A Continuum Theory of Elastic Material Surfaces

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Abstract

A mathematical framework is developed to study the mechanical behavior of material surfaces. The tensorial nature of surface stress is established using the force and moment balance laws. Bodies whose boundaries are material surfaces are discussed and the relation between surface and body stress examined. Elastic surfaces are defined and a linear theory with non-vanishing residual stress derived. The free-surface problem is posed within the linear theory and uniqueness of solution demonstrated. Predictions of the linear theory are noted and compared with the corresponding classical results. A note on frame-indifference and symmetry for material surfaces is appended.

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Introduction

As is well known,[1] surfaces of bodies and interfaces between pairs of bodies exhibit properties quite different from those associated with their interiors. While the literature on surface phenomena is extensive,[2] it is based, for the most part, on molecular considerations. In spite of the importance of surface phenomena, with the exception of some isolated work on fluid films[3] and on the thermodynamics of

[1] *Cf., e.g.*, ADAM (1941) and ADAMSON (1967).

[2] *Cf., e.g.*, the historical background outlined by OROWAN (1970). There the contributions of YOUNG, LAPLACE, GIBBS, and others are discussed.

[3] *Cf., e.g.*, SCRIVEN (1960).

non-deformable interfaces,[1] there does not exist a systematic treatment[2] of material surfaces based on the modern ideas now prevalent in continuum mechanics. That such an approach is valid and of use in the understanding of surface phenomena has been cogently argued by HERRING.[3] In this paper we present a first step towards the development of a rational theory of material surfaces.

Section 1 is concerned with preliminary definitions and notation, while Section 2 develops a general theory of surfaces in Euclidean space. In Section 3 we study the deformation of surfaces and introduce the relevant measures of strain. Section 4 is devoted to a careful definition of the notion of a material surface; roughly speaking, a material surface is defined as a two-dimensional continuous body embedded in a Euclidean space of dimension three. What we believe is a radically new view of the boundary surface of a three-dimensional body is also presented in Section 4 together with a precise definition for the interface between two continuous bodies. Here we are concerned with modelling transition zones between immiscible materials and do not consider diffusion; thus phase interfaces do not fall within the scope of our theory. Surface stress is introduced in Section 5 and its tensorial nature (Cauchy's Theorem) is established, in the usual manner, with the aid of the force and moment balance laws. Here, of course, the ideas are the same as those underlying the classical theory of membranes.[4] We feel, however, that the modern geometric concepts used allow for a more precise and compact theory. In Section 6 we study the consequences of equilibrium for a body whose boundary is a material surface. In particular, we deduce interesting relations between the stress field associated with the body and that associated with the boundary surface. Section 7 is concerned with the only constitutive class we presently consider: that of an elastic surface. Here we study, in the usual manner,[5] the consequences of frame-indifference and material symmetry. In Section 8 we deduce a linearized theory of elastic surfaces. A novel feature of this theory is the linearized stress-strain relation giving the surface stress *tensor* as a residual stress *tensor* plus a linear function of surface strain.[6] This obviously generalizes the usual notion of surface tension and is, in fact, consistent with atomistic calculations indicating the presence of *compressive* surface stresses in certain crystals.[7] Section 9 deals with the formulation of the free-surface problem within the linear theory. This problem models situations, for example, in which a part of a body is removed, thereby exposing a free surface. Thus, while the interior of the body may be initially free of stress, the residual surface stress in the boundary surface generates a stress field in the body. Predictions of the linear theory in several simple cases are noted in Section 10 and compared with the corresponding classical results.

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[1] FISHER & LEITMAN (1968); WILLIAMS (1972).

[2] Of course, there do exist various *ad hoc* theories, such as the theory of fracture, in which cracks are allowed to have surface tension, but in which the effect of this surface tension on the strain field in the body is ignored. In this connection see the remarks by GOODIER (1968), p. 21.

[3] HERRING (1953).

[4] *Cf.*, *e.g.*, TRUESDELL & TOUPIN (1960) and NAGHDI (1972).

[5] *Cf.*, *e.g.*, TRUESDELL & NOLL (1965).

[6] *Cf.* the remarks of HERRING (1953).

[7] SHUTTLEWORTH (1950). See also LENNARD-JONES & DENT (1928), LENNARD-JONES (1930), OROWAN (1932).

1. Preliminary Definitions

Let $\mathscr{U}$ and $\mathscr{W}$ designate finite-dimensional inner product spaces, and let

$$\begin{aligned}\text{Unit}(\mathscr{U}) &= \text{the set of all unit vectors in } \mathscr{U},\\ \text{Lin}(\mathscr{U},\mathscr{W}) &= \text{the space of linear transformations from } \mathscr{U} \text{ into } \mathscr{W},\\ \text{Invlin}(\mathscr{U},\mathscr{W}) &= \{\boldsymbol{F}\in\text{Lin}(\mathscr{U},\mathscr{W}): \boldsymbol{F} \text{ is invertible}\}.\end{aligned}$$

(Of course, $\text{Invlin}(\mathscr{U},\mathscr{W})$ is empty when $\dim\mathscr{U}\neq\dim\mathscr{W}$.) We write $\boldsymbol{S}^T$ for the transpose of $\boldsymbol{S}\in\text{Lin}(\mathscr{U},\mathscr{W})$, so that $\boldsymbol{S}^T\in\text{Lin}(\mathscr{W},\mathscr{U})$ and

$$\boldsymbol{w}\cdot\boldsymbol{S}\boldsymbol{u}=\boldsymbol{S}^T\boldsymbol{w}\cdot\boldsymbol{u}$$

for every $\boldsymbol{u}\in\mathscr{U}$, $\boldsymbol{w}\in\mathscr{W}$. (We use the same symbol "$\cdot$" for the inner product on any finite-dimensional inner product space; the underlying space will always be clear from the context.) We say that $\boldsymbol{S}\in\text{Lin}(\mathscr{U},\mathscr{U})$ is *symmetric* if $\boldsymbol{S}=\boldsymbol{S}^T$, *positive-definite* if $\boldsymbol{u}\neq\boldsymbol{0}$ implies $\boldsymbol{u}\cdot\boldsymbol{S}\boldsymbol{u}>0$. For $\boldsymbol{S}\in\text{Lin}(\mathscr{U},\mathscr{U})$, $\text{tr}\,\boldsymbol{S}$ and $\det\boldsymbol{S}$ denote, respectively, the trace and determinant of $\boldsymbol{S}$, and the inner product on $\text{Lin}(\mathscr{U},\mathscr{W})$ is defined by

$$\boldsymbol{U}\cdot\boldsymbol{F}=\text{tr}(\boldsymbol{U}\boldsymbol{F}^T)$$

for all $\boldsymbol{U},\boldsymbol{F}\in\text{Lin}(\mathscr{U},\mathscr{W})$. Let $\dim\mathscr{U}=\dim\mathscr{W}$. Then $\boldsymbol{Q}\in\text{Lin}(\mathscr{U},\mathscr{W})$ is *orthogonal* if

$$\begin{aligned}\boldsymbol{Q}^T\boldsymbol{Q} &= \text{the identity on } \mathscr{U},\\ \boldsymbol{Q}\boldsymbol{Q}^T &= \text{the identity on } \mathscr{W}.\end{aligned}$$

Given vectors $\boldsymbol{u}\in\mathscr{U}$, $\boldsymbol{w}\in\mathscr{W}$, we define $\boldsymbol{u}\otimes\boldsymbol{w}\in\text{Lin}(\mathscr{W},\mathscr{U})$ by

$$(\boldsymbol{u}\otimes\boldsymbol{w})\,\boldsymbol{a}=(\boldsymbol{w}\cdot\boldsymbol{a})\,\boldsymbol{u}$$

for every $\boldsymbol{a}\in\mathscr{W}$, and, when $\boldsymbol{u},\boldsymbol{w}\in\mathscr{W}$,

$$\boldsymbol{u}\wedge\boldsymbol{w}=\boldsymbol{u}\otimes\boldsymbol{w}-\boldsymbol{w}\otimes\boldsymbol{u}.$$

We will consistently use the following notation:

$$\begin{aligned}\text{Sym}(\mathscr{U}) &= \{\boldsymbol{S}\in\text{Lin}(\mathscr{U},\mathscr{U}): \boldsymbol{S} \text{ is symmetric}\},\\ \text{Sym}^+(\mathscr{U}) &= \{\boldsymbol{S}\in\text{Sym}(\mathscr{U}): \boldsymbol{S} \text{ is positive-definite}\},\\ \text{Orth}(\mathscr{U},\mathscr{W}) &= \{\boldsymbol{Q}\in\text{Lin}(\mathscr{U},\mathscr{W}): \boldsymbol{Q} \text{ is orthogonal}\},\\ \text{Unim}(\mathscr{U}) &= \{\boldsymbol{U}\in\text{Lin}(\mathscr{U},\mathscr{U}): \det\boldsymbol{U}=\pm 1\}.\end{aligned}$$

We now state, without proof, the following slightly generalized version of the polar decomposition theorem:[1]

Theorem 1.1. *Each $\boldsymbol{F}\in\text{Invlin}(\mathscr{U},\mathscr{W})$ admits the unique decomposition*

$$\boldsymbol{F}=\boldsymbol{R}\boldsymbol{U} \tag{1.1}$$

with $\boldsymbol{U}\in\text{Sym}^+(\mathscr{U})$ and $\boldsymbol{R}\in\text{Orth}(\mathscr{U},\mathscr{W})$. Moreover

$$\boldsymbol{U}^2=\boldsymbol{F}^T\boldsymbol{F}. \tag{1.2}$$

[1] *Cf.*, *e.g.*, Halmos (1958), p. 169.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ denote finite-dimensional Euclidean point spaces with corresponding translation spaces $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively, and let Φ be a smooth (*i.e.*, class C^1) function from an open set $\mathscr{D}\subset\mathscr{E}_1$ into $\mathscr{E}_2$. Then $\nabla\Phi(\boldsymbol{x})\in\operatorname{Lin}(\mathscr{V}_1,\mathscr{V}_2)$ is the Fréchet derivative at $\boldsymbol{x}\in\mathscr{D}$:

$$\Phi(\boldsymbol{x}+\boldsymbol{h})=\Phi(\boldsymbol{x})+\nabla\Phi(\boldsymbol{x})\,\boldsymbol{h}+o(|\boldsymbol{h}|) \quad \text{as } \boldsymbol{h}\to\boldsymbol{0}. \tag{1.3}$$

When $\mathscr{E}_2=\mathscr{V}_2=\mathbb{R}$, where $\mathbb{R}$ designates the reals, we identify $\nabla\Phi(\boldsymbol{x})$ with the unique vector in $\mathscr{V}_1$ such that the term linear in $\boldsymbol{h}$ in (1.3) has the form $\nabla\Phi(\boldsymbol{x})\cdot\boldsymbol{h}$.

$\mathscr{E}$ will always denote a three-dimensional Euclidean point space, while $\mathscr{V}$ will designate the associated translation space. Further

$$\operatorname{Bid}(\mathscr{V})=\text{the set of all two-dimensional subspaces of } \mathscr{V}.$$

We write $\mathbf{1}$ for the identity in $\operatorname{Lin}(\mathscr{V},\mathscr{V})$. More generally, elements $\boldsymbol{T}\in\operatorname{Lin}(\mathscr{V},\mathscr{V})$ will be referred to as *tensors*. For a smooth vector field $\boldsymbol{u}$: $\mathscr{D}\to\mathscr{V}$, where $\mathscr{D}\subset\mathscr{E}$ is open,

$$\operatorname{div}\boldsymbol{u}=\operatorname{tr}\nabla\boldsymbol{u}.$$

On the other hand, for a smooth tensor field $\boldsymbol{T}$: $\mathscr{D}\to\operatorname{Lin}(\mathscr{V},\mathscr{V})$, $\operatorname{div}\boldsymbol{T}$ is the unique vector field with the following property:

$$(\operatorname{div}\boldsymbol{T})\cdot\boldsymbol{k}=\operatorname{div}(\boldsymbol{T}^T\boldsymbol{k}) \tag{1.4}$$

for every vector $\boldsymbol{k}\in\mathscr{V}$.

2. Surfaces

A **surface** $\mathscr{S}$ in $\mathscr{E}$ is a subset of $\mathscr{E}$ endowed with a structure defined at each $\boldsymbol{x}\in\mathscr{S}$ by a two-dimensional subspace $\mathscr{T}_x$ of $\mathscr{V}$ and a mapping π_x with properties (S_1)–(S_4) listed below:[1]

(S_1) π_x: $\mathscr{N}_x\to\mathscr{S}$ is a class C^2 mapping with domain $\mathscr{N}_x$ an open neighborhood of zero in $\mathscr{T}_x$.

(S_2) For some neighborhood $\mathscr{M}$ of $\boldsymbol{x}$ in $\mathscr{E}$, $\pi_x(\mathscr{N}_x)\cap\mathscr{M}=\mathscr{S}\cap\mathscr{M}$.

(S_3) $\pi_x(\boldsymbol{\tau})-(\boldsymbol{x}+\boldsymbol{\tau})$ belongs to $\mathscr{T}_x^\perp$ for every $\boldsymbol{\tau}\in\mathscr{N}_x$ and is of order $o(|\boldsymbol{\tau}|)$ as $\boldsymbol{\tau}\to\boldsymbol{0}$.

(S_4) There exists a continuous field $\boldsymbol{m}$: $\mathscr{S}\to\mathscr{V}$ that never vanishes and has $\boldsymbol{m}(x)\in\mathscr{T}_x^\perp$ at each $\boldsymbol{x}\in\mathscr{S}$.

$\mathscr{T}_x$ is called the **tangent space** at $\boldsymbol{x}$; $\pi_x(\boldsymbol{\tau})$ is the projection, perpendicular to $\mathscr{T}_x$, of $\boldsymbol{x}+\boldsymbol{\tau}$ onto $\mathscr{S}$. It follows from (S_3) that π_x is injective, $\pi_x(\boldsymbol{0})=\boldsymbol{x}$, and

$$\nabla\pi_x(\boldsymbol{0})=\mathsf{I}(\boldsymbol{x}),$$

where $\mathsf{I}(\boldsymbol{x})\in\operatorname{Lin}(\mathscr{T}_x,\mathscr{V})$ is the **inclusion** map

$$\mathsf{I}(\boldsymbol{x})\,\boldsymbol{\tau}=\boldsymbol{\tau}$$

for every $\boldsymbol{\tau}\in\mathscr{T}_x$. We write $\mathbf{1}(\boldsymbol{x})\in\operatorname{Lin}(\mathscr{T}_x,\mathscr{T}_x)$ for the **identity** map on $\mathscr{T}_x$. We remark that, although $\mathbf{1}(\boldsymbol{x})\,\boldsymbol{\tau}=\mathsf{I}(\boldsymbol{x})\boldsymbol{\tau}$ for every $\boldsymbol{\tau}\in\mathscr{T}_x$, as functions $\mathbf{1}(\boldsymbol{x})$ and $\mathsf{I}(\boldsymbol{x})$ are different because of the difference in codomains. Let $\boldsymbol{P}(\boldsymbol{x})\in\operatorname{Lin}(\mathscr{V},\mathscr{T}_x)$ denote the **perpen-**

[1] Here we base our work on the unpublished lecture notes of WALTER NOLL in which the local projections π_x define the structure on $\mathscr{S}$. NOLL bases his treatment on (S_1) and (S_3) together with an axiom asserting the local uniqueness of the mappings π_x.

dicular projection from $\mathscr{V}$ onto $\mathscr{T}_x$. Then for every $\tau \in \mathscr{T}_x$, $v \in \mathscr{V}$,

$$\tau \cdot [P(x)v] = [\mathsf{I}(x)\tau] \cdot v,$$

so that

$$\mathsf{I}(x)^T = P(x) \tag{2.1}$$

and

$$P(x)\, P^T(x) = 1(x) = \mathsf{I}(x)^T\, \mathsf{I}(x). \tag{2.2}$$

By (S_1)–(S_3) the topology on $\mathscr{S}$ induced by the standard topology on $\mathscr{E}$ is equivalent to the topology formed by taking images under the local projections π_x of open sets in $\mathscr{T}_x$. Moreover, one can verify that there exists a new family of neighborhoods $\bar{\mathscr{N}}_x \subset \mathscr{N}_x$ of points $x \in \mathscr{S}$ such that each composition $\bar{\pi}_y^{-1} \circ \bar{\pi}_x$ is of class C^2, where, for each $x \in \mathscr{S}$, $\bar{\pi}_x = \pi_x|_{\bar{\mathscr{N}}_x}$.

In view of axiom (S_4), $\mathscr{S}$ is orientable. Each of the two possible continuous normal fields $n_1 = m/|m|$ and $n_2 = -n_1$ is called an **orientation** for $\mathscr{S}$. Of course,

$$\mathsf{I}(x)\, P(x) = \mathbf{1} - n(x) \otimes n(x) \tag{2.3}$$

for either orientation n for $\mathscr{S}$. Further, (S_1) implies that each orientation is smooth on $\mathscr{S}$.

Remark. $\mathsf{I}(x)$ composed with any linear transformation A: $\mathscr{V} \to \mathscr{T}_x$ simply extends the codomain of A from $\mathscr{T}_x$ to $\mathscr{V}$; indeed, $\mathsf{I}(x)A: \mathscr{V} \to \mathscr{V}$ and

$$\mathsf{I}(x)\, A\, v = A\, v$$

for every $v \in \mathscr{V}$.

Smoothness of functions on $\mathscr{S}$ is defined in the standard manner. Thus, for example, φ: $\mathscr{S} \to \mathbb{R}$ is smooth if given any $x \in \mathscr{S}$ the function $\varphi \circ \pi_x$: $\mathscr{N}_x \to \mathbb{R}$ is smooth. When this is the case the **gradient** $\nabla\varphi(x)$ is defined by

$$\nabla\varphi(x) = \nabla(\varphi \circ \pi_x)(\mathbf{0}),$$

so that $\nabla\varphi(x) \in \mathscr{T}_x$. A similar definition applies to a vector or point field u on $\mathscr{S}$; thus $\nabla u(x) \in \mathrm{Lin}(\mathscr{T}_x, \mathscr{V})$. Finally, we shall write $\nabla_{\mathscr{S}}$ rather than ∇ when we wish to emphasize that ∇ is the gradient on $\mathscr{S}$.

By a **tangential vector field** we mean a map t: $\mathscr{S} \to \mathscr{V}$ such that $t(x) \in \mathscr{T}_x$ at each $x \in \mathscr{S}$.

We define the **tangential derivative** $D u$ of a smooth vector field u: $\mathscr{S} \to \mathscr{V}$ by

$$D u = P \nabla u, \tag{2.4}$$

so that $D u(x) \in \mathrm{Lin}(\mathscr{T}_x, \mathscr{T}_x)$ maps each tangent vector $\tau \in \mathscr{T}_x$ into that portion of $\nabla u(x)\tau$ which lies in $\mathscr{T}_x$. When restricted to tangential vector fields D represents the usual *covariant derivative.*

Now let n be an orientation for $\mathscr{S}$. The field

$$\mathsf{L} = -D n \tag{2.5}$$

is called the **Weingarten map** and

$$\bar{\kappa} = \tfrac{1}{2} \operatorname{tr} \mathsf{L} \tag{2.6}$$

is the **mean curvature**. It is not difficult to show that $\mathsf{L} = \mathsf{L}^T$ and $\mathsf{I}\,\mathsf{L} = -\nabla n$.

Let $\boldsymbol{u}$ be a smooth vector field. Then $\boldsymbol{u}$ admits the unique decomposition

$$\boldsymbol{u} = \boldsymbol{u}_{\mathscr{S}} + u_n \boldsymbol{n}$$

with $\boldsymbol{u}_{\mathscr{S}}$ a smooth tangential vector field and u_n a smooth scalar field. We call $\boldsymbol{u}_{\mathscr{S}}$ and u_n, respectively, the **tangential** and **normal components** of $\boldsymbol{u}$ relative to $\mathscr{S}$ (and to the orientation $\boldsymbol{n}$). Clearly,

$$\nabla \boldsymbol{u} = \nabla \boldsymbol{u}_{\mathscr{S}} + u_n \nabla \boldsymbol{n} + \boldsymbol{n} \otimes \nabla u_n,$$

so that, by (2.4) and (2.5),

$$D\boldsymbol{u} = D\boldsymbol{u}_{\mathscr{S}} - u_n \mathsf{L}. \tag{2.7}$$

The **surface divergence** $\operatorname{div}_{\mathscr{S}} \boldsymbol{u}$ of a smooth vector field $\boldsymbol{u}: \mathscr{S} \to \mathscr{V}$ is defined by

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{u} = \operatorname{tr} D\boldsymbol{u}. \tag{2.8}$$

In view of (2.8), (2.7), and (2.6),

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{u} = \operatorname{div}_{\mathscr{S}} \boldsymbol{u}_{\mathscr{S}} - 2\bar{\kappa} u_n. \tag{2.9}$$

Let Σ be a **regular subsurface**[1] of $\mathscr{S}$, *i.e.* a compact subset of $\mathscr{S}$ whose boundary $\partial\Sigma$ is piecewise smooth and for which the *divergence theorem*

$$\int_{\partial\Sigma} \boldsymbol{t} \cdot \boldsymbol{v} = \int_{\Sigma} \operatorname{div}_{\mathscr{S}} \boldsymbol{t} \tag{2.10}$$

holds whenever $\boldsymbol{t}$ is a smooth tangent field on $\mathscr{S}$. Here $\boldsymbol{v}(\boldsymbol{x}) \in \operatorname{Unit}(\mathscr{T}_{\boldsymbol{x}})$ is the outward unit normal to $\partial\Sigma$ at $\boldsymbol{x}$. For any smooth vector field $\boldsymbol{u}: \mathscr{S} \to \mathscr{V}$, $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}_{\mathscr{S}} \cdot \boldsymbol{v}$, and (2.9) leads to the following trivial generalization of (2.10):

$$\int_{\partial\Sigma} \boldsymbol{u} \cdot \boldsymbol{v} = \int_{\Sigma} (\operatorname{div}_{\mathscr{S}} \boldsymbol{u} + 2\bar{\kappa} u_n). \tag{2.11}$$

We call $\mathscr{S}$ a **regular closed surface** if it is a geometrically closed[2] regular subsurface of itself. Note that when Σ is geometrically closed (2.10) reduces to

$$\int_{\Sigma} \operatorname{div}_{\mathscr{S}} \boldsymbol{t} = 0. \tag{2.12}$$

Now let S be a smooth field on $\mathscr{S}$ that assigns to each $\boldsymbol{x} \in \mathscr{S}$ a linear transformation $\mathsf{S}(\boldsymbol{x}): \mathscr{T}_{\boldsymbol{x}} \to \mathscr{V}$. We define the surface divergence of S in a manner completely analogous to (1.4); *i.e.* $\operatorname{div}_{\mathscr{S}} \mathsf{S}$ is the vector field on $\mathscr{S}$ defined by

$$(\operatorname{div}_{\mathscr{S}} \mathsf{S}) \cdot \boldsymbol{k} = \operatorname{div}_{\mathscr{S}} (\mathsf{S}^T \boldsymbol{k}) \tag{2.13}$$

for every vector $\boldsymbol{k} \in \mathscr{V}$. Since $\mathsf{S}^T \boldsymbol{k}$ is a tangent field, (2.10) implies that

$$\boldsymbol{k} \cdot \int_{\partial\Sigma} \mathsf{S}\boldsymbol{v} = \int_{\partial\Sigma} \mathsf{S}^T \boldsymbol{k} \cdot \boldsymbol{v} = \boldsymbol{k} \cdot \int_{\Sigma} \operatorname{div}_{\mathscr{S}} \mathsf{S},$$

so that

$$\int_{\partial\Sigma} \mathsf{S}\boldsymbol{v} = \int_{\Sigma} \operatorname{div}_{\mathscr{S}} \mathsf{S}. \tag{2.14}$$

[1] Note that all sufficiently small compact subsets of $\mathscr{S}$ with piecewise smooth boundaries are regular subsurfaces.

[2] Σ is geometrically closed if $\partial\Sigma = \emptyset$.

A **tangential tensor field** is a field T on $\mathscr{S}$ that assigns to each $x \in \mathscr{S}$ a linear transformation $\mathsf{T}(x)$: $\mathscr{T}_x \to \mathscr{T}_x$. Examples of tangential tensor fields are $\mathbf{1}$, L, and $D\boldsymbol{u}$. The above definition of the divergence cannot be used for such a field T, since $\mathsf{T}^T \boldsymbol{k}$ is defined only for $\boldsymbol{k} \in \mathscr{T}_x$. However, since $\mathsf{I}(x)\mathsf{T}(x)$: $\mathscr{T}_x \to \mathscr{V}$, $\mathsf{S} = \mathsf{I}\mathsf{T}$ is a field of the type discussed in the previous paragraph, and we can define

$$\operatorname{div}_{\mathscr{S}} \mathsf{T} = \operatorname{div}_{\mathscr{S}}(\mathsf{I}\,\mathsf{T}), \tag{2.15}$$

or equivalently, by (2.13),

$$(\operatorname{div}_{\mathscr{S}} \mathsf{T}) \cdot \boldsymbol{k} = \operatorname{div}_{\mathscr{S}}(\mathsf{T}^T \boldsymbol{P} \boldsymbol{k})$$

for every $\boldsymbol{k} \in \mathscr{V}$. Then, if we replace S in (2.14) by $\mathsf{I}\,\mathsf{T}$ and use the fact that $\mathsf{I}\,\mathsf{T}\,\boldsymbol{v} = \mathsf{T}\,\boldsymbol{v}$, we are led to the conclusion that

$$\int_{\partial\Sigma} \mathsf{T}\,\boldsymbol{v} = \int_{\Sigma} \operatorname{div}_{\mathscr{S}} \mathsf{T}\,. \tag{2.16}$$

The following useful identities can be proved in the same way as their more familiar counterparts:

$$\begin{aligned}
\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) &= (\nabla \boldsymbol{u})^T \boldsymbol{v} + (\nabla \boldsymbol{v})^T \boldsymbol{u}\,,\\
\nabla(\varphi\,\boldsymbol{u}) &= \varphi\,\nabla \boldsymbol{u} + \boldsymbol{u} \otimes \nabla\varphi\,,\\
\operatorname{div}_{\mathscr{S}}(\varphi\,\boldsymbol{u}) &= \varphi\,\operatorname{div}_{\mathscr{S}} \boldsymbol{u} + \boldsymbol{u} \cdot \nabla\varphi\,,\\
\operatorname{div}_{\mathscr{S}}(\varphi\,\mathsf{S}) &= \varphi\,\operatorname{div}_{\mathscr{S}} \mathsf{S} + \mathsf{S}\,\nabla\varphi\,,\\
\operatorname{div}_{\mathscr{S}}(\mathsf{S}^T \boldsymbol{u}) &= (\operatorname{div}_{\mathscr{S}} \mathsf{S}) \cdot \boldsymbol{u} + \mathsf{S} \cdot \nabla \boldsymbol{u}\,,
\end{aligned} \tag{2.17}$$

where φ, $\boldsymbol{u}$, $\boldsymbol{v}$, and S are smooth fields on $\mathscr{S}$ with φ scalar-valued, $\boldsymbol{u}$ and $\boldsymbol{v}$ vector-valued, and $\mathsf{S}(x) \in \operatorname{Lin}(\mathscr{T}_x, \mathscr{V})$. If T is a smooth tangential tensor field, then $(2.17)_4$ and $(2.17)_5$ with $\mathsf{S} = \mathsf{I}\,\mathsf{T}$ yield, on use of (2.15),

$$\begin{aligned}
\operatorname{div}_{\mathscr{S}}(\varphi\,\mathsf{T}) &= \varphi\,\operatorname{div}_{\mathscr{S}} \mathsf{T} + \mathsf{T}\,\nabla\varphi\,,\\
\operatorname{div}_{\mathscr{S}}(\mathsf{T}^T \boldsymbol{P} \boldsymbol{u}) &= (\operatorname{div}_{\mathscr{S}} \mathsf{T}) \cdot \boldsymbol{u} + \mathsf{T} \cdot D\boldsymbol{u}\,.
\end{aligned} \tag{2.18}$$

Further, $(2.18)_2$ with $\boldsymbol{u} = \boldsymbol{n}$ and (2.5) imply that

$$(\operatorname{div}_{\mathscr{S}} \mathsf{T}) \cdot \boldsymbol{n} = \mathsf{T} \cdot \mathsf{L}\,. \tag{2.19}$$

Proposition 2.1. $\operatorname{div}_{\mathscr{S}} \mathbf{1} = \operatorname{div}_{\mathscr{S}} \mathsf{I} = 2\bar{\kappa}\boldsymbol{n}$. (2.20)

Proof. For any $\boldsymbol{k} \in \mathscr{V}$ we have

$$(\operatorname{div}_{\mathscr{S}} \mathbf{1}) \cdot \boldsymbol{k} = (\operatorname{div}_{\mathscr{S}} \mathsf{I}) \cdot \boldsymbol{k} = \operatorname{div}_{\mathscr{S}}(\boldsymbol{P}\boldsymbol{k}) = \operatorname{div}_{\mathscr{S}}\big(\boldsymbol{k} - (\boldsymbol{n} \cdot \boldsymbol{k})\,\boldsymbol{n}\big) = (\boldsymbol{n} \cdot \boldsymbol{k})\,2\bar{\kappa}\,,$$

as is clear from (2.15) with $\mathsf{T} = \mathbf{1}$, (2.13), (2.1), (2.3), $(2.17)_3$, and (2.9). □

In addition to the above identities, the following lemma will be useful.

Lemma 2.1. *Let* S *be a smooth tensor field on* $\mathscr{S}$ *with* $\mathsf{S}(x) \in \operatorname{Lin}(\mathscr{T}_x, \mathscr{V})$, *and let* $\boldsymbol{u}$ *be a smooth vector field on* $\mathscr{S}$. *Then*

$$\int_{\partial\Sigma} \mathsf{S}\,\boldsymbol{v} \otimes \boldsymbol{u} = \int_{\Sigma} [\operatorname{div}_{\mathscr{S}} \mathsf{S} \otimes \boldsymbol{u} + \mathsf{S}\,\nabla \boldsymbol{u}^T]\,. \tag{2.21}$$

Further,

$$\int_{\partial\Sigma} \boldsymbol{u}\otimes \mathsf{S}\,\boldsymbol{v} = \int_{\Sigma} [\boldsymbol{u}\otimes \operatorname{div}_{\mathscr{S}} \mathsf{S} + \nabla\boldsymbol{u}\,\mathsf{S}^T],$$
$$\int_{\partial\Sigma} \boldsymbol{u}\wedge \mathsf{S}\,\boldsymbol{v} = \int_{\Sigma} [\boldsymbol{u}\wedge \operatorname{div}_{\mathscr{S}} \mathsf{S} + \nabla\boldsymbol{u}\,\mathsf{S}^T - \mathsf{S}\,\nabla\boldsymbol{u}^T], \tag{2.22}$$
$$\int_{\partial\Sigma} \boldsymbol{u}\cdot \mathsf{S}\,\boldsymbol{v} = \int_{\Sigma} [(\operatorname{div}_{\mathscr{S}} \mathsf{S})\cdot\boldsymbol{u} + \mathsf{S}\cdot\nabla\boldsymbol{u}].$$

Proof. For every $\boldsymbol{k}\in\mathscr{V}$ we have

$$\Big(\int_{\partial\Sigma} \mathsf{S}\,\boldsymbol{v}\otimes\boldsymbol{u}\Big)\boldsymbol{k} = \int_{\partial\Sigma} (\boldsymbol{u}\cdot\boldsymbol{k})\,\mathsf{S}\,\boldsymbol{v} = \int_{\Sigma} \operatorname{div}_{\mathscr{S}}((\boldsymbol{u}\cdot\boldsymbol{k})\mathsf{S}),$$

on using (2.14). However, if we note that $\nabla(\boldsymbol{u}\cdot\boldsymbol{k}) = (\nabla\boldsymbol{u})^T\boldsymbol{k}$ by $(2.17)_1$, $(2.17)_4$ implies that

$$\int_{\Sigma} \operatorname{div}_{\mathscr{S}}((\boldsymbol{u}\cdot\boldsymbol{k})\mathsf{S}) = \int_{\Sigma} [(\boldsymbol{u}\cdot\boldsymbol{k})\operatorname{div}_{\mathscr{S}} \mathsf{S} + \mathsf{S}\,\nabla\boldsymbol{u}^T\boldsymbol{k}] = \Big\{\int_{\Sigma} [(\operatorname{div}_{\mathscr{S}} \mathsf{S})\otimes\boldsymbol{u} + \mathsf{S}\,\nabla\boldsymbol{u}^T]\Big\}\boldsymbol{k},$$

which yields the proof of (2.21). The operators on $\mathrm{Lin}(\mathscr{V},\mathscr{V})$ which deliver the transpose, skew part, and trace are linear and continuous and, when applied to $\boldsymbol{a}\otimes\boldsymbol{b}$, have values $\boldsymbol{b}\otimes\boldsymbol{a}$, $\frac{1}{2}\boldsymbol{a}\wedge\boldsymbol{b}$, and $\boldsymbol{a}\cdot\boldsymbol{b}$, respectively. Thus, if we apply these operators to both sides of (2.21), we are led, at once, to equations (2.22). □

A *curve* γ on $\mathscr{S}$ is a subset of $\mathscr{S}$ with the following property: γ is the range of a smooth one-to-one mapping from $[0,1]$ into $\mathscr{S}$. We write $|\gamma|$ for the *length* of γ. An *oriented curve* is a curve γ together with a choice of unit normal field $\boldsymbol{v}$ to γ; $\boldsymbol{v}$ is then called the *positive unit normal field* for γ. (A unit normal field $\boldsymbol{v}$ for γ is a continuous vector field on γ such that, for each $\boldsymbol{x}\in\gamma$, $\boldsymbol{v}(\boldsymbol{x})\in\mathrm{Unit}(\mathscr{T}_{\boldsymbol{x}})$ is perpendicular to γ at $\boldsymbol{x}$. Of course, each curve γ has exactly two such unit normal fields.) A curve $\gamma'\subset\gamma$ is an *oriented subcurve* of γ if the positive unit normal fields to γ' and γ coincide on γ'.

Definitions analogous to the above apply to curves in $\mathscr{T}_{\boldsymbol{x}}$. In particular, if γ is an oriented curve in $\mathscr{N}_{\boldsymbol{x}}$, we denote by $\gamma^*_{\boldsymbol{x}}$ the image of γ under $\pi_{\boldsymbol{x}}$; $\gamma_{\boldsymbol{x}}$ is assumed to have the orientation induced by $\pi_{\boldsymbol{x}}$. We shall omit the subscript $\boldsymbol{x}$ when the underlying tangent space $\mathscr{T}_{\boldsymbol{x}}$ is clear from the context. If γ is parametrized by $\boldsymbol{f}\colon[0,1]\to\mathscr{N}_{\boldsymbol{x}}$, then γ^* is parametrized by $\pi_{\boldsymbol{x}}\circ\boldsymbol{f}\colon[0,1]\to\mathscr{S}$, and a simple computation leads to the inequality

$$\big||\gamma^*|-|\gamma|\big| \leqq \sup_{\boldsymbol{y}\in\gamma^*}\{|\nabla\pi_{\boldsymbol{x}}(\pi_{\boldsymbol{x}}^{-1}(\boldsymbol{y}))-\mathsf{l}(\boldsymbol{x})|\}\,|\gamma|. \tag{2.23}$$

Given $\boldsymbol{x}\in\mathscr{S}$, we write $\mathscr{L}_{\boldsymbol{x}}(\boldsymbol{v})$ for the family of all line segments in $\mathscr{N}_{\boldsymbol{x}}$ having $\boldsymbol{v}\in\mathrm{Unit}(\mathscr{T}_{\boldsymbol{x}})$ as positive unit normal, and, for $\boldsymbol{\tau}\in\mathscr{N}_{\boldsymbol{x}}$,

$$\mathscr{L}_{\boldsymbol{x}}(\boldsymbol{v},\boldsymbol{\tau}) = \{\ell\in\mathscr{L}_{\boldsymbol{x}}(\boldsymbol{v})\colon \boldsymbol{\tau}\in\ell\}. \tag{2.24}$$

Let Λ_ε $(\varepsilon>0)$ be a one-parameter family of subsets of $\mathscr{S}$. We say that Λ_ε *tends to* $\boldsymbol{x}\in\mathscr{S}$ if, given any neighborhood $\mathscr{N}$ of $\boldsymbol{x}$ in $\mathscr{S}$, there is an $\varepsilon_0>0$ such that $\Lambda_\varepsilon\subset\mathscr{N}$ for all $\varepsilon<\varepsilon_0$.

Let Σ_ε $(\varepsilon>0)$ be a one-parameter family of area-measurable sets in $\mathscr{N}_{\boldsymbol{x}}$ such that $\Sigma^*_\varepsilon = \pi_{\boldsymbol{x}}(\Sigma_\varepsilon)$ tends to $\boldsymbol{x}$. Then, since $\mathscr{T}_{\boldsymbol{x}}$ is the tangent space at $\boldsymbol{x}$, it is not difficult to verify that

$$|\text{area}(\Sigma_\varepsilon^*) - \text{area}(\Sigma_\varepsilon)| \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.25}$$

3. Kinematics

Let Diff($\mathscr{E}$) denote the set of all class C^2 diffeomorphisms of $\mathscr{E}$ onto itself. Let $\mathscr{s}_0$ be a surface in $\mathscr{E}$. By a **deformation** of $\mathscr{s}_0$ (into $\mathscr{s}$) we mean a mapping $\boldsymbol{f}$: $\mathscr{s}_0 \to \mathscr{s}$, where $\mathscr{s}$ is a surface in $\mathscr{E}$, such that $\boldsymbol{f} = \boldsymbol{g}|_{\mathscr{s}_0}$ with $\boldsymbol{g} \in \text{Diff}(\mathscr{E})$. Let $\mathscr{T}_X^0$, $X \in \mathscr{s}_0$, and $\mathscr{T}_x$, $x \in \mathscr{s}$, designate the tangent spaces to $\mathscr{s}_0$ and $\mathscr{s}$, respectively, and let

$$\bar{\mathsf{F}} = \nabla \boldsymbol{f}, \tag{3.1}$$

so that $\bar{\mathsf{F}}(X) \in \text{Lin}(\mathscr{T}_X^0, \mathscr{V})$.

Proposition 3.1. *For each* $X \in \mathscr{s}_0$ *the linear transformation* $\bar{\mathsf{F}}(X)$ *has range in* $\mathscr{T}_x$, *where* $x = \boldsymbol{f}(X)$. *Further, if we let* $\mathsf{F}(X)$ *denote* $\bar{\mathsf{F}}(X)$ *considered as a linear transformation with* $\mathscr{T}_x$ *as codomain, then* $\mathsf{F}(X) \in \text{Invlin}(\mathscr{T}_X^0, \mathscr{T}_x)$.

The field F is called the **deformation gradient** corresponding to $\boldsymbol{f}$. The above proposition and Theorem 1.1 imply that F has the polar decomposition

$$\mathsf{F} = \mathsf{R}\mathsf{U} \tag{3.2}$$

with $\mathsf{U}(X) \in \text{Sym}^+(\mathscr{T}_X^0)$ and $\mathsf{R}(X) \in \text{Orth}(\mathscr{T}_X^0, \mathscr{T}_x)$. The tensor $\mathsf{U}(X)$ is called the **right stretch tensor** at X. For all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathscr{T}_X^0$

$$(\mathsf{F}^T \mathsf{F} \boldsymbol{u}) \cdot \boldsymbol{v} = \mathsf{F}\boldsymbol{u} \cdot \mathsf{F}\boldsymbol{v} = \bar{\mathsf{F}}\boldsymbol{u} \cdot \bar{\mathsf{F}}\boldsymbol{v} = (\bar{\mathsf{F}}^T \bar{\mathsf{F}} \boldsymbol{u}) \cdot \boldsymbol{v},$$

so that, by (1.2),

$$\mathsf{U}^2 = \mathsf{F}^T \mathsf{F} = \bar{\mathsf{F}}^T \bar{\mathsf{F}}. \tag{3.3}$$

The **displacement** $\boldsymbol{u}$: $\mathscr{s}_0 \to \mathscr{V}$ corresponding to $\boldsymbol{f}$ is defined by

$$\boldsymbol{u}(X) = \boldsymbol{f}(X) - X.$$

Then, clearly,

$$\nabla \boldsymbol{u} = \bar{\mathsf{F}} - \mathsf{I}, \tag{3.4}$$

where $\mathsf{I}(X)$: $\mathscr{T}_X^0 \to \mathscr{V}$ is the inclusion map of the tangent space to $\mathscr{s}_0$ at X. By (3.3), (3.4), (2.1), (2.2) and (2.4),

$$\mathsf{U}^2 = \mathbf{1} + 2\mathsf{E} + \nabla \boldsymbol{u}^T \nabla \boldsymbol{u}, \tag{3.5}$$

where $\mathbf{1}(X)$ is the identity on $\mathscr{T}_X^0$ and

$$\mathsf{E} = \tfrac{1}{2}(D\boldsymbol{u} + D\boldsymbol{u}^T). \tag{3.6}$$

The tangential tensor field E is called the **infinitesimal strain**; E is important in theories based on the approximative assumption that $\nabla \boldsymbol{u}$ be small. By (2.7) and the symmetry of L,

$$\mathsf{E} = \tfrac{1}{2}(D\boldsymbol{u}_{\mathscr{s}_0} + D\boldsymbol{u}_{\mathscr{s}_0}^T) - u_n \mathsf{L}, \tag{3.7}$$ [1]

where L is the Weingarten map on $\mathscr{s}_0$, and where $\boldsymbol{u}_{\mathscr{s}_0}$ and u_n are, respectively, the tangential and normal components of $\boldsymbol{u}$ relative to $\mathscr{s}_0$.

---

[1] *Cf.*, *e.g.*, NAGHDI (1972), $(6.18)_1$.

Now let $g \in \mathrm{Diff}(\mathscr{E})$ be given by

$$g(X) = Z_0 + F_0(X - X_0), \tag{3.8}$$

where X_0, $Z_0 \in \mathscr{E}$ and $F_0 \in \mathrm{Invlin}(\mathscr{V}, \mathscr{V})$. Then $\mathscr{s} = g(\mathscr{s}_0)$ is a surface in $\mathscr{E}$, and $f_0 = g|_{\mathscr{s}_0}$ is a deformation of $\mathscr{s}_0$ into $\mathscr{s}$. A deformation f_0 of this form is called a **homogeneous deformation** of $\mathscr{s}_0$ into $\mathscr{s}$.

Proposition 3.2. *Let $\mathscr{U} \in \mathrm{Bid}(\mathscr{V})$ and let $\mathsf{F} \in \mathrm{Invlin}(\mathscr{T}_X^0, \mathscr{U})$. Then there exists a homogeneous deformation f_0 of $\mathscr{s}_0$ whose deformation gradient at X is F.*

Proof. Let n_1 and n_2 be unit vectors with $n_1 \in \mathscr{T}_X^{0\perp}$ and $n_2 \in \mathscr{U}^\perp$, let

$$F_0 \in \mathrm{Invlin}(\mathscr{V}, \mathscr{V})$$

be defined by

$$F_0 u = \mathsf{F} u, \qquad u \in \mathscr{T}_X^0,$$

$$F_0 n_1 = n_2,$$

and let $g: \mathscr{E} \to \mathscr{E}$ be given by (3.8) with X_0, Z_0 arbitrary. Then the corresponding homogeneous deformation f_0 has F as deformation gradient at X. □

4. Bodies. Material Surfaces. Interfaces

We begin by defining a body[1] in sufficient generality to include the notion of a material surface. A **body** is a set $\mathbb{B}$ with a structure defined by a family $\mathbb{C}$ of **configurations** subject to the following axioms:

(i) each $\kappa \in \mathbb{C}$ is an injection of $\mathbb{B}$ into $\mathscr{E}$;

(ii) if $\kappa \in \mathbb{C}$ and $f \in \mathrm{Diff}(\mathscr{E})$, then $f|_{\kappa(\mathbb{B})} \circ \kappa \in \mathbb{C}$;

(iii) if κ, $\mu \in \mathbb{C}$, then there exists an $f \in \mathrm{Diff}(\mathscr{E})$ such that $\mu \circ \kappa^{-1} = f|_{\kappa(\mathbb{B})}$.

The elements $X \in \mathbb{B}$ are called **material points**, and the sets $\kappa(\mathbb{B})$ ($\kappa \in \mathbb{C}$) **images** of $\mathbb{B}$. Let $\mathbb{B}_0 \subset \mathbb{B}$. Then $\mathbb{B}_0$ equipped with the family

$$\mathbb{C}_0 = \{\kappa|_{\mathbb{B}_0} : \kappa \in \mathbb{C}\}$$

is also a body; $\mathbb{B}_0$ is called a **subbody** of $\mathbb{B}$. Two bodies $\mathbb{B}_1$ and $\mathbb{B}_2$ are **compatible** if $\mathbb{B}_1 \cup \mathbb{B}_2$ can be endowed with the structure of a body in such a way that $\mathbb{B}_1$ and $\mathbb{B}_2$ are subbodies of $\mathbb{B}_1 \cup \mathbb{B}_2$.

A **material surface** is a body $\mathscr{S}$ whose images are all surfaces. We will consistently write $\mathscr{T}_X(\kappa)$ for the tangent space to the surface $\kappa(\mathscr{S})$ at the point $X = \kappa(X)$; $\mathscr{T}_X(\kappa)$ is called the **tangent space** at X in κ.

A **three-dimensional body** is a body $\mathscr{B}$ whose images are closures of bounded[2] open sets in $\mathscr{E}$. For such a body $\partial\mathscr{B}$ is the subset of $\mathscr{B}$ with the property that $\kappa(\partial\mathscr{B}) = \partial(\kappa(\mathscr{B}))$ in some (and hence every) configuration κ. We say that $\mathscr{B}$ has a **material boundary** if $\partial\mathscr{B}$ is a material surface.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be compatible three-dimensional bodies. Then $\mathscr{S} = \mathscr{B}_1 \cap \mathscr{B}_2$ is the **interface** between $\mathscr{B}_1$ and $\mathscr{B}_2$ provided $\mathscr{S}$, as a subbody of $\mathscr{B}_1 \cup \mathscr{B}_2$, is a material surface.

[1] *Cf.* NOLL (1973), p. 70.

[2] We assume $\mathscr{B}$ is bounded to avoid repeated regularity assumptions concerning the behavior of the relevant fields at infinity. In Section 10, where we apply our theory, this assumption is tacitly dropped.

5. Surface Stress

Let $\mathscr{S}$ be a surface in $\mathscr{E}$. A **traction field** $\boldsymbol{t}$ for $\mathscr{S}$ is a mapping which assigns to each oriented curve γ in $\mathscr{S}$ a continuous function

$$\boldsymbol{t}_\gamma: \gamma \to \mathscr{V}$$

with properties (P_1), (P_2), and (P_3) listed below.

(P_1) If γ_1 and γ_2 are oriented curves in $\mathscr{S}$ with γ_1 an oriented subcurve of γ_2, then

$$\boldsymbol{t}_{\gamma_1}(\boldsymbol{x}) = \boldsymbol{t}_{\gamma_2}(\boldsymbol{x}) \qquad \text{for every } \boldsymbol{x} \in \gamma_1.$$

Let $\boldsymbol{\tau} \in \mathscr{N}_{\boldsymbol{x}}$ and $\boldsymbol{v} \in \mathrm{Unit}(\mathscr{T}_{\boldsymbol{x}})$. We define (see (2.24))

$$\begin{aligned} \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\tau}) &= \boldsymbol{t}_{\ell^*}(\pi_{\boldsymbol{x}}(\boldsymbol{\tau})), \qquad \ell \in \mathscr{L}_{\boldsymbol{x}}(\boldsymbol{v}, \boldsymbol{\tau}), \\ \boldsymbol{t}_{\boldsymbol{v}}(\boldsymbol{x}) &= \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{0}). \end{aligned} \tag{5.1}$$

In view of (P_1), $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\tau})$ is independent of the choice of $\ell \in \mathscr{L}_{\boldsymbol{x}}(\boldsymbol{v}, \boldsymbol{\tau})$.

(P_2) For each $\boldsymbol{x} \in \mathscr{S}$ and $\boldsymbol{v} \in \mathrm{Unit}(\mathscr{T}_{\boldsymbol{x}})$, the mapping $\boldsymbol{\tau} \mapsto \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\tau})$ is continuous on $\mathscr{N}_{\boldsymbol{x}}$.

(P_3) For each smooth tangential vector field $\boldsymbol{v}$ on $\mathscr{S}$ with $\boldsymbol{v}(\boldsymbol{x}) \in \mathrm{Unit}(\mathscr{T}_{\boldsymbol{x}})$ the mapping $\boldsymbol{x} \mapsto \boldsymbol{t}_{\boldsymbol{v}}(\boldsymbol{x})$ is smooth on $\mathscr{S}$.

Let γ be an oriented curve with positive normal $\boldsymbol{v}$. The vector $\boldsymbol{t}_\gamma(\boldsymbol{x})$ represents the force per unit length at $\boldsymbol{x}$ exerted by γ^+ (the part of $\mathscr{S}$ into which $\boldsymbol{v}$ points) on γ^- (the part of $\mathscr{S}$ away from which $\boldsymbol{v}$ points). Thus

$$\int_\gamma \boldsymbol{t}_\gamma \tag{5.2}$$

is the total force on γ exerted by γ^+ on γ^-. Let Σ be a regular subsurface of $\mathscr{S}$ with boundary $\gamma = \partial\Sigma$. Then γ is the union of smooth oriented curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ whose positive unit normals coincide with the exterior unit normal to $\partial\Sigma$. For such a curve γ we define $\boldsymbol{t}_\gamma$ as follows: at each regular point[1] $\boldsymbol{x}$ of γ we take $\boldsymbol{t}_\gamma(\boldsymbol{x}) = \boldsymbol{t}_{\gamma_i}(\boldsymbol{x})$ for the appropriate γ_i. (For the integral (5.2) to make sense it is only necessary to define $\boldsymbol{t}_\gamma$ at regular points of γ.)

A **body force field** $\boldsymbol{b}$ for $\mathscr{S}$ is a continuous vector field on $\mathscr{S}$; the vector $\boldsymbol{b}(\boldsymbol{x})$ represents the force per unit area exerted on $\mathscr{S}$ at $\boldsymbol{x}$ by the environment. Generally, when $\mathscr{S}$ is an image of an interface between three-dimensional bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ which exert surface forces $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ on $\mathscr{S}$, then

$$\boldsymbol{b} = \boldsymbol{b}^* + \boldsymbol{t}_1 + \boldsymbol{t}_2, \tag{5.3}$$

where $\boldsymbol{b}^*$ is the total body force per unit area on $\mathscr{S}$. (Of course, $\boldsymbol{b}^*$ includes the inertial force on $\mathscr{S}$.) However, most of our results are independent of the specific representation (5.3).

Throughout the remainder of this section $\boldsymbol{t}$ is a traction field and $\boldsymbol{b}$ a body force field for $\mathscr{S}$. We say that $\boldsymbol{t}$ and $\boldsymbol{b}$ are in **equilibrium** if, given any regular subsurface Σ of $\mathscr{S}$ with boundary $\partial\Sigma = \gamma$:

$$\begin{aligned} \int_\gamma \boldsymbol{t}_\gamma + \int_\Sigma \boldsymbol{b} &= \boldsymbol{0}, \\ \int_\gamma \boldsymbol{p} \wedge \boldsymbol{t}_\gamma + \int_\Sigma \boldsymbol{p} \wedge \boldsymbol{b} &= \boldsymbol{0}, \end{aligned} \tag{5.4}$$

[1] $\boldsymbol{x}$ is a regular point of γ if γ is smooth at $\boldsymbol{x}$.

where

$$p(x) = x - x_0 \tag{5.5}$$

is the position vector from an arbitrary fixed point x_0.

Theorem 5.1. *Let t and b be in equilibrium. Then $t_\gamma(x)$ depends on γ only through the positive unit normal v to γ at x. In fact, given $x \in \mathscr{S}$ and $v \in \mathrm{Unit}(\mathscr{T}_x)$,*

$$t_\gamma(x) = t_v(x) \tag{5.6}$$

for every oriented curve γ in $\mathscr{S}$ which passes through x and whose positive unit normal field takes the value v at x.

The proof of this theorem is based on the following three lemmas. The first is a direct consequence of the continuity of t_γ on γ; the second follows from (P_2), (2.23), and $(5.1)_2$; the third is a consequence of (2.25) and the continuity of b on $\mathscr{S}$.

Lemma 5.1. *Let γ be an oriented curve in $\mathscr{S}$, and let γ_ε ($\varepsilon > 0$) be a one-parameter family of oriented subcurves of γ such that γ_ε tends to $x \in \mathscr{S}$ as $\varepsilon \to 0$. Then*

$$\int_{\gamma_\varepsilon} t_{\gamma_\varepsilon} = |\gamma_\varepsilon|\, t_\gamma(x) + o(|\gamma_\varepsilon|) \quad \text{as } \varepsilon \to 0.$$

Lemma 5.2. *Let $v \in \mathrm{Unit}(\mathscr{T}_x)$. Further, let ℓ_ε ($\varepsilon > 0$) be a one-parameter family of line segments with $\ell_\varepsilon \in \mathscr{L}_x(v)$, and suppose that ℓ_ε tends to $x \in \mathscr{S}$ as $\varepsilon \to 0$. Then*

$$\int_{\ell_\varepsilon^*} t_{\ell_\varepsilon^*} = |\ell_\varepsilon|\, t_v(x) + o(|\ell_\varepsilon|) \quad \text{as } \varepsilon \to 0.$$

Lemma 5.3. *Let $x \in \mathscr{S}$, and let Σ_ε ($\varepsilon > 0$) be a one-parameter family of area-measurable sets in $\mathscr{N}_x$ such that $\Sigma_\varepsilon^* = \pi_x(\Sigma_\varepsilon)$ tends to x as $\varepsilon \to 0$. Then*

$$\int_{\Sigma_\varepsilon^*} b = O(\mathrm{Area}(\Sigma_\varepsilon)) \quad \text{as } \varepsilon \to 0.$$

Proof of Theorem 5.1. In what follows it is often convenient to identify the tangent space $\mathscr{T}_x$ with the tangent *plane* at x by means of the correspondence $\tau \leftrightarrow x + \tau$ for every $\tau \in \mathscr{T}_x$. We shall generally make no distinction between the tangent space and tangent plane; it will always be clear from the context which is intended. Our first step will be to prove that

$$t_v(x) = -t_{-v}(x) \tag{5.7}$$

for every $v \in \mathrm{Unit}(\mathscr{T}_x)$. Thus choose such a v and consider the rectangle R_ε in $\mathscr{T}_x$ centered at x with sides of length ε and $\varepsilon\delta$ ($\varepsilon > 0$, $\delta > 0$) and with the sides of length $\varepsilon\delta$ parallel to v. Clearly, for sufficiently small ε, $R_\varepsilon \in \mathscr{N}_x$ and, further, $R_\varepsilon^* = \pi_x(R_\varepsilon)$ as well as each of its sides tends to x as $\varepsilon \to 0$. Thus if we apply $(5.4)_1$ to R_ε^*, divide the resulting equation by ε, and let $\varepsilon \to 0$, we conclude with the aid of Lemmas 5.2 and 5.3 that

$$t_v(x) + t_{-v}(x) + \delta\{t_\tau(x) + t_{-\tau}(x)\} = 0,$$

where $\tau \in \mathrm{Unit}(\mathscr{T}_x)$ and $v \cdot \tau = 0$. Since δ is arbitrary, this clearly implies (5.7).

Now let γ be an oriented curve on $\mathscr{S}$ whose positive unit normal at x is v, and let Γ be a segment of the tangent line to γ at x, lying within $\mathscr{N}_x$, including x, and having orientation opposite to that of γ at x. Then $\Gamma \in \mathscr{L}_x(-v, 0)$ and, by

(5.1) and (5.7),

$$\boldsymbol{t}_{\Gamma^*}(\boldsymbol{x}) = -\boldsymbol{t}_\nu(\boldsymbol{x}).$$

Thus, to complete the proof, it clearly suffices to prove that

$$\boldsymbol{t}_\gamma(\boldsymbol{x}) = -\boldsymbol{t}_{\Gamma^*}(\boldsymbol{x}). \tag{5.8}$$

With this in mind, let

$$\gamma' = \pi_{\boldsymbol{x}}^{-1}(\gamma \cap \pi_{\boldsymbol{x}}(\mathcal{N}_{\boldsymbol{x}})),$$

so that γ' is that curve in $\mathcal{N}_{\boldsymbol{x}}$ which corresponds, under the diffeomorphism $\pi_{\boldsymbol{x}}$, to that part of γ lying in the range of $\pi_{\boldsymbol{x}}$. If γ and Γ^* coincide in a neighborhood of $\boldsymbol{x}$, then (near $\boldsymbol{x}$) γ' must be a line segment through $\boldsymbol{x}$ with orientation opposite to that of Γ, and (5.8) follows at once from (5.7). Thus we assume that γ and Γ^* differ in every neighborhood of $\boldsymbol{x}$. Then locally at $\boldsymbol{x}$ in $\mathcal{N}_{\boldsymbol{x}}$ we shall have, for a subsegment of Γ, one of the two situations in Figure 1.

Case (i). Choose $\boldsymbol{y}$ on Γ with $|\boldsymbol{x}-\boldsymbol{y}|=\varepsilon$ and $\boldsymbol{z}\in\gamma'$ with $\boldsymbol{z}-\boldsymbol{y}$ perpendicular to Γ. Let ℓ_ε denote the line segment from $\boldsymbol{y}$ to $\boldsymbol{z}$, and let γ'_ε and Γ_ε denote, respectively, those parts of γ and Γ which with ℓ_ε form a curvilinear triangle Σ_ε with vertices $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$. Further, let ℓ_ε be equipped with the orientation corresponding to the outward normal to Σ_ε. Then ℓ_ε^*, Γ_ε^*, and $\gamma_\varepsilon = \pi_{\boldsymbol{x}}(\gamma'_\varepsilon)$ form the boundary of the region $\Sigma_\varepsilon^* = \pi_{\boldsymbol{x}}(\Sigma_\varepsilon)$ in $\mathcal{S}$, and therefore, since $\boldsymbol{t}$ and $\boldsymbol{b}$ are in equilibrium, $(5.4)_1$ applied to Σ_ε^* yields

$$\int_{\ell_\varepsilon^*} \boldsymbol{t}_{\ell_\varepsilon^*} + \int_{\Gamma_\varepsilon^*} \boldsymbol{t}_{\Gamma^*} + \int_{\gamma_\varepsilon} \boldsymbol{t}_\gamma + \int_{\Sigma_\varepsilon^*} \boldsymbol{b} = \boldsymbol{0}. \tag{5.9}$$

Clearly, Σ_ε^*, ℓ_ε^*, Γ_ε^*, and γ_ε each tend to $\boldsymbol{x}$ as $\varepsilon\to 0$. Further, since Γ_ε and γ'_ε are tangent at $\boldsymbol{x}$, it follows that $|\Gamma_\varepsilon|=\varepsilon$, $|\gamma'_\varepsilon| = |\Gamma_\varepsilon| + o(\varepsilon)$, $|\ell_\varepsilon| = o(\varepsilon)$, and $\mathrm{Area}(\Sigma_\varepsilon) = o(\varepsilon^2)$

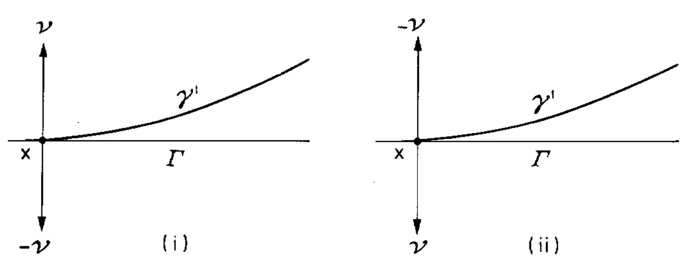

Fig. 1

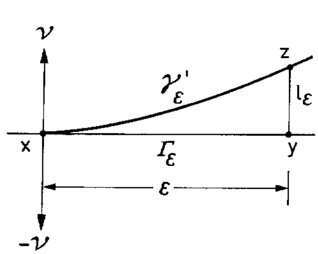

Fig. 2

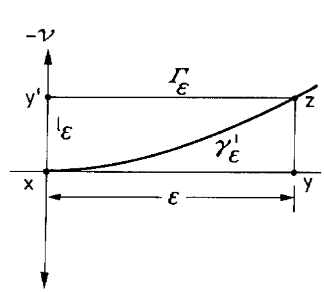

Fig. 3

as $\varepsilon \to 0$. Thus if we divide (5.9) by ε and let $\varepsilon \to 0$, we conclude, with the aid of Lemmas 5.1–5.3, that

$$t_{\gamma}(x)+t_{\Gamma^*}(x)=\mathbf{0},$$

which is the desired result (5.8).

Case (ii). For this case we must choose another curvilinear triangle in $\mathcal{N}_x$ before $(5.4)_1$ can be applied. This time Σ_ε is the curvilinear triangle with vertices at x, y' and z, where $|x-y'|=|y-z|$ and $x-y'$ is perpendicular to $x-y$. With Γ_ε the line segment from y' to z furnished with unit normal $-\nu$, and ℓ_ε the line segment from x to y' furnished with the outward normal to Σ_ε, a completely analogous analysis leads once again to (5.8). □

Theorem 5.2. *Let* t *and* b *be in equilibrium. Then there exists a smooth symmetric tangential tensor field* T *such that*

$$t_\nu(x)=\mathsf{T}(x)\nu \tag{5.10}$$

for every $\nu \in \mathrm{Unit}(\mathscr{T}_x)$ *and every* $x \in \mathscr{S}$. *Further*

$$\mathrm{div}_{\mathscr{S}}\,\mathsf{T}+b=\mathbf{0}. \qquad (5.11)^1$$

Proof. We begin by defining $\hat{\mathsf{T}}(x,\cdot)$ on $\mathscr{T}_x$ by

$$\hat{\mathsf{T}}(x,v)=|v|\,t_\nu(x) \tag{5.12}$$

with

$$\nu=v/|v| \quad \text{for } v \neq \mathbf{0}.$$

If we set $\hat{\mathsf{T}}(x,\mathbf{0})=\mathbf{0}$, it is a simple matter, by use of (5.7), to show that

$$\hat{\mathsf{T}}(x,\lambda v)=\lambda\hat{\mathsf{T}}(x,v) \tag{5.13}$$

for every $v \in \mathscr{T}_x$ and every real number λ. We now show that $\hat{\mathsf{T}}(x,\cdot)\colon \mathscr{T}_x \to \mathscr{V}$ is linear; *i.e.*,

$$\hat{\mathsf{T}}(x,\cdot) \in \mathrm{Lin}(\mathscr{T}_x,\mathscr{V}). \tag{5.14}$$

By virtue of (5.13) it suffices to show additivity on linearly independent vectors. Thus let $u, v \in \mathscr{T}_x$ be linearly independent and construct a triangle in $\mathscr{T}_x$ with sides of lengths $\varepsilon|u|$, $\varepsilon|v|$, and $\varepsilon|u+v|$ having outward normals u, v, and $-(u+v)$, respectively, and with x as circum-center. If we denote this triangle by Δ_ε, it is clear that, for small enough ε, Δ_ε will lie in $\mathcal{N}_x$. Since $\mathrm{Area}(\Delta_\varepsilon)=O(\varepsilon^2)$ as $\varepsilon\to 0$, and since $\Delta^*_\varepsilon=\pi_x(\Delta_\varepsilon)$ and each of its (curvilinear) sides tends to x as $\varepsilon \to 0$, if we apply $(5.4)_1$ to Δ^*_ε and use Lemmas 5.2 and 5.3 we arrive at

$$\varepsilon|u|\,\hat{\mathsf{T}}(x,u/|u|)+\varepsilon|v|\,\hat{\mathsf{T}}(x,v/|v|)+\varepsilon|u+v|\,\hat{\mathsf{T}}(x,-(u+v)/|u+v|)=o(\varepsilon) \quad \text{as } \varepsilon\to 0.$$

Dividing by ε, letting $\varepsilon \to 0$, and using (5.13) yields

$$\hat{\mathsf{T}}(x,u)+\hat{\mathsf{T}}(x,v)-\hat{\mathsf{T}}(x,(u+v))=\mathbf{0},$$

so that $\hat{\mathsf{T}}(x,\cdot)$ is additive on linearly independent vectors and (5.14) holds.

[1] *Cf.*, *e.g.*, TRUESDELL & TOUPIN (1960), Eq. $(212.6)_1$.

We now write

$$\hat{\mathsf{T}}(x, v) = \hat{\mathsf{T}}(x)\, v \tag{5.15}$$

for all $v \in \mathscr{T}_x$, so that $\hat{\mathsf{T}}(x) \in \operatorname{Lin}(\mathscr{T}_x, \mathscr{V})$.

In view of (5.6), (5.12), and (5.15), the balance laws $(5.4)_1$ and $(5.4)_2$ now take the forms

$$\begin{gathered} \int_{\partial\Sigma} \hat{\mathsf{T}}\, v + \int_{\Sigma} b = \mathbf{0}, \\ \int_{\partial\Sigma} p \wedge \hat{\mathsf{T}}\, v + \int_{\Sigma} p \wedge b = \mathbf{0}, \end{gathered} \tag{5.16}$$

where v is the outward unit normal to $\partial\Sigma$. By (P_3) the mapping $x \mapsto \hat{\mathsf{T}}(x)$ is smooth on $\mathscr{S}$. Thus $(5.16)_1$ and the divergence theorem imply that

$$\int_{\Sigma} (\operatorname{div}_{\mathscr{S}} \hat{\mathsf{T}} + b) = \mathbf{0}$$

for every regular subsurface Σ, and therefore, in the usual manner, the continuity of $\operatorname{div}_{\mathscr{S}} \hat{\mathsf{T}}$ and b yield the local relation

$$\operatorname{div}_{\mathscr{S}} \hat{\mathsf{T}} + b = \mathbf{0}. \tag{5.17}$$

Next let Σ be an *arbitrary* regular subsurface of $\mathscr{S}$. Since $\nabla(x - x_0) = \nabla \pi_x(\mathbf{0}) = \mathsf{I}(x)$, (2.1) and (5.5) yield $(\nabla p)^T = P$, and $(2.22)_2$ (with $u = p$ and $\mathsf{S} = \hat{\mathsf{T}}$) implies that

$$\int_{\partial\Sigma} p \wedge \hat{\mathsf{T}}\, v = \int_{\Sigma} (p \wedge \operatorname{div}_{\mathscr{S}} \hat{\mathsf{T}} + P^T \hat{\mathsf{T}}^T - \hat{\mathsf{T}} P).$$

If we use $(5.16)_2$ and (5.17), we arrive at the relation

$$\int_{\Sigma} (P^T \hat{\mathsf{T}}^T - \hat{\mathsf{T}} P) = \mathbf{0};$$

hence

$$\hat{\mathsf{T}} P = P^T \hat{\mathsf{T}}^T. \tag{5.18}$$

The remainder of the proof will make repeated use (without mention) of (2.1) and (2.2). Choose $n \in \mathscr{T}_x^{\perp}$. Then, since $\hat{\mathsf{T}}(x)\colon \mathscr{V} \to \mathscr{T}_x$, (5.18) implies that

$$\hat{\mathsf{T}}^T n = \mathsf{I}\, \hat{\mathsf{T}}^T n = P^T \hat{\mathsf{T}}^T n = \hat{\mathsf{T}} P n = \mathbf{0}$$

(where, for convenience, we have suppressed the argument x). Thus

$$n \cdot \hat{\mathsf{T}}\, v = \hat{\mathsf{T}}^T n \cdot v = 0$$

for every $v \in \mathscr{T}_x$, so that *the range of* $\hat{\mathsf{T}}(x)$ *lies in* $\mathscr{T}_x$. Therefore, if we let

$$\mathsf{T} = P \hat{\mathsf{T}}, \tag{5.19}$$

so that T is a smooth tangential tensor field on $\mathscr{S}$, it follows that

$$\hat{\mathsf{T}} = \mathsf{I}\, \mathsf{T}. \tag{5.20}$$

Thus

$$\hat{\mathsf{T}}\, v = \mathsf{T}\, v$$

for every $v \in \mathscr{T}_x$; hence (5.12) and (5.15) imply (5.10). On the other hand, (5.20) and the definition of the divergence of a tangential tensor field yield (5.11). Thus, to complete the proof, we have only to show that T is symmetric. By (5.18) and (5.19),

$$\mathsf{T}=\mathsf{T}\boldsymbol{P}\boldsymbol{P}^T=\boldsymbol{P}\hat{\mathsf{T}}\boldsymbol{P}\boldsymbol{P}^T=\boldsymbol{P}\boldsymbol{P}^T\hat{\mathsf{T}}^T\boldsymbol{P}^T=\hat{\mathsf{T}}^T\boldsymbol{P}^T=\mathsf{T}^T,$$

and the proof is complete. ▯

For the remainder of this section we assume that $\boldsymbol{t}$ and $\boldsymbol{b}$ are in equilibrium. The symmetric tangential tensor field T, defined in Theorem 5.2, is called the **surface stress**.

Theorem 5.3. *The surface stress* T *and the body force* $\boldsymbol{b}$ *satisfy the relation*[1]

$$\mathsf{T}\cdot\mathsf{L}=-b_n,$$

where $b_n=\boldsymbol{b}\cdot\boldsymbol{n}$ *with* $\boldsymbol{n}$ *a choice of unit normal field for* $\mathscr{s}$ *and* L *is the corresponding Weingarten map* (2.5).

Proof. The result is immediate on using (2.19) and (5.11). ▯

We say that T is a **surface tension** σ if σ is a smooth scalar field on $\mathscr{s}$ and

$$\mathsf{T}=\sigma\mathbf{1}.$$

Theorem 5.4. *Let* T *be a surface tension* σ. *Then*

$$\nabla\sigma=-\boldsymbol{b}_{\mathscr{s}};\qquad 2\bar{\kappa}\sigma=-b_n,$$

where $\boldsymbol{b}_{\mathscr{s}}$ *and* b_n *are, respectively, the tangential and normal components of* $\boldsymbol{b}$.

Proof. By (5.11), (2.18), with $\varphi=\sigma$ and $\mathsf{T}=\mathbf{1}$, and (2.20)

$$\nabla\sigma+2\bar{\kappa}\sigma\boldsymbol{n}+\boldsymbol{b}=\mathbf{0}.$$

The desired conclusions follow from this relation and the fact that $\nabla\sigma$ is a tangential vector field. ▯

Theorem 5.4 has an important corollary when $\mathscr{s}$ is (an image of) the interface between two three-dimensional bodies $\mathscr{B}_1$ and $\mathscr{B}_2$. Indeed, in the absence of inertial forces on $\mathscr{s}$, and when the forces exerted by $\mathscr{B}_1$ and $\mathscr{B}_2$ on $\mathscr{s}$ are derived from (not necessarily constant) pressures p_1 and p_2, then $\boldsymbol{b}$ in (5.3) is given by

$$\boldsymbol{b}=(p_1-p_2)\boldsymbol{n}.$$

Here, of course, $\boldsymbol{n}$ is the outward unit normal to $\mathscr{B}_1$ in the configuration under consideration. This should serve to motivate

Corollary 5.5. *If* T *is a surface tension* σ, *and if* $\boldsymbol{b}=(p_1-p_2)\boldsymbol{n}$, *then* σ *is constant and*

$$2\bar{\kappa}\sigma=p_2-p_1. \tag{5.21}$$

The equation (5.21) is usually referred to as *Laplace's formula*.[2] We remark that Theorem 5.3 is a generalized version of this relation.

In applications T represents the stress field in the current configuration of a material surface $\mathscr{S}$ undergoing deformation. More precisely, if $\boldsymbol{\mu}$ is such a con-

[1] *Cf., e.g.*, TRUESDELL & TOUPIN (1960), Eq. $(212.22)_2$.

[2] *Cf., e.g.*, LANDAU & LIFSHITZ (1959), § 60.

figuration with $\boldsymbol{\mu}(\mathscr{S})=\mathscr{s}$, then for any $\boldsymbol{x}\in\mathscr{s}$ and any unit vector $\boldsymbol{v}\in\mathscr{T}_x(\boldsymbol{\mu})$, $\mathsf{T}(\boldsymbol{x})\boldsymbol{v}$ is the force per unit length at $\boldsymbol{x}$ on any oriented curve γ in $\mathscr{s}$ passing through $\boldsymbol{x}$ and having $\boldsymbol{v}$ as its positive unit normal at $\boldsymbol{x}$. Let $\boldsymbol{\kappa}$ be a second configuration of $\mathscr{S}$ and let $\boldsymbol{f}=\boldsymbol{\mu}\circ\boldsymbol{\kappa}^{-1}$ denote the corresponding deformation of $\mathscr{s}_0=\boldsymbol{\kappa}(\mathscr{S})$ into $\mathscr{s}=\boldsymbol{\mu}(\mathscr{S})$. Then there exists a unique smooth field S on $\mathscr{s}_0$ with $\mathsf{S}(\boldsymbol{X})\in\mathrm{Lin}(\mathscr{T}_X(\boldsymbol{\kappa}),\mathscr{V})$ such that

$$\int_{\gamma_0}\mathsf{S}\boldsymbol{v}_0=\int_{\gamma}\mathsf{T}\boldsymbol{v} \tag{5.22}$$

for every oriented curve γ_0 in $\mathscr{s}_0$, provided $\gamma=\boldsymbol{f}(\gamma_0)$ and $\boldsymbol{v}_0$ and $\boldsymbol{v}$ are, respectively, the positive unit normal fields for γ_0 and γ. S is called the **Piola-Kirchhoff stress** in $\boldsymbol{\mu}$ relative to $\boldsymbol{\kappa}$; by (5.22), $\mathsf{S}\boldsymbol{v}_0$ is the force per unit length of γ_0. If F denotes the deformation gradient of $\boldsymbol{f}$, it may be shown that[1]

$$(\det\mathsf{U})(\mathsf{F}^{-1})^T\boldsymbol{v}_0=j\boldsymbol{v}, \tag{5.23}$$

where U is the right stretch tensor field and j is the Radon-Nikodym derivative of the arc length measure on γ with respect to that on γ_0. Equations (5.22) and (5.23) imply that

$$\mathsf{S}(\boldsymbol{X})=(\det\mathsf{U}(\boldsymbol{X}))\,\mathsf{I}(\boldsymbol{f}(\boldsymbol{X}))\,\mathsf{T}(\boldsymbol{f}(\boldsymbol{X}))(\mathsf{F}(\boldsymbol{X})^{-1})^T, \tag{5.24}$$

where $\mathsf{I}(\boldsymbol{x})$ is the inclusion map of $\mathscr{T}_x(\boldsymbol{\mu})$ into $\mathscr{V}$. In view of (2.1) and (2.2),

$$\mathsf{T}=(\det\mathsf{U})^{-1}\boldsymbol{P}\,\mathsf{S}\,\mathsf{F}^T,$$

where $\boldsymbol{P}(\boldsymbol{x})=\mathsf{I}(\boldsymbol{x})^T$ is the perpendicular projection of $\mathscr{V}$ onto $\mathscr{T}_x(\boldsymbol{\mu})$, and where, for convenience, we have omitted arguments. Thus, since T takes symmetric values, S must satisfy

$$\boldsymbol{P}\,\mathsf{S}\,\mathsf{F}^T=\mathsf{F}\,\mathsf{S}^T\,\mathsf{I}.$$

Let

$$\boldsymbol{b}_0(\boldsymbol{X})=[\det\mathsf{U}(\boldsymbol{X})]\,\boldsymbol{b}(\boldsymbol{x}),$$

where $\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{X})$, so that

$$\int_{\boldsymbol{f}^{-1}(\Sigma)}\boldsymbol{b}_0=\int_{\Sigma}\boldsymbol{b}$$

for any regular subsurface Σ of $\mathscr{s}$. Then balance of forces $(5.16)_1$ is equivalent to the requirement that

$$\int_{\partial\Sigma_0}\mathsf{S}\boldsymbol{v}_0+\int_{\Sigma_0}\boldsymbol{b}_0=\boldsymbol{0}$$

for every regular subsurface Σ_0 of $\mathscr{s}_0$ with $\boldsymbol{v}_0$ the outward unit normal to $\partial\Sigma_0$. The divergence theorem together with the continuity of both $\mathrm{div}_{\mathscr{s}_0}\mathsf{S}$ and $\boldsymbol{b}_0$ therefore yield the local form

$$\mathrm{div}_{\mathscr{s}_0}\mathsf{S}+\boldsymbol{b}_0=\boldsymbol{0}. \tag{5.25}$$

6. Equilibrium for Three-Dimensional Bodies with Material Boundaries

Let $\mathscr{B}$ be a three-dimensional body with a material boundary $\mathscr{S}=\partial\mathscr{B}$, and let $B=\boldsymbol{\mu}(\mathscr{B})$ be the image of $\mathscr{B}$ in a configuration $\boldsymbol{\mu}$. If $\boldsymbol{n}$ denotes the *outward* unit normal field to ∂B, we take $\boldsymbol{t}_i$, the force per unit area exerted on ∂B by the material

[1] *Cf.*, *e.g.*, TRUESDELL (1966), (11.12).

in $\mathring{B}$ (the interior of B) to be given by

$$\boldsymbol{t}_i = -\boldsymbol{T}\boldsymbol{n},$$

where $\boldsymbol{T}$ is the continuous extension to ∂B of the Cauchy stress in $\mathring{B}$. The body force field $\boldsymbol{b}$ on the image surface $\mathscr{s} = \boldsymbol{\mu}(\mathscr{S})$ is then given by (*cf.* (5.3))

$$\boldsymbol{b} = \boldsymbol{b}^* - \boldsymbol{T}\boldsymbol{n} + \boldsymbol{t}_e,$$

where $\boldsymbol{b}^*$ is the inertial force per unit area and $\boldsymbol{t}_e$ is the force per unit area exerted on ∂B by the environment. Thus, if the traction and body force fields are in equilibrium, (5.3) and (5.11) imply that

$$\operatorname{div}_{\mathscr{s}} \mathsf{T} + \boldsymbol{b}^* - \boldsymbol{T}\boldsymbol{n} + \boldsymbol{t}_e = \boldsymbol{0} \quad \text{on } \partial B, \tag{6.1}$$

and, when $\boldsymbol{b}^*$ and $\boldsymbol{t}_e$ vanish,

$$\operatorname{div}_{\mathscr{s}} \mathsf{T} = \boldsymbol{T}\boldsymbol{n} \quad \text{on } \partial B. \tag{6.2}$$

If, in addition, $\mathscr{B}$ is in equilibrium in $\boldsymbol{\mu}$ with zero body force, then

$$\operatorname{div} \boldsymbol{T} = \boldsymbol{0} \quad \text{on } B, \tag{6.3}$$

where the divergence is here as defined in (1.4).

Theorem 6.1. *Let B be an image of a three-dimensional body with a material boundary. Further, let T be a smooth symmetric tangential tensor field on $\mathscr{s}$, let $\boldsymbol{T}$ be a smooth symmetric tensor field on B, and assume that* (6.2) *and* (6.3) *are satisfied. Then*

$$\int_B \boldsymbol{T} + \int_{\mathscr{s}} \mathsf{I}\,\mathsf{T}\,\boldsymbol{P} = \boldsymbol{0}. \tag{6.4}$$

Proof. By the divergence theorem on B,

$$\int_{\mathscr{s}} \boldsymbol{p} \otimes \boldsymbol{T}\boldsymbol{n} = \int_B (\boldsymbol{p} \otimes \operatorname{div} \boldsymbol{T} + \boldsymbol{T}^T),$$

where $\boldsymbol{p}$ is given by (5.5). Since $\operatorname{div} \boldsymbol{T} = \boldsymbol{0}$ and $\boldsymbol{T} = \boldsymbol{T}^T$, this reduces to

$$\int_{\mathscr{s}} \boldsymbol{p} \otimes \boldsymbol{T}\boldsymbol{n} = \int_B \boldsymbol{T}. \tag{6.5}$$

On the other hand, (6.2), $(2.22)_1$ with $\boldsymbol{u} = \boldsymbol{p}$ and $\mathsf{S} = \mathsf{I}\,\mathsf{T}$, and $\nabla \boldsymbol{p} = \mathsf{I}$ imply that

$$\int_{\mathscr{s}} \boldsymbol{p} \otimes \boldsymbol{T}\boldsymbol{n} = \int_{\mathscr{s}} \boldsymbol{p} \otimes \operatorname{div}_{\mathscr{s}} \mathsf{T} = -\int_{\mathscr{s}} \mathsf{I}(\mathsf{I}\,\mathsf{T})^T. \tag{6.6}$$

By use of (2.1) and the symmetry of T the desired result follows from (6.5) and (6.6). □

The quantities

$$\bar{\boldsymbol{T}}(B) = \frac{1}{\operatorname{Vol}(B)} \int_B \boldsymbol{T}$$

and

$$\bar{\mathsf{T}}(\mathscr{s}) = \frac{1}{\operatorname{Area}(\mathscr{s})} \int_{\mathscr{s}} \mathsf{I}\,\mathsf{T}\,\boldsymbol{P}$$

represent, respectively, the **mean stress** in B and the **mean surface stress** on ∂B. Thus Theorem 6.1 gives a simple relation between the mean value of the body stress, the mean value of the surface stress, and the geometry of the equilibrium configuration. Further, it is interesting to note that this relation is completely independent of specific constitutive assumptions. In particular, if $\boldsymbol{T}$ is a pressure, $-p\mathbf{1}$, and T a surface tension, $\sigma\mathbf{1}$, on taking the trace of (6.4) we have

$$3\bar{p}\,\mathrm{Vol}(B)=2\bar{\sigma}\,\mathrm{Area}(\mathscr{s}),$$

so that the mean pressure $\bar{p}$ and mean surface tension $\bar{\sigma}$ must have the same sign.

One can ask the question: if T is a *constant* surface tension, will the corresponding mean stress $\bar{\boldsymbol{T}}(B)$ be a pressure? As will be clear from what follows, the answer to this question is, in general, no. For the remainder of this section we assume that the hypotheses of Theorem 6.1 are satisfied.

Theorem 6.2. *Let* T *be a constant surface tension* $\sigma\neq 0$. *Then a necessary and sufficient condition that the mean stress* $\bar{\boldsymbol{T}}(B)$ *be a pressure is that*

$$\int_{\mathscr{s}} \boldsymbol{n}\otimes\boldsymbol{n}=\alpha\mathbf{1}. \tag{6.7}$$

Proof. $\mathsf{T}=\sigma\mathbf{1}$ with σ constant and (2.3) imply that

$$\int_{\mathscr{s}} \mathsf{I}\,\mathsf{T}\boldsymbol{P}=\sigma\int_{\mathscr{s}} \mathsf{I}\boldsymbol{P}=\sigma\int_{\mathscr{s}}(\mathbf{1}-\boldsymbol{n}\otimes\boldsymbol{n})=\sigma\,\mathrm{Area}(\mathscr{s})\,\mathbf{1}-\sigma\int_{\mathscr{s}}\boldsymbol{n}\otimes\boldsymbol{n};$$

since $\sigma\neq 0$, this result, in conjunction with (6.4), leads to the desired conclusion. □

An example of a region in $\mathscr{E}$ in which (6.7) is *not* satisfied is furnished by a rectangular box, R. Indeed, let a_i $(i=1,2,3)$ denote the areas of the sides, and let $\{\boldsymbol{e}_i\}$ be an orthonormal basis for $\mathscr{V}$ with each $\boldsymbol{e}_i$ perpendicular to the face with area a_i. Then, clearly,

$$\int_{\partial R} \boldsymbol{n}\otimes\boldsymbol{n}=2\sum_{i=1}^{3} a_i(\boldsymbol{e}_i\otimes\boldsymbol{e}_i), \tag{6.8}$$

so that (6.7) holds if and only if $a_1=a_2=a_3$. Of course, for this example ∂R is not smooth. However, if $\mathscr{A}_\varepsilon$ $(\varepsilon>0)$ is a sequence of sufficiently smooth "approximate rectangular boxes" such that R and $\mathscr{A}_\varepsilon$ differ only within sets of area less than ε, then

$$\Big|\int_{\partial R}\boldsymbol{n}\otimes\boldsymbol{n}-\int_{\partial\mathscr{A}_\varepsilon}\boldsymbol{n}\otimes\boldsymbol{n}\Big|<\varepsilon.$$

Thus, in view of (6.8), for ε sufficiently small $\mathscr{s}=\partial\mathscr{A}_\varepsilon$ cannot possibly satisfy (6.7).

We now give a class of regions for which (6.7) is satisfied. Choose an origin $\boldsymbol{o}\in\mathscr{E}$ and identify $\mathscr{E}$ with $\mathscr{V}$ in the natural way. Then the *geometric symmetry group* $g_{\boldsymbol{o}}(B)$ for B relative to $\boldsymbol{o}$ is the group of all orthogonal transformations $\boldsymbol{Q}$ such that $\boldsymbol{Q}(B)=B$. We shall say that B has **weak geometrical symmetry** if there exists an origin $\boldsymbol{o}$ such that for any $\boldsymbol{e}\in\mathscr{V}$, $\boldsymbol{e}\neq\mathbf{0}$, the set $\{\boldsymbol{Q}\boldsymbol{e}:\boldsymbol{Q}\in g_{\boldsymbol{o}}(B)\}$ spans $\mathscr{V}$. Simple examples of regions with weak geometrical symmetry are furnished by spheres and cubes.

Theorem 6.3. *Let* T *be a constant surface tension, and assume that* B *has weak geometrical symmetry. Then the mean stress* $\bar{\boldsymbol{T}}(B)$ *is a pressure.*

Proof. Let A be the *symmetric* tensor

$$A = \int_{\mathscr{A}} \boldsymbol{n} \otimes \boldsymbol{n}.$$

Then, clearly, $\boldsymbol{Q} \in g_o(B)$ implies that

$$\boldsymbol{Q}A\boldsymbol{Q}^T = \int_{\mathscr{A}} \boldsymbol{Q}\boldsymbol{n} \otimes \boldsymbol{Q}\boldsymbol{n} = \int_{\boldsymbol{Q}(\mathscr{A})} \boldsymbol{Q}\boldsymbol{n} \otimes \boldsymbol{Q}\boldsymbol{n} = \int_{\mathscr{A}} \boldsymbol{n} \otimes \boldsymbol{n} = A.$$

Thus $\boldsymbol{Q}A = A\boldsymbol{Q}$, so that $\boldsymbol{Q}$ leaves each of the characteristic spaces of A invariant. Let $\boldsymbol{e}$ be a characteristic vector of A corresponding to a characteristic value α. Since B has weak geometrical symmetry, $\{\boldsymbol{Q}\boldsymbol{e}: \boldsymbol{Q} \in g_o(B)\}$ spans $\mathscr{V}$. Thus the characteristic space for α is all of $\mathscr{V}$. Hence $A = \alpha \mathbf{1}$, and the desired conclusion follows from Theorem 6.2. □

It is often convenient to refer all quantities to a reference configuration $\boldsymbol{\kappa}$ of $\mathscr{B}$. Thus consider the Piola-Kirchhoff stress S in $\boldsymbol{\mu}$ relative to $\boldsymbol{\kappa}$, as defined in (5.24):

$$\mathsf{S} = (\det \mathsf{U})\, \mathsf{I}\, \mathsf{T}(\mathsf{F}^T)^{-1}.$$

We then have, from (5.25) and, respectively, (6.1) and (6.2),

$$\operatorname{div}_{\mathscr{A}_0} \mathsf{S} + \boldsymbol{b}_0^* - \boldsymbol{S}\boldsymbol{n}' + \boldsymbol{s}_e = \mathbf{0} \quad \text{on } \partial\boldsymbol{\kappa}(\mathscr{B}) \tag{6.9}$$

and

$$\operatorname{div}_{\mathscr{A}_0} \mathsf{S} = \boldsymbol{S}\boldsymbol{n}' \quad \text{on } \partial\boldsymbol{\kappa}(\mathscr{B}). \tag{6.10}$$

Here $\boldsymbol{S}$ denotes the continuous extension to $\partial\boldsymbol{\kappa}(\mathscr{B})$ of the Piola-Kirchhoff stress over $\boldsymbol{\kappa}(\mathring{\mathscr{B}}) = \widehat{\boldsymbol{\kappa}(\mathscr{B})}^{\circ}$, $\boldsymbol{n}'$ is the outward unit normal to $\partial\boldsymbol{\kappa}(\mathscr{B})$, and $\boldsymbol{b}_0^*$ and $\boldsymbol{s}_e$ represent the inertial force and external traction, respectively.

7. Elastic Surfaces

An **elastic surface** is a material surface $\mathscr{S}$ together with a constitutive equation

$$\mathsf{T} = \hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\mathsf{F}, X)$$

giving the surface stress T at the material point X in any deformation from an arbitrary configuration $\boldsymbol{\kappa}$, provided, of course, F is the deformation gradient at $\boldsymbol{\kappa}(X)$. Thus, letting $\boldsymbol{\mu}$ be the deformed configuration and $\mathscr{U} = \mathscr{T}_X(\boldsymbol{\mu})$, we have $\mathsf{F} \in \operatorname{Invlin}(\mathscr{T}_X(\boldsymbol{\kappa}), \mathscr{U})$, and, since T is the stress in $\boldsymbol{\mu}$, $\mathsf{T} \in \operatorname{Sym}(\mathscr{U})$. By Proposition 3.1, the domain of $\hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\cdot, X)$ is $\bigcup_{\mathscr{U} \in \operatorname{Bid}(\mathscr{V})} \operatorname{Invlin}(\mathscr{T}_X(\boldsymbol{\kappa}), \mathscr{U})$ (since every element F in that set can be obtained as the gradient of a (homogeneous) deformation from $\boldsymbol{\kappa}$). Thus

$$\hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\cdot, X): \bigcup_{\mathscr{U} \in \operatorname{Bid}(\mathscr{V})} \operatorname{Invlin}(\mathscr{T}_X(\boldsymbol{\kappa}), \mathscr{U}) \to \bigcup_{\mathscr{U} \in \operatorname{Bid}(\mathscr{V})} \operatorname{Sym}(\mathscr{U})$$

with

$$\hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\mathsf{F}, X) \in \operatorname{Sym}(\mathscr{U}) \quad \text{whenever } \mathsf{F} \in \operatorname{Invlin}(\mathscr{T}_X(\boldsymbol{\kappa}), \mathscr{U}).$$

For convenience, we now choose a material point X and reference configuration $\boldsymbol{\kappa}$ and write

$$\mathscr{T} \text{ for } \mathscr{T}_X(\boldsymbol{\kappa}) \quad \text{and} \quad \hat{\mathsf{T}}(\cdot) \text{ for } \hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\cdot, X).$$

If we assume that the surface stress is frame-indifferent, the principle of **material frame-indifference**[1] implies that

$$\mathsf{Q}\hat{\mathsf{T}}(\mathsf{F})\mathsf{Q}^T = \hat{\mathsf{T}}(\mathsf{Q}\,\mathsf{F}) \tag{7.1}$$

for every $\mathsf{F} \in \mathrm{Invlin}(\mathscr{T}, \mathscr{U})$, $\mathsf{Q} \in \mathrm{Orth}(\mathscr{U}, \mathscr{W})$, and $\mathscr{U}, \mathscr{W} \in \mathrm{Bid}(\mathscr{V})$.

Proposition 7.1. $\hat{\mathsf{T}}$ *is completely determined by its restriction to* $\mathrm{Sym}^+(\mathscr{T})$. *In fact,*

$$\hat{\mathsf{T}}(\mathsf{F}) = \mathsf{F}\,\mathsf{U}^{-1}\,\hat{\mathsf{T}}(\mathsf{U})\,\mathsf{U}^{-1}\,\mathsf{F}^T. \tag{7.2}$$

Proof. Proceeding in the usual manner,[2] we take, in (7.1), $\mathscr{W} = \mathscr{T}$ and $\mathsf{Q} = \mathsf{R}^T$, where R is the orthogonal transformation in the polar decomposition (3.2). Thus

$$\hat{\mathsf{T}}(\mathsf{F}) = \mathsf{R}\,\hat{\mathsf{T}}(\mathsf{U})\,\mathsf{R}^T, \tag{7.3}$$

and, since $\mathsf{R} = \mathsf{F}\,\mathsf{U}^{-1}$, we can write (7.3) in the form (7.2). □

The **symmetry group**[3] $\mathscr{G}_\kappa$ for X relative to κ is the group of all $\mathsf{H} \in \mathrm{Unim}(\mathscr{T})$ such that

$$\hat{\mathsf{T}}(\mathsf{F}) = \hat{\mathsf{T}}(\mathsf{F}\,\mathsf{H}) \tag{7.4}$$

for every $\mathsf{F} \in \mathrm{Invlin}(\mathscr{T}, \mathscr{U})$ and $\mathscr{U} \in \mathrm{Bid}(\mathscr{V})$. By (7.1) and (7.4),

$$\mathsf{Q}\,\hat{\mathsf{T}}(\mathsf{F})\,\mathsf{Q}^T = \hat{\mathsf{T}}(\mathsf{Q}\,\mathsf{F}\,\mathsf{Q}^T) \tag{7.5}$$

whenever $\mathsf{Q} \in \mathscr{G}_\kappa \cap \mathrm{Orth}(\mathscr{T}, \mathscr{T})$ and $\mathsf{F} \in \mathrm{Invlin}(\mathscr{T}, \mathscr{T})$. If $\mathrm{Orth}(\mathscr{T}, \mathscr{T}) \subset \mathscr{G}_\kappa$, then X is **isotropic** relative to κ. If $\mathscr{G}_\kappa = \mathrm{Unim}(\mathscr{T})$ for some (and hence every[4]) configuration κ, then X is a **fluid**.

Theorem 7.2. *If the material point X is a fluid, then the surface stress T is a surface tension. In fact, the constitutive equation reduces to*

$$\mathsf{T} = \sigma_\kappa(\det \mathsf{U})\,1 \tag{7.6}$$

with σ_κ scalar-valued.

Proof. $\mathscr{G}_\kappa = \mathrm{Unim}(\mathscr{T})$ implies

$$\hat{\mathsf{T}}(\mathsf{U}) = \hat{\mathsf{T}}(\mathsf{U}(\det \mathsf{U})\,\mathsf{U}^{-1}) = \hat{\mathsf{T}}((\det \mathsf{U})\,1). \tag{7.7}$$

Since $\mathrm{Orth}(\mathscr{T}, \mathscr{T}) \subset \mathrm{Unim}(\mathscr{T})$, (7.5) and (7.7) yield

$$\mathsf{Q}\hat{\mathsf{T}}(\mathsf{U})\,\mathsf{Q}^T = \hat{\mathsf{T}}(\mathsf{U})$$

for every $\mathsf{Q} \in \mathrm{Orth}(\mathscr{T}, \mathscr{T})$. This and the symmetry of $\hat{\mathsf{T}}(\mathsf{U})$ imply that $\hat{\mathsf{T}}(\mathsf{U})$ is a scalar multiple of the identity 1 on $\mathscr{T}$. Thus (7.7) reduces to

$$\hat{\mathsf{T}}(\mathsf{U}) = \sigma_\kappa(\det \mathsf{U})\,1. \quad □$$

Thus, by Corollary 5.5, if $\mathscr{s}$ is the image of an elastic fluid interface with zero inertia, lying between two three-dimensional bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ whose correspond-

[1] This is discussed in the Appendix.

[2] *Cf., e.g.*, TRUESDELL & NOLL (1965), § 29.

[3] For further details consult the Appendix.

[4] It is shown in the Appendix that, as in the three-dimensional case, the symmetry groups for any given X relative to two different configurations are conjugate, so that $\mathscr{G}_\kappa = \mathrm{Unim}(\mathscr{T})$ implies that the symmetry group of X relative to any configuration is the relevant unimodular group.

ing stress fields are pressures, then σ_κ must be constant on $\mathcal{S}$ and satisfy Laplace's formula (5.21).

The tensor

$$\mathsf{M} = \hat{\mathsf{T}}_\kappa(1, X) \tag{7.8}$$

is the **residual stress** at X in the configuration κ. An immediate consequence of (7.5) is:

Proposition 7.3. *If the material is isotropic relative to* κ, *then the residual stress in* κ *is a surface tension.*

Recall the Piola-Kirchhoff surface stress defined by (5.24):

$$\mathsf{S} = (\det \mathsf{U})\, \mathsf{IT}(\mathsf{F}^T)^{-1}.$$

By virtue of (7.2)

$$\mathsf{S} = (\det \mathsf{U})\, \mathsf{IFU}^{-1}\, \hat{\mathsf{T}}(\mathsf{U})\, \mathsf{U}^{-1} = \mathsf{IF}\hat{\mathsf{S}}(\mathsf{U}).$$

Thus, since $\bar{\mathsf{F}} = \mathsf{IF}$ (see Proposition (3.1)),

$$\mathsf{S} = \bar{\mathsf{F}}\hat{\mathsf{S}}(\mathsf{U}), \tag{7.9}$$

where

$$\hat{\mathsf{S}}(\mathsf{U}) = (\det \mathsf{U})\, \mathsf{U}^{-1}\, \hat{\mathsf{T}}(\mathsf{U})\, \mathsf{U}^{-1}. \tag{7.10}$$

Further, since $\hat{\mathsf{T}}$ has symmetric values,

$$\hat{\mathsf{S}}: \mathrm{Sym}(\mathscr{T}) \to \mathrm{Sym}(\mathscr{T}); \tag{7.11}$$

and, if $\mathsf{Q} \in \mathrm{Orth}(\mathscr{T}, \mathscr{T}) \cap \mathscr{G}_\kappa$, then (7.5) implies

$$\hat{\mathsf{S}}(\mathsf{QUQ}^T) = \mathsf{Q}\hat{\mathsf{S}}(\mathsf{U})\, \mathsf{Q}^T. \tag{7.12}$$

Also, by (7.8) and (7.10),

$$\hat{\mathsf{S}}(1) = \mathsf{M}.$$

Assume now that $\hat{\mathsf{S}}$ is differentiable at $\mathbf{1}$ with Fréchet derivative C, so that

$$\hat{\mathsf{S}}(1+\mathsf{E}) = \mathsf{M} + \mathsf{C}[\mathsf{E}] + o(|\mathsf{E}|) \tag{7.13}$$

as $\mathsf{E} \to \mathbf{0}$ with $\mathsf{E} \in \mathrm{Sym}(\mathscr{T})$. We call C the **elasticity tensor.** Of course, M and C depend on the material point X, or equivalently, on the position $\mathbf{X}$ of X in $\mathcal{S}_0$. If $\mathsf{Q} \in \mathrm{Orth}(\mathscr{T}, \mathscr{T}) \cap \mathscr{G}_\kappa$, then (7.12) and (7.13) imply

$$\mathsf{QC}[\mathsf{E}]\,\mathsf{Q}^T = \mathsf{C}[\mathsf{QEQ}^T] + o(|\mathsf{E}|) \quad \text{as } \mathsf{E} \to \mathbf{0}.$$

Dividing by $|\mathsf{E}|$, proceeding to the limit as $\mathsf{E} \to \mathbf{0}$, and using the linearity of C, we have

$$\mathsf{C}[\mathsf{QEQ}^T] = \mathsf{QC}[\mathsf{E}]\,\mathsf{Q}^T \tag{7.14}$$

for every $\mathsf{Q} \in \mathrm{Orth}(\mathscr{T}, \mathscr{T}) \cap \mathscr{G}_\kappa$ and every $\mathsf{E} \in \mathrm{Sym}(\mathscr{T})$.

Using a standard theorem for linear, isotropic, symmetric tensor-valued functions of a symmetric tensor variable,[1] we find that when the material point X is isotropic relative to κ,

$$\mathsf{C}[\mathsf{E}] = \lambda_0(\mathrm{tr}\, \mathsf{E})1 + 2\mu_0 \mathsf{E}. \tag{7.15}$$

[1] *Cf.*, *e.g.*, GURTIN (1972), p. 76.

In this case we also have, from Proposition 7.3, that

$$\mathsf{M}=\sigma_0 \mathsf{1}. \tag{7.16}$$

Of course, λ_0, μ_0, and σ_0 depend on the material point X.

8. Linearized Theory

Consider now an elastic surface $\mathscr{S}$. The linear theory for $\mathscr{S}$ is based on the assumption that the deformation relative to a fixed reference configuration be small. Thus let $\boldsymbol{\kappa}$ be a fixed reference configuration, let $\boldsymbol{f}$ be a deformation of $\mathscr{A}_0=\boldsymbol{\kappa}(\mathscr{S})$, and assume that

$$\varepsilon=\sup_{\mathscr{A}_0}|\nabla \boldsymbol{u}| \tag{8.1}$$

is small. In view of (3.5) and (3.6), the fields U and E can be considered functions of $\nabla \boldsymbol{u}$. It therefore follows from (3.5) that

$$\mathsf{U}=\mathsf{1}+\mathsf{E}+O(\varepsilon^2) \tag{8.2}$$

as $\varepsilon \to 0$, where E is the infinitesimal strain given by (3.6) and $\mathsf{1}(X)$ is the identity on $\mathscr{T}_X=\mathscr{T}_X(\boldsymbol{\kappa})$.

If Σ is a regular domain in $\mathscr{A}_0$, then its change of area due to the deformation $\boldsymbol{f}$ is

$$\int_\Sigma (J-1), \tag{8.3}$$

where

$$J=\det \mathsf{U}$$

is the Jacobian associated with $\boldsymbol{f}$. Thus

$$J-1=\det \mathsf{U}-1=\det \mathsf{U}-\det \mathsf{1}=\operatorname{tr}(\mathsf{U}-\mathsf{1})+O(\varepsilon^2),$$

so that by (8.2),

$$J-1=\operatorname{tr} \mathsf{E}+O(\varepsilon^2)$$

as $\varepsilon \to 0$. Since $\operatorname{tr} \mathsf{E}=\operatorname{div}_{\mathscr{A}_0} \boldsymbol{u}$, and since the bound (8.1) is uniform,

$$\int_\Sigma J-1=\int_\Sigma \operatorname{div}_{\mathscr{A}_0} \boldsymbol{u}+O(\varepsilon^2)$$

as $\varepsilon \to 0$. For this reason we call

$$\delta a(\Sigma)=\int_\Sigma \operatorname{div}_{\mathscr{A}_0} \boldsymbol{u} \tag{8.4}$$

the **infinitesimal area change** under the displacement $\boldsymbol{u}$. Note that if Σ is geometrically closed, then (2.11) implies that

$$\delta a(\Sigma)=-2\int_\Sigma \bar{\kappa} u_n,$$

where u_n is the normal component of $\boldsymbol{u}$ on $\mathscr{A}_0$.

Consider next the Piola-Kirchhoff stress as given in (7.9). Then (3.4), (8.2), (7.13), and (8.1) imply that

$$\mathsf{S}=\mathsf{I}\mathsf{M}+\mathsf{I}\mathsf{C}[\mathsf{E}]+\nabla \boldsymbol{u}\, \mathsf{M}+o(\varepsilon) \tag{8.5}$$

as $\varepsilon \to 0$, where $\mathsf{I}(X): \mathscr{T}_X \to \mathscr{V}$ is the inclusion map. The basic equations of the linear theory are the strain-displacement relation (3.6), the constitutive equation (8.5) with terms of order $o(\varepsilon)$ neglected, and the balance law (5.25):

$$\begin{aligned} \mathsf{E} &= \tfrac{1}{2}(D\boldsymbol{u} + D\boldsymbol{u}^T), \\ \mathsf{S} &= \mathsf{I}\mathsf{M} + \mathsf{I}\mathsf{C}[\mathsf{E}] + \nabla\boldsymbol{u}\,\mathsf{M}, \\ \operatorname{div}_{\mathscr{S}_0} \mathsf{S} &+ \boldsymbol{b}_0 = \mathbf{0}. \end{aligned} \tag{8.6}$$

We remark that the linearized version $(8.6)_2$ of the Piola-Kirchhoff surface stress does not take symmetric values because of the presence of the term $\nabla\boldsymbol{u}\,\mathsf{M}$.

When the material point concerned is isotropic (in the reference configuration) we have, from (7.15), (7.16), and $(8.6)_2$,

$$\mathsf{S} = \sigma_0 \mathsf{I} + \lambda_0 (\operatorname{tr} \mathsf{E})\, \mathsf{I} + 2\mu_0 \mathsf{I}\mathsf{E} + \sigma_0 \nabla\boldsymbol{u}. \tag{8.7}$$

Remark. It should be noted that the stress S in $(8.6)_2$ is not invariant under an infinitesimal rigid displacement. Indeed, an infinitesimal rigid displacement of the surface is a displacement field of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \boldsymbol{W}(\boldsymbol{x} - \boldsymbol{x}_0) \tag{8.8}$$

with $\boldsymbol{W} \in \operatorname{Lin}(\mathscr{V}, \mathscr{V})$ skew. For such a field $\nabla_{\mathscr{S}_0} \boldsymbol{u} = \boldsymbol{W}\mathsf{I}$ and (2.4), (3.6) imply that $\mathsf{E} = \mathbf{0}$. Thus, by $(8.6)_2$,

$$\mathsf{S} = \mathsf{I}\mathsf{M} + \boldsymbol{W}_0 \mathsf{I}\mathsf{M}.$$

This is as it should be: it is simply the transformation law $\mathsf{I}\mathsf{M} \to \mathsf{I}\mathsf{M} + \boldsymbol{W}_0 \mathsf{I}\mathsf{M}$ for the residual stress in an infinitesimal rigid rotation of the entire surface.

Since $(8.6)_2$ can also be written in the form

$$\mathsf{S} = \mathsf{I}\mathsf{M} + O(\varepsilon),$$

we might also wish to consider situations in which S obeys a constitutive relation of the form

$$\mathsf{S} = \mathsf{I}\mathsf{M}.$$

With this in mind we now trivially generalize $(8.6)_2$ as follows:

$$\mathsf{S} = \mathsf{I}\mathsf{M} + \mathsf{I}\mathsf{C}[\mathsf{E}] + \nabla\boldsymbol{u}\,\overline{\mathsf{M}}, \tag{8.9}$$

so that we have the possibility of setting $\overline{\mathsf{M}}$ and C equal to zero. Of course (8.9) also allows for situations in which we wish to neglect the term $\nabla\boldsymbol{u}\,\overline{\mathsf{M}}$ but retain the term $\mathsf{C}[\mathsf{E}]$, and vice versa. We observe that if $\overline{\mathsf{M}}$ is present, then it is equal to M. Thus, we are adopting a convenient notation for dealing simultaneously with three cases of possible interest. Similarly, we rewrite (8.7) as follows:

$$\mathsf{S} = \sigma_0 \mathsf{I} + \lambda_0 (\operatorname{tr} \mathsf{E})\, \mathsf{I} + 2\mu_0 \mathsf{I}\mathsf{E} + \bar{\sigma}_0 \nabla\boldsymbol{u}. \tag{8.10}$$

9. Linearized Theory of a Body with a Free Surface

Consider a three-dimensional elastic body $\mathscr{B}$ with a material boundary $\partial\mathscr{B}$, and assume that, in a fixed reference configuration, $\mathscr{B}$ occupies the region B.

Then, in the absence of body forces, the system of *equilibrium* equations appropriate to small deformations from B consists of

$$\left.\begin{aligned} &\boldsymbol{E}=\tfrac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T),\\ &\boldsymbol{S}=\boldsymbol{C}[\boldsymbol{E}],\\ &\operatorname{div}\boldsymbol{S}=\mathbf{0}, \end{aligned}\right\} \quad \text{on } B \tag{9.1}$$

where $\boldsymbol{u}: B\to\mathscr{V}$ is the displacement, $\boldsymbol{E}: B\to\mathrm{Sym}(\mathscr{V})$ the strain, $\boldsymbol{S}: B\to\mathrm{Sym}(\mathscr{V})$ the stress, and $\boldsymbol{C}: B\to\mathrm{Lin}(\mathrm{Sym}(\mathscr{V}),\mathrm{Sym}(\mathscr{V}))$ the elasticity tensor. We assume that $\partial\mathscr{B}$ is a material surface $\mathscr{S}$ which we here take to be elastic. We further assume that $\mathscr{S}$ is free from external loads. Then, for infinitesimal deformations of $\mathscr{S}$ from the reference configuration, we must adjoin to (9.1) the system (8.6) for the surface $\mathscr{S}=\partial\mathscr{B}$. Under equilibrium conditions this consists of $(8.6)_1$, (8.9), and the appropriate modification to $(8.6)_3$ as in (6.10):

$$\left.\begin{aligned} &\mathsf{E}=\tfrac{1}{2}(D\boldsymbol{u}+D\boldsymbol{u}^T),\\ &\mathsf{S}=\mathsf{IM}+\mathsf{IC}[\mathsf{E}]+\nabla_{\!s}\boldsymbol{u}\overline{\mathsf{M}},\\ &\operatorname{div}_{s}\mathsf{S}=\boldsymbol{S}\boldsymbol{n} \end{aligned}\right\} \quad \text{on } \partial B. \tag{9.2}$$

Here $\boldsymbol{n}$ is the outward unit normal to $\mathscr{s}=\partial B$. Of course, in (9.1) ∇ and div are the gradient and divergence operators in $\mathscr{E}$, while in (9.2) $D=\boldsymbol{P}\nabla_{\!s}$ and div_s involve the gradient $\nabla_{\!s}$ and divergence div_s with respect to $\mathscr{s}$. Further $\boldsymbol{u}$ is the restriction of the displacement $\boldsymbol{u}$ to $\mathscr{s}$.

We now state the **free-surface problem** for an elastic body.

Given: At each $X\in B$ the (body) elasticity tensor $\boldsymbol{C}(X)\in\mathrm{Sym}(\mathrm{Sym}(\mathscr{V}))$; at each $X\in\partial B$ the (surface) elasticity tensor $\mathsf{C}(X)\in\mathrm{Sym}(\mathrm{Sym}(\mathscr{T}_X))$, the residual stress $\mathsf{M}(X)\in\mathrm{Sym}(\mathscr{T}_X)$, and the tensor $\overline{\mathsf{M}}(X)\in\mathrm{Sym}(\mathscr{T}_X)$. We assume that $\boldsymbol{C}(X)$ is positive-definite, that $\mathsf{C}(X)$ and $\overline{\mathsf{M}}(X)$ are positive semi-definite,[1] and that $\boldsymbol{C}$, C, M and $\overline{\mathsf{M}}$ are smooth fields.

Find: A class C^2 displacement field $\boldsymbol{u}: B\to\mathscr{V}$ such that (9.1) and (9.2) are satisfied.

The quantities

$$\begin{aligned} &U_B\{\boldsymbol{u}\}=\tfrac{1}{2}\int_B \boldsymbol{C}[\boldsymbol{E}]\cdot\boldsymbol{E},\\ &U_{\partial B}\{\boldsymbol{u}\}=\tfrac{1}{2}\int_{\partial B}[\mathsf{C}[\mathsf{E}]\cdot\mathsf{E}+(\nabla_{\!s}\boldsymbol{u}\overline{\mathsf{M}})\cdot\nabla_{\!s}\boldsymbol{u}] \end{aligned} \tag{9.3}$$

represent, respectively, the **strain energy** of B and of ∂B in the displacement $\boldsymbol{u}$. In view of the properties of $\boldsymbol{C}$, C, and $\overline{\mathsf{M}}$,

$$U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}=0 \quad\Rightarrow\quad \boldsymbol{E}=\mathbf{0} \text{ on } B. \tag{9.4}$$

Theorem 9.1. *Let $\boldsymbol{u}$ be a solution of the free-surface problem. Then*

$$-\tfrac{1}{2}\int_{\partial B}\mathsf{M}\cdot\mathsf{E}=U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}.$$

---

[1] Notice that we do not assume *a priori* that M is positive semi-definite. Thus we include (at least for situations in which the term $\nabla_{\!s}\boldsymbol{u}\overline{\mathsf{M}}$ can be neglected) the possibility of a compressive residual stress (*cf.* the discussion by HERRING (1953), pp. 15–16).

Proof. By the divergence theorem,

$$\int_{\partial B} \boldsymbol{u}\cdot\boldsymbol{S}\boldsymbol{n}=\int_B \operatorname{div}(\boldsymbol{S}^T\boldsymbol{u})=\int_B [(\operatorname{div}\boldsymbol{S})\cdot\boldsymbol{u}+\boldsymbol{S}\cdot\nabla\boldsymbol{u}].$$

By use of $(9.1)_3$, $(9.1)_2$, and the symmetry of $\boldsymbol{S}$, this becomes

$$\int_{\partial B} \boldsymbol{u}\cdot\boldsymbol{S}\boldsymbol{n}=\int_B \boldsymbol{S}\cdot\boldsymbol{E}=\int_B \boldsymbol{C}[\boldsymbol{E}]\cdot\boldsymbol{E}=2\,U_B\{\boldsymbol{u}\}. \tag{9.5}$$

On the other hand, since $\partial B=\mathscr{S}$ is a regular geometrically closed surface, $\partial\mathscr{S}=\emptyset$ and $(9.2)_3$ and $(2.17)_5$ yield

$$\int_{\partial B} \boldsymbol{u}\cdot\boldsymbol{S}\boldsymbol{n}=\int_{\partial B} \boldsymbol{u}\cdot\operatorname{div}_{\mathscr{S}}\mathsf{S}=-\int_{\partial B}\mathsf{S}\cdot\nabla_{\mathscr{S}}\boldsymbol{u}. \tag{9.6}$$

Equations $(9.2)_2$, (9.5), and (9.6), together with the symmetry of M and $\mathsf{C}[\mathsf{E}]$, imply that

$$2\,U_B\{\boldsymbol{u}\}=-\int_B [\mathsf{M}\cdot\mathsf{E}+\mathsf{C}[\mathsf{E}]\cdot\mathsf{E}+(\nabla_{\mathscr{S}}\boldsymbol{u}\,\bar{\mathsf{M}})\cdot\nabla_{\mathscr{S}}\boldsymbol{u}],$$

and this relation with $(9.3)_2$ yields the desired result. □

Corollary 9.2. *Any two solutions of the free-surface problem differ at most by an infinitesimal rigid displacement of the entire body.*

Proof. Suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the free-surface problem. Then $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$ gives rise to fields $\boldsymbol{E}, \mathsf{E}, \boldsymbol{S}$, and S, satisfying equations (9.1) and (9.2) with $\mathsf{M}=\mathbf{0}$. Thus we may conclude from Theorem 9.1 that

$$U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}=0,$$

and (9.4) yields $\boldsymbol{E}=\mathbf{0}$ on B; hence $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ can differ at most by an infinitesimal rigid displacement of the entire body.[1] □

Corollary 9.3. *Assume that* M *is a constant surface tension* $\sigma_0\neq 0$. *Then the area change* $\delta a(\partial B)$ *corresponding to a solution* $\boldsymbol{u}$, *if one exists, of the free-surface problem is given by*

$$\delta a(\partial B)=-\frac{2}{\sigma_0}[U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}].$$

Proof. This is an immediate consequence of (8.4) and Theorem 9.1 upon noting that

$$\mathsf{M}\cdot\mathsf{E}=\sigma_0\operatorname{div}_{\mathscr{S}}\boldsymbol{u}. \quad \square \tag{9.7}$$

Note that, by Corollary 9.3 and since $U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}\geqq 0$, if $\delta a(\partial B)\neq 0$, then the surface area decreases or increases according as σ_0 is positive or negative.

As is true in classical elasticity theory, the foregoing problem can also be characterized by a minimum principle. Let $\mathscr{A}$ denote the class of all C^2 vector fields on B. We call

$$\Phi\{\boldsymbol{u}\}=U_B\{\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}+\int_{\partial B}\mathsf{M}\cdot\mathsf{E} \tag{9.8}$$

[1] *Cf., e.g.*, GURTIN (1972), Theorem 13.2.

the **potential energy** of $\boldsymbol{u}\in\mathscr{A}$. Φ is the total strain energy plus the work done by the residual stress. By Theorem 9.1, when $\boldsymbol{u}$ is a solution $\Phi\{\boldsymbol{u}\}$ equals one-half of the work done by the residual stress.

Theorem 9.4. *If* $\boldsymbol{u}$ *is a solution of the free-surface problem, then*

$$\Phi\{\boldsymbol{u}\}\leqq\Phi\{\bar{\boldsymbol{u}}\}$$

for every $\bar{\boldsymbol{u}}\in\mathscr{A}$.

Proof. It is a simple matter to show that

$$\begin{aligned} U_B\{\bar{\boldsymbol{u}}\}&=U_B\{(\bar{\boldsymbol{u}}-\boldsymbol{u})+\boldsymbol{u}\}=U_B\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}+U_B\{\boldsymbol{u}\}+\int_B \boldsymbol{C}[\boldsymbol{E}]\cdot(\bar{\boldsymbol{E}}-\boldsymbol{E}),\\ U_{\partial B}\{\bar{\boldsymbol{u}}\}&=U_{\partial B}\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}+U_{\partial B}\{\boldsymbol{u}\}+\int_{\partial B}\left[\mathsf{C}[\mathsf{E}]\cdot(\bar{\mathsf{E}}-\mathsf{E})+(\nabla_{\!s}\boldsymbol{u}\,\bar{\mathsf{M}})\cdot(\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u}))\right]. \end{aligned} \tag{9.9}$$

Thus, from the definition of Φ and equations (9.9), we have

$$\begin{aligned} &\Phi\{\bar{\boldsymbol{u}}\}-\Phi\{\boldsymbol{u}\}-U_B\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}-U_{\partial B}\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}\\ &\quad=\int_B \boldsymbol{C}[\boldsymbol{E}]\cdot(\bar{\boldsymbol{E}}-\boldsymbol{E})+\int_{\partial B}\left[\mathsf{C}[\mathsf{E}]\cdot(\bar{\mathsf{E}}-\mathsf{E})+(\nabla_{\!s}\boldsymbol{u}\,\bar{\mathsf{M}})\cdot\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u})+\mathsf{M}\cdot(\bar{\mathsf{E}}-\mathsf{E})\right]. \end{aligned} \tag{9.10}$$

Now by $(9.1)_2$, the symmetry of $\boldsymbol{S}$, $(9.1)_3$, and the divergence theorem,

$$\begin{aligned} \int_B \boldsymbol{C}[\boldsymbol{E}]\cdot(\bar{\boldsymbol{E}}-\boldsymbol{E})&=\int_B \boldsymbol{S}\cdot(\bar{\boldsymbol{E}}-\boldsymbol{E})=\int_B \boldsymbol{S}\cdot\nabla(\bar{\boldsymbol{u}}-\boldsymbol{u})\\ &=\int_B\left[\operatorname{div}(\boldsymbol{S}^T(\bar{\boldsymbol{u}}-\boldsymbol{u}))-(\operatorname{div}\boldsymbol{S})\cdot(\bar{\boldsymbol{u}}-\boldsymbol{u})\right]\\ &=\int_{\partial B}\boldsymbol{S}\boldsymbol{n}\cdot(\bar{\boldsymbol{u}}-\boldsymbol{u}). \end{aligned} \tag{9.11}$$

By use of $(9.2)_3$ and $(2.17)_5$ with $\boldsymbol{u}$ replaced by $(\bar{\boldsymbol{u}}-\boldsymbol{u})$ and $\Sigma=\partial B$ (observing that since ∂B is geometrically closed, $\partial\Sigma=\emptyset$) (9.11) implies that

$$\int_B \boldsymbol{C}[\boldsymbol{E}]\cdot(\bar{\boldsymbol{E}}-\boldsymbol{E})=\int_{\partial B}(\operatorname{div}_s\mathsf{S})\cdot(\bar{\boldsymbol{u}}-\boldsymbol{u})=-\int_{\partial B}\mathsf{S}\cdot\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u}). \tag{9.12}$$

However, by $(9.2)_2$,

$$\begin{aligned} \mathsf{S}\cdot\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u})&=(\mathsf{I}\mathsf{M}+\mathsf{I}\mathsf{C}[\mathsf{E}]+\nabla_{\!s}\boldsymbol{u}\,\bar{\mathsf{M}})\cdot(\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u}))\\ &=\mathsf{M}\cdot(\bar{\mathsf{E}}-\mathsf{E})+\mathsf{C}[\mathsf{E}]\cdot(\bar{\mathsf{E}}-\mathsf{E})+\nabla_{\!s}\boldsymbol{u}\bar{\mathsf{M}}\cdot\nabla_{\!s}(\bar{\boldsymbol{u}}-\boldsymbol{u}), \end{aligned}$$

on using the symmetry of M and of $\mathsf{C}[\mathsf{E}]$. This together with (9.10) and (9.12) yields

$$\Phi\{\bar{\boldsymbol{u}}\}-\Phi\{\boldsymbol{u}\}=U_B\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}+U_{\partial B}\{\bar{\boldsymbol{u}}-\boldsymbol{u}\}\geqq 0. \quad \square$$

Upon noting that S is not necessarily symmetric, and using the same steps as were used in the proof of (6.4), we see that $(9.1)_3$ and $(9.2)_3$ yield the identity

$$\int_B \boldsymbol{S}+\int_{\partial B}\mathsf{I}\mathsf{S}^T=\boldsymbol{0}. \tag{9.13}$$

Since S is symmetric, this implies that

$$\int_{\partial B} (\mathsf{I}\mathsf{S}^T - \mathsf{S}P) = \mathbf{0},$$

and this result, together with the symmetry of M, $\overline{\mathsf{M}}$, and C, yields the identity

$$\int_{\partial B} (\mathsf{I}\overline{\mathsf{M}}\, \nabla_{\!\scriptscriptstyle\mathcal{S}} u^T - \nabla_{\!\scriptscriptstyle\mathcal{S}} u\, \overline{\mathsf{M}} P) = \mathbf{0}. \tag{9.14}$$

Use of (2.21) of Lemma 2.1 with $\mathsf{S} = \mathsf{I}\overline{\mathsf{M}}$ and $\Sigma = \partial B$ (so that $\partial\Sigma = \emptyset$) yields

$$\int_{\partial B} \mathsf{I}\overline{\mathsf{M}}\, \nabla_{\!\scriptscriptstyle\mathcal{S}} u^T = -\int_{\partial B} \operatorname{div}_{\scriptscriptstyle\mathcal{S}}(\mathsf{I}\overline{\mathsf{M}}) \otimes u;$$

hence (9.14) becomes, by use of (2.15),

$$\int_{\partial B} u \wedge \operatorname{div}_{\scriptscriptstyle\mathcal{S}} \overline{\mathsf{M}} = \mathbf{0}. \tag{9.15}$$

The relation (9.15) must be satisfied by the solution u of the free-surface problem. When $\overline{\mathsf{M}} = \bar{\sigma}_0 \mathbf{1}$ with $\bar{\sigma}_0 \neq 0$ and constant, (9.15) reduces to the requirement that

$$\int_{\partial B} \bar{\kappa}\, u \wedge n = \mathbf{0}.$$

When $\mathscr{B}$ is homogeneous and isotropic (relative to the reference configuration, $(9.1)_2$ reduces to[1]

$$S = \lambda(\operatorname{tr} E)\,\mathbf{1} + 2\mu E, \tag{9.16}$$

$(9.2)_2$ takes the form (8.10), and λ, μ, σ_0, λ_0, μ_0, and $\bar{\sigma}_0$ are constants. Thus, if we take the trace of (9.13) and use (8.10) and (9.16), we find that the infinitesimal change of area $\delta a(\partial B)$ and the infinitesimal change[2] of volume

$$\delta v(B) = \int_B \operatorname{div} u$$

are related by

$$2\sigma_0 \operatorname{Area}(\partial B) + (3\lambda + 2\mu)\,\delta v(B) + (\bar{\sigma}_0 + 2\lambda_0 + 2\mu_0)\,\delta a(\partial B) = 0.$$

We now discuss briefly the more general *mixed boundary-value problem* for an elastic body with a material surface. For convenience, we omit all proofs; these are completely analogous to their counterparts in the free-surface problem. Using the notation of this section, we suppose that $\mathcal{S} = \partial B$ is a regular geometrically closed surface with

$$\partial B = \Sigma \cup \hat{\Sigma},$$

where Σ and $\hat{\Sigma}$ are regular, complementary, subsurfaces of ∂B with

$$\Gamma = \Sigma \cap \hat{\Sigma},$$

a closed, piecewise smooth curve. Then the problem may be stated as follows:

Given: Everything as in the free-surface problem, except that now C, M, and $\overline{\mathsf{M}}$ are defined on Σ rather than ∂B, and, in addition, continuous vector fields b, $\hat{s}$, and $\hat{u}$ defined, respectively, in B, on Σ, and on $\hat{\Sigma}$.

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[1] *Cf., e.g.,* GURTIN (1972), Theorem 22.2.

[2] *Cf., e.g.,* GURTIN (1972), § 12.

Find: A class C^2 displacement field $\boldsymbol{u}: B \to \mathscr{V}$ such that

$$\left.\begin{aligned} &\boldsymbol{E} = \tfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T), \\ &\boldsymbol{S} = \boldsymbol{C}[\boldsymbol{E}], \\ &\operatorname{div} \boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}, \end{aligned}\right\} \quad \text{on } B \tag{9.17}$$

$$\left.\begin{aligned} &\mathsf{E} = \tfrac{1}{2}(D\boldsymbol{u} + D\boldsymbol{u}^T), \\ &\mathsf{S} = \mathsf{IM} + \mathsf{IC}[\mathsf{E}] + \nabla_{\!s} \boldsymbol{u} \overline{\mathsf{M}}, \\ &\operatorname{div}_s \mathsf{S} + \hat{\boldsymbol{s}} = \boldsymbol{S}\boldsymbol{n}, \end{aligned}\right\} \quad \text{on } \Sigma \tag{9.18}$$

and

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \hat{\Sigma}. \tag{9.19}$$

The quantities $U_B\{\boldsymbol{u}\}$ and $U_\Sigma\{\boldsymbol{u}\}$ are defined as in (9.3) with ∂B replaced by Σ. We then have the following analog of Theorem 9.1:

Theorem 9.5. *Let $\boldsymbol{u}$ be a solution of the mixed boundary-value problem. Then*

$$-\tfrac{1}{2}\int_{\partial B} \mathsf{M} \cdot \mathsf{E} = U_B\{\boldsymbol{u}\} + U_\Sigma\{\boldsymbol{u}\} - \tfrac{1}{2}\int_B \boldsymbol{b} \cdot \boldsymbol{u} - \tfrac{1}{2}\int_\Sigma \hat{\boldsymbol{s}} \cdot \boldsymbol{u} - \tfrac{1}{2}\int_{\hat{\Sigma}} \hat{\boldsymbol{u}} \cdot \boldsymbol{S}\boldsymbol{n} - \tfrac{1}{2}\int_\Gamma \mathsf{S}\boldsymbol{v} \cdot \hat{\boldsymbol{u}},$$

where $\boldsymbol{v}$ is the outward unit normal to Σ.

This can be used as before to obtain the appropriate uniqueness result:

Corollary 9.6. *Any two solutions of the mixed boundary-value problem differ at most by an infinitesimal rigid displacement of the entire body.*

We remark that if $\hat{\Sigma} \neq \emptyset$ then there is at most one solution; infinitesimal rigid rotations of the entire body are excluded by the prescription of the displacement field on $\hat{\Sigma}$.

The potential energy is now defined by

$$\Phi\{\bar{\boldsymbol{u}}\} = U_B\{\bar{\boldsymbol{u}}\} + U_\Sigma\{\bar{\boldsymbol{u}}\} - \int_B \boldsymbol{b} \cdot \bar{\boldsymbol{u}} + \int_\Sigma (\mathsf{M} \cdot \bar{\mathsf{E}} - \hat{\boldsymbol{s}} \cdot \bar{\boldsymbol{u}}).$$

The counterpart of Theorem 9.4 is:

Theorem 9.7. *If $\boldsymbol{u}$ is a solution of the mixed boundary-value problem, and if $\bar{\boldsymbol{u}}$ is a class C^2 vector field on B satisfying $\bar{\boldsymbol{u}} = \hat{\boldsymbol{u}}$ on $\hat{\Sigma}$, then*

$$\Phi\{\boldsymbol{u}\} \leqq \Phi\{\bar{\boldsymbol{u}}\}.$$

10. Simple Applications of the Linear Theory

We now use the linear theory developed in Section 8 for a three-dimensional body with material boundary and examine, in some simple cases, the predicted departure from classical results. This departure is accounted for by the presence in (6.1) of $(\operatorname{div}_s \mathsf{T} + \boldsymbol{b}^*)$, which is absent in classical treatments.

10.1. *Infinite cylindrical surface bounding a homogeneous isotropic body.* Consider the removal of an infinite circular cylinder of radius a from a homogeneous isotropic body $\mathscr{B}$ occupying in its natural configuration all of $\mathscr{E}$. The surface of the cylindrical hole so exposed is a material surface. We assume that this, too, is homogeneous and isotropic. Thus, the corresponding free-surface problem consists in equations (9.1) and (9.2) with $(9.1)_2$ given by (9.16) and with $(9.2)_2$ taking the form (8.10). This problem admits of the following solution: the radius of the cylindrical hole decreases by

$$\frac{a\sigma_0}{\gamma}, \quad \gamma = 2\mu a + \bar{\sigma}_0 + \lambda_0 + 2\mu_0, \tag{10.1}$$

while the corresponding stresses are

$$S_{rr} = -S_{r\theta} = \frac{2\mu a^2 \sigma_0}{\gamma r^2},$$
$$S_{\theta\theta} = 0.$$

Here, of course, (r, θ) denotes the usual polar coordinates.

The associated problem of removing from the same body the exterior to an infinite cylinder of radius a results in a solution in which the radius again decreases, this time by

$$\frac{a\sigma_0}{2\lambda a + \gamma}, \tag{10.2}$$

while the stress is a uniform pressure of magnitude

$$\frac{2(\lambda+\mu)\sigma_0}{2\lambda a + \gamma}.$$

It is of interest to note that (for $\lambda, \mu, \sigma_0, \bar{\sigma}_0, \lambda_0, \mu_0 > 0$) the cylinder decreases in radius by an amount less than the decrease in radius of the cylindrical hole. Further, as $(\bar{\sigma}_0 + \lambda_0 + 2\mu_0)/a \to 0$, (10.1) and (10.2) tend, respectively, to

$$\frac{\sigma_0}{2\mu} \quad \text{and} \quad \frac{\sigma_0}{2(\lambda+\mu)}.$$

10.2. *Plane waves in a half-space.*[1] We indicate the departure from the classical result in the simplest case, that of solenoidal waves in which the displacements are parallel to the plane surface of the body. Classically the field equations to be solved consist in

$$\operatorname{div} \boldsymbol{S} = \rho \ddot{\boldsymbol{u}} \quad \text{in } z > 0$$

in conjunction with the strain-displacement relation $(9.1)_1$, and the stress-strain relation (9.16). The relevant boundary condition is

$$\boldsymbol{S}\boldsymbol{e}_3 = \boldsymbol{0} \quad \text{on } z = 0,$$

where the body occupies the region $z > 0$ and $\boldsymbol{e}_3$ is perpendicular to $z = 0$. This problem admits the solution $\boldsymbol{u} = u(x, y, z, t)\,\boldsymbol{e}_2$,

$$u = \cos\{k(x \sin\alpha - z\cos\alpha - ct)\} + \cos\{k(x\sin\alpha + z\cos\alpha - ct)\},$$

where $\boldsymbol{e}_2$ is parallel to the y axis and $c^2 = \mu/\rho$. This solution consists in an incident wave together with a reflected wave, the angle α of incidence clearly being equal

[1] For a fuller discussion of the classical theory, *cf.*, *e.g.*, NADEAU (1964), p. 241.

to the angle of reflection. Further, there is no change of phase or wave length between the two component waves.

Treating $z=0$ as a (homogeneous and isotropic) material surface introduces the new boundary condition

$$S e_3 + \mathrm{div}_{\mathscr{s}} \mathsf{S} = \mathbf{0} \quad \text{on } z=0,$$

where S is given by (8.10) with E defined as in (3.6). Here $\mathscr{s}$ is the plane surface $z=0$. The corresponding solution is now

$$u = \cos\{k(x \sin\alpha - z\cos\alpha - ct)\} + \cos\{k(x\sin\alpha + z\cos\alpha - ct) - \delta\},$$

where

$$\delta = 2\tan^{-1}(l \sin\alpha \tan\alpha),$$

with

$$l = \frac{k(\bar{\sigma}_0 + \mu_0)}{\mu}.$$

Again the angles of incidence and reflection are equal and there is no change in wavelength. The departure from the classical result is contained in the phase change δ, which now depends upon wavelength. Thus a superposition of such waves would imply a distortion of the composite incident wave upon reflection. Energy flux considerations show that the material surface is periodically storing and releasing energy. Figure 4 gives the phase change δ as a function of the angle of incidence α for various values of the dimensionless wavelength l.

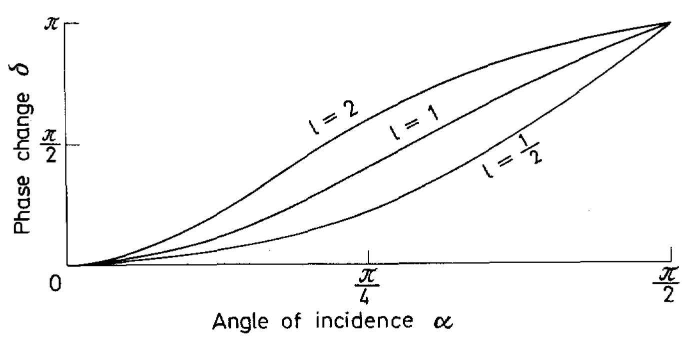

Fig. 4. Phase change, δ, as a function of angle of incidence, α, for various values of dimensionless wave length $l = k(\bar{\sigma}_0 + \mu_0)/\mu$.

Appendix

Material Frame-indifference for Elastic Surfaces

We wish here to justify our assertion (7.1) and hence use the notation of Section 7. Let $\mathscr{S}$ be an elastic material surface, and let $\boldsymbol{\kappa}$ be a configuration of $\mathscr{S}$. Let $X \in \mathscr{S}$ and set $\mathscr{T} = \mathscr{T}_X(\boldsymbol{\kappa})$, $\boldsymbol{\kappa}(X) = \boldsymbol{X}$. If $\boldsymbol{f}$ denotes a deformation from $\boldsymbol{\kappa}$, let $\nabla \boldsymbol{f}(\boldsymbol{X}) = \bar{\mathsf{F}}$, where the range of F is $\mathscr{U} \in \mathrm{Bid}(\mathscr{V})$, so that $\mathsf{F} \in \mathrm{Invlin}(\mathscr{T}, \mathscr{U})$. (Recall that F is $\bar{\mathsf{F}}$ with codomain restricted to the range.) Set $\boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{x}$. The surface stress T is $\hat{\mathsf{T}}_{\boldsymbol{\kappa}}(\mathsf{F}, X) \in \mathrm{Sym}(\mathscr{U})$. We write this as $\mathsf{T} \equiv \hat{\mathsf{T}}(\mathsf{F})$.

Consider a change of frame[1]

$$x^* = c + Q(x - x_0),$$

where $x_0, c \in \mathscr{E}$ and $Q \in \mathrm{Orth}(\mathscr{V}, \mathscr{V})$ are arbitrary. This induces a deformation f^* from κ defined by

$$f^*(X) = c + Q(f(X) - x_0).$$

The deformation gradient of f^* at X is F^*, where

$$\mathsf{F}^* = \mathsf{Q}^- \mathsf{F},$$

where $\mathsf{Q}^- \in \mathrm{Orth}(\mathscr{U}, \mathscr{W})$ is $Q|_{\mathscr{U}}$ with codomain equal to the range $\mathscr{W} \in \mathrm{Bid}(\mathscr{V})$. Thus the corresponding stress is

$$\mathsf{T}^* = \hat{\mathsf{T}}(\mathsf{F}^*) = \hat{\mathsf{T}}(\mathsf{Q}^- \mathsf{F}), \tag{A.1}$$

where $\mathsf{T}^* \in \mathrm{Sym}(\mathscr{W})$. Assuming that the surface stress is frame-indifferent yields

$$\mathsf{T}^* Q s = Q \mathsf{T} s \tag{A.2}$$

for every $s \in \mathscr{U}$. Equations (A.1) and (A.2) imply that

$$\hat{\mathsf{T}}(\mathsf{Q}^- \mathsf{F})\, \mathsf{Q}^- = \mathsf{Q}^- \hat{\mathsf{T}}(\mathsf{F}),$$

so that

$$\mathsf{Q}^- \hat{\mathsf{T}}(\mathsf{F})(\mathsf{Q}^-)^T = \hat{\mathsf{T}}(\mathsf{Q}^- \mathsf{F}).$$

The result (7.1) follows on observing that given $\mathscr{W} \in \mathrm{Bid}(\mathscr{V})$ and $\mathsf{Q}^- \in \mathrm{Orth}(\mathscr{U}, \mathscr{W})$, we can construct $Q \in \mathrm{Orth}(\mathscr{V}, \mathscr{V})$ such that $Q|_{\mathscr{U}}$ (with codomain restricted to the range) is Q^-.

Material Symmetry for Elastic Surfaces

Let κ and μ be two configurations of an elastic surface $\mathscr{S}$, and let f be a deformation from $\mu(\mathscr{S}) = \mathscr{s}$. Then f induces a deformation $f \circ \lambda$ from $\kappa(\mathscr{S}) = \mathscr{s}_0$, where $\lambda = \mu \circ \kappa^{-1}$. If T denotes the surface stress in the deformed configuration at a given surface material point, then, by the definition of an elastic surface, we have

$$\hat{\mathsf{T}}_\kappa(\nabla_{\mathscr{s}_0}(f \circ \lambda)) = \mathsf{T} = \hat{\mathsf{T}}_\mu(\nabla_{\mathscr{s}} f),$$

where we have suppressed the material point dependence. Writing $\nabla_{\mathscr{s}} f = \bar{\mathsf{F}}$ and F for $\bar{\mathsf{F}}$ when the codomain is restricted to the range, we have

$$\hat{\mathsf{T}}_\mu(\mathsf{F}) = \hat{\mathsf{T}}_\kappa(\mathsf{F}\mathsf{H}), \tag{A.3}$$

where $\mathsf{H} = \nabla_{\mathscr{s}_0} \lambda$. Thus to determine the surface stress T in any configuration it suffices to know the response function $\hat{\mathsf{T}}_\kappa$ relative to just one configuration κ. We define the symmetry group $\mathscr{G}_\kappa$ relative to κ as essentially those configurations in which the response is the same as that in κ. In view of (A.3), it suffices to consider only those configurations that share the same tangent space at the material point under consideration. Thus we define the symmetry group of the material point X in the configuration κ by

$$\mathscr{G}_\kappa(X) = \{\mathsf{H} \in \mathrm{Unim}(\mathscr{T}_X(\kappa)) : \hat{\mathsf{T}}_\kappa(\mathsf{F}, X) = \hat{\mathsf{T}}_\kappa(\mathsf{F}\mathsf{H}, X)$$
$$\text{for every } \mathsf{F} \in \mathrm{Invlin}(\mathscr{T}_X(\kappa), \mathscr{U}) \text{ with } \mathscr{U} \in \mathrm{Bid}(\mathscr{V})\},$$

[1] *Cf., e.g.*, TRUESDELL & NOLL (1965), p. 42.

where, for the usual reasons, we have restricted our attention to unimodular tensors. Clearly $\mathscr{G}_\kappa(X)$ is a subgroup of $\text{Unim}(\mathscr{T}_X(\kappa))$.

From (A.3) we can deduce that, if κ and μ are arbitrary configurations of $\mathscr{S}$, then

$$\mathscr{G}_\mu(X) = \mathsf{H}\,\mathscr{G}_\kappa(X)\,\mathsf{H}^{-1},$$

where $\mathsf{H} = \nabla_{\sigma_0}(\mu \circ \kappa^{-1})(\kappa(X))$, with $\sigma_0 = \kappa(\mathscr{S})$.

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