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THE CONSISTENCY OF THE AXIOM OF COMPREHENSION IN THE INFINITE-VALUED PREDICATE LOGIC OF ŁUKASIEWICZ

It is natural to suggest that the paradoxes of naive set theory can be avoided not by asserting only special cases of the axiom of comprehension $(Ex)(y)(y \in x \leftrightarrow A(y))$, as in set theories like ZF formulated in classical predicate logic, but rather by retaining the unrestricted comprehension axiom while weakening the underlying logic. Various experiments have been made with such "type-free" logics by Ackermann [1], Fitch [3], Schütte [6], and others. One of the most interesting proposals is Skolem's. After showing that versions of Russell's paradox can be produced from the unrestricted comprehension axiom in any finite-valued Lukasiewicz predicate logic, Skolem conjectured that the axiom can however be consistently added to the infinite-valued Lukasiewicz predicate logic [7]. He also suggested that it may be possible to derive a significant amount of mathematics in a set theory based on this logic.

Skolem's conjecture about the consistency of the axiom of comprehension in the infinite-valued logic has been partially confirmed by Skolem himself and by Chang and others [2]. In this paper I shall prove Skolem's conjecture by using some simple modifications of methods from classical proof theory and model theory: normalization of proofs in a natural-deduction calculus and the use of maximally consistent sets of formulas. It remains to investigate the mathematical strength of the system, and I conclude the paper with some remarks on this subject.

In outline the consistency proof proceeds as follows. Louise Hay [4] has provided a complete axiomatization of the infinite-valued predicate logic, although her formalization is not an axiomatization in the strict sense since it requires an infinitary inference rule. (It follows from work by Scarpellini [5] that this is the most one can hope for.) The result of adding the comprehension axiom to Hay's predicate logic, with only the dyadic predicate "ϵ" is here called H; it is therefore sufficient to prove that H is consistent. For this purpose H is enlarged by adding certain Hilbert τ-terms to obtain a system H_1. Every theorem of H is a theorem of H_1. H_1 is then

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proved consistent by the use of a natural-deduction calculus G in the language of H_1. G is shown to be consistent, and therefore there exist sets of formulas which are maximally consistent with respect to G. Not every theorem of H_1 is a theorem of G, but every closed theorem of H_1 belongs to every maximally G-consistent set. Therefore a contradiction cannot be a theorem of H_1 or H, and H is consistent. The consistency of the natural-deduction system G follows from the fact that certain proofs in G can be normalized in the sense explained below; it is for this normalization that the τ-terms are required. (ϵ-terms could have been used instead, but since the universal quantifier is primitive in G it is slightly more natural to use τ-terms.)

The language of H has free variables $a_0, a_1, \ldots$ and bound variables $v_0, v_1, \ldots$. The terms and formulas of H are determined recursively as follows. Every free variable is a term. The constant f is a formula, read "the false." If A and B are formulas, $(A \to B)$ is a formula. If s and t are terms, $s \in t$ is a formula. For any free variable a, bound variable x, and formula $A(a)$, the class abstract $(x\colon A(x))$ is a term and the universalization $(x)A(x)$ is a formula. ($A(x)$ is the expression which results from replacing all occurrences of a in $A(a)$ with occurrences of x, changing bound variables in $A(a)$ as necessary to avoid unintended bindings. Similarly, for any term t, $A(t)$ is the result of substituting t for a in $A(a)$.) A formula or term is *closed* if it contains no free variables.

The following abbreviations are convenient

$-A$ is $(A \to f)$.

$(A \vee B)$ is $((A \to B) \to B)$.

$(A \,\&\, B)$ is $-(-A \vee -B)$.

$(A \longleftrightarrow B)$ is $((A \to B) \,\&\, (B \to A))$.

$(A + B)$ is $(-A \to B)$.

$(A \cdot B)$ is $-(A \to -B)$.

${}^nA$ is $(A \cdot (A \cdots A))$ (n conjuncts).

$(A^1 \to B)$ is $(A \to B)$, and $(A^{n+1} \to B)$ is $(A \to (A^n \to B))$ for $n \geqslant 1$.

$s \notin t$ is $-s \in t$.

Parentheses may be omitted from terms and formulas in accordance with the usual conventions. It will also sometimes be necessary to distinguish symbol tokens from symbol types. In particular, "formula," unless qualified, will be taken to mean "formula token." Thus in the sequence A, B, A there are three formulas; two of these are of the same type, type A as we shall say.

The axiomatic system H has the following axiom schemes.

H1. $A \to (B \to A)$.

H2. $(A \to B) \to ((B \to C) \to (A \to C))$.

H3. $A \vee B \to B \vee A$.

H4. $(-A \to -B) \to (B \to A)$.

H5. $(Ex)A(x) \cdot (Ex)A(x) \to (Ex)(A(x) \cdot A(x))$.

H6. $A(t) \to (Ex)A(x)$.

H7. $(Ex)A(x) \to (Ey)A(y)$.

H8. $(x)(A(x) \to B) \to ((Ex)A(x) \to B)$, if no occurrence of x in B is bound to the initial quantifier of $(x)(A(x) \to B)$.

H9. $(A \to (Ex)B(x)) \to (Ex)(A \to B(x))$, if no occurrence of x in A is bound to the initial quantifier of $(Ex)(A \to B(x))$.

H10. $A(t) \to t \in (x\colon A(x))$.

H11. $t \in (x\colon A(x)) \to A(t)$.

The inference rules for H are modus ponens, generalization, i.e., $\vdash (x)A(x)$ if $\vdash A(a)$, and the infinitary rule (inf): $\vdash A$ if $\vdash (A^n \to f) \to A$ for every $n \geqslant 1$. (Instead of inf Hay used the rule: $\vdash A$ if $\vdash {}^nA + A$ for every $n \geqslant 1$. Since in the infinite-valued sentential logic ${}^1A + A$ is $-A \to A$, ${}^2A + A$ is $--(A \to -A) \to A$ which is equivalent to $(A^2 \to f) \to A$, and in general ${}^nA + A$ is equivalent to $(A^n \to f) \to A$, rule inf is equivalent to Hays' infinitary rule.)

From the axioms $a \in (x\colon A(x)) \to A(a)$ and $A(a) \to a \in (x\colon A(x))$ one easily derives $a \in (x\colon A(x)) \longleftrightarrow A(a)$ by use of H1–H4 and modus ponens. Therefore $\vdash (y)(y \in (x\colon A(x)) \longleftrightarrow A(y))$ by generalization, and $\vdash (Ex)(y)(y \in x \longleftrightarrow A(y))$ by H6 and modus ponens. The principles of comprehension are therefore theorems in H.

The language of the systems H_1 and G extends that of H by enlarging the definition of "term" with the clause: for any bound variable x and formula $A(a)$ *whose only free variables are of type* a, $\tau_x^n A(x)$ is a term for every positive integer n. The set of formulas is of course thereby also enlarged, since formulas of H_1 may contain τ-terms. (The reason for supplying infinitely many τ-terms for each formula $A(a)$ with only free variables of type a will become clear in the consistency proof for G.)

The system H_1 has axiom schemes $H_1 1$–$H_1 11$ which are just H1–H11 for formulas of H_1. The inference rules of H_1 are again modus ponens, generalization, and inf. Every axiom of H is also an axiom of H_1, and so H_1 is an extension of H. (It is a conservative extension, but this fact is not needed for the theorems below.)

The *scope* of τ in a term $\tau_x^n B(x)$ is the expression $B(x)$. Notice that in an instance of $H_1 6$, i.e., $A(t) \rightarrow (Ex)A(x)$, the term t cannot occur in the scope of a τ in $A(t)$, since otherwise $(Ex)A(x)$ would not be a formula of H_1. For example, if $A(t)$ were $t \in \tau_y^1 t \in y$, $(Ex)A(x)$ would be $(Ex)x \in \tau_y^1 x \in y$ which by the recursive definition of "formula" would be a formula of H_1 only if $a \in \tau_y^1 a \in y$ were a formula for some free variable a. This is impossible since $\tau_y^1 a \in y$ is not a term. (A similar observation about the natural-deduction rule $-ui$ to be introduced below is important in the consistency proof for G.)

The crucial contribution of G and its consistency proof is to provide maximally G-consistent sets for Theorem 9 below. The τ-terms were introduced to extend H to H_1, because without them the H theorem $(A \rightarrow (Ex)B(x)) \rightarrow (Ex)(A \rightarrow B(x))$, for example, is not provable in G. Proofs in G are upwardly branching finite trees. Each node of a proof tree is a formula, and the topmost formulas are assumption formulas. Each assumption formula is assigned a positive integer which is written directly above it, and if two assumption formulas in a proof have the same number they must be formulas of the same type. When reference is made to the type of an *assumption*, as opposed to an assumption formula, the number of the assumption is to be taken into account. Thus $\overset{2}{A}$ and $\overset{3}{A}$ as assumptions in a proof are assumptions of different type. Certain of the inference rules of G may *close* assumptions in a proof. An assumption which has not been closed in a proof is *open* in that proof. A proof without open assumptions is a *categorical* proof. A formula is a *theorem* of G if it is the conclusion of a categorical proof in G. A categorical proof whose conclusion is of type f is a *contradictive* proof.

Capital sigma and pi, sometimes with subscripts, will be used to denote proofs in G. $\begin{matrix}\Sigma\\ A\end{matrix}$ is a proof whose conclusion is A. $\begin{matrix}k\\ (A)\\ \Sigma\end{matrix}$ is a proof with zero or more open assumptions of type $\overset{k}{A}$, so $\begin{matrix}k\\ (A)\\ \Sigma\\ B\end{matrix}$ is a proof with conclusion B and perhaps open assumptions of type $\overset{k}{A}$. For any proofs $\begin{matrix}\Sigma\\ A\end{matrix}$ and $\begin{matrix}k\\ (A)\\ \Pi\end{matrix}$, $\begin{matrix}\Sigma\\ (A)\\ \Pi\end{matrix}$ is the proof which results from first forming the proof $\begin{matrix}\Sigma_1\\ A\end{matrix}$ by replacing each assumption $\overset{n}{C}$ in $\begin{matrix}\Sigma\\ A\end{matrix}$ with $\overset{m+n}{C}$, where m is the least natural number sufficient to insure that no assumption number in $\begin{matrix}\Sigma_1\\ A\end{matrix}$ is an assumption number in $\begin{matrix}k\\ (A)\\ \Pi\end{matrix}$, and then replacing each open assumption of type $\overset{k}{A}$ in $\begin{matrix}k\\ (A)\\ \Pi\end{matrix}$ with the conclusion of a proof of type $\begin{matrix}\Sigma_1\\ A\end{matrix}$. Two proofs Σ and Π are said to be *disjoint* if for no A and k do Σ and Π both have an open assumption of type $\overset{k}{A}$.

The inference rules for G are the eight following.

Implication Introduction ($\to i$)

$\dfrac{\begin{matrix}\Sigma\\ B\end{matrix}}{A\to B}$, closing, if desired, all open assumptions of type $\overset{k}{A}$ in $\begin{matrix}\Sigma\\ B\end{matrix}$, for some k. (If an application of $\to i$ closes assumptions of type $\overset{k}{A}$ this is indicated in a proof by writing "*k*" to the right of the line above the conclusion of the inference.)

Implication Elimination ($\rightarrow e$)

$$\frac{\overset{\Sigma}{A} \quad \overset{\Pi}{A \rightarrow B}}{B}$$, provided that $\overset{\Sigma}{A}$ and $\overset{\Pi}{A \rightarrow B}$ are disjoint.

Disjunction ($\vee$)

$$\frac{\overset{\Sigma}{A \vee B}}{B \vee A}.$$

Ex falso quodlibet (q)

$$\frac{\overset{\Sigma}{f}}{A}$$, provided A is not of type f.

Universal Quantifier Introduction (ui)

$$\frac{\overset{\Sigma}{A(\tau_x^n A(x))}}{(x)A(x)}$$, provided of course that $A(a)$ has only one free variable.

Negative Universal Quantifier Introduction ($-ui$)

$$\frac{\overset{\Sigma}{-A(t)}}{-(x)A(x)}$$, provided that t is a closed term.

Class Introduction (ci)

$$\frac{\overset{\Sigma}{A(t)}}{t \in (x: A(x))}.$$

Negative Class Introduction $(-ci)$

$$\frac{\begin{matrix}\Sigma\\ -A(t)\end{matrix}}{t \notin (x: A(x))}.$$

In addition to these primitive rules the following four derived rules will be used in the proof of Theorem 1 below.

Double Negation (dn)

$$\begin{matrix}\Sigma\\ \dfrac{f}{A}\end{matrix}, \text{ closing, if desired, all open assumptions of type } \overset{k}{-A} \text{ in } \begin{matrix}\Sigma\\ f\end{matrix}.$$

If no assumptions of type $\overset{k}{-A}$ are closed by an application of dn and A is not of type f, dn is simply an application of the rule q. If A is of type f the rule is obviously justified. If some assumptions are closed by the rule and A is not of type f the following proof justifies dn:

$$\dfrac{\dfrac{\dfrac{\overset{m}{f}}{A}\,q}{f \rightarrow A}\rightarrow i, m \qquad \dfrac{\dfrac{\begin{matrix}k\\(-A)\\ \Sigma\\ f\end{matrix}}{A \vee f}\rightarrow i, k}{f \vee A}\vee}{A}\rightarrow e.$$

Universal Quantifier Elimination (ue)

$$\begin{matrix}\Sigma\\ \dfrac{(x)A(x)}{A(t)}\end{matrix}, \text{ provided that } t \text{ is closed.}$$

This is justified by

$$\dfrac{\dfrac{\overset{\Sigma}{(x)A(x)} \qquad \dfrac{\overset{k}{-A(t)}}{-(x)A(x)}\,ui}{f}\to e.}{A(t)}\,dn, k$$

Existential Quantifier Introduction (*ei*)

$$\dfrac{\overset{\Sigma}{A(t)}}{(Ex)A(x)}, \text{ provided } t \text{ is closed.}$$

This is justified by:

$$\dfrac{\dfrac{\overset{\Sigma}{A(t)} \qquad \dfrac{\overset{k}{(x)-A(x)}}{-A(t)}\,ue}{f}\to e.}{(Ex)A(x)}\to i, k$$

Existential Quantifier Elimination (*ee*)

$$\dfrac{\overset{\Sigma}{(Ex)A(x)}}{A(\epsilon_x^n A(x))}, \text{ where } \epsilon_x^n A(x) \text{ is an abbreviation for } \tau_x^n - A(x)$$

and $(Ex)A(x)$ is closed.

This rule is justified by

$$\dfrac{\dfrac{\dfrac{\overset{k}{-A(\epsilon_x^n A(x))}}{(x)-A(x)}\,ui \qquad \overset{\Sigma}{(Ex)A(x)}}{f}\to e.}{A(\epsilon_x^n A(x))}\,dn, k$$

Not every axiom of H_1 can be proved in G, but every closed axiom is provable in G, as our first theorem shows.

THEOREM 1. *Every closed axiom of* H_1 *is a theorem of* G.

Proof. Axioms $H_1 1$–$H_1 4$ are easily proved in G as follows.

$$\cfrac{\cfrac{\overset{1}{A}}{B\to A}\to i}{A\to(B\to A)}\to i, 1 \qquad \cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{A}\quad \overset{2}{A\to B}}{B}\to e \qquad \overset{3}{B\to C}}{C}}{A\to C}\to i, 1}{\cfrac{(B\to C)\to(A\to C)}{(A\to B)\to((B\to C)\to(A\to C))}\to i, 2.}\to i, 3$$

$$\cfrac{\cfrac{\overset{1}{A\vee B}}{B\vee A}\vee}{A\vee B\to B\vee A}\, i, 1$$

A closed instance of $H_1 5$ is proved in G as follows, since if $(Ex)A(x)$ is closed the term $\epsilon^1_x A(x)$ must be closed.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{3}{(Ex)A(x)}}{A(\epsilon^1_x A(x))}ee \quad \cfrac{\cfrac{\overset{1}{(Ex)A(x)}}{A(\epsilon^1_x A(x))}ee \quad \overset{2}{A(\epsilon^1_x A(x))\to -A(\epsilon^1_x A(x))}}{-A(\epsilon^1_x A(x))}\to e}{f}\to e}{-(Ex)A(x)}\to i, 1}{(Ex)A(x)\to -(Ex)A(x)}\to i, 3 \quad \overset{4}{(Ex)A(x)\cdot(Ex)A(x)}}{\cfrac{\cfrac{\cfrac{f}{A(\epsilon^1_x A(x))\cdot A(\epsilon^1_x A(x))}\to i, 2}{(Ex)(A(x)\cdot A(x))}ei}{(Ex)A(x)\cdot(Ex)A(x)\to(Ex)(A(x)\cdot A(x))}\to i, 4.}\to e$$

A closed instance of $H_1 6$ is trivially proved by use of *ei*, since t must be closed:

$$\cfrac{\cfrac{\overset{1}{A(t)}}{(Ex)A(x)}ei}{A(t)\to(Ex)A(x)}\to i, 1.$$

Closed instances of $H_1 7, H_1 8$, and $H_1 9$ are proved as follows.

$$\dfrac{\dfrac{\dfrac{\overset{1}{(Ex)A(x)}}{A(\epsilon_x^1 A(x))}\,ee}{(Ey)A(y)}\,ei}{(Ex)A(x)\to(Ey)A(y)}\to i,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{(Ex)A(x)}}{A(\epsilon_x^1A(x))}\,ee \quad \dfrac{\overset{1}{(x)(A(x)\to B)}}{A(\epsilon_x^1A(x))\to B}\,ue}{B}\to e}{(Ex)A(x)\to B}\to i,2}{(x)(A(x)\to B)\to((Ex)A(x)\to B)}\to i,1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{A}\quad \overset{1}{A\to(Ex)B(x)}}{(Ex)B(x)}\to e}{B(\epsilon_x^1B(x))}\,ee}{A\to B(\epsilon_x^1B(x))}\to i,2}{(Ex)(A\to B(x))}\,ei}{(A\to(Ex)B(x))\to(Ex)(A\to B(x))}\to i,1.$$

Proofs of $H_1 10$ and $H_1 11$ are:

$$\dfrac{\dfrac{\overset{1}{A(t)}}{t\in(x\colon A(x))}\,ci}{A(t)\to t\in(x\colon A(x))}\to i,1 \qquad \dfrac{\dfrac{\dfrac{\overset{2}{t\in(x\colon A(x))}\quad \dfrac{\overset{1}{-A(t)}}{t\notin(x\colon A(x))}-ci}{f}}{A(t)}\,dn,1}{t\in(x\colon A(x))\to A(t)}\to i,2.$$

This completes the proof of Theorem 1.

A τ-term $\tau_x^n A(x)$ is said to be *used* in a proof Σ if Σ has a part

$\dfrac{\overset{\Pi}{A(\tau_x^n A(x))}}{(x)A(x)}\,ui$. A proof is *pure* if for every part $\dfrac{\overset{\Pi}{A(\tau_x^n A(x))}}{(x)A(x)}\,ui$ in the proof

the term $\tau_x^n A(x)$ is not used in $\overset{\Pi}{A(\tau_x^n A(x))}$. Clearly every proof can be converted to a pure proof, for a part

$$\begin{array}{c} \Pi_1 \\ \dfrac{A(\tau_x^n A(x))}{((x)A(x))}\, ui \\ \Pi_2 \\ \dfrac{A(\tau_x^n A(x))}{(x)A(x)}\, ui \end{array}$$

can be changed to

$$\begin{array}{c} \Pi_3 \\ \dfrac{A(\tau_x^k A(x))}{((x)A(x))}\, ui \\ \Pi_2 \\ \dfrac{A(\tau_x^n A(x))}{(x)A(x)}\, ui \end{array}$$

where k is a number such that no τ-term $\tau_y^k B(y)$ was in the original proof and Π_3 is the result of substituting $\tau_x^k A(x)$ for $\tau_x^n A(x)$ throughout Π_1. By a series of such changes any proof which is not pure can be transformed to a pure proof, and so henceforth it will be assumed that every proof is pure.

The next two theorems are lemmas for the proof of the consistency of G.

THEOREM 2. *If a proof* $\overset{k}{\underset{\Sigma}{(A)}}$ *has an open assumption of type* $\overset{k}{A}$, *then it has exactly one open assumption of type* $\overset{k}{A}$.

This theorem results from the restriction on $\rightarrow e$. The proof is by induction on the length of $\overset{k}{\underset{\Sigma}{(A)}}$, where the length of a proof is simply the number of formulas occurring in it. The shortest proofs $\overset{k}{\underset{\Sigma}{(A)}}$ consist of a single assumption $\overset{k}{A}$, so the theorem is obviously true of these. Assume the theorem true for proofs $\overset{k}{\underset{\Pi}{(A)}}$ shorter than a proof $\overset{k}{\underset{\Sigma}{(A)}}$. If $\overset{k}{\underset{\Sigma}{(A)}}$ is $\dfrac{\overset{k}{(A)}\atop \Pi}{B}\,r$, where r is $\rightarrow i, q, \vee, ui, -ui, ci$, or $-ci$, the theorem follows obviously from the induction hypothesis. If $\overset{k}{\underset{\Sigma}{(A)}}$ is

$$\dfrac{\overset{k}{(A)}\atop \Sigma_1 \atop B \qquad \Sigma_2 \atop B\rightarrow C}{C}\rightarrow e \quad \text{or} \quad \dfrac{\Sigma_3 \atop B \qquad \overset{k}{(A)}\atop \Sigma_4 \atop B\rightarrow C}{C}\rightarrow e$$

the theorem also follows from the induction hypothesis, since by the restriction on $\rightarrow e$ there cannot be open assumptions of type $\overset{k}{A}$ in $\overset{\Sigma_2}{B\rightarrow C}$ or $\overset{\Sigma_3}{B}$.

For any proof Σ let $\Sigma_t^{\tau_x^n A(x)}$ be the result of substituting terms of type t for all terms of type $\tau_x^n A(x)$ in Σ, i.e., the result of replacing each formula $B(\tau_x^n A(x))$ in Σ with $B(t)$. The following theorem shows that sometimes the result of such a substitution is a proof.

THEOREM 3. *If t is a closed term and $\tau_x^n A(x)$ is not used in Σ, then $\Sigma_t^{\tau_x^n A(x)}$ is a proof.*

Proof. The proof is by induction on the length of Σ. "$\Pi_t^{\tau_x^n A(x)}$" will be abbreviated to "Π_t". If Σ is an assumption $\overset{k}{B(\tau_x^n A(x))}$, then since t is closed all τ-terms in $B(\tau_x^n A(x))$ remain τ-terms in $B(t)$ and $B(t)$ is a formula, so $\overset{k}{B(t)}$ is a proof. Suppose the theorem true for proofs shorter than a proof Σ. If Σ is $\dfrac{\Pi}{B(\tau_x^n A(x))}\,r$ in which r is $\rightarrow i, q, \vee, ci$ or $-ci$ the theorem clearly follows by the induction hypothesis, as it also does if Σ is

$$\dfrac{\overset{\Sigma_1}{B(\tau^n_x A(x))} \quad \overset{\Sigma_2}{B(\tau^n_x A(x)) \to C(\tau^n_x A(x))}}{C(\tau^n_x A(x))} \to e.$$

If Σ is $\dfrac{\overset{\Pi}{-B(s(\tau^n_x A(x)))}}{-(y)B(y)} - ui$, where $s(\tau^n_x A(x))$ is a term which may contain $\tau^n_x A(x)$, then no occurrence of y in $-(y)B(y)$ can lie within a τ-term $\tau^m_z C(z)$ in $-(y)B(y)$ unless y is of type z. By the induction hypothesis $\overset{\Pi_t}{-B'(s(t))}$ is a proof where $B'(s(t))$ is the result of substituting t for $\tau^n_x A(x)$ in $B(s(\tau^n_x A(x))$. But then Σ_t is a proof. (In general this would not be the case if free variables were permitted in τ-terms, for then, for example, $(y)\tau^n_x x \in y \in y$ would be a formula and a proof $\dfrac{\overset{\Pi}{\tau^n_x x \in s \notin s}}{-(y)\tau^n_x x \in y \in y} - ui$ would by the substitution of t for $\tau^n_x x \in s$ become $\dfrac{\overset{\Pi_t}{t \notin s}}{-(y)\tau^n_x x \in y \in y} - ui$, which is no longer a proof because $-(y)\tau^n_x x \in y \in y$ cannot be inferred from $t \notin s$ by $-ui$.)

If Σ is $\dfrac{\overset{\Pi}{B(\tau^m_y B(y))}}{(y)B(y)}\ ui$ then $\tau^n_x A(x)$ is not of type $\tau^m_y B(y)$, since $\tau^n_x A(x)$ is not used in Σ. By the induction hypothesis $\overset{\Pi_t}{B'(\tau^m_y B'(y))}$ is a proof, where $B'(y)$ is the result of substituting t for $\tau^n_x A(x)$ in $B(y)$. Then $\dfrac{\overset{\Pi}{B'(\tau^m_y B'(y))}}{(y)B'(y)}\ ui$ is also a proof. This completes the proof of Theorem 3.

The consistency of G will follow from the fact that every contradictive proof in G can be reduced to a proof which is "normal" in a certain sense. It is then easy to see that there can be no normal contradictive proof. To prove the consistency of G first the following operations called "reductions" to be performed on contradictive proofs are required. Let $\overset{\Sigma}{f}$ be any contradictive proof.

f-reduction: If $\begin{smallmatrix}\Sigma\\ f\end{smallmatrix}$ has a proper contradictive part $\begin{smallmatrix}\Pi\\ f\end{smallmatrix}$, then $\begin{smallmatrix}\Sigma\\ f\end{smallmatrix}$ is replaced with such a proper contradictive part which itself has no proper contradictive part.

$\rightarrow$ *reduction*: If $\begin{smallmatrix}\Sigma\\ f\end{smallmatrix}$ has a part

$$\dfrac{\begin{array}{c}\Sigma_1\\ A\end{array} \qquad \dfrac{\begin{array}{c}k\\ (A)\\ \Sigma_2\\ B\end{array}}{A \rightarrow B}\rightarrow i, k}{B}\rightarrow e\ ,$$

that part is replaced with $\begin{smallmatrix}\Sigma_1\\ (A)\\ \Sigma_2\\ B\end{smallmatrix}$. If $\begin{smallmatrix}\Sigma\\ f\end{smallmatrix}$ has a part

$$\dfrac{\begin{array}{c}\Sigma_1\\ A\end{array} \qquad \dfrac{\begin{array}{c}\Sigma_2\\ B\end{array}}{A \rightarrow B}\rightarrow i}{B}\rightarrow e$$

in which the $\rightarrow i$ closes no assumption, then that part is replaced with $\begin{smallmatrix}\Sigma_2\\ B\end{smallmatrix}$.

u-reduction: If $\begin{smallmatrix}\Sigma\\ f\end{smallmatrix}$ is

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ A(\tau_x^n A(x))\end{array}}{(x)A(x)}ui \qquad \dfrac{\begin{array}{c}\Pi_1\\ -A(t)\end{array}}{-(x)A(x)}-ui}{f}\rightarrow e\ ,$$

it is replaced with

$$\dfrac{\begin{array}{c}\Pi_t^{\tau_x^n A(x)}\\ A(t)\end{array} \qquad \begin{array}{c}\Pi_1\\ -A(t)\end{array}}{f}\rightarrow e.$$

By Theorem 3 the result of this reduction is a proof, since t must be closed and we may assume that $\overset{\Sigma}{f}$ is pure.

c-reduction: If $\overset{\Sigma}{f}$ is

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ A(t)\end{matrix}}{t\in(x\colon A(x))}\,ci \qquad \dfrac{\begin{matrix}\Sigma_2\\ -A(t)\end{matrix}}{t\notin(x\colon A(x))}\,-ci}{f}\to e$$

then it is replaced with $\dfrac{\begin{matrix}\Sigma_1\\ A(t)\end{matrix}\quad\begin{matrix}\Sigma_2\\ -A(t)\end{matrix}}{f}\to e.$

A proof is *weakly normal* if no reduction can be made in it. By Theorem 2 any $\to$ reduction in a contradictive proof results in a shorter proof, since it is clear from that theorem that the length of a proof $\begin{matrix}\Sigma\\ (A)\\ \Pi\\ B\end{matrix}$ must be less than the sum of the lengths of $\begin{matrix}\Sigma\\ A\end{matrix}$ and $\begin{matrix}\Sigma\\ (A)\\ \Pi\\ B\end{matrix}$. It is obvious that the other reductions shorten proofs. Therefore we have

THEOREM 4. *Every contradictive proof can by a finite number of reductions be reduced to a weakly normal contradictive proof.*

An application of the rule $\vee$ will be referred to simply as "a $\vee$." An instance $\dfrac{\begin{matrix}\Sigma\\ A\vee B\end{matrix}}{B\vee A}$ of $\vee$ is *categorical* if the proof $\begin{matrix}\Sigma\\ A\vee B\end{matrix}$ is categorical. A contradictive proof is *normal* if it is weakly normal and contains no categorical $\vee$s.

THEOREM 5. *If there is a contradictive proof, then there is a normal contradictive proof.*

An example suffices to show that every contradictive proof can be converted to a normal contradictive proof. Let

$$
\begin{array}{ccc}
 & & \Sigma_2 \\
 & & \underline{F \vee E} \\
 & & (E \vee F) \\
\Sigma_1 & & \Sigma_3 \\
\dfrac{A \vee B}{B \vee A}\,\vee & & \dfrac{D \vee C}{C \vee D}\,\vee \\
 & \ddots \quad \vdots \quad .\!\cdot^{\cdot} & \\
 & f &
\end{array}
$$

be a weakly normal proof with only the three $\vee$'s as indicated. If none of these $\vee$'s is categorical, the proof is already normal. If not, suppose for example that $\dfrac{\overset{\Sigma_1}{A \vee B}}{B \vee A}$ is categorical. Then form the categorical proof

$$
\begin{array}{ccc}
 & & \Sigma_2 \\
 & & \underline{F \vee E} \\
 & & (E \vee F) \\
m & & \Sigma_3 \\
\dfrac{A}{B \vee A} \to i & & \dfrac{D \vee C}{C \vee D} \\
 & \ddots \quad \vdots \quad .\!\cdot^{\cdot} & \\
 & \dfrac{\dfrac{\dfrac{f}{B}\,q}{A \to B} \to i, m \qquad \overset{\Sigma_1}{A \vee B}}{B} \to e. & \text{(assuming } B \text{ is not of type } f)
\end{array}
$$

(The disjointness restriction for the $\to e$ is satisfied because $\overset{\Sigma_1}{A \vee B}$ is categorical.) Let this proof be $\overset{\Pi}{B}$. Now form the contradictive proof

$$\begin{array}{ccc} & \Sigma_2 & \\ & \dfrac{F \vee E}{(E \vee F)} & \\ \Pi & \Sigma_3 & \\ \dfrac{B}{B \vee A} & \dfrac{D \vee C}{C \vee D}, & \\ \quad\cdot & \cdot\quad & \\ & f & \end{array} \quad \text{where } \begin{array}{c}\Pi \\ \dfrac{B}{\overline{B \vee A}}\end{array} \text{ is short for } \begin{array}{c} \Pi \qquad j \\ \dfrac{\dfrac{B \quad B \to A}{A} \to e,}{B \vee A} \to i, j \end{array}$$

for appropriate j. Reduce this contradictive proof to a weakly normal contradictive proof which may be symbolized as

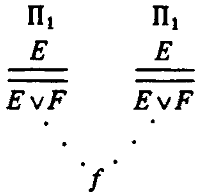
$$\begin{array}{ccc} \Sigma_4 & & \Sigma_5 \\ \dfrac{F \vee E}{E \vee F}\vee & & \dfrac{F \vee E}{E \vee F}\vee \\ \vdots & & \vdots \\ \dfrac{D \vee C}{C \vee D}\vee & & \dfrac{D \vee C}{C \vee D}\vee. \\ \quad\cdot & & \cdot\quad \\ & f & \end{array}$$

If none of the $\vee$'s here is categorical, the proof is normal. On the other hand, if for example $\begin{array}{c}\Sigma_5 \\ F \vee E\end{array}$ is categorical form the categorical proof

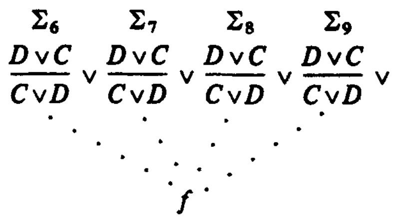
$$\begin{array}{ccc} j_1 & & j_2 \\ \dfrac{F}{E \vee F} \to i & & \dfrac{F}{E \vee F} \to i \\ \vdots & & \vdots \\ \dfrac{D \vee C}{C \vee D}\vee & & \dfrac{D \vee C}{C \vee D}\vee \\ \quad\cdot & & \cdot\quad \\ & f & \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f}{E}\,q}{F \to E} \to i, j_1 \qquad \begin{array}{c}\Sigma_5 \\ F \vee E\end{array}}{E} \to e}{F \to E} \to i, j_2 \qquad \begin{array}{c}\Sigma_5 \\ F \vee E\end{array}}{E} \to e$$

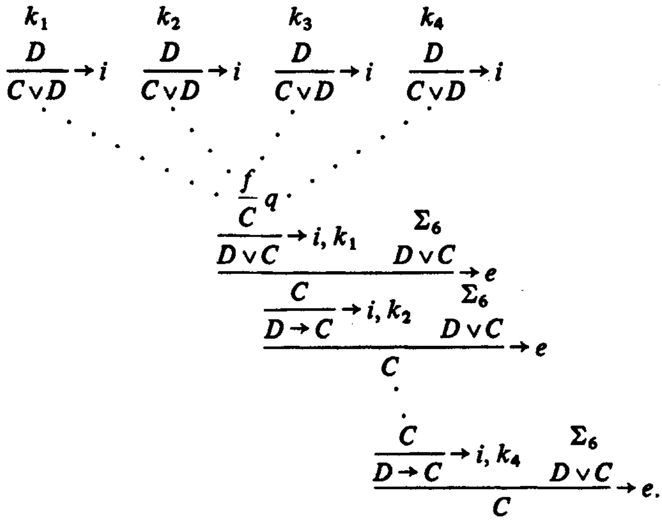
Let this be $\begin{matrix}\Pi_1\\ E\end{matrix}$. Next form the contradictive proof



and reduce this to a weakly normal proof, which in general will have the form



If none of the ∨'s in this proof is categorical, the proof is normal. Otherwise, suppose that $\begin{matrix}\Sigma_6\\ D\vee C\end{matrix}$ is categorical. Form the categorical proof



Let this be $\begin{matrix}\Pi_2\\ C\end{matrix}$. Then form the contradictive proof

$$\begin{matrix} \Pi_2 & \Pi_2 & \Pi_2 & \Pi_2 \\ \underline{\underline{C}} & \underline{\underline{C}} & \underline{\underline{C}} & \underline{\underline{C}} \\ C \vee D & C \vee D & C \vee D & C \vee D \\ \vdots & \vdots & \vdots & \vdots \\ & & f & \end{matrix}.$$

This proof no longer contains any $\vee$'s, and so when it is reduced to a weakly normal proof that proof must be normal. It is evident that the procedure of this example will work in general.

The consistency proof for G is completed by the next theorem.

THEOREM 6. *There can be no normal contradictive proof in G.*

The theorem is proved by showing that a normal contradictive proof must have an infinite path, which is impossible. Suppose $\begin{matrix}\Sigma\\ f\end{matrix}$ is a normal contradictive proof. $\begin{matrix}\Sigma\\ f\end{matrix}$ must be $\dfrac{\begin{matrix}\Sigma_1\\ A\end{matrix} \quad \begin{matrix}\Sigma_2\\ -A\end{matrix}}{f} \to e$, since $\begin{matrix}\Sigma\\ f\end{matrix}$ can only be inferred by $\to e$. Since $\begin{matrix}\Sigma\\ f\end{matrix}$ is normal, the conclusion of $\begin{matrix}\Sigma_2\\ -A\end{matrix}$ cannot be by $\to i$, q, or $\vee$. $-A$ cannot be inferred by *ui* or *ci*. Therefore the conclusion of $\begin{matrix}\Sigma_2\\ -A\end{matrix}$ must be by $-ui$, $-ci$, or $\to e$. Suppose the conclusion of $\begin{matrix}\Sigma_2\\ -A\end{matrix}$ is by $-ui$. Then A is $(x)B(x)$, which cannot be inferred by $\to i$, $\vee$, $-ui$, *ci*, or $-ci$ because of its form, and which also cannot be inferred by *ui* or q because the proof is normal. $\begin{matrix}\Sigma_1\\ A\end{matrix}$ must therefore be $\dfrac{\begin{matrix}\Sigma_3\\ C\end{matrix} \quad \begin{matrix}\Sigma_4\\ C \to A\end{matrix}}{C} \to e$. The conclusion of $\begin{matrix}\Sigma_4\\ C \to A\end{matrix}$ cannot be by $\to i$, $\vee$, or q because the proof is normal, and because of its form $C \to A$ cannot be inferred by *ui*, $-ui$, *ci*, or $-ci$.

Therefore $\overset{\Sigma_4}{C \to A}$ must be $\dfrac{\overset{\Sigma_5}{D} \quad \overset{\Sigma_6}{D \to (C \to A)}}{C \to A} \to e$. By continuing in this fashion we see that $\overset{\Sigma_1}{A}$ must contain an infinite path, which is impossible. A similar argument shows that $\overset{\Sigma_1}{A}$ would have an infinite path if the conclusion of $\overset{\Sigma_2}{-A}$ were by $-ci$. Therefore $\overset{\Sigma_2}{-A}$ must be $\dfrac{\overset{\Pi_1}{B} \quad \overset{\Pi_2}{B \to -A}}{-A} \to e$.

Since $\overset{\Sigma}{f}$ is normal $B \to -A$ cannot be inferred by $\to i$, q, or $\vee$, and it cannot be inferred by ui, $-ui$, ci, or $-ci$ because of its form. Therefore $\overset{\Pi_2}{B \to -A}$ must be $\dfrac{\overset{\Pi_3}{C} \quad \overset{\Pi_4}{C \to (B \to -A)}}{-A} \to e$, and again we see that $\overset{\Sigma}{f}$ must have an infinite path. Therefore there cannot be a normal contradictive proof, and the theorem is proved.

A set X of formula types is *G-inconsistent* if there is a proof $\overset{\Sigma}{f}$ in G all of whose open assumption formula types are in X. X is *G-consistent* if X is not G-inconsistent. A set M of formula types is *maximally G-consistent* if M is G-consistent and, for any formula type A, if $M \cup \{A\}$ is G-consistent then A is in M. G-consistency is a property of finite character – if every finite subset of a set X is G-consistent then X is G-consistent. Therefore as in classical logic it follows that any G-consistent set can be included in a maximally G-consistent set.

THEOREM 7. *Any maximally G-consistent set M has the following properties*:

M1. *If A is a theorem of G then the formula type A is in M.*

M2. *If the types A and $A \to B$ are in M, then the type B is in M.*

M3. *If $(x)A(x)$ is a closed formula, so that $\tau_x^n A(x)$ is a term, then if the type $A(\tau_x^n A(x))$ is in M the type $(x)A(x)$ is in M.*

M4. *For any formula A, if the type A is not in M then $A^n \to f$ is in M for some positive integer n.*

To prove M1, suppose that there is a categorical G-proof of A but that the type A is not in M. Then $M \cup \{A\}$ must be inconsistent, and there is a proof $\genfrac{}{}{0pt}{}{\Pi}{f}$ whose open assumption formula types are in M or of type A. Replacing each open assumption formula of type A in $\genfrac{}{}{0pt}{}{\Pi}{f}$ with the conclusion of a categorical proof $\genfrac{}{}{0pt}{}{\Sigma}{A}$ then gives a proof $\genfrac{}{}{0pt}{}{\Pi_1}{f}$ all of whose open assumption formula types belong to M, and M is inconsistent. To prove M2, assume that the type B is not in M while the types A and $A \to B$ are in M. Then $M \cup \{B\}$ is inconsistent and there is a proof $\genfrac{}{}{0pt}{}{\Sigma}{f}$ all of whose open assumption formula types are B or are in M. Replacing each open assumption formula of type B in $\genfrac{}{}{0pt}{}{\Sigma}{f}$ with the conclusion of a proof $\dfrac{\overset{i}{A} \quad \overset{j}{A \to B}}{B} \to e$, for appropriate i's and j's, then gives a proof which makes M inconsistent. The proof of M3 is similar, replacing open assumption formulas of type $(x)A(x)$ with the conclusions of proofs $\dfrac{\overset{k}{A(\tau_x^n A(x))}}{(x)A(x)}\ ui$. If A is not in M, then there is a proof $\genfrac{}{}{0pt}{}{\Sigma}{f}$ with open assumptions $\overset{k_1}{B_1}, \ldots, \overset{k_n}{B_n}, \overset{p_1}{A}, \ldots, \overset{p_m}{A}$ with the B_i's, if any, belonging to M. $n + m$ applications of $\to i$ to $\genfrac{}{}{0pt}{}{\Sigma}{f}$ then yield a categorical proof of $B_1 \to (\cdots \to (B_n \to (A^m \to f))\cdots)$, and $A^m \to f$ is in M by n applications of M2. This proves M4.

It remains now to be shown that every closed theorem of H_1 is in every maximally G-consistent set. To do this, it is necessary to describe H_1 proofs more precisely and to assign an ordinal number, for use in a transfinite induction, to each H_1 proof. Proofs in H_1 may be regarded as trees determined recursively as follows. An H_1 axiom is a proof of height zero. If $\overset{\Sigma}{A(a)}$ is an H_1 proof of height α, $\dfrac{\overset{\Sigma}{A(a)}}{(x)A(x)}$ is an H_1 proof of height $\alpha + 1$. If $\overset{\Sigma}{A}$ and $\overset{\Pi}{A \to B}$ are H_1 proofs of heights α and β, then $\dfrac{\overset{\Sigma}{A} \quad \overset{\Pi}{A \to B}}{B}$ is an H_1 proof of height max $(\alpha, \beta) + 1$. If $\left\{\overset{\Sigma_i}{(A^1 \to f) \to A}\right\}_{1<i<\omega}$ is a sequence of H_1 proofs of heights $\alpha_1, \alpha_2, \ldots$, etc., then $\dfrac{\left\{\overset{\Sigma_i}{(A^i \to f) \to A}\right\}_{1<i<\omega}}{A}$ is an H_1 proof of height $\mathrm{Sup}\{\alpha_i/1 < i < \omega\}$, the least ordinal greater than all the α_i's.

An H_1 proof is *pure* if for every subproof $\dfrac{\overset{\Sigma}{A(a)}}{(x)A(x)}$ in the proof the variable a occurs only in $\overset{\Sigma}{A(a)}$. If a proof is not pure it clearly can be converted to a pure proof. For example, even if all the free variables $a_0, a_1, a_2, \ldots$ appear in the proof, these can be replaced with $a_0, a_2, a_4, \ldots$ leaving infinitely many free variables not appearing in the proof. Then a subproof $\dfrac{\overset{\Pi}{A(a)}}{(x)A(x)}$ in which the variable a occurs in the main proof outside $\overset{\Pi}{A(a)}$ can be replaced with $\dfrac{\overset{\Pi^a_b}{A(b)}}{(x)A(x)}$, where b is a free variable not appearing in the original proof and Π^a_b is the result of replacing the free variables of type a with free variables of type b throughout Π. By a series of such changes any proof can be converted to a pure proof.

Corresponding to Theorem 1.3 for G we have for H_1 the following.

THEOREM 8. *For any H_1 proof Σ and closed term t, Π^a_b is a proof if the free variable a is not the variable of any generalization* $\overset{\Pi}{\dfrac{A(a)}{(x)A(x)}}$ *in Σ.*

The straightforward proof by transfinite induction on the height of Σ is omitted.

The next theorem implies the consistency of H.

THEOREM 9. *Let $\underset{A}{\Sigma}$ be a pure H_1 proof in which the conclusion A is closed. Then the type A is in any maximally G-consistent set.*

The proof uses transfinite induction on the height of $\underset{A}{\Sigma}$. Let M be maximally consistent, and suppose the theorem true for proofs whose heights are less than the height of $\underset{A}{\Sigma}$. If $\underset{A}{\Sigma}$ consists only of a closed axiom of H_1, then by Theorem 1 A is a theorem of G and so by Theorem 7 is in M. If $\underset{A}{\Sigma}$ is $\dfrac{\overset{\Sigma_i}{(A^i \to f) \to A} \quad 1 < i < \omega}{A}$, then since A is closed each $(A^n \to f) \to A$ is closed and by the induction hypothesis is in M. Therefore A must be in M, since by property M4 of Theorem 7 if A is not in M then $A^m \to f$ is in M for some M, and so again by Theorem 7 A is in M. If $\underset{A}{\Sigma}$ is $\dfrac{\overset{\Sigma_1}{B} \quad \overset{\Sigma_2}{B \to A}}{B}$, then B need not be closed. Let $b_1, \ldots, b_n$ be the free variables of B, so that B may be designated by "$B(b_1, \ldots, b_n)$". Since $\underset{A}{\Sigma}$ is pure, no b_i is the variable for an application of the generalization rule in $\underset{B}{\Sigma_1}$ or $\underset{B \to A}{\Sigma_2}$.

Therefore by Theorem 3 there are proofs $\begin{array}{c}\Sigma_3\\ B(t,\ldots,t)\end{array}$ and $\begin{array}{c}\Sigma_4\\ B(t,\ldots,t)\to A\end{array}$, where t is any fixed closed term, which are the results of substituting t for the b_i's in $\begin{array}{c}\Sigma_1\\ B\end{array}$ and $\begin{array}{c}\Sigma_2\\ B\to A\end{array}$. The heights of $\begin{array}{c}\Sigma_3\\ B(t,\ldots,t)\end{array}$ and $\begin{array}{c}\Sigma_4\\ B(t,\ldots,t)\to A\end{array}$ are equal to those of $\begin{array}{c}\Sigma_1\\ B\end{array}$ and $\begin{array}{c}\Sigma_2\\ B\to A\end{array}$, respectively, and so by the induction hypothesis the closed formula types $B(t,\ldots,t)$ and $B(t,\ldots,t)\to A$ are in M. By property M2 of Theorem 7 it follows that A is in M. Finally, if $\begin{array}{c}\Sigma\\ A\end{array}$ is $\begin{array}{c}\Pi\\ \dfrac{B(b)}{(y)B(y)}\end{array}$, then b is the only free variable of $B(b)$, and $\tau^n_t B(y)$ is a term for every n. Since $\begin{array}{c}\Sigma\\ A\end{array}$ is pure $\Pi^b_{\tau^n_y B(y)}$ is a proof of height less than the height of $\begin{array}{c}\Sigma\\ f\end{array}$, and so by the induction hypothesis $B(\tau^n_y B(y))$ is in M. By property M3 of Theorem 7 $(y)B(y)$ is then in M.

COROLLARY. *H is simply consistent in the sense that for no A are A and $-A$ both theorems of H.*

Proof. If A and $-A$ are theorems of H, f is a theorem of H and so also of H_1. The formula f is closed, so if f is a theorem of H, f is in every maximally G-consistent set. Obviously f cannot be in any G-consistent set, and there do exist maximally G-consistent sets since the set of all theorems of G is consistent.

As a set theory H has some serious disadvantages. First, the axiom of extensionality $(x)(y)((z)(z\in x\longleftrightarrow z\in y)\to x=y)$, where $x=y$ is defined to be $(z)(x\in z\longleftrightarrow y\in z)$, cannot be consistently added to H. The reason is that $x=y\vee x\neq y$ is easily proved in H; here is a proof in G, letting $\begin{array}{c}\Sigma\\ y=y\end{array}$ be a categorical proof of $y=y$ and t be $(z\colon z=y\to z\neq y)$:

$$\dfrac{\dfrac{\overset{1}{x=y\to x\neq y}}{x\in t}\,ci \qquad \dfrac{\overset{2}{x=y}}{x\in t\to y\in t}\,ue}{\Sigma \qquad y\in t}\to e$$

$$\dfrac{y\in t}{y=y\to y\neq y}\quad \text{class elimination, easily derived in } G$$

$$\dfrac{\overset{\Sigma}{y=y} \qquad y=y\to y\neq y}{\Sigma}\to e$$

$$\dfrac{\dfrac{\dfrac{y=y \qquad y\neq y}{f}\to e}{x\neq y}\to i,2}{x=y\vee x\neq y}\to i,1.$$

There is an analogous proof in H. It follows that $A \vee -A$ can be proved for arbitrary A from the axiom of extensionality by using $(x\colon x=x \;\&\; A)$, call it s, and the provable formula $(x\colon x=x)=s \vee (x\colon x=x)\neq s$. Second, it is apparently not possible to derive even classical first-order number theory in H. This is because the failure of the distributive law $(A\to(B\to C))\to((A\to B)\to(A\to C))$ makes it impossible to prove that the class of natural numbers N is closed under the successor operation, if N is defined in the natural way as $(x\colon (y)(0\in y \;\&\; (z)(z\in y\to z'\in y)\to x\in y))$, where z' is the successor of z.

It therefore seems reasonable to look for a rather different way of extending the Lukasiewicz sentential logic to a set theory. A natural course is to add another implication $A\Rightarrow B$ which may be thought of as the infinite disjunction $(A\to B)\vee(A^2\to B)\vee(A^3\to B)\vee\cdots$. It is evident that this weak implication distributes over the Lukasiewicz implication. I have investigated the following system S, and have proved that it is consistent and contains at least classical first-order number theory. The axioms of S are:

1. $A\to(B\to A)$.
2. $(A\to B)\to((B\to C)\to(A\to C))$.
3. $A\vee B\to B\vee A \qquad (A\vee B \text{ is still } (A\to B)\to B)$.
4. $f\to A$.
5. $(A\to B)\to(A\Rightarrow B)$.
6. $(A\Rightarrow(B\to C))\to((A\Rightarrow B)\to(A\Rightarrow C))$.

7. $(x)(A \to B) \to (A \to (x)B)$, x not free in A.

8. $(x)A(x) \to A(t)$.

9. $(x_1 \cdots x_n : A(x_1 \cdots X_n))t_1 \cdots t_n \to A(t_1 \cdots t_n)$.

10. $A(t_1 \cdots t_n \to (x_1 \cdots x_n : A(x_1 \cdots x_n))t_1 \cdots t_n$.

(It is convenient to use relation abstracts and to dispense with $\in$.) The inference rules are modus ponens, generalization, and the rule

$$\vdash (A \Rightarrow B) \to (A \to B) \text{ if } \vdash A \vee -A.$$

N is defined as $(x: (y)((y0 \,\&\, (z)(yz \to yz')) \Rightarrow yx))$, and by using Axiom 6 one easily proves $(x)(Nx \to Nx')$. It is also not difficult to prove $(x)(Nx \vee -Nx)$. In fact, excluded middle is provable in S for many classes. The axiom of extensionality in the weaker form $(x)(y)((z)(xz \leftrightarrow yz) \Rightarrow x = y)$ is probably consistent with S, but I have not found a consistency proof.

Of course, the chief open problem for any set theory based on the infinite-valued logic is to find a natural interpretation for it, an interpretation which justifies the formal system in the way in which the cumulative type structure justifies the axioms of ZF.

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