The Riemann Problem for a van der Waals Fluid with Entropy Rate Admissibility Criterion - Isothermal Case

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1. Introduction

The system of equations governing isothermal flow of an inviscid and compressible fluid in Lagrangian coordinates is given by

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0, \end{aligned} \tag{1.1}$$

where u, v, and p are the velocity, the specific volume, and the pressure of the fluid, respectively. For the isothermal flow of an ideal gas, $p(v) = RT/v$, where R is a universal constant, and T is the constant absolute temperature. In this case, the system (1.1) is hyperbolic. On the other hand, the equation of state of a van der Waals fluid is

$$p(v) = \frac{RT}{v-b} - \frac{a}{v^2}, \tag{1.2}$$

where a and b are the characteristic constants of the fluid. Thus when the temperature is sufficiently low, $p'(v)$ is positive, and thus the system (1.1) is elliptic on a certain interval (α, β), while $p'(v)$ is negative on (b, α) (liquid phase) as well as on (β, ∞) (vapor phase) on which (1.1) is hyperbolic.

The Riemann problem for (1.1) is a special initial-value problem in which the initial data are of the form

$$(u(x,0), v(x,0)) = \begin{cases} (u_0, v_0) & x < 0 \\ (u_1, v_1) & x > 0. \end{cases} \tag{1.3}$$

Since both the system (1.1) and the data (1.3) are invariant under the transformation $(x, t) \to (\gamma x, \gamma t)$, $\gamma > 0$, the solution of (1.1), (1.3) is a function of x/t, *i.e.*, it is a fan of waves that emanate from the origin and propagate with individual speeds. For hyperbolic problems the structure of solutions of the Riemann problem has been investigated thoroughly (see LEIBOVICH [1], and LIU [2]). In particular, it is known that the problem specified by (1.1), and (1.3) generally admits several solutions. Admissibility criteria, such as the LAX entropy condition [3], the viscosity criterion [4], the entropy rate admissibility criterion [5], [6], and the extended entropy condition [2], have been employed in order to single out a physically admissible solution.

Failure of uniqueness in the problem set by (1.1), and (1.3) arises also when the system is not hyperbolic, for instance for the van der Waals fluid (1.2), and so admissibility criteria have to be postulated there, too. I refer to the work of JAMES [7], SHEARER [8], and SLEMROD [9] for a discussion of the solution of the Riemann problem in that case. In [9] admissibility is decided by means of the "capillarity argument." Here I employ the entropy rate admissibility criterion proposed by DAFERMOS [5], [6]. Since the system (1.1) describes isothermal motions, I refer to this criterion as the energy rate admissibility criterion. The main interest this paper offers is its proof that that criterion can be applied to a nonhyperbolic system.

This paper has five sections. In Section 2 I formulate the problem; in Section 3 I discuss the fundamental properties of the energy rate along the shock curve and the phase boundary curve. In Section 4 I treat the possible solutions which join two constant states lying in different phases by backward waves, phase boundaries, and forward waves. Also, I compare the energy rate for different solutions. Finally, in Section 5 I discuss a special Riemann problem and show that the well known Maxwell construction is admissible according to the energy rate criterion.

2. Preliminaries

Consider a van der Waals fluid in which the equation of state is given by (1.2). If the temperature is sufficiently low, a typical isotherm is presented in Figure 1. The p-v curve is not monotone; thus the system (1.1) is not hyperbolic. Since we do not need to restrict attention to the specific equation (1.2), we will assume in what follows that $p(v)$ is any function which satisfies the following conditions:

$$p'(v) < 0 \quad \text{for } 0 < b < v < \alpha, \quad \beta < v, \tag{C1}$$

$$p'(\alpha) = p'(\beta) = 0, \tag{C2}$$

$$p'(v) > 0 \quad \text{for } \alpha < v < \beta, \tag{C3}$$

$$p(v) > 0 \quad \text{for } b < v < \infty, \tag{C4}$$

$$p''(v) > 0 \quad \text{for } b < v < v_*, \; v^* < v < \infty, \tag{C5}$$

$$(\alpha < v_* < \beta, \; \beta < v^* < \infty),$$

$$p''(v) < 0 \quad \text{for } v_* < v < v^*. \tag{C6}$$

The intervals (b, α) and (β, ∞) will be called the α-phase (liquid phase) and the β-phase (vapor phase), respectively. The horizontal line for which the areas A

and B are equal is called the Maxwell line. We denote the pressure at the Maxwell line by p_m. The values of v in the α-phase and the β-phase at which the pressure is equal to p_m are denoted by α_m and β_m, respectively. The fluid may behave in totally different ways at pressures just below and just above p_m. Various properties change discontinuously in the transition. This fact motivates the following definitions; the homogeneous solution $v(x) =$ constant is

(S1) stable if $b < v < \alpha_m$ or $\beta_m < v$.

(S2) neutrally stable if $v = \alpha_m$ or $v = \beta_m$.

(S3) metastable if $\alpha_m < v \leqq \alpha$ or $\beta \leqq v < \beta_m$.

(S4) unstable if $\alpha < v < \beta$.

The state (S3) may be observed, yet a small perturbation will change the phase drastically.

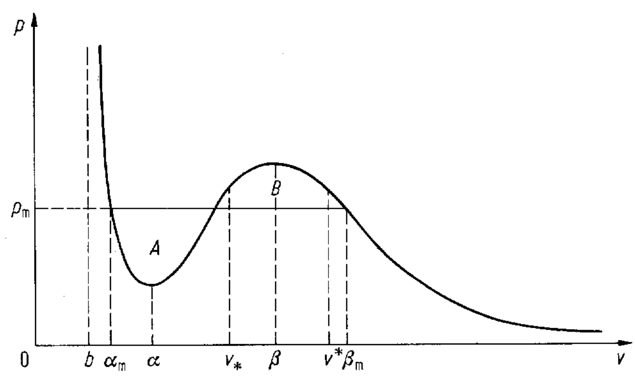

Fig. 1. A typical isotherm of a van der Waals fluid.

We discuss the Riemann problem (1.1), (1.3) with $p(v)$ satisfying (C1)–(C6). The solution of the Riemann problem consists of constant states joined by rarefaction waves or jump discontinuites. I explain them very briefly.

A rarefaction wave is a wedge in the x-t plane in which one of the Riemann invariants

$$u \mp \int^{v} \sqrt{p'(w)}\, dw$$

is constant. It is clear that rarefaction waves are defined only in the hyperbolic region. When two constant states (u_-, v_-) and (u_+, v_+) are adjacent to a forward (backward) rarefaction wave on the left and right, then

$$u_+ - u_- = \underset{(+)}{-} \int_{v_-}^{v_+} \sqrt{-p'(w)}\, dw. \tag{2.1}$$

Using the fact that the characteristic curves expand in the rarefaction wave region, we have the following relation between v_- and v_+, if $p'' \neq 0$.

$$
\begin{aligned}
&\text{forward rarefaction wave:} && v_- > v_+ \quad \text{if } p'' > 0, \\
& && v_- < v_+ \quad \text{if } p'' < 0, \\
&\text{backward rarefaction wave:} && v_- < v_+ \quad \text{if } p'' > 0, \\
& && v_- > v_+ \quad \text{if } p'' < 0.
\end{aligned}
\tag{2.2}
$$

A jump discontinuity satisfies the Rankine-Hugoniot condition, which takes the form

$$\sigma[u_+ - u_-] = [p(v_+) - p(v_-)], \quad \sigma[v_+ - v_-] = -[u_+ - u_-], \tag{2.3}$$

where σ is the speed of propagation of the jump discontinuity, and (u_-, v_-) and $u_+, v_+)$ are the states on the left and the right of the jump discontinuity. Solving (2.3) for σ, we obtain

$$\sigma = \pm \sqrt{-\frac{p_+ - p_-}{v_+ - v_-}}. \tag{2.4}$$

Substituting (2.4) into (2.3b), we obtain the following relation between (u_-, v_-) and (u_+, v_+):

$$u_+ - u_- = \mp \sqrt{-\frac{p_+ - p_-}{v_+ - v_-}}(v_+ - v_-). \tag{2.5}$$

We observe that there are two families of jump discontinuities, the forward one when $\sigma > 0$, and the backward one when $\sigma < 0$. In particular, if both v_- and v_+ lie in the same phase, jump discontinuities will be called *shocks*, while if v_- and v_+ lie in different phases, jump discontinuities will be called *phase boundaries*.

As in the case of rarefaction waves there is a physical restriction on the relation between v_- and v_+ for shocks. This restriction is obtained by applying an appropriate admissibility criterion such as the energy (entropy) rate criterion, which will be discussed later in this section. If $p'' \neq 0$, we have the following relation between v_- and v_+.

$$
\begin{aligned}
&\text{forward shock:} && v_- < v_+ \quad \text{if } p'' > 0, \\
& && v_- > v_+ \quad \text{if } p'' < 0, \\
&\text{backward shock:} && v_- > v_+ \quad \text{if } p'' > 0, \\
& && v_- < v_+ \quad \text{if } p'' < 0.
\end{aligned}
\tag{2.6}
$$

In (2.1) or (2.5) it is easily seen that the set of (u_+, v_+) $((u_-, v_-))$ which can be connected to (u_-, v_-) $((u_+, v_+))$ on the right (left), by a rarefaction wave, a shock, or a phase boundary forms a one-parameter family of states.

We define a forward wave curve to be the set of (u, v) which is connected to a given (u_G, v_G), on the right, by a forward shock, a forward rarefaction wave, or a combination of them. We define a backward wave curve in the same manner.

In both cases the set of (u, v) on the wave curve through (u_G, v_G) belongs to the same phase. We should notice that as p'' changes sign in the β-phase, if v_G is in the β-phase and $p''(v_G) \neq 0$, there will be a set of (u, v) on the forward (backward) wave curve through (u_G, v_G) connected by a combination of a forward (backward) shock and a forward (backward) rarefaction wave (see GREENBERG [10] for the details). The forward and backward wave curves pass through (u_G, v_G) and the case in which $p'' > 0$ is depicted in Figure 2.

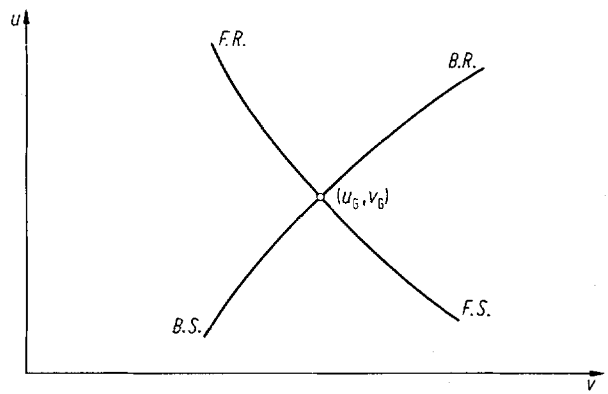

Fig. 2. The backward and forward wave curves ($p'' > 0$) through (u_G, v_G). Here B.S. and F.R., *etc.*, stand for a backward shock and a forward rarefaction wave, *etc.*

In the same manner we define the phase boundary curve as the set of (u, v) connected to (u_G, v_G), on the right, by a backward or a forward phase boundary. Unlike the backward (or forward) wave curve, the phase boundary curve for a given (u_G, v_G) does not pass through (u_G, v_G). The phase boundary curve for the case where v_G is close to α_m is drawn in Figure 3.

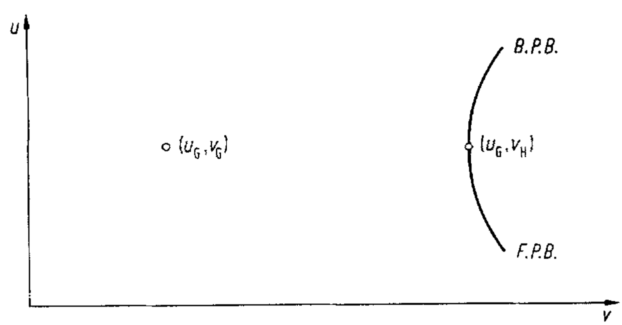

Fig. 3. The backward and forward phase boundary curves associated with (u_G, v_G). Here B.P.B. (F.P.B.) stands for a backward (forward) phase boundary, and v_H satisfies $p(v_H) = p(v_G)$.

We shall see that the Riemann Problem generally admits several weak (piecewise smooth selfsimilar) solutions. As an attempt to pick out the physically meaningful solution in the class of solutions discussed in Section 4, we assume that the admissible solution satisfies the energy rate admissbility criterion. This criterion is called the entropy rate admissibility criterion by DAFERMOS [5], [6], and is based on the following argument.

The density of mechanical energy for the fluid is given by

$$E(u, v) = \tfrac{1}{2} u^2 + \int^{v} (-p(w))\, dw.$$

As a consequence of the second law of thermodynamics, mechanical energy is dissipated during isothermal or isentropic flow of the fluid. In particular, in the Riemann problem the rate of decay of the total mechanical energy is given by

$$D_+ E(u, v) = \sum_{\substack{\text{jump}\\ \text{discontinuities}}} \sigma(\tau)\, A(v_-, v_+), \tag{2.7}$$

where $\sigma(\tau)$ is the speed of the jump discontinuity and

$$A(v_-, v_+) = \tfrac{1}{2}[p(v_-) + p(v_+)]\,(v_+ - v_-) + \int_{v_-}^{v_+} p(w)\, dw.$$

Different solutions generally dissipate energy at different rates. According to the energy rate admissibility criterion a solution is admissible if it minimizes (2.7). Note that in the following sections we shall apply this admissibility criterion to a small class of solution, not to all possible solutions.

3. Fundamental Properties of Energy Rate

We discuss here the fundamental properties of the energy rate for backward shocks, forward shocks, and phase boundaries.

Lemma 3.1. *Assume that $p'(v) < 0$, $p''(v) > 0$ or $(p''(v) < 0)$ for v in an interval (a, b). Let (u_-, v_-) be a given point with $v_- \in (a, b)$, and let (u_+, v_+) be a point on the forward shock curve through (u_-, v_-). Then the energy rate is a monotonically decreasing (increasing) function of v_+ along the forward shock curve between v_- and v_+. For the backward shock curve through (u_-, v_-) the energy rate is a monotonically increasing (decreasing) function of v_+.*

Proof. Along the forward shock curve through (u_-, v_-), the energy rate is given by

$$E = -\sigma \left\{ \tfrac{1}{2}(p_+ + p_-)(v_+ - v_-) - \int_{v_-}^{v_+} p(w)\, dw \right\}, \tag{3.1}$$

where

$$\sigma = \sqrt{-\frac{p_+ - p_-}{v_+ - v_-}}\,.$$

Differentiation of E with respect to v_+ yields

$$\frac{dE}{dv_+} = \frac{1}{4\sigma}(\sigma^2 - \lambda_+^2)\left(3p_+ - p_- - \frac{2\int_{v_-}^{v_+} p(w)\,dw}{v_+ - v_-}\right), \tag{3.2}$$

where $\lambda_+ = \sqrt{-p'_+}$. If $p'' > 0$,

$$\lambda_+^2 \underset{(<)}{>} \sigma^2,\ p_+ \underset{(<)}{>} p_-,\ p_+ \underset{(<)}{>} \frac{\int_{v_-}^{v_+} p(w)\,dw}{v_+ - v_-}, \qquad \text{for } v_- \underset{(<)}{>} v_+.$$

We see that dE/dv_+ in (3.2) has negative sign and E is a decreasing function on v_+. On the other hand, if $p'' < 0$,

$$\lambda_+^2 \underset{(<)}{>} \sigma^2,\ p_+ \underset{(>)}{<} p_-,\ p_+ \underset{(>)}{<} \frac{\int_{v_-}^{v_+} p(w)\,dw}{v_+ - v_-}, \qquad \text{for } v_- \underset{(>)}{<} v_+.$$

In this case dE/dv_+ in (3.2) has negative sign, so that (3.1) is an increasing function of v_+.

Since the energy rate along the forward shock curve through (u_-, v_-) is obtained by reversing the sign of the right-hand side in (3.1), the corresponding result can be easily obtained. Q.E.D.

Remark 3.1. If we apply Lemma 3.1 to shocks with the physical requirement that the energy be dissipated across shocks, we verify the relation (2.6) between v_- and v_+.

Next, consider the energy rate for phase boundaries. Suppose (u_-, v_-) is a given point with v_- in the α-phase. When it exists, we let v_a be the point in the β-phase at which $p(v_a) = p(v_-)$. Similarly, we denote by v_b the point in the β-phase at which

$$\tfrac{1}{2}(v_b - v_-)(p_b + p_-) - \int_{v_b}^{v_-} p(w)\,dw = 0, \tag{3.3}$$

provided, of course, such a point exists. The left-hand side of (3.3) corresponds to the signed area between the chord joining $(v_-, p(v_-))$ to $(v_b, p(v_b))$ and the graph of $p(v)$ between v_- and v_b.

Lemma 3.2. *Let $v_- \geqq \alpha_m$. Then, the energy rate is an increasing (decreasing) function of v_+ along the forward (backward) phase boundary curve associated with (u_-, v_-). On the other hand, if $v_- < \alpha_m$, then the energy rate has only one local minimum (maximum) in the interval (v_a, v_b) along the forward (backward) phase boundary curve associated with (u_-, v_-).*

Proof. Set

$$Q = (v_+ - v_-)(3p_+ - p_-) - 2\int_{v_-}^{v_+} p(w)\,dw.$$

Then, if $v_- \geqq \alpha_m$, Q is negative for $v_+ \geqq v_a$. Hence, the energy rate is an increasing function of v_+ along the forward phase boundary curve associated with (u_-, v_-). If $v_- < a_m$, E in (3.1) is zero when $v_+ = v_a, v_b$, and is negative, between v_a and v_b. Since dQ/dv_+ is negative, and Q is positive at $v_+ = v_a$ and is negative at $v_+ = v_b$, Q changes sign from positive to negative once between v_a and v_b. Hence dE/dv^+ in (3.2) changes sign from negative to positive in the interval (v_a, v_b) along the forward phase boundary curve associated with (u_-, v_-). The corresponding result for backward phase boundaries can be obtained in the same manner. Q.E.D.

Lemmas 3.1 and 3.2 show that shocks and phase boundaries exhibit different behavior.

4. Possible Solutions to the Riemann Problem and Comparison of their Energy Rate

As a preliminary step in obtaining the admissible solution to the Riemann problem (1.1), (1.3), which will be discussed in the next section, we study here all possible solutions to the Riemann problem, *i.e.*, all connections of the states (u_0, v_0) and (u_1, v_1) in (1.3) by means of shock waves, rarefaction waves, and phase boundaries. Here we will only consider solutions to the Riemann problem with the form depicted in Figure 4, that is, with v_0 close to α_m and v_1 close to β_m. The case where v_0 is close to β_m and v_1 is close to α_m can be treated in a similar manner. In Figure 4 the forward (backward) wave could be either a forward (backward) shock or a rarefaction wave, and the phase boundary could be either forward or backward. As p'' changes sign in the β-phase, the forward wave may be a combination of shocks and rarefaction waves. We denote by (u_L, v_L) the intermediate constant state on the left of the phase boundary and by (u_R, v_R) the intermediate constant state on the right of the phase boundary. Then v_L is in the α-phase and v_R is in the β-phase.

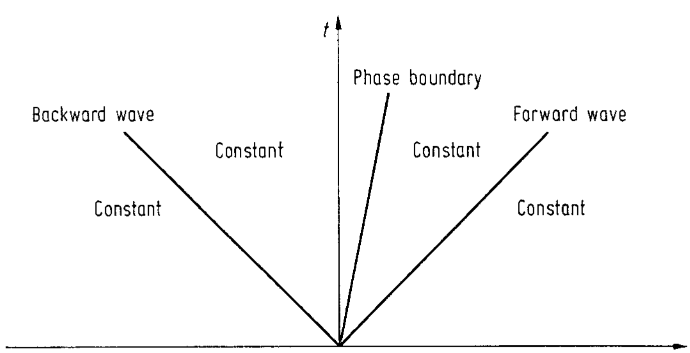

Fig. 4. A typical solution configuration of the Riemann problem.

We note that there might be solutions of other types, for instance having more than one phase boundary (an odd number) between the forward and backward waves and that if v_0 and v_1 are far from α_m and β_m, the connection may be as in Figure 5, with a phase boundary connecting the α-phase and the β-phase, as shown in Figure 6. However, I do not treat these cases in this paper.

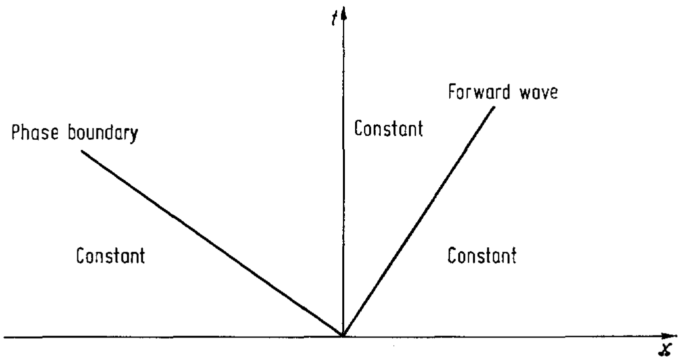

Fig. 5. Another solution configuration of the Riemann problem.

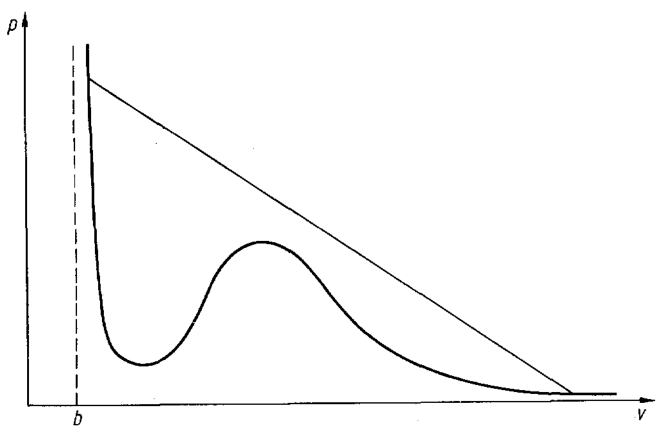

Fig. 6. The connection between α-phase and β-phase corresponding to the solution in Figure 5.

As the backward and forward waves may be either shocks or rarefaction waves and there are backward and forward phase boundaries, there are 8 possible combinations of connections, namely,

(1) B.R.—F.P.B.—F.R.,	(5) B.R.—B.R.B.—F.R.,
(2) B.R.—F.P.B.—F.S.,	(6) B.R.—B.P.B.—F.S.,
(3) B.S.—F.P.B.—F.R.,	(7) B.S.—B.P.B.—F.R.,
(4) B.S.—F.P.B.—F.S.,	(8) B.S.—B.P.B.—F.S.

Here, for example, B.S., F.P.B., and F.R. stand for "backward shock", "forward phase boundary", and "forward rarefaction wave", respectively. Therefore, (1) means that (u_0, v_0) is joined to (u_L, v_L) by a backward rarefaction wave, then (u_L, v_L) is joined to (u_R, v_R) by a forward phase boundary, and (u_R, v_R) is joined to (u_1, v_1) by a forward rarefaction wave. We derive the differential equations which are satisfied by (1)–(8). To avoid complications, we treat these cases only, although p'' changes sign in the β-phase.

Case (1) (and (5)). In this case u_0 is joined to u_L by a backward rarefaction wave, hence,

$$u_L = u_0 + \int_{v_1}^{v_L} \lambda(w)\, dw, \tag{4.1}$$

where $\lambda(w) = \sqrt{-p'(w)}$. Then, u_L is joined to u_R by the forward (backward for (5)) phase boundary, namely,

$$u_R = u_L \underset{(+)}{-} \sqrt{-\frac{p_R - p_L}{v_R - v_L}}\,(v_R - v_L). \tag{4.2}$$

Also u_R is joined to u_1 by the forward rarefaction wave; therefore

$$u_1 = u_R - \int_{v_R}^{v_1} \lambda(w)\, dw. \tag{4.3}$$

From (4.1)–(4.3) we deduce

$$u_0 + \int_{v_0}^{v_L} \lambda(w)\, dw \underset{(+)}{-} \sigma_p(v_R - v_L) - \int_{v_R}^{v_1} \lambda(w)\, dw = u_1, \tag{4.4}$$

here $\sigma_p = \sqrt{-\dfrac{p_R - p_L}{v_R - v_L}}$. If we differentiate (4.4) with respect to v_L, regarding v_R as a function of v_L, we obtain the differential equation

$$\frac{dv_R}{dv_L} = \frac{(\lambda_L \underset{(-)}{+} \sigma_p)^2}{(\lambda_R \underset{(+)}{-} \sigma_p)^2}, \tag{4.5}$$

where $\lambda_L = \sqrt{-p_L'}$ and $\lambda_R = \sqrt{-p_R'}$.

Case (2) (and (6)). As above we have the following relations for u_L and v_R:

$$u_L = u_0 + \int_{v_0}^{v_L} \lambda(w)\, dw, \tag{4.6}$$

$$u_R = u_L \underset{(+)}{-} \sigma_p(v_R - v_L), \tag{4.7}$$

$$u_1 = u_R - \sigma_F(v_1 - v_R), \tag{4.8}$$

where $\sigma_F = \sqrt{-\dfrac{p_R - p_1}{v_R - v_1}}$. From (4.6)–(4.8) we see

$$u_0 + \int_{v_0}^{v_L} \lambda(w)\, dw \underset{(+)}{-} \sigma_p(v_R - v_L) - \sigma_F(v_1 - v_R) = u_1. \tag{4.9}$$

Differentiating (4.9) with respect to v_L, we obtain

$$\frac{dv_R}{dv_L}=\frac{\sigma_F(\lambda_L \underset{(-)}{+} \sigma_p)^2}{(\sigma_F \underset{(-)}{+} \sigma_p)\,(\lambda_R^2 \underset{(-)}{+} \sigma_F\sigma_p)}. \tag{4.10}$$

Case (*3*) (*and* (*7*)). As relations for v_L, v_R we have

$$u_L = u_0 + \sigma_B(v_1 - v_0), \tag{4.11}$$

$$u_R = u_L \underset{(-)}{+} \sigma_p(v_R - v_L), \tag{4.12}$$

$$u_1 = u_R - \int_{v_R}^{v_1} \lambda(w)\, dw, \tag{4.13}$$

where $\sigma_B = \sqrt{-\dfrac{p_L - p_0}{v_L - v_0}}$. From (4.11)–(4.13) we obtain

$$\frac{dv_r}{dv_L}=\frac{(\sigma_B \underset{(+)}{-} \sigma_p)\,(\lambda_L^2 \underset{(+)}{-} \sigma_B\sigma_p)}{\sigma_B(\lambda_R \underset{(-)}{+} \sigma_p)^2}. \tag{4.14}$$

Case (*4*) (*and* (*8*)). In a similar manner we derive the differential equation

$$\frac{dv_R}{dv_L}=\frac{\sigma_F(\sigma_B \underset{(+)}{-} \sigma_p)\,(\lambda_L^2 \underset{(+)}{-} \sigma_B\sigma_p)}{\sigma_B(\sigma_F \underset{(-)}{+} \sigma_p)\,(\lambda_L^2 \underset{(-)}{+} \sigma_F\sigma_p)}. \tag{4.15}$$

We examine the sign of (4.5), (4.10), (4.14), and (4.15). We have the following relations among σ_B, σ_F, σ_p, λ_L, and λ_R.

$$0 \leqq \sigma_p < \sigma_B, \quad 0 \leqq \sigma_p < \sigma_F,$$

$$\sigma_B < \lambda_L, \quad \sigma_F < \lambda_R.$$

Hence we easily see that dv_R/dv_L is positive in all cases. In particular, if $\sigma_p = 0$, namely, $p_L = p_R$, then

$$\frac{dv_R}{dv_L}=\frac{\lambda_L^2}{\lambda_R^2}=\frac{p_L'}{p_R'}. \tag{4.16}$$

Next we consider how the energy rate varies as a function of v_L. The energy rate for the backward shock is given by

$$E_B = \sigma_B\left\{\tfrac{1}{2}(v_L - v_0)(p_L + p_0) - \int_{v_0}^{v_L} p(w)\, dw\right\}. \tag{4.17}$$

If we differentiate (4.17) with respect to v_L, we obtain

$$\frac{dE_B}{dv_L} = -\frac{1}{4\sigma_B}(\sigma_B^2 - \lambda_L^2)\left(3p_L - p_0 - \frac{2\int_{v_0}^{v_L} p(w)\, dw}{v_L - v_0}\right). \tag{4.18}$$

The energy rate for the forward (backward) phase boundary is given by

$$E_p = \underset{(-)}{+} \sigma_p\left\{\tfrac{1}{2}(v_R - v_L)(p_R + p_L) - \int_{v_L}^{v_R} p(w)\, dw\right\}. \tag{4.19}$$

Differentiating (4.19) with respect to v_L, regarding v_R as a function of v_L, we get

$$\frac{dE_p}{dv_L} = \genfrac{}{}{0pt}{}{+}{(-)} \frac{1}{4\sigma_p}\left\{\frac{dv_R}{dv_L}(\sigma_p^2 - \lambda_p^2)\left(3p_R - p_L - \frac{2\int_{v_L}^{v_R} p(w)\,dw}{v_R - v_L}\right)\right.$$
$$\left. - (\sigma_p^2 - \lambda_L^2)\left(3p_L - p_R - \frac{2\int_{v_L}^{v_R} p(w)\,dw}{v_R - v_L}\right)\right\}. \tag{4.20}$$

For the forward shock the energy rate is given by

$$E_F = -\sigma_F\left\{\tfrac{1}{2}(v_1 - v_R)(p_R + p_1) - \int_{v_R}^{v_1} p(w)\,dw\right\}. \tag{4.21}$$

Differentiation of (4.21) yields

$$\frac{dE_F}{dv_L} = -\frac{1}{4\sigma_F}\frac{dv_R}{dv_L}(\sigma_F^2 - \lambda_R^2)\left(3p_R - p_1 - \frac{2\int_{v_1}^{v_R} p(w)\,dw}{v_R - v_1}\right). \tag{4.22}$$

Set

$$A = 3p_R - p_L - \frac{2\int_{v_L}^{v_R} p(w)\,dw}{v_R - v_L}, \quad B = 3p_L - p_R - \frac{2\int_{v_L}^{v_R} p(w)\,dw}{v_R - v_L},$$

$$C = 3p_L - p_0 - \frac{2\int_{v_0}^{v_L} p(w)\,dw}{v_L - v_0}, \quad D = 3p_R - p_1 - \frac{2\int_{v_1}^{v_R} p(w)\,dw}{v_R - v_1}.$$

The signs of A and B will play important roles in the next section, so we discuss it in the following

Lemma 4.1. *If p_R and p_L lie above the Maxwell line and are equal $(p_R = p_L > p_m)$, then A and B are positive. On the other hand, if p_R and p_L lie below the Maxwell line and are equal $(p_R = p_L < p_m)$, then A and B are negative. If $p_R = p_L = p_m$, then $A = B = 0$, and dA/dv_L and dB/dv_L have negative sign.*

Proof. The first two statements are easy to verify, so we prove only the third statement. If $p_R = p_L = p_m$, then

$$\frac{dv_R}{dv_L} = \frac{\lambda_L^2}{\lambda_R^2},$$

$$\int_{v_L}^{v_R} p(w)\,dw = \tfrac{1}{2}(v_R - v_L)(p_R + p_L) = (v_R - v_L)\,p_m.$$

Since $A = B = 0$ at the Maxwell line $(p_R = p_L = p_m)$, A and B are expressed as

$$A = -2\lambda^2(\alpha_m)(v_L - \alpha_m) + 0((v_L - \alpha_m)),$$
$$B = -2\lambda^2(\alpha_m)(v_L - \alpha_m) + 0((v_L - \alpha_m)),$$

provided v_L is close to α_m. Q.E.D.

We can obtain an analogous result for C and D. It is easy to see from (2.6) that C and D are positive if $p'' > 0$ and negative if $p'' < 0$. As v_L and v_R approach v_0 and v_1, respectively, C and D approach zero. Furthermore, since v_L and v_R satisfy the system of differential equations (4.5), (4.10), (4.14), or (4.15), we deduce

$$\frac{dC}{dv_L} = \frac{dD}{dv_L} = -2\lambda^2$$

at $v_L = v_0$.

5. Special Riemann Problem

In this section we discuss a rather special Riemann problem. Specifically, we treat the case where $u_0 = u_1$ and $p_0 = p_1$. We have the following result concerning the sign of dE/dv_L (E denotes the total energy rate).

Theorem 5.1. *Suppose that* $u_0 = u_1, p_0 = p_1$, *and* v_R *obeys one of the differential equations* (4.5), (4.10), (4.14), *or* (4.15), *as appropriate. Then, as* v_L *approaches* v_0, dE/dv_L *approaches a negative number if* $p_0 < p_m$, *a positive number if* $p_0 > p_m$, *and zero if* $p_0 = p_m$.

Proof. From (4.18) and (4.22) we easily see that dE_B/dv_L and dE_F/dv_L approach zero as v_L approaches v_0. Therefore, it remains to examine the sign of dE_p/dv_L as v_L approaches v_0. Combining the differential equations (4.5), (4.10), (4.14), and (4.15) with (4.20), and taking the limit of (4.20) as v_L approaches v_0, we obtain

from (4.5): $$\lim_{v_L \to v_0} \frac{dE_p}{dv_L} = -\frac{\lambda_0}{2\lambda_1}(\lambda_1 A + \lambda_0 B),$$

from (4.10): $$\lim_{v_L \to v_0} \frac{dE_p}{dv_L} = -\frac{\lambda_0}{4\lambda_1}\{\lambda_0(A + B) + 2\lambda_1 A\},$$

from (4.14): $$\lim_{v_L \to v_0} \frac{dE_p}{dv_L} = -\frac{\lambda_0}{4\lambda_1}\{\lambda_1(A + B) + 2\lambda_0 B\},$$

and from (4.15): $$\lim_{v_L \to v_0} \frac{dE_p}{dv_L} = -\frac{\lambda_0}{4\lambda_1}(\lambda_0 + \lambda_1)(A + B).$$

Recalling Lemma 4.1, we conclude that the limit of dE/dv_L as v_L approaches v_0 is negative if $p_0 < p_m$, positive if $p_0 > p_m$, and zero if $p_0 = p_m$. Q.E.D.

As a consequence of Theorem 5.1 we deduce the following

Corollary 5.1. *If* $u_0 = u_1$ *and* $p_0 = p_1 < p_m$ *or* $p_0 = p_1 > p_m$, *the connection between* (u_0, v_0) *and* (u_1, v_1) *by the stationary phase boundary* $((u_0, v_0) = (u_L, v_L)$, $(u_1, v_1) = (u_R, v_R))$ *is not admissible according to the energy rate admissibility criterion.*

The case where $p_0 = p_1 = p_m$ is more delicate, yet we can show the following

Theorem 5.2. *Suppose* $u_0 = u_1$ *and* $p_0 = p_1 = p_m$. *Then the connection between* (u_0, v_0) *and* (u_1, v_1) *by the stationary phase boundary is admissible in the sense that it minimizes the energy rate for* v_L *close to* v_0.

Proof. If we draw the backward wave curve through (u_0, v_0) and the forward wave curve through (u_1, v_1) in the u-v plane, there are three possibilities, depicted in Figures 7, 8, 9, depending on the value of v^* (the v coordinate of the inflection

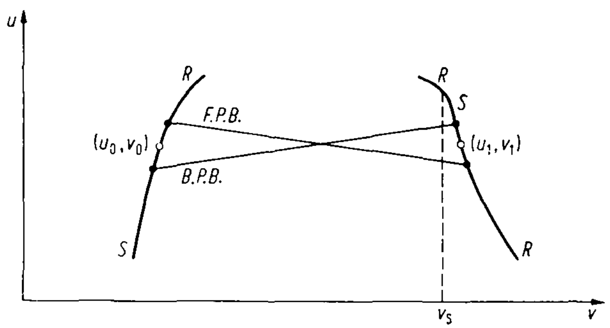

Fig. 7. The backward and forward wave curves ($p'' > 0$ in the neighborhood of v_1).

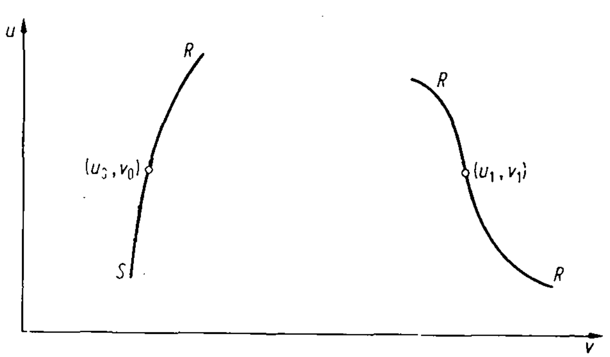

Fig. 8. The backward and forward wave curves ($p'' = 0$ at v_1).

point in the β-phase). Since the proof is similar in all three cases, we shall only consider the case of Figure 7 and will show that dE/dv_L is positive if $v_L > v_0$ and negative if $v_L < v_0$, for v_L close to v_0.

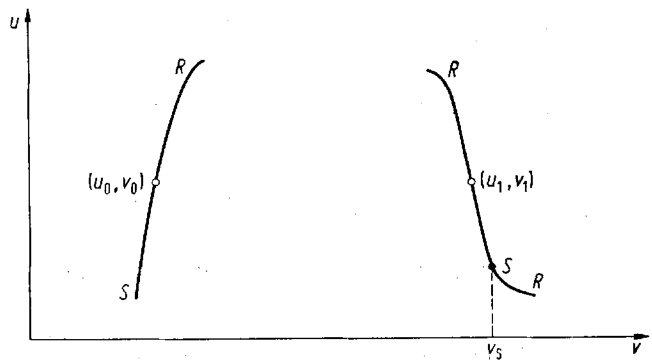

Fig. 9. The backward and forward wave curves ($p'' < 0$ in the neighborhood of v_1).

Using the fact that dv_R/dv_L is positive, we find that the connection between (u_0, v_0) and (u_1, v_1) is of type (1) if $v_L < v_0$, and of type (8) if $v_L < v_0$ and v_L is close to v_0. If $v_1 \ll v_0$, the forward wave is a combination of a shock and a rarefaction wave (see GREENBERG [10] for the details).

In case (1), dv_R/dv_L is given by (4.5) and dE/dv_L is

$$\frac{dE}{dv_L} = \frac{1}{4\sigma_p}\left\{\frac{dv_R}{dv_L}(\sigma_p^2 - \lambda_R^2)\,A - (\sigma_p^2 - \lambda_L^2)\,B\right\}. \tag{5.1}$$

Combining (4.5) with (5.1), we find

$$\frac{dE}{dv_L} = -\frac{\lambda_L + \sigma_p}{\lambda_R - \sigma_p}\{(\lambda_L + \lambda_R)(A + B) - (v_R - v_L)(\lambda_L \lambda_R + \sigma_p^2)\,\sigma_p\}. \tag{5.2}$$

Since, for $v_L > v_0$,

$$A + B = 4\left\{\tfrac{1}{2}(p_R + p_L) - \frac{\int_{v_L}^{v_R} p(w)\,dw}{v_R - v_L}\right\} < 0,$$

dE/dv_L in (5.2) is positive for $v_L > v_0$.

In case (8), dv_R/dv_L is given by (4.15) and dE/dv_L is

$$\frac{dE}{dv_L} = -\frac{1}{4\sigma_B}(\sigma_B^2 - \lambda_L^2)\,C - \frac{1}{4\sigma_F}\frac{dv_R}{dv_L}(\sigma_F^2 - \lambda_R^2)\,D$$
$$+ \frac{1}{4\sigma_p}\left\{\frac{dv_R}{dv_L}(\sigma_p^2 - \lambda_R^2)\,A + (\sigma_p^2 - \lambda_L^2)\,B\right\}. \tag{5.4}$$

It is no longer easy to find a relation like (5.2). We expand (5.4) in series of powers of $(v_L - v_0)$ and examine the terms of lowest order. Then

$$\begin{aligned}\frac{dE_S}{dv_L} &= -\frac{1}{4\sigma_B(\sigma_F+\sigma_p)(\lambda_R^2+\sigma_F\sigma_p)}\{\sigma_F\lambda_R^2(\sigma_B^2-\lambda_L^2)\,C \\ &\quad - \sigma_B\lambda_L^2(\sigma_F^2-\lambda_R^2)\,D + O(\sigma_p(C+D))\} \\ &= \frac{\lambda_0}{4}\left\{p_0'' + p_1''\left(\frac{\lambda_0}{\lambda_1}\right)^5\right\}(v_L-v_0)^2 + o((v_L-v_0)^2), \end{aligned} \tag{5.5}$$

$$\begin{aligned}\frac{dE_p}{dv_L} &= -\frac{1}{4\sigma_B(\sigma_F+\sigma_p)(\lambda_R^2+\sigma_F\sigma_p)\,\sigma_p}\{-\sigma_F\sigma_B\lambda_L^2\lambda_R^2(A-B) \\ &\quad - \sigma_p(\sigma_F\lambda_R^2(\lambda_L^2+\sigma_B^2)\,A + \sigma_B\lambda^2(\lambda_L^2+\sigma_F^2)\,B) + O(\sigma_p^2)\} \\ &= -\lambda_0^2(v_R-v_L)\,\sigma_p + \lambda_0^3\left(1+\frac{\lambda_0}{\lambda_1}\right)(v_L-v_0) + o((v_L-v_0)), \end{aligned} \tag{5.6}$$

where E_S and E_p denote the energy rate of shocks and phase boundary, respectively. From (5.5) and (5.6) we find that the terms of lowest order in (5.5) and (5.6) are different, and that the terms of lowest order in (5.6) are negative for $v_L < v_0$. This fact indicates that dE_p/dv_L dominates dE/dv_L in a neighborhood of v_0 and dE/dv_L is negative if v_L is close to v_0 and $v_L < v_0$. Combining the above statements, we infer that the energy rate attains a local minimum at $v_L = v_0$. Q.E.D.

Remark 5.1. Notice that the terms of lowest order in the expressions for dE_p/dv_L and dE_S/dv_L are different. This indicates another essential difference between shocks and phase boundaries.

Remark 5.2. Theorems 5.1 and 5.2 seem to indicate the following behavior. If $u_0 = u_1$ and $p_0 = p_1 > p_m$, v_L will be greater than v_0 because dE/dv_L is negative at $v_L = v_0$. From Figures 7, 8, and 9 we see $u_L > u_R$. Using the Rankine-Hugoniot conditions for the phase boundary namely,

$$\sigma[u_R - u_L] = [p_R - p_L], \quad \sigma[v_R - v_L] = -[u_R - u_L],$$

we conclude that σ is positive. Hence the phase boundary will move forward. On the other hand if $u_0 = u_1$ and $p_0 = p_1 < p_0$, the phase boundary will move backward, by the same argument.

Remark 5.3. As a matter of fact, if $u_0 = u_1$, $p_0 = p_1 > p_m$, $p''(v_1) > 0$ (Figure 7), and v_0 is close to α_m, we can justify Remark 5.2. In this case

$$E = E_p, \quad \frac{dv_R}{dv_L} = \frac{(\lambda_L - \sigma_p)^2}{(\lambda_R - \sigma_p)^2}.$$

Substituting dv_R/dv_L from the above expression into Equation (5.1), we obtain (5.2). As we increase v_L, $(A + B)$ in (5.2) will change sign from positive to negative. Let us denote by $\tilde{v}_L$ the first point at which $A + B = 0$, provided a solution exists for the above dv_R/dv_L on the interval $[v_0, \tilde{v}_L]$. This $\tilde{v}_L$ should be less than a_m because when $v_L = \alpha_m$, $A + B$ is negative (observe that $p_R < p_L$). Hence dE/dv_L changes sign from negative to positive in $(v_0, \tilde{v}_L)$. In this case the state v_L is stable in the sense that every point $x_0 > 0$ will undergo a phase transition to the liquid state v_L in a finite time (the state v_R may be stable or metastable).

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