On the General Theory of Fading Memory

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1. Introduction

Often in physics one encounters a constitutive assumption asserting that the value of a variable $\boldsymbol{h}$ at time t is given by a functional $\mathfrak{h}$ of the history up to t of another variable $\boldsymbol{f}$:

$$\boldsymbol{h}(t)=\mathfrak{h}(\boldsymbol{f}^t). \tag{1.1}$$

Here $\boldsymbol{f}^t$ is a function on $[0,\infty)$ with the property that $\boldsymbol{f}^t(s)$ equals the value of $\boldsymbol{f}$ at time $t-s$:

$$\boldsymbol{f}^t(s)=\boldsymbol{f}(t-s), \qquad 0\leqq s<\infty. \tag{1.2}$$

The functional $\mathfrak{h}$ characterizes the *material*, or class of materials, under consideration. It is frequently possible to prove theorems in a branch of physics without completely specifying the form of $\mathfrak{h}$, but usually one must know something in advance about the smoothness of $\mathfrak{h}$. For this reason several norms have been proposed for sets of histories $\boldsymbol{f}^t$.[1]

A norm $\|\cdot\|$ assigned to spaces of histories $\boldsymbol{f}^t$ is subject to three elementary physical requirements:[2]

(1) The history $\boldsymbol{f}^{t+\sigma}$ of $\boldsymbol{f}$ up to time $t+\sigma$, $\sigma\geqq 0$, in a process for which $\boldsymbol{f}$ has the history $\boldsymbol{f}^t$ up to time t and is held constant in the interval $[t,t+\sigma]$ is called the "static continuation of $\boldsymbol{f}^t$ by amount σ". It is required that *if the norm of $\boldsymbol{f}^t$ is finite, then the norm of each static continuation of $\boldsymbol{f}^t$ also must be finite. Moreover, if the distance $\|\boldsymbol{f}_1^t-\boldsymbol{f}_2^t\|$ between two histories is zero, then there must be zero distance between their static continuations by any given amount.*

(2) If $\boldsymbol{f}^t$ is the history of $\boldsymbol{f}$ up to time t, then the history of $\boldsymbol{f}$ up to the earlier time $t-\sigma$, $\sigma\geqq 0$, is called the "σ-section of $\boldsymbol{f}^t$". It is required that *if $\boldsymbol{f}^t$ has finite norm, then the norm of each σ-section of $\boldsymbol{f}^t$ must be finite.*

(3) In physical theories "equilibrium states" must have finite norm; that is, *if $\boldsymbol{f}^t(s)\equiv\boldsymbol{a}$, a constant, then we require* $\|\boldsymbol{f}^t\|<\infty$.

[1] *E.g.* COLEMAN & NOLL [1960, 1], [1961, 1], [1964, 2]; COLEMAN [1964, 1]; WANG [1965, 3]; PERZYNA [1967, 2].

[2] For a discussion of the physical significance of the requirements (1)—(3), see COLEMAN & MIZEL [1966, 1, pp. 87—89].

In a recent study[3] of $\mathscr{L}_p$-norms of the form

$$\|f^t\|^p = \int_{[0,\infty)} |f^t|^p \, d\mu, \qquad 1 \leqq p < \infty, \tag{1.3}$$

with μ a positive, regular, Borel measure on $[0,\infty)$, we found the restrictions placed on μ by the requirements enumerated above. Here we show that many of the results obtained in our study of $\mathscr{L}_p$-spaces are valid also for general Banach function spaces.

A theory of materials with memory which rests upon Banach function spaces, rather than upon $\mathscr{L}_p$-spaces, has not only the advantage of greater generality but also the advantage of eliminating the arbitrary numbers p and de-emphasizing the "influence functions" k,[4] which are not experimentally determinable and appear in a physical theory as analytical encumbrances rather than as aids to understanding. We believe that these advantages are illustrated in our recent paper[5] on the thermodynamics of materials with memory.

We have attempted to keep the present paper sufficiently self-contained that it can be read independently of earlier articles on materials with memory. When a proof already given in our article on $\mathscr{L}_p$-spaces[6] applies without modification to more general Banach function spaces, we state the generalized theorem but do not repeat the proof. Many of the results assembled here will be applied in an essay, now in preparation, on asymptotic stability of solutions of functional-differential equations.

2. Properties of General Influence Measures

Let μ be an *influence measure*; that is, a non-trivial, Σ-finite, positive, regular Borel measure on $[0,\infty)$, and let $\mathscr{S}$ be the set of all μ-measurable functions ϕ mapping $[0,\infty)$ into $[0,\infty)$. Let ν be a function on $\mathscr{S}$ such that for all ϕ (or ϕ_i) in $\mathscr{S}$:

(i) $0 \leqq \nu(\phi) \leqq \infty$, and $\nu(\phi)=0$ if and only if $\phi \overset{\circ}{=} 0$;[7]

(ii) $\nu(\phi_1+\phi_2) \leqq \nu(\phi_1)+\nu(\phi_2)$, and $\nu(a\phi) = a\,\nu(\phi)$ for all numbers $a \geqq 0$;

(iii) if $\phi_1 \overset{\circ}{\leqq} \phi_2$, then $\nu(\phi_1) \leqq \nu(\phi_2)$;

(iv) there is at least one function ψ in $\mathscr{S}$ with $0<\nu(\psi)<\infty$;

(v) if $\psi, \phi_1, \phi_2, \ldots$ are in $\mathscr{S}$ and if $\phi_n \uparrow \psi$ pointwise μ-a.e., then $\nu(\phi_n) \uparrow \nu(\psi)$.

Such a function ν is called *a non-trivial function norm, relative to μ, with the sequential Fatou property.*[8]

Let V be a non-trivial[9], separable, real Banach space with norm $|\cdot|$, and let $\bar{\mathscr{V}}$ be the set of μ-measurable functions ϕ mapping $[0,\infty)$ into V.[10] We define

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[3] [1966, 1].

[4] *Cf.* Coleman & Noll [1960, 1].

[5] [1967, 1]. Our present Theorem 3.1 motivates the formula (3.10) used in that essay to define the norm $\|\cdot\|$ on histories.

[6] [1966, 1].

[7] A superposed $\circ$ indicates that the given relation holds pointwise μ-a.e., *i.e.* for all s in $[0,\infty)$ except for a set X with $\mu(X)=0$.

[8] This is the terminology used by Luxemburg & Zaanen [1966, 2].

[9] *I.e.* with at least one non-zero element.

[10] In many applications the space V has finite dimension. In fact we believe that the results to follow have interest even in the case in which V is the set of real numbers.

a function $\|\cdot\|$ on $\bar{\mathscr{V}}$ by writing, for each ϕ in $\bar{\mathscr{V}}$,

$$\|\phi\| = \nu(|\phi|), \tag{2.1}$$

and we denote by $\mathscr{V}$ the set of all functions ϕ in $\bar{\mathscr{V}}$ with $\|\phi\| < \infty$.

If ψ is a function on $[0,\infty)$ and σ a positive number, then the *static continuation* of ψ by amount σ is the function $\psi^{(\sigma)}$ on $[0,\infty)$ defined by[11]

$$\psi^{(\sigma)}(s) = \begin{cases} \psi(0), & 0 \leqq s \leqq \sigma, \\ \psi(s-\sigma), & \sigma < s < \infty; \end{cases} \tag{2.2}$$

the *σ-section* of ψ is the function $\psi_{(\sigma)}$ on $[0,\infty)$ given by

$$\psi_{(\sigma)}(s) = \psi(s+\sigma), \qquad 0 \leqq s < \infty. \tag{2.3}$$

We now lay down an assumption about ν.

Postulate 1. *If a given function ϕ is in $\mathscr{V}$, then all the static continuations $\phi^{(\sigma)}$, $\sigma \geqq 0$, of ϕ are also in $\mathscr{V}$. Furthermore, if ϕ and ψ in $\mathscr{V}$ are such that $\|\phi - \psi\| = 0$, then $\|\phi^{(\sigma)} - \psi^{(\sigma)}\| = 0$ for all $\sigma \geqq 0$.*

Theorem 2.1. *Postulate 1 requires that the influence measure μ have an atom at $s=0$ and be absolutely continuous on $(0,\infty)$ with respect to Lebesgue measure λ.*

Proof. Let $\mathbf{e}$ be a vector in V with $|\mathbf{e}|=1$. To show that $s=0$ is a μ-atom, we suppose that $\mu(\{0\})=0$ and let χ_0 denote the characteristic function of $\{0\}$. Then, $\chi_0 \mathrel{\mathring{=}} 0$; that is, $\mathbf{e}\,\chi_0(s) = \mathbf{0}$ μ-almost everywhere, and, by (2.1) and property (i) of ν, $\|\mathbf{e}\,\chi_0\| = 0$. But, by Postulate 1, this yields $\|(\mathbf{e}\,\chi_0)^{(\sigma)}\| = 0$ for all $\sigma \geqq 0$, and since (2.2) implies that $(\mathbf{e}\,\chi_0)^{(\sigma)} = \mathbf{e}\,\chi_{[0,\sigma]}$, we have $\|\mathbf{e}\,\chi_{[0,\sigma]}\| = 0$ for all $\sigma \geqq 0$. Thus, by (2.1),

$$\nu(\chi_{[0,\sigma]}) = 0 \qquad \text{for each } \sigma \geqq 0. \tag{2.4}$$

Property (i) of ν and (2.4) imply that $\mu\{[0,\sigma]\} = 0$ for all $\sigma \geqq 0$, which contradicts the non-triviality of μ. Hence $\mu(\{0\}) \neq 0$; i.e., $s=0$ is a μ-atom.

To complete the proof, we must show that the restriction of μ to $(0,\infty)$ is absolutely continuous relative to Lebesgue measure λ on $(0,\infty)$, *i.e.* that, for subsets X of $(0,\infty)$, $\lambda(X)=0$ implies $\mu(X)=0$. To do this by contradiction, we let $X \subset (0,\infty)$ be a purported λ-null Borel set with non-zero μ-measure:

$$X \subset (0,\infty), \qquad \lambda(X)=0, \qquad \mu(X)>0. \tag{2.5}$$

Since we then have

$$0 < \mu(X) = \lim_{\sigma\to\infty} \mu(X \cap [\sigma^{-1},\sigma]),$$

we may suppose, without loss of generality, that X is contained in $[\alpha^{-1},\alpha]$ for some $\alpha > 1$. We now let $\chi_{X-\sigma}$ be the characteristic function of the set $X-\sigma$ defined as follows:

$$X-\sigma = \{s-\sigma \mid s \in X\} = \{s \mid s+\sigma \in X\}.$$

According to Postulate 1, none of the sets $X-\sigma$, $0 \leqq \sigma < \alpha^{-1}$, is a μ-null set; for if one were, then we would have, by (2.1) and (i),

$$\|\mathbf{e}\,\chi_{X-\sigma}\| = \nu(\chi_{X-\sigma}) = 0,$$

[11] *Cf.* COLEMAN & NOLL (1962) [1964, 2, Eq.(3.8)].

which implies that

$$0=\|(e\chi_{X-\sigma})^{(\sigma)}\|=\|e\chi_X\|=\nu(\chi_X)$$

and hence that X itself is a μ-null set. Thus, (2.5) yields

$$\mu(X-\sigma)>0 \qquad \text{for all } \sigma \text{ in } [0,\alpha^{-1}). \tag{2.6}$$

However, (2.6), when combined with our assertion that $\lambda(X)=0$, contradicts a theorem of WEINER, YOUNG, and SAKS[12] which asserts: If μ is a positive, regular, Σ-finite measure defined on Borel subsets of $(0,\infty)$ and if $\lambda(X)=0$, then the set of σ values for which $\mu(X-\sigma)>0$ must have zero Lebesgue measure. Thus, (2.5) is impossible, and on $(0,\infty)$ μ is absolutely continuous relative to λ; q.e.d.

Let us now add to Postulate 1 the following assumption.

Postulate 2. *If ϕ is in $\mathscr{V}$, then so also are all its σ-sections, $\phi_{(\sigma)}$, $\sigma\geqq 0$.*

Remark 2.1. If χ_X is the characteristic function of a μ-measurable, bounded subset X of $[0,\infty)$, then

$$\nu(\chi_X)<\infty. \tag{2.7}$$

Proof. Since X is a bounded subset of $[0,\infty)$, we may pick a positive number β such that X is contained in $[0,\beta)$. It clearly follows from properties (i)–(iv) of ν that there exists a function ψ in $\mathscr{S}$ such that $0<\nu(\psi)<\infty$ and $\psi(\alpha)\geqq 1$ for some $\alpha\geqq 0$. Hence, by (2.1), for any vector e in V with $|e|=1$ we have $\|e\psi\|<\infty$; that is, $e\psi$ is in $\mathscr{V}$ and, by Postulates 1 and 2, $(e\psi)_{(\alpha)}$ and $((e\psi)_{(\alpha)})^{(\beta)}$ are also in $\mathscr{V}$. However, since $\psi(\alpha)\geqq 1$ it follows from (2.1)–(2.3) that

$$|(e\psi)_{(\alpha)}(0)|\geqq 1, \quad \text{and} \quad |(e\psi)_{(\alpha)}{}^{(\beta)}(s)|\geqq 1 \quad \text{for all } s\in[0,\beta).$$

Thus,

$$|(e\psi)_{(\alpha)}{}^{(\beta)}(s)|\geqq\chi_X(s) \qquad \text{for all } s\in[0,\infty),$$

and

$$\nu\big(|(e\psi)_{(\alpha)}{}^{(\beta)}|\big)<\infty,$$

which, by property (i) of ν, implies (2.7); q.e.d.

Theorem 2.2. *Postulates 1 and 2, together, imply that either $\mu((0,\infty))=0$ or λ is absolutely continuous on $(0,\infty)$ with respect to μ.*

Proof. We must show that if

$$\mu((0,\infty))>0, \tag{2.8}$$

then, for subsets X of $(0,\infty)$, $\mu(X)=0$ implies $\lambda(X)=0$. To do this by contradiction we assume (2.8) and let a set X be such that

$$X\subset(0,\infty), \quad \mu(X)=0, \quad \lambda(X)>0. \tag{2.9}$$

Since

$$0<\lambda(X)=\lim_{\sigma\to\infty}\lambda(X\cap[\sigma^{-1},\sigma])$$

without loss of generality we can assume that X is contained in an interval of the form $[\alpha^{-1},\alpha]$ for some $\alpha>0$; furthermore, by Theorem 2.1, we can assume that X is a G_δ set.[13] Let e be a unit vector in V, and let σ be $\geqq 0$. Since $\mu(X)=0$,

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[12] *Vid.* SAKS [1937, 1, Theorem 11.1, p. 91].

[13] *Vid. e.g.* [1937, 1, p. 41].

we have, by property (i) of ν and (2.1),

$$0=\nu(\chi_X)=\|e\,\chi_X\|,$$

and, by Postulate 1 and (2.2),

$$0=\|e\,\chi_X^{(\sigma)}\|=\|e\,\chi_{X+\sigma}\|=\nu(\chi_{X+\sigma}),$$

with $X+\sigma$ the set defined by

$$X+\sigma=\{s+\sigma\,|\,s\in X\}=\{s\,|\,s-\sigma\in X\}.$$

Therefore,

$$\mu(X+\sigma)=0 \qquad \text{for all } \sigma\geqq 0. \tag{2.10}$$

It follows from FUBINI's theorem that

$$\int_0^\infty f(\sigma)\,d\lambda(\sigma)\geqq\int_\alpha^\infty g(\sigma)\,d\mu(\sigma),$$

where

$$f(\sigma)=\int_{X+\sigma} d\mu, \qquad g(\sigma)=\int_{X+\sigma} d\lambda.$$

Therefore, since (2.10) asserts that $f(\sigma)=0$, and the translation-invariance of Lebesgue measure implies that $g(\sigma)=\lambda(X)$, we have

$$0=\int_0^\infty f(\sigma)\,d\lambda(\sigma)\geqq\int_\alpha^\infty \lambda(X)\,d\mu(\sigma)=\lambda(X)\,\mu([\alpha,\infty)). \tag{2.11}$$

In (2.9) we assumed $\lambda(X)>0$; hence (2.11) yields

$$\mu([\alpha,\infty))=0, \tag{2.12}$$

and consequently (2.8) implies

$$\mu((0,\alpha))>0. \tag{2.13}$$

Let $\mathscr{S}_{(0,\alpha)}$ be the set of all functions ϕ in $\mathscr{S}$ which vanish outside of $(0,\alpha)$. We note that (i)–(v) imply that $\mathscr{S}_{(0,\alpha)}$ contains at least one function $\tilde{\phi}$ with $\nu(\tilde{\phi})=\infty$; *i.e.* with

$$\|e\tilde{\phi}\|=\infty. \tag{2.14}$$

Since $\tilde{\phi}(0)=0$, (2.2) and (2.12) yield

$$(e\tilde{\phi})^{(\alpha)}\stackrel{\circ}{=}\mathbf{0}, \qquad \text{i.e. } \|(e\tilde{\phi})^{(\alpha)}\|=0,$$

and $(e\tilde{\phi})^{(\alpha)}$ is in $\mathscr{V}$. But then, by Postulate 2, $(e\tilde{\phi})^{(\alpha)}{}_{(\alpha)}$ is also in $\mathscr{V}$, and since (2.2) and (2.3) yield $(e\tilde{\phi})^{(\alpha)}{}_{(\alpha)}=e\tilde{\phi}$, we have $\|e\tilde{\phi}\|<\infty$ which contradicts (2.14). Thus (2.12) and (2.13), when taken together, yield a contradiction, and we may conclude that if (2.8) is assumed, (2.9) is impossible, q.e.d.

Let us now adopt the usual terminology in which the words *measurable, locally summable*, and *almost everywhere (a.e.)*, without further modifiers, all refer to Lebesgue measure.

It follows from Theorem 2.1 that there exists a locally summable function k on $(0,\infty)$ such that for every interval (a,b) in $(0,\infty)$

$$\mu((a,b))=\int_a^b k(s)\,ds. \tag{2.15}$$

This function k, the Lebesgue-Radon-Nikodým derivative of μ, is the analogue in our present more general theory of the *influence function* occurring in COLEMAN & NOLL's theory [1960, 1], [1961, 1], [1964, 2][14] of fading memory. Theorem 2.2 tells us that k is essentially positive; *i.e.*

$$k(s)>0 \qquad \text{a.e. in } (0,\infty). \tag{2.16}$$

Furthermore, it is now clear that (2.7) can be strengthened to the following assertion: If X is a bounded measurable subset of $[0,\infty)$, then $\nu(\chi_X)<\infty$ and

$$\nu(\chi_X)>0 \quad \text{if either } 0\in X, \quad \text{or } \lambda(X)>0 \text{ and } \mu((0,\infty))>0. \tag{2.17}$$

In particular, for χ_0, the characteristic function of $\{0\}$, we have

$$0<\nu(\chi_0)<\infty. \tag{2.18}$$

3. Translations in Space of Past Histories

In applications[15] of the present theory, the functions ϕ in $\mathscr{V}$ are called *histories*; their independent variable is usually denoted by s and is called the *elapsed time*. The value $\phi(0)$ of a history ϕ at $s=0$ is the *present value* of ϕ, and the *past values* $\phi(s)$ are those for which $0<s<\infty$. In terms more suggestive than precise, Theorems 2.1 and 2.2 tell us that $\|\phi\|=\nu(|\phi|)$ places greater emphasis on the present value of ϕ than on any individual past value but does not "ignore" any interval of past time.

The function space $\mathfrak{B}$ obtained by calling two functions ϕ_1, ϕ_2 in $\mathscr{V}$ the same whenever $\|\phi_1-\phi_2\|=0$ is easily shown to be a Banach space. Here we examine the structure of $\mathfrak{B}$, employing, of course, the Postulates 1 and 2 and their consequences given in Theorems 2.1 and 2.2.

If ϕ is a function in $\mathscr{V}$, the restriction of ϕ to $(0,\infty)$ is called the *past history* of ϕ and is denoted by ϕ_r. We employ the symbol $\mathscr{V}_r$ for the set of all functions ϕ_r obtained by restricting members of $\mathscr{V}$ to $(0,\infty)$. The function $\|\cdot\|_r$ on $\mathscr{V}_r$ defined by

$$\|\phi_r\|_r=\|\phi\,\chi_{(0,\infty)}\|=\nu(|\phi\,\chi_{(0,\infty)}|) \tag{3.1}$$

is clearly a semi-norm. The *space of past histories* is the function space $\mathfrak{B}_r$ obtained by calling the same those past histories ϕ_r, ψ_r for which $\|\phi_r-\psi_r\|_r=0$; $\mathfrak{B}_r$, like V and $\mathfrak{B}$, is a Banach space.

Theorem 3.1. *The Banach space $\mathfrak{B}$ is algebraically and topologically the direct sum of V and $\mathfrak{B}_r$; that is*

$$\mathfrak{B}=V\oplus\mathfrak{B}_r, \tag{3.2}$$

[14] See also COLEMAN & MIZEL [1966, 1].

[15] *Cf.* COLEMAN & MIZEL [1967, 1].

and the norm $\|\cdot\|$ *on* $\mathfrak{B}$ *is equivalent to the norm* $\|\cdot\|'$ *defined by*

$$\|\phi\|' = |\phi(0)| + \|\phi_r\|_r. \tag{3.3}$$

Here $|\cdot|$ is our original norm on V, $\|\cdot\|$ is the norm on $\mathfrak{B}$ defined by (2.1), and $\|\cdot\|_r$ is the norm on $\mathfrak{B}_r$ defined by (3.1). The equivalence of $\|\cdot\|'$ to $\|\cdot\|$ shows that the present value $\phi(0)$ of a history ϕ in $\mathfrak{B}$ has approximately the same importance to ϕ as its entire past history ϕ_r.

Proof of Theorem 3.1. Since $\mathfrak{B}$ is formed from functions ϕ mapping $[0,\infty)$ into V, and $\mathfrak{B}_r$ is formed from the restrictions of the functions ϕ to $(0,\infty)$, (3.2) is trivial as an algebraic statement. To show, however, that $\|\cdot\|$ is equivalent to $\|\cdot\|'$, we must produce two positive numbers c_1 and c_2 such that

$$\|\phi\| \leqq c_1 \|\phi\|', \tag{3.4}$$

and

$$\|\phi\|' \leqq c_2 \|\phi\|. \tag{3.5}$$

Now, for each function ϕ in $\mathscr{V}$ we have

$$\phi = \phi(0)\,\chi_0 + \phi\,\chi_{(0,\infty)}, \tag{3.6}$$

and, by (2.1), property (iii) of ν, and (3.1),

$$\|\phi\| = \nu(|\phi|) \leqq \nu(|\phi(0)\,\chi_0|) + \nu(|\phi\,\chi_{(0,\infty)}|) = |\phi(0)|\,\nu(\chi_0) + \|\phi_r\|.$$

Therefore (3.4) holds with

$$c_1 = \max\{\nu(\chi_0), 1\}. \tag{3.7}$$

Equation (3.6) also yields

$$\nu(|\phi|) \geqq \nu(|\phi(0)\,\chi_0|) = |\phi(0)|\,\nu(\chi_0), \qquad \nu(|\phi|) \geqq \nu(|\phi\,\chi_{(0,\infty)}|) = \|\phi\|_r.$$

Thus

$$2\,\|\phi\| \geqq |\phi(0)|\,\nu(\chi_0) + \|\phi_r\|_r,$$

and (3.5) holds with

$$c_2^{-1} = \tfrac{1}{2}\min\{\nu(\chi_0), 1\}. \tag{3.8}$$

The assertion (2.18) insures that c_1 and c_2, given by (3.7) and (3.8), are finite and positive; q.e.d.

We now concentrate our attention on the Banach space $\mathfrak{B}_r$.

Let us define translation operators by the following formulae which hold for each function ϕ_r in $\mathscr{V}_r$:

$$T^{(\sigma)}\phi_r = \begin{cases} \mathbf{0}, & 0 < s \leqq \sigma, \\ \phi_r(s-\sigma), & \sigma < s < \infty; \end{cases} \tag{3.9}$$

$$T_{(\sigma)}\phi_r = \phi_r(s+\sigma), \qquad 0 < s < \infty. \tag{3.10}$$

Our Postulates 1 and 2 imply that for each $\sigma \geqq 0$, $T^{(\sigma)}$ and $T_{(\sigma)}$ map $\mathscr{V}_r$ into $\mathscr{V}_r$. Furthermore, $T^{(\sigma)}$ and $T_{(\sigma)}$, when regarded as functions on $\mathfrak{B}_r$, are well defined; in fact, we have the following theorem.

Theorem 3.2. *For each* $\sigma \geqq 0$, $T^{(\sigma)}$ *and* $T_{(\sigma)}$ *are bounded linear operators mapping* $\mathfrak{B}_r$ *into itself.*

As the proof of this theorem is precisely the same as that which we employed for Lemma 1 of our 1966 essay,[16] we do not repeat it here.

Since

$$T^{(\sigma_1+\sigma_2)}=T^{(\sigma_1)}\,T^{(\sigma_2)},$$
$$T_{(\sigma_1+\sigma_2)}=T_{(\sigma_1)}\,T_{(\sigma_2)}, \tag{3.11}$$

the sets $\{T^{(\sigma)}\}$, $\{T_{(\sigma)}\}$ of operators $T^{(\sigma)}$ and $T_{(\sigma)}$, $\sigma \geqq 0$, form Abelian semi-groups. We denote the norm of $T^{(\sigma)}$ by $\bar{N}(\sigma)$ and the norm of $T_{(\sigma)}$ by $\underline{N}(\sigma)$:

$$\bar{N}(\sigma)=\sup_{\|\phi_r\|_r>0}\frac{\|T^{(\sigma)}\phi_r\|_r}{\|\phi_r\|_r}, \tag{3.12}$$

$$\underline{N}(\sigma)=\sup_{\|\phi_r\|_r>0}\frac{\|T_{(\sigma)}\phi_r\|_r}{\|\phi_r\|_r}. \tag{3.13}$$

It follows from (3.11) that $\bar{N}(\cdot)$ and $\underline{N}(\cdot)$ are submultiplicative functions on $[0,\infty)$:

$$\bar{N}(\sigma_1+\sigma_2)\leqq\bar{N}(\sigma_1)\,\bar{N}(\sigma_2),\quad \underline{N}(\sigma_1+\sigma_2)\leqq\underline{N}(\sigma_1)\,\underline{N}(\sigma_2),\qquad \sigma_1,\sigma_2\in[0,\infty). \tag{3.14}$$

In theories of materials with fading memory, one assumes that bounded functions of compact support are dense in $\mathfrak{V}_r$.[17] One also desires that LEBESGUE's theorem on dominated convergence hold in $\mathfrak{V}_r$. These familiar properties can be obtained by laying down the following postulate.

Postulate 3. *The space $\mathfrak{V}_r$ is separable.*

Remark 3.1. It follows from known results in the theory of Banach function spaces that if $\mathfrak{V}_r$ is separable then every element of $\mathfrak{V}_r$ is of absolutely continuous norm.[18] By a theorem of LUXEMBURG & ZAANEN,[19] ϕ_r in $\mathfrak{V}_r$ is of absolutely continuous norm if, and only if, for every sequence ψ_r^n in $\mathfrak{V}_r$ such that $|\psi_r^n| \overset{a}{\leqq} |\phi_r|$ for all n and $\psi_r^n \to \psi_r^0$ pointwise *a.e.*, we have $\|\psi_r^n - \psi_r^0\|_r \to 0$. Thus, *Postulate 3 does give $\mathfrak{V}_r$ the dominated convergence property familiar in $\mathscr{L}_p$-spaces.*

Remark 3.2. In a Banach function space the closure of the set of bounded functions of bounded support contains all functions of absolutely continuous norm in the space.[20] Therefore, Postulate 3 implies that bounded functions of bounded support are dense in $\mathfrak{V}_r$, and this, in turn,[21] implies that *continuous functions of compact support are dense in $\mathfrak{V}_r$.*

If Postulates 1–3 are assumed, Remark 3.2 and a proof we have given elsewhere[22] may be invoked to establish the following two theorems.

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16 [1966, 1, pp. 95, 96]. (The proof of the boundedness of the operators $T^{(\sigma)}$ and $T_{(\sigma)}$ is a straightforward application of the closed graph theorem.)

17 *Vid. e.g.* [1967, 1].

18 *Vid.* LUXEMBURG [1965, 2, Theorem 46.2, p. 241] and LORENTZ & WERTHEIM [1953, 1, Proof of Theorem 1, pp. 570, 571].

19 [1963, 1, Theorem 2.2, p. 157].

20 LUXEMBURG & ZAANEN [1956, 1, Theorem 4, p. 117], [1963, 1, Theorem 2.4, p. 157].

21 *Cf.* DUNFORD & SCHWARTZ [1958, 1, Exercise 17, p. 170].

22 [1967, 1, Appendix 1]. Since we have not yet assumed that $\mathfrak{V}_r$ contains non-trivial constant functions, that proof must be slightly modified: The bounded uniformly continuous functions used in the proof, and particularly the constant function ω occurring in (I-9), should here be taken to have compact support.

Theorem 3.3. *The functions* $\bar{N}(\cdot)$ *and* $\underline{N}(\cdot)$ *are lower semi-continuous and are bounded on each closed interval of* $[0,\infty)$.

Theorem 3.4. *The semi-groups* $\{T^{(\sigma)}\}$ *and* $\{T_{(\sigma)}\}$ *are strongly continuous in the sense that for each* ϕ_r *in* $\mathfrak{V}_r$ *and each* $\sigma \geqq 0$

$$\lim_{\zeta\to\sigma} \|T^{(\zeta)}\phi_r - T^{(\sigma)}\phi_r\|_r = 0, \quad \lim_{\zeta\to\sigma} \|T_{(\zeta)}\phi_r - T_{(\sigma)}\phi_r\|_r = 0. \tag{3.15}$$

Let us now turn to equation (1.1) of the Introduction. If f is a function on $(-\infty, a)$ and if $t \leqq a$, the *history of* f *up to* t is the function f^t on $[0,\infty)$ defined by (1.2). We say that a V-valued function f over $(-\infty,\infty)$ is *admissible* if f^t is in $\mathscr{V}$ for each t, *i.e.* if

$$\|f^t\| < \infty \qquad \text{for all } t \in (-\infty,\infty). \tag{3.16}$$

By Postulate 2, for (3.16) to hold it suffices that $\|f^t\|$ be finite for all large t.

Let $\mathfrak{h}$ in (1.1) be a continuous function mapping $\mathfrak{V}$ into some metric space $\mathscr{M}$. It follows from Theorem 3.1 that $\mathfrak{h}$ can be regarded equally well as a function of ordered pairs $(\phi(0), \phi_r)$ with $\phi(0)$ in V and ϕ_r in $\mathfrak{V}_r$, *i.e.*

$$\mathfrak{h}(\phi) = \mathfrak{h}(\phi(0), \phi_r),$$

and hence (1.1) can be written

$$\mathfrak{h}(t) = \mathfrak{h}(f^t) = \mathfrak{h}(f^t(0), f_r^t) = \mathfrak{h}(f(t), f_r^t) \tag{3.17}$$

where $f(t) = f^t(0)$ is the present value of f^t, and f_r^t, called the *past history*[23] corresponding to f^t, is the restriction of f^t to $(0,\infty)$. Moreover, the continuity of $\mathfrak{h}$ over $\mathfrak{V}$ implies that $\mathfrak{h}(f(t), f_r^t)$ is jointly continuous in its two variables $f(t) \in V$ and $f_r^t \in \mathfrak{V}_r$.

Now, (1.1) can be viewed as a functional transformation $f \to h$ mapping admissible functions f on $(-\infty,\infty)$ into functions h on $(-\infty,\infty)$ with values in $\mathscr{M}$; that is, if we are given $f(\tau)$ for all τ in $(-\infty,\infty)$ we can use (1.1) to calculate $h(t)$ for each t in $(-\infty,\infty)$. It is clear from (3.17), the assumed continuity of $\mathfrak{h}$, and Theorem 3.4 that this functional transformation $f \to h$ preserves regularity in the following sense.

Remark 3.3.[24] Let $\mathfrak{h}$ be a continuous functional mapping $\mathfrak{V}$ into a metric space, and suppose that f is an admissible function on $(-\infty,\infty)$. If f is a regulated function, *i.e.* a function for which the limits

$$\lim_{t\to t^+} f(\tau) \quad \text{and} \quad \lim_{\tau\to t^-} f(\tau)$$

exist for each t in $(-\infty,\infty)$, then h, given by (1.1), is also a regulated function. Furthermore, h can suffer discontinuities only at those times t_i at which f is discontinuous; at all other times h must be continuous.

By Remark 3.2, our assumption that $\mathfrak{V}_r$ is separable implies that functions that are zero for large s form an everywhere dense subset of $\mathfrak{V}$. Postulate 3 therefore gives a type of "fading memory" to any functional $\mathfrak{h}$ that is continuous over $\mathfrak{V}$: the response $\mathfrak{h}(f^t)$ to an arbitrary history f^t in $\mathscr{V}$ must be approximately

[23] *Cf.* COLEMAN [1964, 1, p. 251].

[24] *Cf.* [1966, 1, Remark 5.1].

the same as the response $\mathfrak{h}({}_\tau\boldsymbol{f}^t)$ to a history ${}_\tau\boldsymbol{f}^t$ in $\mathscr{V}$ with

$$ {}_\tau\boldsymbol{f}^t(s)=\begin{cases}\boldsymbol{f}^t(s), & 0\leqq s\leqq\tau,\\ \boldsymbol{0}, & s>\tau,\end{cases}$$

provided, of course, that τ is chosen sufficiently large. In the following two sections we consider a stronger concept of fading memory, called the "relaxation property".

4. Conditions Equivalent to the Relaxation Property

The space $\mathfrak{V}_r$ is said to have the *relaxation property*[25] if

$$\lim_{\sigma\to\infty}\|\boldsymbol{T}^{(\sigma)}\boldsymbol{\phi}_r\|_r=0, \qquad \text{for each } \boldsymbol{\phi}_r\in\mathfrak{V}_r. \tag{4.1}$$

Postulates 1–3 yield

Theorem 4.1. *If $\mathfrak{V}_r$ has the relaxation property, then $\bar{N}(\cdot)$ is bounded on* $[0,\infty)$.

Proof. By Theorem 3.4, for each fixed element $\boldsymbol{\phi}_r$ of $\mathfrak{V}$ the non-negative number $\|\boldsymbol{T}^{(\sigma)}\boldsymbol{\phi}_r\|$ depends continuously on σ for $\sigma\geqq 0$. Hence, (4.1) implies

$$\sup_{\sigma\geqq 0}\|\boldsymbol{T}^{(\sigma)}\boldsymbol{\phi}_r\|=K(\boldsymbol{\phi}_r)<\infty.$$

However, since this argument holds for *each* $\boldsymbol{\phi}_r$ in $\mathfrak{V}_r$, for the norms (3.21) of the linear operators $\boldsymbol{T}^{(\sigma)}$ we have, by the uniform boundedness principle,[26]

$$\sup_{\sigma\geqq 0}\bar{N}(\sigma)=M<\infty; \tag{4.2}$$

q.e.d.

Let us now assume

Postulate 4. *The space $\mathfrak{V}_r$ contains non-trivial constant functions. That is, for each vector $\boldsymbol{a}$ in V, the function $\boldsymbol{a}_r^*$, defined by*

$$\boldsymbol{a}_r^*(s)\equiv\boldsymbol{a}, \qquad 0<s<\infty, \tag{4.3}$$

is in $\mathscr{V}_r$.

By (3.1) and property (ii) of ν,

$$\|\boldsymbol{a}_r^*\|_r=\nu(|\boldsymbol{a}\,\chi_{(0,\infty)}|)=|\boldsymbol{a}|\,\nu(\chi_{(0,\infty)}).$$

Thus Postulate 4 is equivalent to assuming

$$\nu(\chi_{(0,\infty)})<\infty. \tag{4.4}$$

Using Postulates 1–4, we may easily prove the converse to Theorem 4.1:

Theorem 4.2. *If $\bar{N}(\cdot)$ is bounded on* $[0,\infty)$, *then $\mathfrak{V}_r$ has the relaxation property.*

Proof. Here (4.2) holds by hypothesis, and to each $\varepsilon>0$ and each function $\boldsymbol{\phi}_r$, there corresponds, by Remark 3.2, a bounded function $\tilde{\boldsymbol{\phi}}_r$ in $\mathscr{V}_r$ such that

$$\|\tilde{\boldsymbol{\phi}}_r-\boldsymbol{\phi}_r\|_r<\frac{\varepsilon}{2M}, \tag{4.5}$$

[25] *Cf.* [1966, 1, § 6].

[26] *Vid. e.g.* Hille & Phillips [1957, 1, Theorem 2.5.5, p. 26].

with M the number in (4.2). Clearly, (3.12), (4.2), and (4.5) yield

$$\| \boldsymbol{T}^{(\sigma)}(\boldsymbol{\phi}_r - \tilde{\boldsymbol{\phi}}_r) \|_r \leqq \bar{N}(\sigma) \, \| \boldsymbol{\phi}_r - \tilde{\boldsymbol{\psi}}_r \|_r < \frac{\varepsilon}{2} \tag{4.6}$$

for all $\sigma \geqq 0$. It follows from (3.9) that $\boldsymbol{T}^{(\sigma)} \tilde{\boldsymbol{\phi}}_r$ approaches the zero function pointwise as $\sigma \to \infty$, and, by Postulate 4 and the boundedness of $\tilde{\boldsymbol{\phi}}_r$, the functions $\boldsymbol{T}^{(\sigma)} \boldsymbol{\phi}_r$ are dominated in $\mathscr{V}$ in the sense that there is a constant function $\boldsymbol{a}_r^*$ in $\mathscr{V}$ satisfying

$$|\boldsymbol{a}_r^*(s)| \geqq |\boldsymbol{T}^{(\sigma)} \tilde{\boldsymbol{\phi}}_r(s)|, \qquad s \in (0, \infty),$$

for all $\sigma \geqq 0$. Thus, since $\mathfrak{V}_r$ has the dominated convergence property, there exists a number β such that for $\sigma > \beta$

$$\| \boldsymbol{T}^{(\sigma)} \boldsymbol{\phi}_r \|_r < \frac{\varepsilon}{2}. \tag{4.7}$$

Of course the triangle inequality implies

$$\| \boldsymbol{T}^{(\sigma)} \boldsymbol{\phi}_r \|_r \leqq \| \boldsymbol{T}^{(\sigma)} \tilde{\boldsymbol{\phi}}_r \|_r + \| \boldsymbol{T}^{(\sigma)} (\boldsymbol{\phi}_r - \tilde{\boldsymbol{\phi}}_r) \|_r,$$

and therefore (4.6) and (4.7) yield

$$\| \boldsymbol{T}^{(\sigma)} \boldsymbol{\phi}_r \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $\sigma > \beta$; q.e.d.

It is clear from Theorem 3.1 that Postulate 4 may be formulated as follows: For each vector $\boldsymbol{a}$ in V, the set $\mathscr{V}$ contains the function $\boldsymbol{a}^*$ defined by

$$\boldsymbol{a}^*(s) = \boldsymbol{a}, \qquad 0 \leqq s < \infty. \tag{4.8}$$

If $\boldsymbol{\phi}$ is in $\mathscr{V}$, we denote by $\boldsymbol{\phi}^\dagger$ the constant function in $\mathscr{V}$ obeying

$$\boldsymbol{\phi}^\dagger(s) \equiv \boldsymbol{\phi}(0), \qquad 0 \leqq s < \infty,$$

and we call $\boldsymbol{\phi}^\dagger$ the *equilibrium history corresponding* to $\boldsymbol{\phi}$. It is easily shown[27] that, under the assumption of Postulate 4, the relaxation property is equivalent to the assertion that for each $\boldsymbol{\phi}$ in $\mathscr{V}$,

$$\lim_{\sigma \to \infty} \| \boldsymbol{\phi}^{(\sigma)} - \boldsymbol{\phi}^\dagger \| = 0. \tag{4.9}$$

That is, $\mathfrak{V}_r$ *has the relaxation property if and only if the static continuations of a given history approach, in* $\mathfrak{V}$, *the corresponding equilibrium history as the amount of continuation is increased without limit.*

5. Some Direct Consequences of the Relaxation Property

We here consider some easily proved propositions showing that if $\mathfrak{V}_r$ has the relaxation property then, for large t, the asymptotic behavior of $\boldsymbol{f}^t$ in $\mathfrak{V}$ follows,

[27] *Cf.* [1966, 1, Remark 6.1]. In that article it is shown by counter example that Postulates 1–4 alone do not imply the relaxation property. Of course, for the counter example given,

$$\overline{\lim_{\sigma \to \infty}} \; \bar{N}(\sigma) = \infty.$$

in a natural way, the asymptotic behavior of $f(t)$ in V.[28] Our first theorem presupposes our assumption of Postulates 1, 2, and 4, but does not require Postulate 3:

Theorem 5.1. *If $\mathfrak{V}_r$ has the relaxation property, if f and g are admissible functions on $(-\infty,\infty)$, and if*

$$\lim_{t\to\infty} |f(t)-g(t)|=0, \tag{5.1}$$

then

$$\lim_{t\to\infty} \|f^t-g^t\|=0. \tag{5.2}$$

Proof. It follows from (1.2) that (5.1) may be written

$$|f^t(0)-g^t(0)|\to 0 \quad \text{as} \quad t\to\infty.$$

Therefore, by Theorem 3.1, to establish (5.6) it suffices to show that

$$\|f_r^t-g_r^t\|_r\to 0 \quad \text{as} \quad t\to\infty, \tag{5.3}$$

where f_r^t and g_r^t are the restrictions of f^t and g^t to $(0,\infty)$. Let $\varepsilon>0$ be given. By (5.1) and (4.4) there exists a t_ε such that

$$|f(t)-g(t)|<\frac{\varepsilon}{3\nu(\chi_{(0,\infty)})}, \qquad \text{for all } t\geqq t_\varepsilon;$$

that is, for each $t>t_\varepsilon$

$$|f^t(s)-g^t(s)|<\frac{\varepsilon}{3\nu(\chi_{(0,\infty)})}, \qquad \text{for } 0\leqq s\leqq t-t_\varepsilon. \tag{5.4}$$

By (3.9),

$$f_r^t=T^{(t-t_\varepsilon)}f_r^{t_\varepsilon}+f_r^t\,\chi_{(0,t-t_\varepsilon)},$$

$$g_r^t=T^{(t-t_\varepsilon)}g_r^{t_\varepsilon}+g_r^t\,\chi_{(0,t-t_\varepsilon)},$$

and the triangle inequality yields

$$\|f_r^t-g_r^t\|_r\leqq\|(f_r^t-g_r^t)\,\chi_{(0,t-t_\varepsilon)}\|_r+\|T^{(t-t_\varepsilon)}f_r^{t_\varepsilon}\|_r+\|T^{(t-t_\varepsilon)}g_r^{t_\varepsilon}\|_r. \tag{5.5}$$

By (3.1), (5.4), and properties (ii) and (iii) of ν,

$$\begin{aligned}\|(f_r^t-g_r^t)\,\chi_{(0,t-t_\varepsilon)}\|_r&=\nu(|f^t-g^t|\,\chi_{(0,t-t_\varepsilon)})\\ &\leqq\nu\left(\frac{\varepsilon}{3\nu(\chi_{(0,\infty)})}\,\chi_{(0,t-t_\varepsilon)}\right)=\frac{\varepsilon\,\nu(\chi_{(0,t-t_\varepsilon)})}{3\nu(\chi_{(0,\infty)})}\leqq\frac{\varepsilon}{3}.\end{aligned} \tag{5.6}$$

Since $f_r^{t_\varepsilon}$ and $g_r^{t_\varepsilon}$ both belong to $\mathscr{V}_r$ and since we are assuming (4.1), once t_ε and ε are fixed there exists a t' sufficiently large that

$$\|T^{(t-t_\varepsilon)}f_r^{t_\varepsilon}\|_r<\frac{\varepsilon}{3} \quad \text{and} \quad \|T^{(t-t_\varepsilon)}g_r^{t_\varepsilon}\|_r<\frac{\varepsilon}{3}, \qquad \text{for all } t>t'. \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5), we find that for $t>t'$ we have $\|f_r^t-g_r^t\|<\varepsilon$, which means that (5.3) holds; q.e.d.

[28] Our present Theorems 5.1—5.3 generalize Theorems 7—9 of [1966, 1].

Theorem 5.1 has the following immediate corollary:

Theorem 5.2.[29] *If $\mathfrak{V}_r$ has the relaxation property, if $\boldsymbol{f}$ is an admissible function on $(-\infty,\infty)$, and if for some vector $\boldsymbol{a}$ in V*

$$\lim_{t\to\infty} |\boldsymbol{f}(t)-\boldsymbol{a}|=0,$$

then

$$\lim_{t\to\infty} \|\boldsymbol{f}^t-\boldsymbol{a}^*\|=0,$$

and thus for any continuous function $\mathfrak{h}$ on $\mathfrak{V}$

$$\lim_{t\to\infty} \mathfrak{h}(\boldsymbol{f}^t)=\mathfrak{h}(\boldsymbol{a}^*).$$

Here $\boldsymbol{a}^*$ is the constant function (4.8) *with value $\boldsymbol{a}$.*

Another corollary to Theorem 5.1 is

Theorem 5.3. *Let $\mathfrak{h}$ be a constitutive functional;* i.e., *a function mapping $\mathfrak{V}$ into a metric space $\mathscr{M}$ with metric ρ, and suppose that $\mathfrak{V}_r$ has the relaxation property. For two admissible functions $\boldsymbol{f}$ and $\boldsymbol{g}$ on $(-\infty,\infty)$ obeying*

$$\lim_{t\to\infty} |\boldsymbol{f}(t)-\boldsymbol{g}(t)|=0, \tag{5.8}$$

we have

$$\lim_{t\to\infty} \rho\big(\mathfrak{h}(\boldsymbol{f}^t),\mathfrak{h}(\boldsymbol{g}^t)\big)=0, \tag{5.9}$$

provided either (α) $\mathfrak{h}$ is continuous, and, for some number c, the set

$$\mathscr{L}_c=\{\boldsymbol{g}^t \mid t\in[c,\infty)\}$$

is compact in $\mathfrak{V}$, or (β) $\mathfrak{h}$ is uniformly continuous on each bounded subset of $\mathfrak{V}$, and, for some c, the set $\mathscr{L}_c$ has finite diameter in $\mathfrak{V}$.

Proof. If (α) or (β) holds, $\mathfrak{h}$ is uniformly continuous over $\mathscr{L}_c$ in the following strong sense: To each $\varepsilon>0$ there corresponds a $\delta>0$ such that

$$\rho\big(\mathfrak{h}(\boldsymbol{f}^t),\mathfrak{h}(\boldsymbol{g}^t)\big)<\varepsilon,$$

whenever $\boldsymbol{g}^t$ is in $\mathscr{L}_c$ and $\boldsymbol{f}^t$ in $\mathfrak{V}$ obeys

$$\|\boldsymbol{f}^t-\boldsymbol{g}^t\|<\delta. \tag{5.10}$$

But, by Theorem 5.2, there clearly exists a t' greater than c and such that (5.10) holds for all $t>t'$; q.e.d.

If we assume Postulates 1–4, then we have the following examples of applications of Theorem 5.3:[30]

Cases meeting the hypothesis (α). For any number c, $\mathscr{L}_c$ is compact in $\mathfrak{V}$ if (1) $\boldsymbol{g}$ is continuous and periodic on $(-\infty,\infty)$,[31] or if (2) the dimension of V is finite and the V-valued function $\boldsymbol{g}$ is bounded, measurable, and periodic on $(-\infty,\infty)$. Hence, in these cases, the relaxation property, (5.8), and an assumption

[29] *Cf.* Theorem 7 of [1966, 1].

[30] *Cf.* [1966, 1, p. 118].

[31] This case is discussed in detail in Theorem 8 and Remark 8.2 of [1966, 1].

of admissibility for f together suffice for (5.9), provided $\mathfrak{h}$ is a *continuous* function on $\mathfrak{V}$.

A case meeting the hypothesis ($\mathscr{S}$). Clearly, since we here assume Postulate 4, $\mathscr{L}_c$ has finite diameter in $\mathfrak{V}$ whenever g is measurable and bounded on $(-\infty, \infty)$. Hence, boundedness and measurability for g, the relaxation property, (5.8), and an assumption of admissibility for f together yield (5.9), provided $\mathfrak{h}$ is *uniformly continuous on bounded subsets of* $\mathfrak{V}$.

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