classes of theorems in the Lindenbaum algebra and hence modify the completeness proof of §6. Details are left as an exercise.

8. WHAT'S WRONG WITH *MODUS PONENS?*

The weakness of the account of entailment given can, paradoxically enough, be a strength. In their note [1979], Routley, Meyer and Dunn show that any naive set theory (i.e., set theory with an unrestricted abstraction scheme) based on a logic which contains the thesis $A \wedge (A \rightarrow B) \rightarrow B$ (which they call the *modus ponens* axiom) is trivial. They conclude that naive set theory is not on, since "giving up" this principle is too high a price to pay. Their conclusion is wrong and I now wish to explain why.

First, this way of putting it prejudices the issue by claiming that the *modus ponens* axiom is something we have "got" which we will have to lose if we want to adopt naive set theory. Possession, as everyone knows, is nine tenths of the logical law. However, the issue is not as simple as that. Our problem is to provide the best account of certain forms of reasoning which include set theoretic forms. We cannot, it seems, have everything. At least we cannot hold both the abstraction scheme and the *modus ponens* axiom. However, both are candidates for rejection. It is not true to say that we already "have" one which we have to give up. We have both in exactly the same sense, i.e. we appear to start out with a belief in both. Moreover, there are a number of good arguments for supposing that it should be the *modus ponens* axiom which is to get its marching orders. Here is one. The truth scheme is the following: ⌜ α is true iff p⌝ (where α is the name of p) – symbolically:

(1) $\mathrm{Tr}(\alpha) \leftrightarrow p.$

Now let (β) be the sentence 'if this sentence is true then q is, where q is arbitrary', i.e.,

(β) $\mathrm{Tr}(\beta) \rightarrow q.$

Now if we have the *modus ponens* axiom

(2) $A \wedge (A \rightarrow B) \rightarrow B$

we have by making appropriate substitutions

(3) $\mathrm{Tr}(\beta) \wedge (\mathrm{Tr}(\beta) \rightarrow q) \rightarrow q$

but by taking β for α in (1) we get

(4) $\mathrm{Tr}(\beta) \leftrightarrow (\mathrm{Tr}(\beta) \rightarrow q)$

and from (3) and (4) by the properties of conjunction we have

(5) $\mathrm{Tr}(\beta) \rightarrow q.$

Hence by (4) and (5) we get

(6) $\mathrm{Tr}(\beta)$

and by (5) and (6)

$q.$

This shows that the combination of the *modus ponens* axiom and the truth scheme is also unsatisfactory. (In fact the derivation is a trivial modification of the Routley–Meyer–Dunn derivation.) Now the point is that if we accept the *modus ponens* axiom we have to give up the abstraction scheme *and* the *T*-scheme, both of which are clear *a priori* truths. However, if we reject the *modus ponens* axiom we can accept both. Hence, since we are all good democrats, the *modus ponens* axiom loses by a 2 to 1 vote.

Of course, this sort of argument is hardly conclusive. An obvious and highly pertinent question is whether there are independent grounds for doubting either the abstraction/truth schemes or the *modus ponens* axiom. Some will urge against the abstraction/truth schemes that they lead to contradiction. However, this is hardly a cogent argument when we take paraconsistency seriously. It is very difficult to find other reasons for doubting these schemes. On the other hand there are independent reasons for doubting the truth of the *modus ponens* axiom. In particular, the *modus ponens* axiom is not logically true on the account of entailment presented in this paper. (As a counter model let L be the lattice of the powerset of natural numbers, T the principal ultrafilter generated by $\{0\}$, and $\Rightarrow$ any entailment function such that $\{1\} \Rightarrow \{2\} = \{1\}$. Let $\nu(A) = \{1\}$ and $\nu(B) = \{2\}$. Then $\nu(A \wedge (A \rightarrow B) \rightarrow B) = \{1\} \notin T$.)

There is no particular reason to suppose that the sense of $A \wedge (A \rightarrow B)$ will contain that of B. However, this is not to say that the *rule modus ponens* is not acceptable. Indeed it is easy to see that $A \wedge (A \rightarrow B) \models B$. Thus giving up the *modus ponens* axiom does not force us to give up the genuine *modus ponens* viz the rule. It could be objected that if rule *modus ponens* is acceptable then axiom *modus ponens* ought to be true and so the semantics of this paper are incomplete. However, this will not work. The acceptability of a rule amounts to its truth preservation, and, as we know, truth preservation, though a necessary condition, is not sufficient for the truth of the corresponding entailment, i.e., $A \models B$ does not imply $\models A \rightarrow B$. (Just consider $B \models A \rightarrow A$. This is truth preserving.) What is both necessary and sufficient for $\models A \rightarrow B$ is precisely $A \int B$. (See §6.6.)

An objection to the whole semantics presented here might go as follows. Any theoretical account of validity must be assessed against the facts, the facts in this case being our intuitions concerning logical truth and validity. It might be thought that we are more sure of the truth of the *modus ponens* axiom than of any theoretical account of logical truth. Hence *any* account according to which the *modus ponens* axiom is not logically true is thereby refuted. The objection is somewhat naive methodologically. It is well known that a good theory can successfully overturn incompatible "facts". Moreover, in the context of a debate concerning which of two "obvious" claims has to go, it clearly begs the question. However, it is worth considering further what grounds we have for supposing the *modus ponens* axiom true.

One ground is the acceptability of rule *modus ponens* which we have discussed. Another is the following. If $\ulcorner A \rightarrow B \urcorner$ is true then certainly the inference from A to B is valid (by definition), though $\ulcorner A \rightarrow B \urcorner$ is not needed as an extra premise (as Lewis Carroll taught us). I suspect that we mistakenly record this fact by saying that if A is true and $\ulcorner A \rightarrow B \urcorner$ is true then B is true, which by the properties of the T-scheme gives us the *modus ponens* axiom. However, if we were careful, what we should say is that if $\ulcorner A \rightarrow B \urcorner$ is true then if A is true B is true, which by the properties of the T-scheme is just the law of identity $(A \rightarrow B) \rightarrow (A \rightarrow B)$. This is, in fact, the genuine *modus ponens* axiom; the other is a counterfeit.

Hence I conclude that Routley, Meyer and Dunns writing off of naive set theory is incorrect.

9. CONCLUSION

I conclude that the semantics presented here are the first semantics for entailment which avoid the blunder of identifying validity with truth preservation and are not *ad hoc*. This vindicates the theory of sense, based on the content metaphor, which is where we started and reprieves naive set theory which is where we finish.

10. APPENDIX

10. In Priest [1979a] I argued that the paraconsistent semantics I gave there, LP, might be thought to be adequate for the truth functional connectives $\wedge$, $\vee$, $\neg$, but need to be augmented for $\rightarrow$. The theory of sense presented here is just such an augmentation. For the notions of sense lattice and truth filter are a generalization of the semantics of LP. The proofs of this section will make clear exactly how.

10.1 Let me start by recapping on LP.
An LP evaluation is a map μ from P into $\{t, p, f\}$. μ is extended to an evaluation of all formulas of the $\vee$, $\wedge$, $\neg$ fragment F^-, of F according to the following truth tables:

$\neg$	
t	f
p	p
f	t

$\vee$	t	p	f
t	t	t	t
p	t	p	p
f	t	p	f

$\wedge$	t	p	f
t	t	p	f
p	p	p	f
f	f	f	f

$\Gamma \models_{LP} A$ iff for no μ, $\mu(A) = f$ but $\mu(B) = t$ or p for all $B \in T$.

10.2 We can reformulate LP as follows. Let $t = \{1\}, p = \{01\}$ and $f = \{0\}$. The conditions given by the truth tables then become:

$$1 \in \mu(\neg A) \iff 0 \in \mu(A)$$

$$0 \in \mu(\neg A) \iff 1 \in \mu(A),$$

$$1 \in \mu(A \wedge B) \iff 1 \in \mu(A) \text{ and } 1 \in \mu(B),$$

$$0 \in \mu(A \wedge B) \iff 0 \in \mu(A) \text{ or } 0 \in \mu(B).$$

Dually for $\vee$.

$\Gamma \models_{\mathbf{LP}} A$ iff for no μ $1 \notin \mu(A)$ but $1 \in \mu(B)$ for all $B \in \Gamma$.

10.3 THEOREM. Let $\Gamma \cup \{A\} \subseteq F^{-}$. Then if $\Gamma \models_{\mathbf{LP}} A, \Gamma \models A$.

Proof. We will prove the contrapositive.
Suppose $\Gamma \not\models A$. Let $I = \langle\langle LT \Rightarrow \rangle\nu\rangle$ be an interpretation such that

$$\nu(A) \notin T \text{ but } \nu(B) \in T \text{ for all } B \in T.$$

Define an LP evaluation μ thus:

$$\left.\begin{array}{l} 1 \in \mu(p) \iff \nu(p) \in T \\ 0 \in \mu(p) \iff \nu(\neg p) \in T \end{array}\right\} \quad (*)$$

for all $p \in P$.
It is easy to check that μ is indeed an LP evaluation and moreover that condition $(*)$ holds when 'p' is replaced by an arbitrary member of F^{-}. The LP evaluation μ shows that $\Gamma \not\models_{\mathbf{LP}} A$.

10.4 THEOREM. Let $\Gamma \cup \{A\} \subseteq F^{-}$. Then if $\Gamma \models A, \Gamma \models_{\mathbf{LP}} A$.

Proof. We will prove the contrapositive.
Suppose that $\Gamma \not\models_{\mathbf{LP}} A$. Let μ be an evaluation such that $1 \in \mu(B)$ for all $B \in \Gamma$ but $1 \notin \mu(A)$.
Let $L = \langle C \cap \cup * \leqslant \rangle$ where $C = \{\{1\}\{10\}\{0\}\}$,
$\leqslant$ is the ordering given by $\{0\} \leqslant \{0, 1\} \leqslant \{1\}$ and $\cap, \cup, *$ are the LP operations corresponding to conjunction, disjunction and negation respectively. It is easily checked that L is a sense lattice.
Let $T = \{\{1\}\{1, 0\}\}$ then T is a truth filter on L.
Let $\Rightarrow$ be any entailment function on L, T. Then $\langle L, T, \Rightarrow\rangle$ is a model structure.

Define the evaluation $\nu : P \to C$ by

$$\nu(p) = \mu(p) \quad (*)$$

It is easy to check that $(*)$ holds if 'p' is replaced by an arbitrary member of F^-. The evaluation ν shows that $\Gamma \not\models A$.

10.5 COROLLARY. If $A \in F^-$, $\models A$ iff $\models_{\mathbf{LP}} A$ iff A is a two valued tautology.

Proof. This follows from 10.3, 10.4 and §3.8 of Priest [1979a].

The University of Western Australia

NOTE

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