Controllable States of Elastic Dielectrics

M. SINGH & A. C. PIPKIN

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Contents

Part I. Catalog of Controllable States

Page

I.1. Introduction . . . 170
I.2. Statics . . . 171
I.3. Constitutive Equations . . . 173
I.4. Formally Equivalent Physical Theories . . . 175
I.5. Summary of the Basic Equations . . . 176
I.6. Homogeneous Deformation in a Uniform Field . . . 176
I.7. Expansion and Eversion of a Spherical Shell in a Radial Field . . . 177
I.8. Cylindrically Symmetrical Deformations of a Tube in a Radial Field of Flux . . . 179
I.9. Cylindrically Symmetrical Deformations of a Tube Sector in a Helical Electric Field 180
I.10. Deformations of a Cuboid in a Uniform Field of Flux . . . 182
I.11. Deformations of a Cuboid in a Uniform Electric Field . . . 183
I.12. Flexural Deformations of a Block in a Radial Field of Flux. . . . 183
I.13. Flexural Deformations of a Block in a Helical Electrical Field. . . . 184
I.14. Azimuthal Shear of a Cuboid in a Uniform Axial Field . . . 184

Part II. States with Specified Electric Field

II.1. Introduction . . . 186
II.2. Notation . . . 187
II.3. Controllable States . . . 187
II.4. Permissible Fields . . . 188
II.5. Permissible Strains . . . 189
II.6. Preliminary Analysis. I. . . . 190
II.7. Preliminary Analysis. II . . . 191
II.8. States with Invariants not all Constant . . . 192
II.9. Cases with g_1 Non-Degenerate . . . 193
II.10. Cases with g_1 Degenerate. I . . . 194
II.11. Cases with g_1 Degenerate. II . . . 194
II.12. Homogeneous Deformations . . . 195
II.13. States with all Invariants Constant . . . 195
II.14. The Case $a_3 b_3 c_3 \neq 0$ . . . 197
II.15. The Case $c_3 = 0$, $a_3 b_3 \neq 0$ . . . 198
II.16. The Case $b_3 = 1$. Basic Equations . . . 199
II.17. The Case $b_3 = 1$. Solution of Basic Equations . . . 200
II.18. The Case $b_3 = 1$. Deformations . . . 201

Part III. States with Specified Dielectric Displacement Field

III.1. Introduction . . . 202
III.2. Notation . . . 203
III.3. Controllable States . . . 203
III.4. Permissible Fields . . . 204
III.5. Invariants not all Constant. General Analysis . . . 205
III.6. Solutions with Invariants not all Constant . . . 206
III.7. Homogeneous Deformations . . . 207
III.8. All Invariants Constant . . . 207

Part I. Catalog of Controllable States

I.1. Introduction

In the theory of finite deformations of elastic solids, there are certain problems which can be solved exactly, by the inverse method. The deformation is prescribed at the outset, and it is verified that the deformation can be supported without body force, in every homogeneous, isotropic, incompressible, elastic material. In the present paper we describe a similar set of exact solutions in the theory of elastic dielectrics. The deformation and either the electric field or the dielectric displacement field are prescribed initially, and it is shown that the resulting state can be supported without mechanical body force or distributed charge in every homogeneous, isotropic, incompressible, elastic dielectric. We call such states *controllable states* of these materials.

The modern development of finite elasticity theory was stimulated by RIVLIN's observation that problems of the kind just described could be solved exactly without any detailed knowledge of the form of the strain-energy function for the material. These deformations could then be used in the experimental determination of the strain-energy function. There is a moderately large number of such deformations, including pure homogeneous deformations (RIVLIN [*1*]*), simultaneous extension, inflation, azimuthal and axial dislocation, torsion, and eversion of tubes (RIVLIN [*2*], [*3*], ERICKSEN & RIVLIN [*4*]), inflation and eversion of spherical shells (GREEN & SHIELD [*5*], ERICKSEN [*6*]), and some deformations involving flexure (RIVLIN [*2*], ADKINS, GREEN & SHIELD [*7*], ERICKSEN [*6*]). The only additions to this list since ERICKSEN's [*6*] thorough search are the cylindrical shearing solutions found by KLINGBEIL & SHIELD [*8*] and the generalizations of these solutions obtained by superposing extension and flexure on them [*9*], which we found in the course of the present work.

No comparably large body of exact solutions exists in the theory of elastic dielectrics. TOUPIN [*10*], in his original work on the subject, considered some cases of homogeneous deformation and polarization. ERINGEN [*12*] has discussed the extension of a tube, combined with a radial field. VERMA [*13*] has considered the expansion of a spherical shell in a radial field. PIPKIN & RIVLIN [*14*] have discussed the problem of electrical conduction in a stretched and twisted tube in an axial electric field, which is formally equivalent to a problem of polarization. So far as we know, no other exact solutions are recorded in the literature.

In the present part of this paper, Part I, we list a number of controllable states of initially homogeneous, isotropic, incompressible elastic dielectrics. In Part II we show that every controllable state with a prescribed non-zero electric field is among those listed in Part I. In Part III we show that Part I also contains every controllable state with a prescribed non-zero dielectric displacement field.

Because our aim is complete coverage rather than detailed examination of each solution, we usually carry the solution only so far as is necessary to verify that the state considered can be supported by surface tractions alone. Furthermore, although we place each state in the context of a body of definite shape and

* Numbers in square brackets indicate references at the end of the paper.

find an external field compatible with the boundary data implied by the interior state, we do not show that these are the only body shapes which might be used.

In Sections 2 and 3 we outline the basic theory, which we believe to be equivalent to TOUPIN'S [10], [11] theory of elastic dielectrics. The theory differs formally from TOUPIN'S theory in that we do not decompose the stress, the electric field, or the dielectric displacement into sums of various parts. Such decompositions are irrelevant to our present purpose, and also not of any empirical significance, so far as we can see, except in linearized theories. A second difference is that we do not use the expressions for response coefficients in terms of derivatives of the stored-energy density.

The latter omission is mainly for notational convenience, although it also has the side effect of making the solutions directly applicable to formally equivalent problems in the theories of electrical conduction and heat conduction in deformed isotropic materials [15], where no representation in terms of stored-energy is to be expected. The relevant analogies are outlined in Section 4.

The controllable states are described in Sections 6 to 14. Homogenous deformation in the presence of a uniform electric or dielectric displacement field is considered in Section 6. Symmetrical expansion of a spherical shell, with or without eversion, and with a prescribed radial dielectric displacement field, is considered in Section 7. We list these known solutions for the sake of completeness. The solution given by ERINGEN [12] is a special case of the five-parameter family of deformations of a tube, combined with a prescribed radial dielectric displacement field, which we describe in Section 8. The combination of this family of deformations with helical electric fields is described in Section 9.

The deformation of a cuboid into a rectangular block by flexure, extension, and shear, combined with a uniform dielectric displacement field along one of the principal directions of strain, is described in Section 10. The combination of these deformations with uniform electric fields in one of the principal planes is described in Section 11. The somewhat similar deformations which carry a rectangular block into a segment of the wall of a cylindrical tube, combined with a radial dielectric displacement field or helical electric field, are described in Sections 12 and 13, respectively.

The deformation of a cuboid by extension, flexure, and azimuthal shear, combined with a uniform axial electric or dielectric displacement field, is described in Section 14. This completes the enumeration of controllable states.

I.2. Statics

In the present section we summarize those basic assumptions and equations of continuum electrostatics and mechanics which are independent of the composition of the material media which may be involved.

We assume that there exists a macroscopic electric field E_i, with the dimensions of force per unit charge, which is conservative:

$$\oint_C E_i \, dx_i = 0. \tag{2.1}$$

Here C is an arbitrary closed curve, and dx_i is the vector element of arc along it. We next assume that there exists a macroscopic field of flux, or dielectric displacement, D_i, with the dimensions of charge per unit area, which has charge at its sources:

$$\oiint_S D_i n_i \, dS = Q. \tag{2.2}$$

Here S is an arbitrary closed surface with a real element dS and unit outward normal n_i, and Q is the total charge enclosed by S.

We also assume that the resultant force F_i and moment G_i exerted on the material contained in an arbitrary volume V, not including gravitational or inertial forces and moments, can be expressed entirely in terms of a stress vector t_i acting over the surface S of the volume V, in the forms

$$F_i = \oiint_S t_i \, dS, \qquad G_i = \oiint_S e_{ijk} x_j t_k \, dS. \tag{2.3}$$

Here x_i are Cartesian coordinates, and G_i is the moment about the origin. The stress vector t_i is intended to account for all electro-mechanical effects except the gravitational and inertial body forces which we have excluded, and which will be set equal to zero in the work to follow. We specifically exclude surface couples and body couples from the theory, except insofar as such quantities might be made to appear by suitable manipulation and reinterpretation of the basic equations. We also specifically exclude body forces, other than those forces of gravitational or inertial origin which one might wish to include in the theory. We note that electrostatic body forces can be made to appear by suitable manipulation and reinterpretation of the basic equations, if one has any desire to do so, which we do not.

The differential form of (2.1) is

$$E_{i,j} = E_{j,i}, \quad \text{or} \quad E_i = -V_{,i}, \tag{2.4}$$

where V is the electric potential. It also follows from (2.1) that the tangential component of E_i is continuous across a surface of discontinuity of E_i:

$$e_{ijk}(E_j^+ - E_j^-) n_k = 0. \tag{2.5}$$

Here n_i is a unit vector normal to the surface, and E_i^+ is evaluated on the side toward which n_i points, E_i^- being evaluated on the other side. Excluding charged double layers, the condition (2.5) can be satisfied by requiring the potential V to be continuous across surfaces of discontinuity of its derivatives.

If the total charge Q in (2.2) is distributed with the density q per unit volume, then the differential form of (2.2) is

$$D_{i,i} = q. \tag{2.6}$$

The normal component of D_i is discontinuous across a surface with charge ω per unit area:

$$(D_i^+ - D_i^-) n_i = \omega. \tag{2.7}$$

The conventions on superscripts plus and minus are as in (2.5). In the applications which we shall consider, we set $q=0$ in dielectrics and $\omega=0$ on the surfaces of dielectrics. The flux D_i is then solenoidal.

From the assumptions (2.3), with the macroscopic equations of translational and rotational equilibrium or motion, it follows exactly as in continuum mechanics (see, for example, LOVE [16]) that the stress vector t_i depends linearly on the unit normal n_i to the surface S,

$$t_i = \sigma_{ij} n_j, \tag{2.8}$$

that the stress matrix is symmetric,

$$\sigma_{ij} = \sigma_{ji}, \tag{2.9}$$

and that the differential form of the equations of translational equilibrium or motion in the presence of gravitational or interial body forces ρf_i per unit volume is

$$\sigma_{ij,j} + \rho f_i = 0. \tag{2.10}$$

We take ρf_i to be zero in the applications which we consider. It also follows that certain components of stress are discontinuous across a surface on which a force T_i per unit area is imposed, the traction T_i being given by

$$T_i + (\sigma_{ij}^+ - \sigma_{ij}^-) n_j = 0, \tag{2.11}$$

where our conventions on the superscripts plus and minus are as in (2.5) and (2.7). In the absence of electrical effects, at the surface of a body one ordinarily takes the stress σ_{ij}^+ in the surrounding medium to be zero. In the present theory there will in general be a non-zero stress everywhere.

I.3. Constitutive Equations

The validity of the assumptions (2.1) to (2.3) for free space is a fundamental physical postulate. In free space, the flux D_i is directly proportional to the electric field strength,

$$D_i = \varepsilon E_i, \tag{3.1}$$

the dielectric constant ε for free space being a basic physical constant. The stress σ_{ij} in free space is the Maxwell stress M_{ij} defined by

$$M_{ij} = \varepsilon E_i E_j - (\varepsilon/2) E_k E_k \delta_{ij}. \tag{3.2}$$

If it is assumed that (3.1) and (3.2) (*i.e.* $\sigma_{ij} = M_{ij}$) are also valid in the idealized medium called a continuous charge distribution, then, with the relations given in Section 2, it is easy to show that the force F_i and moment G_i in (2.3) for such a medium can be expressed entirely as the force and moment due to a body force $q E_i$ per unit volume.

In the applications which follow, we shall suppose that the dielectric bodies which we consider are surrounded by a charge-free medium described by constitutive equations of the forms (3.1) and (3.2).

Aside from free space and the continuous charge distribution, the most familiar idealized material considered in electrostatics is the dumbbell-model polarizable medium. There is an old and continuing controversy which has its roots in the ascription of physical reality to this model. If one regards it as exact for all dielectrics, then one may well argue for the validity of one or another set of definitions of its properties. Because we have no wish to take part in this debate, we will not attempt to define this model.

We consider materials which are described by constitutive equations, replacing (3.1) and (3.2), which express D_i and σ_{ij} in terms of E_i and quantities describing the state of deformation of the medium. The deformation of a material body can be described by specifying the position $x_i(X_A)$, in the deformed state, of the generic particle which was located at X_A in the undeformed state, all coordinates being measured with respect to a single fixed rectangular Cartesian frame x. The deformation gradients $x_{i,A}$ $(=\partial x_i/\partial X_A)$ provide measures of the deformation. We suppose that the values of D_i and σ_{ij} at a given point are determined by the values of E_i and $x_{i,A}$ at the same point:

$$D_i=f_i(x_{p,A}, E_p), \quad \sigma_{ij}=f_{ij}(x_{p,A}, E_p). \tag{3.3}$$

Thus, we assume that only one state of flux and stress is compatible with given values of the deformation gradients and electric field. This assumption can be modified to allow dependence on such scalar quantities as temperature without affecting the results described in the remainder of this section.

We assume that if a deformed body is rotated rigidly, together with the field E_i in it, into a new orientation with respect to the coordinate frame x, then the stress and flux will undergo the same rotation, so as to remain fixed with respect to the body. From this assumption it follows that the constitutive equations (3.3) must be expressible in the forms [17]

$$D_i=x_{i,A} F_A(x_{p,P} x_{p,Q}, x_{p,P} E_p), \quad \sigma_{ij}=x_{i,A} x_{j,B} F_{AB}(x_{p,P} x_{p,Q}, x_{p,P} E_p). \tag{3.4}$$

If the medium is holohedral isotropic in its undeformed, field-free state, then the relations (3.4) can be further restricted to the forms [17], [18],

$$D_i=(\Lambda_0 \delta_{ij}+\Lambda_1 g_{ij}+\Lambda_2 g_{ik} g_{kj}) E_j \tag{3.5}$$

and

$$\sigma_{ij}=\Phi_0 \delta_{ij}+S_{ij}, \tag{3.6}$$

where

$$\begin{aligned} S_{ij}=\Phi_1 g_{ij}+\Phi_2 g_{ik} g_{kj}+E_i(\Phi_3 \delta_{jk}+\Phi_4 g_{jk}+\Phi_5 g_{jn} g_{nk}) E_k+ \\ +E_j(\Phi_3 \delta_{ik}+\Phi_4 g_{ik}+\Phi_5 g_{in} g_{nk}) E_k. \end{aligned} \tag{3.7}$$

Here g_{ij} is the Finger strain, defined by

$$g_{ij}=x_{i,A} x_{j,A}. \tag{3.8}$$

The coefficients Λ in (3.5) and Φ in (3.6) and (3.7) are functions of the following orthogonal invariants:

$$I_1=g_{ii}, \quad I_2=g_{ij} g_{ij}, \quad I_3=E_i E_i, \quad I_4=E_i g_{ij} E_j, \quad I_5=E_i g_{ij} g_{jk} E_k, \tag{3.9}$$

and

$$I_6=\det g_{ij}. \tag{3.10}$$

Relations of the general types (3.5) to (3.7) were obtained by TOUPIN [10] by making use of a stored-energy function. MARRIS & VILLANUEVA [19] have obtained a canonical form of the type (3.6), (3.7), without using a stored-energy function, in a formally equivalent theory involving the dependence of stress on temperature gradient rather than on E_i.

We shall consider only incompressible materials. The invariant (3.10) is then unity in all deformations, and is accordingly not a variable. The scalar coefficients in (3.5) to (3.7) are then functions of the invariants (3.9) only. In conservative systems, a pressure p arises as a reaction to the constraint of no volume change. Our assumption that the stress is completely determined by $x_{p,A}$ and E_p is then not permissible, and in place of (3.6) we obtain a relation of the form

$$\sigma_{ij} = -p\,\delta_{ij} + S_{ij}, \tag{3.11}$$

where p is arbitrary, and the extra stress S_{ij} is of the form (3.7). We assume that an arbitrary pressure is also present in incompressible materials which are not conservative.

In some of the problems which we shall consider, it is convenient to take D_i rather than E_i as independent variables. By interchanging E_i with D_i in the preceding results, in place of (3.5) we obtain a relation of the form

$$E_i = (\Omega_0\,\delta_{ij} + \Omega_1\,g_{ij} + \Omega_2\,g_{ik}\,g_{kj})\,D_j, \tag{3.12}$$

and in place of (3.7) we arrive at

$$\begin{aligned} S_{ij} = \Psi_1\,g_{ij} + \Psi_2\,g_{ik}\,g_{kj} + D_i(\Psi_3\,\delta_{jk} + \Psi_4\,g_{jk} + \Psi_5\,g_{jn}\,g_{nk})\,D_k + \\ + D_j(\Psi_3\,\delta_{ik} + \Psi_4\,g_{ik} + \Psi_5\,g_{in}\,g_{nk})\,D_k. \end{aligned} \tag{3.13}$$

In incompressible materials, the coefficients Ω and Ψ are functions of the following orthogonal invariants:

$$J_1 = g_{ii}, \quad J_2 = g_{ij}\,g_{ij}, \quad J_3 = D_i\,D_i, \quad J_4 = D_i\,g_{ij}\,D_j, \quad J_5 = D_i\,g_{ij}\,g_{jk}\,D_k. \tag{3.14}$$

I.4. Formally Equivalent Physical Theories

The solutions which we exhibit in the remainder of this paper can be used in physical contexts other than that of the theory of elastic dielectrics. For example, the theories of steady-state heat conduction and electrical conduction in deformable media [15] are formally equivalent to the theory outlined in Sections 2 and 3. In place of a dielectric displacement vector satisfying the equation $D_{i,i} = 0$, in these theories one has a heat flux q_i or an electrical current density J_i satisfying the same equation. The electric potential V and the field strength E_i are replaced by the temperature and its negative gradient, respectively, in the theory of heat conduction. In the case of electrical conduction, V and E_i retain their present meanings.

Constitutive equations of the form (3.5) have been explicitly formulated in connection with such theories [15], and, at least in the case of electrical conduction, it is an empirical fact that conductivity can be strongly affected by deformation. Dependence of stress on temperature gradient, in the form (3.6), (3.7), has been suggested by Marris & Villanueva [19]. However, Gurtin [20] has shown that such dependence is not compatible with the Clausius-Duhem inequality.

I.5. Summary of the Basic Equations

We summarize here the equations and boundary conditions to be satisfied by the solutions in Sections 6 to 14. The deformation must be isochoric, *i.e.* $\det g_{ij}=1$. In the assumed absence of distributed charge, the flux D_i is solenoidal:

$$D_{i,i}=0. \tag{5.1}$$

The electric field strength is conservative:

$$E_{i,j}=E_{j,i}, \quad \text{i.e.} \quad E_i=-V_{,i}. \tag{5.2}$$

In the assumed absence of mechanical body force, the divergence of the stress tensor is zero. With (3.11), for an incompressible dielectric this condition can be written as

$$p_{,i}=S_{ij,j}. \tag{5.3}$$

The constitutive equations for an incompressible dielectric are (3.5) and (3.7), or (3.12) and (3.13), with (3.11). The constitutive equations for the medium surrounding the dielectric are (3.1) and $\sigma_{ij}=M_{ij}$, where M_{ij} is defined in (3.2). We note that the equilibrium equations $M_{ij,j}=0$ are satisfied identically in this medium if (5.1), (5.2), and (3.1) are satisfied. Accordingly, we need not verify the condition $M_{ij,j}=0$ separately.

At the charge-free surface of a dielectric with outward normal n_i, the continuity conditions (2.5) and (2.7) are

$$e_{ijk}(E_j^{(0)}-E_j)\,n_k=0, \qquad (D_i^{(0)}-D_i)\,n_i=0, \tag{5.4}$$

where $E_i^{(0)}$ and $D_i^{(0)}$ are evaluated in the outside medium, and E_i and D_i are evaluated in the dielectric. When a potential V is known to exist, the condition (5.4a) need not be considered separately. With $\sigma_{ij}=-p\,\delta_{ij}+S_{ij}$ the stress in the dielectric and M_{ij}, the stress in the surrounding medium, the traction T_i which must be applied to the surface is

$$T_i=-p\,n_i+S_{ij}\,n_j-M_{ij}\,n_j. \tag{5.5}$$

I.6. Homogeneous Deformation in a Uniform Field

The simplest controllable states are those in which the strain components and field components are constants. As a general example of such a state, we consider the homogeneous deformation of an infinite slab, bounded by the surfaces $X_3=\pm h$ in the undeformed state. The slab is subjected to extensions in the coordinate directions, with extension ratios λ_1, λ_2, and λ_3, respectively, and is then sheared by the amounts κ_1 and κ_2 in the x_1 and x_2 directions. In the total deformation, the particle initially at X_A moves to the point x_i given by

$$x_1=\lambda_1\,X_1+\kappa_1\,\lambda_3\,X_3, \qquad x_2=\lambda_2\,X_2+\kappa_2\,\lambda_3\,X_3, \qquad x_3=\lambda_3\,X_3. \tag{6.1}$$

If the material is incompressible, then $\lambda_1\,\lambda_2\,\lambda_3=1$.

We suppose that a uniform field E_i exists in the slab. By using (6.1) in (3.8), we verify that the strain components are constants. The invariants (3.9) are then also constants, and it follows from (3.5) and (3.7), respectively, that D_i and S_{ij}

are constants. Then (5.1) and (5.2) are satisfied, and (5.3) is satisfied by a constant pressure p.

Alternatively, we can let a uniform flux D_i be prescribed initially. The constancy of E_i and S_{ij} then follows from (3.12) and (3.13), respectively, and again all conditions are satisfied if the pressure p is constant.

Knowing both D_i and E_i in the dielectric, from (5.4) we obtain $E_\alpha^{(0)}$ ($\alpha=1, 2$) and $D_3^{(0)}$ at the surfaces $x_3=\pm\lambda_3 h$ of the deformed slab. With (3.1), the remaining components of $E_i^{(0)}$ and $D_i^{(0)}$ are then determined. Uniform external fields with these values satisfy (5.1) and (5.2).

With S_{ij} known from the constitutive equation, M_{ij} determined from $E_i^{(0)}$ by (3.2), and p an arbitrary constant, from (5.5) we find the tractions T_i which must be exerted on the surfaces in order to maintain the state specified. The normal component T_3 of this traction can be nullified by suitable choice of p. The tangential components T_α ($\alpha=1, 2$) on the upper surface $x_3=\lambda_3 h$, found by following the procedure outlined, are of the forms

$$T_1=a_1\,\kappa_1+(b_1-D_3)\,E_1\,,\qquad T_2=a_2\,\kappa_2+(b_2-D_3)\,E_2\,,\tag{6.2}$$

where the constants a_α and b_α ($\alpha=1, 2$) are given in terms of the coefficients Φ in (3.7). We will not write out the expressions for these constants, which are lengthy but easily obtained. Because of the continuity conditions (5.4), the components E_1, E_2, and D_3 in (6.2) take equal values in the slab and in the surrounding medium.

We note that if no tractions are applied to the faces of the slab, so that $T_1=T_2=0$ in (6.2), then the unsheared state $\kappa_1=\kappa_2=0$ cannot be maintained if the external field is oblique to the slab, unless b_1-D_3 and b_2-D_3 should fortuitiously vanish. In other words, the Maxwell stress due to an oblique external field tends to shear the slab, and will do so unless tangential tractions are applied to prevent such shearing. In the case of a normal field, E_1 and E_2 are zero, and if no tractions are applied the shears κ_1 and κ_2 will in general be zero. The hypothesis that the effective shear moduli a_1 and a_2 for this case cannot be zero might be imposed as restrictions on the form of the constitutive equation.

I.7. Expansion and Eversion of a Spherical Shell in a Radial Field

The expansion or contraction of a spherical shell, in the absence of applied fields, has been discussed by Green & Shield [*5*], and the generalization to cases in which the shell is first turned inside out has been pointed out by Ericksen [*6*]. The superposition of a radial field on the first of these deformations has been discussed by Verma [*13*].

The shell initially has internal radius R_a and external radius R_b. The particle initially at the point R, Θ, Φ in a system of spherical coordinates moves to the point r, ϑ, φ given by [*6*]

$$r(R)=\pm(R^3-R_a^3\pm r_a^3)^{\frac{1}{3}},\qquad \vartheta=\pm\Theta\,,\qquad \varphi=\Phi\,,\tag{7.1}$$

where the plus sign is used in the case of simple expansion or contraction, and the minus sign if the shell is first everted. The constant r_a is the interior radius of the deformed shell in the first case, and the exterior radius in the second. It is easy to verify that the mapping (7.1) is volume-preserving. The physical components of strain in the spherical system are

$$g_{rr}=(r')^2, \quad g_{\vartheta\vartheta}=g_{\varphi\varphi}=(r/R)^2, \quad g_{r\vartheta}=g_{\vartheta\varphi}=g_{\varphi r}=0, \tag{7.2}$$

where $r'=dr/dR$.

We suppose that the physical components of the dielectric displacement field are given both within the dielectric and in the space surrounding it by

$$D_r=Q/4\pi r^2, \quad D_\vartheta=D_\varphi=0. \tag{7.3}$$

The normal component of flux is then trivially continuous across the surfaces $r=r_a$ and $r=r_b=r(R_b)$, and (5.1) is satisfied.

The invariants J_μ ($\mu=1,\ldots,5$) defined in (3.14) are, with (7.2) and (7.3), functions of r (or R) only. From (3.12) we then find that the physical components of the electric field within the dielectric are also functions of r only, given by

$$E_r=[\Omega_0+\Omega_1(r')^2+\Omega_2(r')^4]\,D_r, \quad E_\vartheta=E_\varphi=0. \tag{7.4}$$

The field in the surrounding medium is, from (7.3) and (3.1),

$$E_r^{(0)}=D_r/\varepsilon, \quad E_\vartheta^{(0)}=E_\varphi^{(0)}=0. \tag{7.5}$$

The tangential component of the electric field is trivially continuous across the surfaces of the deformed body. The requirement that the electric field be conservative is satisfied, because the field (7.4) and (7.5) is consistent with an electric potential $V(r)$. If the field is produced by a spherical condenser, the potential difference between the plates is found by integrating E_r from one plate to the other. In (7.3), the constant Q represents the total charge on the inner plate.

With the aid of (3.13) we find that the physical components of the extra stress are functions of r only, given by

$$\begin{aligned} S_{rr}&=\Psi_1(r')^2+\Psi_2(r')^4+2[\Psi_3+\Psi_4(r')^2+\Psi_5(r')^4]\,D_r^2,\\ S_{\vartheta\vartheta}&=S_{\varphi\varphi}=\Psi_1(r/R)^2+\Psi_2(r/R)^4, \quad S_{r\vartheta}=S_{\vartheta\varphi}=S_{\varphi r}=0. \end{aligned} \tag{7.6}$$

The equilibrium equations (5.3) then imply that p is a function of r only, given by

$$p(r)=S_{rr}(r)+2\int_{r_b}^{r}(1/\rho)[S_{rr}(\rho)-S_{\vartheta\vartheta}(\rho)]\,d\rho+p(r_b)-S_{rr}(r_b). \tag{7.7}$$

The tractions which must be applied to the surfaces $r=r_a$ and $r=r_b$ to support the deformation (7.1) in the field (7.3) are obtained by using the preceding results in (5.5). No tangential traction is required. The normal component on one of the surfaces, say $r=r_b$, can be nullified by appropriate choice of the arbitrary constant $p(r_b)$ in (7.7):

$$p(r_b)=S_{rr}(r_b)-M_{rr}(r_b). \tag{7.8}$$

The normal traction on the surface $r=r_a$ is then not zero, in general, but is given by

$$\pm T_r(r=r_a)=-2\int_{r_a}^{r_b}(1/\rho)[S_{rr}(\rho)-S_{\vartheta\vartheta}(\rho)]\,d\rho+M_{rr}(r_a)-M_{rr}(r_b), \tag{7.9}$$

where the plus sign or the minus sign is used according to whether the shell is not, or is, everted. The Maxwell stress component M_{rr} is found by using (7.5) in (3.2) to be

$$M_{rr}=D_r^2/2\varepsilon. \tag{7.10}$$

We note, with (7.3) and (7.5), that the Maxwell stress exerts a larger tension on the inner surface than on the outer surface. If no traction is applied to either surface, so that $T_r=0$ in (7.9), then (7.9) is an implicit equation for the parameter r_a in (7.1), which is involved in the expressions for S_{rr} and $S_{\vartheta\vartheta}$. The choice $r_a=R_a$ corresponding to no deformation can satisfy (7.9) only fortuitously if $T_r=0$, and we can expect that the field will cause the sphere to contract.

I.8. Cylindrically Symmetrical Deformations of a Tube in a Radial Field of Flux

The extension, inflation, torsion, azimuthal and axial dislocation, and eversion of tubes in the absence of electrostatic effects have been discussed by RIVLIN [2], [3], and ERICKSEN & RIVLIN [4]. Some of these solutions have also been discussed by ADKINS, GREEN, & SHIELD [7]. We now consider the superposition of a radial dielectric displacement field on these deformations. The special case of extension has been discussed by ERINGEN [12].

We consider a tube which initially has interior radius R_a and exterior radius R_b. The particle initially at R, Θ, Z in a system of cylindrical polar coordinates moves to the point r, ϑ, z given by [4]

$$r=(AR^2+B)^{\frac{1}{2}}, \quad \vartheta=C\Theta+DZ, \quad z=E\Theta+FZ, \tag{8.1}$$

where the constants $A, B, \dots, F$ satisfy the incompressibility condition

$$A(CF-DE)=1. \tag{8.2}$$

If the deformed tube is to form a complete tube again, supplementary material must be added if $C<1$, and material must be deleted if $C>1$. Even if $C=1$, the tube must be severed in order to perform the deformation if $E\neq 0$.

The physical components of strain are then

$$\begin{gathered} g_{rr}=(AR/r)^2, \quad g_{\vartheta\vartheta}=(C\,r/R)^2+(D\,r)^2, \quad g_{zz}=(E/R)^2+F^2, \\ g_{r\vartheta}=g_{rz}=0, \quad g_{\vartheta z}=(CE\,r/R^2)+FD\,r. \end{gathered} \tag{8.3}$$

We suppose that a radial field of flux is imposed by placing the tube between the plates of a coaxial cylindrical condenser. The flux, satisfying (5.1) and the continuity condition (5.4b), is given both within the dielectric and in the space surrounding it by

$$D_r=Q/2\pi\,r, \quad D_\vartheta=D_z=0, \tag{8.4}$$

where Q represents the charge per unit length on the interior plate.

From (8.3) and (8.4), with $r(R)$ given by (8.1), it follows that the invariants J_μ defined in (3.14) are functions of r (or R) only, and thus that the coefficients in the constitutive equations (3.12) and (3.13) are also functions of r only.

The electric field in the dielectric is found with the aid of (3.12) to be

$$E_r=(\Omega_0+\Omega_1\, g_{rr}+\Omega_2\, g_{rr}^2)\, D_r\,, \qquad E_\vartheta=E_z=0\,. \tag{8.5}$$

The field in the surrounding medium is

$$E_r^{(0)}=D_r/\varepsilon\,, \qquad E_\vartheta^{(0)}=E_z^{(0)}=0\,. \tag{8.6}$$

The field (8.5) and (8.6) is consistent with a potential $V(r)$ and is thus conservative.

The physical components of the extra stress, found by using (8.3) and (8.4) in (3.13), are

$$\begin{aligned} &S_{rr}=\Psi_1\, g_{rr}+\Psi_2\, g_{rr}^2+2(\Psi_3+\Psi_4\, g_{rr}+\Psi_5\, g_{rr}^2)\, D_r^2,\\ &S_{\vartheta\vartheta}=\Psi_1\, g_{\vartheta\vartheta}+\Psi_2\, g_{\vartheta\vartheta}^2, \qquad S_{zz}=\Psi_1\, g_{zz}+\Psi_2\, g_{zz}^2,\\ &S_{r\vartheta}=S_{rz}=0\,, \qquad S_{\vartheta z}=[\Psi_1+\Psi_2(g_{\vartheta\vartheta}+g_{zz})]\, g_{\vartheta z}\,, \end{aligned} \tag{8.7}$$

and each component is a function of r only. The equilibrium equations (5.3) then imply that p is a function of r only, given by

$$p(r)=S_{rr}(r)+\int_{r_b}^{r}(1/\rho)[S_{rr}(\rho)-S_{\vartheta\vartheta}(\rho)]\, d\rho+p(r_b)-S_{rr}(r_b)\,, \tag{8.8}$$

where $r_b=r(R_b)$. The Maxwell stress in the surrounding medium is found by using (8.6) in (3.2):

$$M_{rr}=-M_{\vartheta\vartheta}=-M_{zz}=D_r^2/2\varepsilon\,, \qquad M_{r\vartheta}=M_{\vartheta z}=M_{zr}=0\,. \tag{8.9}$$

The tractions which must be applied to the surfaces $r=r_a=r(R_a)$ and $r=r_b=r(R_b)$ to support the deformation (8.1) in the field (8.4) can now be calculated by using (5.5). No tangential components are required. The normal component on one surface, say $r=r_b$, can be nullified by proper choice of the arbitrary constant $p(r_b)$ in (8.8). This yields a relation of the form (7.8). Then, if the surface $r=r_a$ is the interior radius of the deformed tube (*i.e.* if $A>0$ in (8.1)), the normal traction which must be applied to it is

$$T_r(r=r_a)=-\int_{r_a}^{r_b}(1/\rho)[S_{rr}(\rho)-S_{\vartheta\vartheta}(\rho)]\, d\rho+M_{rr}(r_a)-M_{rr}(r_b)\,. \tag{8.10}$$

We note from (8.7) that if the tube is of finite length, normal and azimuthal tractions must be applied to its ends in order to maintain the given state.

I.9. Cylindrically Symmetrical Deformations of a Tube Sector in a Helical Electric Field

The deformations described in Section 8 can also be supported without body force or distributed charge in the presence of a helical electric field, given both within the dielectric and in the medium surrounding it by

$$E_r=0\,, \qquad E_\vartheta=H/r\,, \qquad E_z=\text{constant}\,. \tag{9.1}$$

If $H\neq 0$, this field is conservative only if it is restricted to some sector $0\leqq\vartheta\leqq\vartheta_0<2\pi$, say. If $E_z=0$, the field might be produced by condenser plates on the planes $\vartheta=0$ and $\vartheta=\vartheta_0$, the potential difference between them being $H\vartheta_0$. The case $H=0$, $E_z\neq 0$, in which the field is a uniform axial field, can be produced in the complete tube $0\leqq\vartheta\leqq 2\pi$. The combined case $H\neq 0$, $E_z\neq 0$, is not likely to be easy to produce experimentally, but we include it for the sake of completeness.

The strain components are given by (8.3). With (9.1), it follows that the invariants I_μ defined in (3.9) are functions of r only. The physical components of flux in the dielectric are found with the aid of (3.5) to be

$$\begin{aligned} D_r &= 0, \\ D_\vartheta &= [\Lambda_0+\Lambda_1 g_{\vartheta\vartheta}+\Lambda_2(g_{\vartheta\vartheta}^2+g_{\vartheta z}^2)]E_\vartheta+g_{\vartheta z}[\Lambda_1+\Lambda_2(g_{\vartheta\vartheta}+g_{zz})]E_z, \\ D_z &= [\Lambda_0+\Lambda_1 g_{zz}+\Lambda_2(g_{zz}^2+g_{\vartheta z}^2)]E_z+g_{\vartheta z}[\Lambda_1+\Lambda_2(g_{\vartheta\vartheta}+g_{zz})]E_\vartheta, \end{aligned} \tag{9.2}$$

where the coefficients Λ depend only on r, since they are functions of the invariants I_μ. The field (9.2) is thus solenoidal. The flux in the medium surrounding the tube is

$$D_r^{(0)}=0, \qquad D_\vartheta^{(0)}=\varepsilon E_\vartheta, \qquad D_z^{(0)}=\varepsilon E_z, \tag{9.3}$$

and again this field is solenoidal. The normal component D_r is trivially continuous across the cylindrical surfaces of the tube.

The physical components of the extra stress are found by using (8.3) and (9.1) in (3.7). We obtain

$$\begin{aligned} S_{rr} &= \Phi_1 g_{rr}+\Phi_2 g_{rr}^2, \qquad S_{r\vartheta}=S_{rz}=0, \\ S_{\vartheta\vartheta} &= \Phi_1 g_{\vartheta\vartheta}+\Phi_2(g_{\vartheta\vartheta}^2+g_{\vartheta z}^2)+ \\ &\quad +2E_\vartheta\{E_\vartheta[\Phi_3+\Phi_4 g_{\vartheta\vartheta}+\Phi_5(g_{\vartheta\vartheta}^2+g_{\vartheta z}^2)]+E_z g_{\vartheta z}[\Phi_4+\Phi_5(g_{\vartheta\vartheta}+g_{zz})]\}, \\ S_{zz} &= \Phi_1 g_{zz}+\Phi_2(g_{\vartheta z}^2+g_{zz}^2)+ \\ &\quad +2E_z\{E_z[\Phi_3+\Phi_4 g_{zz}+\Phi_5(g_{\vartheta z}^2+g_{zz}^2)]+E_\vartheta g_{\vartheta z}[\Phi_4+\Phi_5(g_{\vartheta\vartheta}+g_{zz})]\}, \\ S_{\vartheta z} &= g_{\vartheta z}[\Phi_1+\Phi_2(g_{\vartheta\vartheta}+g_{zz})+\Phi_4(E_\vartheta^2+E_z^2)+\Phi_5(g_{\vartheta\vartheta}+g_{zz})(E_\vartheta^2+E_z^2)]+ \\ &\quad +E_\vartheta E_z[2\Phi_3+\Phi_4(g_{\vartheta\vartheta}+g_{zz})+\Phi_5(g_{\vartheta\vartheta}^2+2g_{\vartheta z}^2+g_{zz}^2)]. \end{aligned} \tag{9.4}$$

Each component depends only on r. The equilibrium equations (5.3) are satisfied if p is a function of r only, of the form (8.8), where S_{rr} and $S_{\vartheta\vartheta}$ are now given by (9.4).

The Maxwell stress in the surrounding medium, found by using (9.1) in (3.2), is

$$\begin{aligned} M_{rr} &= -(\varepsilon/2)(E_\vartheta^2+E_z^2)=M_{\vartheta\vartheta}-\varepsilon E_\vartheta^2=M_{zz}-\varepsilon E_z^2, \\ M_{r\vartheta} &= M_{rz}=0, \qquad M_{\vartheta z}=\varepsilon E_\vartheta E_z. \end{aligned} \tag{9.5}$$

The remainder of the analysis is as in Section 8. No tangential tractions need be applied to the cylindrical boundaries of the dielectric, and the normal component on the surface $r=r_b$ can be made to vanish by proper choice of $p(r_b)$ in (8.8). The normal traction which must then be applied to the surface $r=r_a$ is given by (8.10), where now S_{rr} and $S_{\vartheta\vartheta}$ are given by (9.4) and M_{rr} by (9.5).

The extension of a rod in an axial field is the special case in which $B=D=E=0$ and $C=1$ in (8.1), and $H=0$ in (9.1). In this case $r_a=0$, and there is no interior surface. The deformation and field can be maintained with no traction on the surface $r=r_b$.

The special case of extension, inflation, and torsion of a tube in an axial field is the special case in which $C=1$ and $E=0$ in (8.1), and $H=0$ in (9.1). For this case, the formally equivalent problem of electrical conduction, in which D_i is replaced by the current density J_i, has been considered in more detail in an earlier paper [*14*]. The stress was not considered.

I.10. Deformations of a Cuboid in a Uniform Field of Flux

ERICKSEN [*6*] has discussed certain deformations of a tube wall section, initially bounded by the surfaces $R=R_a$, $R=R_b$, $\Theta=\pm\Theta_0$, and $Z=\pm Z_0$ in cylindrical polar coordinates. In these deformations, the particle initially at the point R, Θ, Z moves to the location x, y, z in a Cartesian system, given by

$$x=AR^2, \quad y=B\Theta, \quad z=(Z/2AB)+C\Theta. \tag{10.1}$$

From (10.1) we obtain

$$\begin{aligned} &g_{xx}=4A^2R^2, \quad g_{xy}=g_{xz}=0, \quad g_{yy}=(B/R)^2, \\ &g_{yz}=BC/R^2, \quad g_{zz}=(C/R)^2+(1/2AB)^2. \end{aligned} \tag{10.2}$$

The mapping (10.1) is isochoric, as required. We note that the strain components are functions of R^2, and thus, with (10.1), functions of x.

We now consider the superposition of a uniform field of flux in the x direction on the deformation (10.1):

$$D_x=\text{constant}, \quad D_y=D_z=0. \tag{10.3}$$

The flux, given by (10.3) both in the dielectric and in the medium surrounding it, is solenoidal.

From (10.2) and (10.3) it follows that the invariants J_μ, defined in (3.14), are functions of x only. The electric field in the dielectric, found by using (10.2) and (10.3) in (3.12), is

$$E_x=[\Omega_0+\Omega_1(4Ax)+\Omega_2(16A^2x^2)]D_x, \quad E_y=E_z=0. \tag{10.4}$$

The electric field in the surrounding space is

$$E_x^{(0)}=D_x/\varepsilon, \quad E_y^{(0)}=E_z^{(0)}=0. \tag{10.5}$$

If we neglect fringe effects at the edges initially bounded by the surfaces $\Theta=\pm\Theta_0$ and $Z=\pm Z_0$, the field is given by (10.4) for $AR_a^2<x<AR_b^2$ (assuming $A>0$ and $R_a<R_b$) and by (10.5) outside this interval, and is thus compatible with a potential $V(x)$. If the field is supplied by a parallel-plate condenser, the potential difference between the plates is found by integrating E_x from one plate to the other, and D_x is the charge per unit area on one plate.

The components of the extra stress S_{ij}, found by using (10.2) and (10.3) in (3.13), are functions of x only, and $S_{xy}=S_{xz}=0$. The equilibrium equations (5.3) are then satisfied if p is a function of x only, given by

$$-p(x)+S_{xx}(x)=-p(x_a)+S_{xx}(x_a)=\sigma_{xx}(x), \tag{10.6}$$

where $x_a = A R_a^2$. Thus σ_{xx} is constant. By using (10.5) in (3.2) we find that the Maxwell stress in the surrounding medium is constant, and that $M_{xy} = M_{xz} = 0$. The tractions which must be applied to the surfaces $x = x_a$ and $x = x_b = A R_b^2$ are found with the aid of (5.5). No tangential traction is required. The normal components on the two faces are $\pm(\sigma_{xx} - M_{xx})$. If the traction on one face is made to vanish by appropriate choice of $p(x_a)$ in (10.6), then, because σ_{xx} and M_{xx} are constants, the traction also vanishes on the other face. The deformation (10.1) and field (10.3) can thus be supported by tractions on the edges initially bounded by the planes $\Theta = \pm\Theta_0$ and $Z = \pm Z_0$. Because we do not have accurate expressions for the fields near the edges, we will not calculate these tractions.

I.11. Deformations of a Cuboid in a Uniform Electric Field

The deformation (10.1) can also be combined with a uniform electric field, given both within the dielectric and in the space surrounding it by

$$E_x = 0, \quad E_y = \text{constant}, \quad E_z = \text{constant}. \tag{11.1}$$

In this case the flux in the dielectric is found, by using (11.1) and (10.2) in (3.5), to be

$$\begin{aligned} D_x &= 0, \\ D_y &= [\Lambda_0 + \Lambda_1 g_{yy} + \Lambda_2 (g_{yy}^2 + g_{yz}^2)] E_y + g_{yz} [\Lambda_1 + \Lambda_2 (g_{yy} + g_{zz})] E_z, \\ D_z &= [\Lambda_0 + \Lambda_1 g_{zz} + \Lambda_2 (g_{yz}^2 + g_{zz}^2)] E_z + g_{yz} [\Lambda_1 + \Lambda_2 (g_{yy} + g_{zz})] E_y. \end{aligned} \tag{11.2}$$

In the surrounding medium, the flux is

$$D_x^{(0)} = 0, \quad D_y^{(0)} = \varepsilon E_y, \quad D_z^{(0)} = \varepsilon E_z. \tag{11.3}$$

The coefficients Λ in (11.2) are functions of the invariants I_μ defined in (3.9), which, according to (10.2) and (11.1), are functions of x only. Consequently, the field (11.2) and (11.3) is solenoidal if the deformed body is bounded by planes $x =$ constant. We again neglect edge effects on the remaining surfaces.

The extra stress components S_{ij} in the dielectric are found by using (10.2) and (11.1) in (3.7), and, as in Section 10, they are functions of x only, with $S_{xy} = S_{xz} = 0$. The equilibrium equations are again satisfied by a pressure $p(x)$ of the form (10.6), where S_{xx} now has a different form, but again with the result that σ_{xx} is constant. Again as in Section 10, it is found that the Maxwell stress in the surrounding medium is constant, and that the components M_{xy} and M_{xz} are zero. Consequently, we again find as in Section 10 that no traction need be applied to the plane surfaces $x = x_a$ and $x = x_b$.

I.12. Flexural Deformations of a Block in a Radial Field of Flux

The flexure and extension of a block has been discussed by RIVLIN [2], and the generalization to include a shear, or axial dislocation, has been discussed by ERICKSEN [6]. In this family of deformations, the particle initially at the point X, Y, Z in a Cartesian system is brought to the position r, ϑ, z in cylindrical polar coordinates, given by

$$r = A X^{\frac{1}{2}}, \quad \vartheta = B Y, \quad z = (2Z/A^2 B) + C Y. \tag{12.1}$$

We suppose that the block is initially bounded by two plane surfaces X=constant and two plane surfaces Y=constant, and we suppose that the block is infinitely long in the Z direction, or equivalently, we will neglect fringe effects at the ends facing the Z direction. If two of the surfaces initially bounding the block are the planes $Y=\pm\pi/B$, then the deformation (12.1) carries these surfaces into the coincident planes $\vartheta=\pm\pi$, where the surfaces may be bonded together to form a tube. The initially plane surfaces X=constant become the interior and exterior cylindrical boundaries of the tube.

The physical components of strain in the cylindrical system r, ϑ, z are found from (12.1) to be given by

$$\begin{aligned} g_{rr}&=A^4/4r^2, \quad g_{r\vartheta}=g_{rz}=0, \\ g_{\vartheta\vartheta}&=B^2 r^2, \quad g_{\vartheta z}=BCr, \quad g_{zz}=C^2+(2/A^2 B)^2. \end{aligned} \tag{12.2}$$

It is easy to verify that the deformation is isochoric. We note that each strain component is a function of r only.

We superpose on the deformation (12.1) the radial field of flux given both in the dielectric and in the surrounding medium by (8.4). The electric field in the dielectric is then of the form (8.5), where g_{rr} is now given by (12.2). Similarly, the remaining considerations in Section 8 are valid for the present problem, if the strain components used in Section 8 are replaced by the components (12.2).

I.13. Flexural Deformations of a Block in a Helical Electric Field

We now consider the superposition of a helical electric field of the form (9.1) on the deformation (12.1). The discussion in Section 9 is applicable to the present problem, the strain components in the formulae of Section 9 now being given by (12.2). We need not consider this case in any further detail.

I.14. Azimuthal Shear of a Cuboid in a Uniform Axial Field

The azimuthal shearing deformation of a cuboid has been considered, under another guise, by KLINGBEIL & SHIELD [*8*], and we have generalized this deformation by combining it with extension and flexure [*9*]. The particle initially located at the point R, Θ, Z in a cylindrical polar system is carried to the point r, ϑ, z given by

$$r=AR, \quad \vartheta=B\log R+C\Theta, \quad z=Z/A^2 C. \tag{14.1}$$

The physical components of strain are constants:

$$\begin{aligned} g_{rr}&=A^2, \quad g_{r\vartheta}=A^2 B, \quad g_{rz}=0, \\ g_{\vartheta\vartheta}&=A^2(B^2+C^2), \quad g_{\vartheta z}=0, \quad g_{zz}=1/A^4 C^2. \end{aligned} \tag{14.2}$$

The incompressibility condition is satisfied. The deformation (14.1) maps a body initially bounded by the surfaces

$$R=R_a, \quad R=R_b, \quad \Theta=C^{-1}(\pm\vartheta_0-B\log R), \quad z=\pm Z_0, \tag{14.3}$$

into a body bounded by the surfaces

$$r=r_a=AR_a, \quad r=r_b=AR_b, \quad \vartheta=\pm\vartheta_0, \quad z=\pm z_0=\pm Z_0/A^2 C. \tag{14.4}$$

We combine the deformation (14.1) with a uniform axial electric field, given both within the dielectric and in the medium surrounding it by

$$E_r = E_\vartheta = 0, \qquad E_z = \text{constant}. \tag{14.5}$$

This field can be imposed, with no fringe effects, by letting condenser plates occupy the planes $z = \pm z_0$. The potential difference between the plates is then $2 z_0 E_z$.

By using (14.2) and (14.5) in the definition (3.9) of the invariants I_μ, we find that these invariants are constants. The flux in the dielectric, found by using (14.2) and (14.5) in (3.5), is a uniform axial field:

$$D_r = D_\vartheta = 0, \qquad D_z = (\Lambda_0 + \Lambda_1 g_{zz} + \Lambda_2 g_{zz}^2) E_z. \tag{14.6}$$

In the medium surrounding the dielectric, the flux is

$$D_r^{(0)} = D_\vartheta^{(0)} = 0, \qquad D_z^{(0)} = \varepsilon E_z. \tag{14.7}$$

The flux given by (14.6) and (14.7) is solenoidal.

The physical components of the extra stress in the dielectric are found by using (14.2) and (14.5) in (3.7), and they are constants:

$$\begin{aligned} &S_{rr} = \Phi_1 A^2 + \Phi_2 A^4(1+B^2), \qquad S_{\vartheta\vartheta} = \Phi_1 A^2 K^2 + \Phi_2 A^4 (K^4 + B^2), \\ &S_{zz} = \Phi_1 (A^4 C^2)^{-1} + \Phi_2 (A^4 C^2)^{-2} + 2E_z^2 [\Phi_3 + \Phi_4 (A^4 C^2)^{-1} + \Phi_5 (A^4 C^2)^{-2}], \\ &S_{r\vartheta} = A^2 B [\Phi_1 + \Phi_2 A^2 (1+K^2)], \qquad S_{rz} = S_{\vartheta z} = 0 \qquad (K^2 = B^2 + C^2). \end{aligned} \tag{14.8}$$

The equilibrium equations (5.3) then take the forms

$$\partial p / \partial r = (S_{rr} - S_{\vartheta\vartheta})/r, \qquad \partial p / \partial \vartheta = 2 S_{r\vartheta}, \qquad \partial p / \partial z = 0, \tag{14.9}$$

giving

$$p = (S_{rr} - S_{\vartheta\vartheta}) \log r + 2\vartheta S_{r\vartheta} + p_0, \tag{14.10}$$

where p_0 is an arbitrary constant.

The components of the Maxwell stress in the surrounding medium are

$$M_{zz} = -M_{rr} = -M_{\vartheta\vartheta} = \varepsilon E_z^2/2, \qquad M_{r\vartheta} = M_{\vartheta z} = M_{zr} = 0. \tag{14.11}$$

The tractions which must be applied to the surfaces (14.4) to support the deformation (14.1) in the field (14.5) can now be found with the aid of (5.5). Because the pressure p in (14.10) depends on both r an ϑ, in general, the normal tractions on the surfaces $r = r_a$ and $r = r_b$ depend on ϑ, and the normal tractions on the surfaces $\vartheta = \pm \vartheta_0$ depend on r. Tangential tractions of magnitude $S_{r\vartheta}$ must be applied to these surfaces. No tangential tractions in the z direction are required. The tractions on the ends $z = \pm z_0$ are purely normal tractions, whose magnitudes depend on both r and ϑ in general. In the special case considered by KLINGBEIL & SHIELD [*8*], $K^2 = 1$ and thus, with (14.8), $S_{rr} - S_{\vartheta\vartheta} = 0$. In this case p is a function of ϑ only, and the tractions become somewhat simpler.

Part II. States with Specified Electric Field

II.1. Introduction

In Part I we have described a number of controllable states of homogeneous, isotropic, incompressible, elastic dielectrics. We have asserted that every controllable state with deformation and non-zero field strength prescribed, or with deformation and non-zero dielectric displacement field prescribed, is among those listed in Part I. We now verify the first part of this assertion, by determining every controllable state which involves a prescribed non-zero electric field.

The analysis is similar to that which ERICKSEN [6] used for the determination of controllable states in finite elasticity theory, and we make direct use of ERICKSEN's results in that part of the problem which pertains only to the deformations. It should be remarked that although ERICKSEN'S analysis of possible deformations is incomplete, we are able to determine every controllable deformation which is consistent with a prescribed non-zero electric field, because the controllability conditions are more restrictive in the latter case. As an accidental by-product of the analysis, we have found a family of controllable states of elastic materials (with zero electric field), which is not among those which ERICKSEN obtained. In Section 3 we follow ERICKSEN'S methods in setting up the conditions to be satisfied by controllable states.

The restrictions imposed on the electric field by the controllability conditions are deduced immediately, in Section 4, by using a result due to HAMEL [21]. These restrictions imply in particular that the field must have cylindrical symmetry, with no radial component. Certain obviously controllable states, involving radial fields in cylindrical or spherical coordinates, are excluded from the present analysis because it is the dielectric displacement field which is prescribed in such solutions.

Those controllability conditions which involve only the deformation are equivalent to the conditions which ERICKSEN [6] used. In Section 5 we list some of his results.

Some general consequences of those controllability conditions which involve both deformation and field are deduced in Sections 6 and 7. The remainder of the analysis is divided into two parts. In the first part, in Sections 8 to 12, we consider cases in which the strain and field invariants (defined in Section 2) are not all constant in space. The remainder of Part II, Sections 13 to 18, deals with states in which all invariants are constant in space.

The analysis of cases in which not all invariants are constant is relatively short in spite of the fact that these cases include almost all of the controllable states, because we can rely heavily on ERICKSEN'S analysis of controllable deformations in these cases. The longer analysis of cases for which all invariants are constant yields, aside from homogeneous deformations with uniform fields, only one family of solutions (Section 18). This is the combined extension, dislocation, and cylindrical shearing deformation with axial field which we have described earlier (Section I.14).

The body of the analysis is outlined in a little more detail at the end of Section 5, after the pertinent background material has been recited.

II.2. Notation

We use the notation g^N_{ij} for the ij-component of the matrix $\boldsymbol{g}^N$. The matrices $\boldsymbol{h}^{(N)} = \| h^{(N)}_{ij} \|$ are defined by

$$h^{(N)}_{ij} = E_i (g^N_{jk} E_k) + (g^N_{ik} E_k) E_j . \tag{2.1}$$

In this notation, the constitutive equations (I.3.5)* and (I.3.7) are respectively

$$D_i = (\Lambda_0 \delta_{ij} + \Lambda_1 g_{ij} + \Lambda_2 g^2_{ij}) E_j \tag{2.2}$$

and

$$S_{ij} = \Phi_1 g_{ij} + \Phi_2 g^2_{ij} + \Phi_3 h^{(0)}_{ij} + \Phi_4 h^{(1)}_{ij} + \Phi_5 h^{(2)}_{ij}. \tag{2.3}$$

The coefficients Λ and Φ are functions of the invariants (I.3.9), which are now written as

$$I_\mu = g^\mu_{ii} \quad (\mu = 1, 2), \qquad I_{3+\mu} = E_i g^\mu_{ij} E_j \quad (\mu = 0, 1, 2). \tag{2.4}$$

II.3. Controllable States

Certain problems in the theory of incompressible elastic dielectrics can be solved by an inverse method, in which an isochoric deformation $x_i(X_A)$ and an electric potential $V(x_i)$ are specified at the outset. The dielectric displacement D_i and the extra stress S_{ij} are then determined from the constitutive equations (2.2) and (2.3), respectively (together with the subsidiary definitions in Section I.3). If the specified deformation $x_i(X_A)$ and potential $V(x_i)$ do indeed provide a solution, the flux D_i derived from (2.2) will satisfy (I.5.1), and the extra stress S_{ij} will be such that there exists a solution p of (I.5.3). The integrability condition for (I.5.3) is

$$S_{ij,jk} = S_{kj,ji}. \tag{3.1}$$

It can happen that $x_i(X_A)$ and $V(x_i)$ are of such forms that (I.5.1) and (3.1) are satisfied identically, regardless of the forms of the functions Λ and Φ in (2.2) and (2.3). In such a case, the state defined by $x_i(X_A)$ and $V(x_i)$ is controllable.

Conditions satisfied by the strain g_{ij} and field E_i in controllable states can be derived by following the procedure which ERICKSEN [6] used to determine controllable states of purely elastic materials. We first observe, from (2.3), that

$$\begin{aligned} S_{ij,jk} = \Phi_1 g_{ij,jk} &+ \sum_{\mu=1}^{5} (\partial \Phi_1 / \partial I_\mu) \left[g_{ij,j} I_{\mu,k} + (g_{ij} I_{\mu,j})_{,k} \right] + \\ &+ \sum_{\lambda=1}^{5} \sum_{\mu=1}^{5} (\partial^2 \Phi_1 / \partial I_\lambda \partial I_\mu) g_{ij} I_{\lambda,j} I_{\mu,k} + \cdots , \end{aligned} \tag{3.2}$$

where the dots indicate analogous terms arising from the terms with coefficients Φ_2 to Φ_5 in (2.3). From (3.2) we see that (3.1) is satisfied identically, without regard to the forms of the coefficients Φ, provided that each of the following tensors is symmetric with respect to interchange of i and k:

$$g^N_{ij,jk}, \tag{3.3}$$

$$g^N_{ij,j} I_{\mu,k} + (g^N_{ij} I_{\mu,j})_{,k}, \tag{3.4}$$

$$g^N_{ij} (I_{\lambda,j} I_{\mu,k} + I_{\mu,j} I_{\lambda,k}), \tag{3.5}$$

* Equations and Sections in Part I are identified by prefixing "I" to the appropriate number.

$$h_{ij,jk}^{(N)}, \tag{3.6}$$

$$h_{ij,j}^{(N)} I_{\mu,k} + (h_{ij}^{(N)} I_{\mu,j})_{,k}, \tag{3.7}$$

$$h_{ij}^{(N)}(I_{\lambda,j} I_{\mu,k} + I_{\mu,j} I_{\lambda,k}). \tag{3.8}$$

Here, and in the remainder of the paper, $N=0, 1, 2$, and $\lambda, \mu=1, \ldots, 5$. Conversely, each of the tensors (3.3) to (3.8) must be symmetric if (3.1) is to be satisfied identically for every choice of the functions Φ in (2.3).

By using (2.2) in (I.5.1), we obtain

$$\sum_{N=0}^{2} \left[A_N (g_{ij}^N E_j)_{,i} + \sum_{\mu=1}^{5} (\partial A_N / \partial I_\mu) I_{\mu,i}\, g_{ij}^N E_j \right] = 0. \tag{3.9}$$

In order for this equation to be satisfied identically, whatever forms the functions A may take, it is necessary and sufficient that the following conditions be satisfied:

$$(g_{ij}^N E_j)_{,i} = I_{\mu,i}\, g_{ij}^N E_j = 0. \tag{3.10}$$

The strain g_{ij} and field E_i for a controllable state must be such that the tensors (3.3) to (3.8) are symmetric, and such that the conditions (3.10) are satisfied. In addition, E_i must satisfy the irrotationality condition (I.5.2), and g_{ij} must be symmetric and positive definite, with unit determinant. Finally, in order to ensure that g_{ij} can be expressed in terms of deformation gradients $x_{i,A}$ in the form (I.3.8), it is necessary and sufficient for g_{ij} to satisfy the following compatibility conditions [22]:

$$4 R_{ijkl} = 2(\overset{-1}{g}_{il,kj} + \overset{-1}{g}_{jk,il} - \overset{-1}{g}_{ik,lj} - \overset{-1}{g}_{jl,ik}) + g_{mn}(A_{jkm} A_{iln} - A_{jlm} A_{ikn}) = 0. \tag{3.11}$$

Here the matrix g_{ij}^{-1} is the inverse of the matrix g_{ij}, and

$$A_{ijk} = A_{jik} = \overset{-1}{g}_{ik,j} + \overset{-1}{g}_{jk,i} - \overset{-1}{g}_{ij,k}. \tag{3.12}$$

II.4. Permissible Fields

The electric fields E_i which can be involved in controllable states can be determined immediately from the conditions that the curl and divergence of the field must vanish and that the magnitude of the field strength must be constant along each line of force. These are the conditions (I.5.2), (3.10a) (with $N=0$), and (3.10b) (with $N=0$ and $\mu=3$), respectively:

$$E_{i,j} - E_{j,i} = E_{i,i} = E_i (E_j E_j)_{,i} = 0. \tag{4.1}$$

HAMEL [21] (see also PRIM [23]) has shown that every field E_i satisfying (4.1) can be expressed in cylindrical polar coordinates r, ϑ, z, in the form

$$\boldsymbol{E} = \nabla(p\,\vartheta + q\,z), \tag{4.2}$$

where p and q are constants. The physical components of $\boldsymbol{E}$ are

$$E_r = 0, \quad E_\vartheta = p/r, \quad E_z = q. \tag{4.3}$$

We note that if $E_i E_i$ is constant, then $p=0$ and the field is uniform.

II.5. Permissible Strains

Certain of the tensors (3.3) to (3.5) are independent of the electric field strength. Ericksen [6] has deduced the restrictions imposed upon controllable strains g_{ij} by symmetry of a set of tensors equivalent to these. We summarize here some of his results.

First, each of the gradients $I_{1,i}$ and $I_{2,i}$ is either zero or an eigenvector of g_{ij}. If both are non-zero, they have a common eigenvalue g_1, say:

$$g_{ij} I_{\mu,j} = g_1 I_{\mu,i}. \tag{5.1}$$

In Section 6 we show that (5.1) holds for $\mu=3, 4, 5$ as well. Next, there exists a non-constant function B such that I_1 and I_2 are both functions of B:

$$I_\mu = I_\mu(B). \tag{5.2}$$

In Section 7 we show that (5.2) is also true for $\mu=3, 4, 5$.

Let g_1, g_2, and g_3 be the eigenvalues of g_{ij}. The incompressibility condition is expressed in terms of these eigenvalues as $g_1 g_2 g_3 = 1$. With this condition, (5.2) implies that each eigenvalue is a function of B. Let a_i, b_i, and c_i be a corresponding orthonormal system of eigenvectors. Then the spectral representation of g^N_{ij} is

$$g^N_{ij} = g^N_1(B)\, a_i a_j + g^N_2(B)\, b_i b_j + g^N_3(B)\, c_i c_j, \tag{5.3}$$

where

$$a_i a_i = b_i b_i = c_i c_i = 1, \qquad a_i b_i = b_i c_i = c_i a_i = 0. \tag{5.4}$$

If either $I_1'(B)$ or $I_2'(B)$ is not zero, where primes denote differentiation with respect to B, it follows from (5.1) that

$$g_{ij} B_{,j} = g_1(B)\, B_{,i}, \tag{5.5}$$

and we can take a_i to be in the direction of $B_{,i}$:

$$a_i = B_{,i}/(B_{,j} B_{,j})^{\frac{1}{2}}. \tag{5.6}$$

Furthermore, if $I_1'(B)$ or $I_2'(B)$ is not zero, the controllability conditions imply that

$$g^N_{ij,j} = C_N(B)\, B_{,i}. \tag{5.7}$$

In Section 8 we show that (5.7) must be satisfied if any invariant I_μ ($\mu=1, \ldots, 5$) is not constant.

Let us establish the convention that if g_1 is a degenerate eigenvalue, then $g_3 = g_1$. If $g_2 = g_1$ as well, then incompressibility requires that $g_1 = g_2 = g_3 = 1$, and thus $g_{ij} = \delta_{ij}$. In this case there is no deformation. More generally, with the convention just mentioned, it follows from the relations (5.3) to (5.7) that

$$(a_i a_j)_{,j} = F_a(B)\, B_{,i} \quad \text{if} \quad g_1 \neq g_3, \tag{5.8}$$

i.e. if g_1 is non-degenerate, and that

$$(b_i b_j)_{,j} = F_b(B)\, B_{,i} \quad \text{if} \quad g_2 \neq g_3, \tag{5.9}$$

i.e. if g_2 is non-degenerate.

We have mentioned that (5.5) and (5.7) will be shown to be valid if any invariant is not constant. In such cases, (5.8) is valid if g_1 is non-degenerate. ERICKSEN [6] has determined all of the controllable deformations which satisfy (5.8), and we will use the results which he obtained for this case in Section 9.

In cases for which g_1 is degenerate but g_2 is not (i.e. $g_1=g_3 \neq g_2$), (5.9) must be satisfied if any invariant is not constant. In these cases we will use the controllability conditions which involve both E_i and g_{ij} to show that either (5.8) is satisfied in spite of the fact that $g_1=g_3$, or the field b_i is normal to the equipotential surfaces for the electric field. In the former case, which we consider in Section 11, the analysis is similar to that for g_1 non-degenerate, and we obtain no new solutions. In the latter case we can rely on ERICKSEN'S results for the case in which g_1 is degenerate. His analysis of this case was restricted to circumstances in which the field b_i is normal to some family of surfaces. In the present problem, with a non-zero electric field, the latter condition is a positive requirement rather than a restrictive assumption. We use ERICKSEN'S results for this case in Section 10.

Homogeneous deformations are of no interest in ERICKSEN'S analysis, but in the present problem homogeneous deformations with non-constant electric fields are of interest. To ensure that no such states have been overlooked in Sections 9 to 11, where we rely on ERICKSEN'S results, in Section 12 we consider such states separately.

If all of the invariants I_μ ($\mu=1, \ldots, 5$) are constant, neither (5.8) nor (5.9) is necessarily valid, and we cannot use ERICKSEN'S results. The results for this case, which are obtained in Sections 13 to 18, involve deformations which are not among those ERICKSEN found.

II.6. Preliminary Analysis. I

In the present section we show that in order for each of the tensors (3.5) to be symmetric, it is necessary and sufficient for those gradients $I_{\mu,i}$ which are not zero to be eigenvectors of g_{ij} with a common eigenvalue g_1 (say):

$$g_{ij} I_{\mu,j} = g_1 I_{\mu,i} \qquad (\mu=1, \ldots, 5). \tag{6.1}$$

The method of proof, which is given in ERICKSEN'S [6] paper, is based on the fact that if $u_i v_j = u_j v_i$ and the vector v_i is not zero, then $u_i = \lambda v_i$.

With the lemma just mentioned, symmetry of the tensors (3.5) with $\mu=\lambda$ and $N=1$ implies that the vector $g_{ij} I_{\mu,j}$ is a multiple of $I_{\mu,i}$ if the latter is not zero, and the result follows trivially if $I_{\mu,i}$ is the null vector:

$$g_{ij} I_{\mu,j} = A_{\boldsymbol{\mu}} I_{\boldsymbol{\mu},i}. \tag{6.2}$$

Here the boldface subscript indicates suspension of the summation convention.

Next, symmetry of the tensors (3.5) with $N=1$ and $\mu \neq \lambda$ requires, with (6.2), that

$$(A_{\boldsymbol{\lambda}} - A_{\boldsymbol{\mu}})(I_{\boldsymbol{\lambda},i} I_{\boldsymbol{\mu},k} - I_{\boldsymbol{\lambda},k} I_{\boldsymbol{\mu},i}) = 0. \tag{6.3}$$

It follows that either $A_\lambda = A_\mu$ or $I_{\lambda,i}$ is a multiple of $I_{\mu,i}$, or one of these gradients is zero. In the latter case the corresponding coefficient A_μ in (6.2) is arbitrary, and we can set $A_\mu = A_\lambda$ without loss of generality. If both gradients are non-zero, they are parallel eigenvectors of g_{ij} according to (6.2), and the corresponding eigenvalues are thus equal. Hence, we obtain $A_\lambda = A_\mu$ in every case. Denoting the common value by g_1, we obtain (6.1) as desired. Conversely, (6.1) implies that the tensors (3.5) are symmetric, for every choice of N.

II.7. Preliminary Analysis. II

We now show that there is a non-constant function B such that each invariant I_μ is a function of B, i.e. $I_\mu = I_\mu(B)$. It is equivalent to show that those gradients $I_{\mu,i}$ which are not zero are parallel.

If all of the invariants I_μ are constants, then each is a constant function of any arbitrarily chosen function B. If only one invariant is not constant, then we can let B be that invariant, and the result $I_\mu = I_\mu(B)$ $(\mu = 1, \ldots, 5)$ again follows. A third trivial case is that in which g_1 is a non-degenerate eigenvalue of g_{ij}, in which case it follows from (6.1) that those gradients $I_{\mu,i}$ which are not zero are parallel.

We can accordingly restrict our attention to cases in which g_1 is a degenerate eigenvalue, and at least two of the invariants are not constant. If g_1 is triply degenerate, then incompressibility requires that $g_1 = g_2 = g_3 = 1$, and thus $g_{ij} = \delta_{ij}$. In this case it follows from (2.4) that I_1 and I_2 are constants, while I_3, I_4, and I_5 are equal, and so we can obtain the desired result by taking $B = I_3$, say.

If g_1 is doubly degenerate, we take $g_1 = g_3 \neq g_2$, as stated in Section 5. From (3.10b) with $N=0$ we obtain

$$E_i I_{\mu,i} = 0. \tag{7.1}$$

Now, the non-zero gradients $I_{\mu,i}$ must lie in the plane of principal directions with principal value g_1, according to (6.1). They must also lie in the plane perpendicular to E_i, according to (7.1) (and assuming $E_i \neq 0$). Hence, the non-zero gradients lie along the intersection of these two planes, and are thus parallel unless the two planes coincide.

Consequently, the only case left to consider is that in which the two planes mentioned above are coincident. In that case, E_i is an eigenvector of g_{ij} corresponding to the non-degenerate eigenvalue g_2, and g_{ij} can be expressed in the form

$$g_{ij} = g_1 \delta_{ij} + (g_1^{-2} - g_1) I_3^{-1} E_i E_j. \tag{7.2}$$

Here we have set $g_2 = 1/g_1^2$, which follows from incompressibility if $g_1 = g_3$. With (7.2), the invariants (2.4) are

$$\begin{aligned} &I_1 = 2g_1 + g_1^{-2}, \quad I_2 = 2g_1^2 + g_1^{-4}, \quad I_3 = E_i E_i, \\ &I_4 = E_i E_i / g_1^2, \quad I_5 = E_i E_i / g_1^4. \end{aligned} \tag{7.3}$$

From (7.3) we see that if g_1 is constant, the desired result is obtained by taking $B = E_i E_i$, say. Suppose that g_1 is not constant, so that $I_{1,i} \neq 0$. Then ERICKSEN's results (5.2) and (5.7) hold, with $B = g_1$, say. By using (7.2) in (5.7), with $B = g_1$, we obtain

$$g_{1,i} + [(g_1^{-2} - g_1) I_3^{-1}]_{,j} E_i E_j + (g_1^{-2} - g_1) I_3^{-1} (E_i E_j)_{,j} = C_N(g_1)\, g_{1,i}. \tag{7.4}$$

By taking into account the facts that $E_i I_{3,i}$ and $E_i g_{1,i}$ are zero according to (7.1) and (5.2), and that $2(E_i E_j)_{,j} = I_{3,i}$, according to (4.1) and (7.3), from (7.4) we obtain

$$(g_1^{-2}-g_1)I_3^{-1}I_{3,i}=2[C_N(g_1)-1]g_{1,i}. \tag{7.5}$$

Thus $I_{3,i}$ is a multiple of $g_{1,i}$, as desired. The case $g_1=1$, for which also $g_2=g_3=1$, has already been covered. This completes the proof that all of the invariants can be represented as functions of a single non-constant function B. The representation of the invariants I_1 and I_2 as functions of B, together with the incompressibility condition $g_1 g_2 g_3=1$, implies that

$$g_i=g_i(B) \qquad (i=1,2,3). \tag{7.6}$$

II.8. States with Invariants not all Constant

From this point onward, through Section 12, we use the assumption that at least one invariant I_μ is not constant. Cases in which all invariants are constant are considered in Sections 13 to 18.

We have shown that (5.1) and (5.2) are valid for $\mu=1,\ldots,5$. Furthermore, (5.1) implies that all of the tensors (3.5) are symmetric. From these results, it also follows that the tensors (3.4) are of the forms

$$g_{ij,j}^N I'_\mu B_{,k}+(g_1^N)' I'_\mu B_{,i} B_{,k}+g_1^N I_{\mu,ik}. \tag{8.1}$$

Here primes denote differentiation with respect to B. Assuming that $I'_\mu \neq 0$ for some μ, the tensors (8.1) are all symmetric if and only if $g_{ij,j}^N$ is a multiple of $B_{,i}$, say $C_N B_{,i}$.

Symmetry of the tensors (3.3) requires that $g_{ij,j}^N$ be the gradient of a scalar, and thus that $C_N B_{,i}$ be the gradient of a scalar. Hence, C_N is a function of B, and we have

$$g_{ij,j}^N=C_N(B)B_{,i}. \tag{8.2}$$

With this result, the tensors (3.3) and (3.4) are all symmetric.

If $I'_\mu \neq 0$ for some μ, it follows from (3.10b) with $N=0$ (or from (7.1)) that

$$E_i B_{,i}=0. \tag{8.3}$$

Conversely, it follows from (8.3), (6.1), and (7.6) that all of the conditions (3.10b) are satisfied.

We next consider the tensors (3.8). By using (6.1), (8.3), and the definition (2.1) of $h_{ij}^{(N)}$, we obtain

$$h_{ij}^{(N)} I_{\mu,j}=(g_1^N E_i+g_{ik}^N E_k) I'_\mu E_j B_{,j}=0. \tag{8.4}$$

Hence, all of the tensors (3.8) are zero, and thus trivially symmetric.

With (8.4), symmetry of the tensors (3.7) requires that

$$I'_\mu h_{ij,j}^{(N)} B_{,k}=I'_\mu h_{kj,j}^{(N)} B_{,i}. \tag{8.5}$$

With $I'_\mu \neq 0$ for some μ, (8.5) is satisfied if and only if $h_{ij,j}^{(N)}$ is a multiple of $B_{,i}$. Then, by using the symmetry of the tensors (3.6), we obtain

$$h_{ij,j}^{(N)}=D_N(B)B_{,i}. \tag{8.6}$$

Conversely, from (8.6) and our previous results it follows that all of the tensors (3.6) and (3.7) are symmetric.

Let us represent E_i in terms of the eigenvectors of g_{ij} (see Section 5). Because a_i is in the direction of $B_{,i}$ (see (5.6)), it follows from (8.3) that E_i has no component in the direction of a_i:

$$E_i = E_b\, b_i + E_c\, c_i. \tag{8.7}$$

With (5.3), the invariants I_3, I_4, and I_5 then take the forms

$$I_{3+\mu}(B) = g_2^\mu(B)\, E_b^2 + g_3^\mu(B)\, E_c^2. \tag{8.8}$$

We note that if $g_2 = g_3$, the eigenvectors b_i and c_i are undefined to the extent of an arbitrary common rotation about the direction of a_i. We can fix the directions of b_i and c_i by adopting the convention that c_i is perpendicular to E_i, i.e. $E_c = 0$, if $g_2 = g_3$. With this convention covering the degenerate case, it follows from (8.8) that E_b and E_c are functions of B:

$$E_b = E_b(B), \qquad E_c = E_c(B). \tag{8.9}$$

We have not yet considered (3.10a) in any detail. By using the representations (5.3) and (8.7) in (3.10a), we obtain

$$\begin{aligned} 0 = (g_{ij}^N E_j)_{,i} &= [g_2^N(B)\, E_b(B)\, b_i + g_3^N(B)\, E_c(B)\, c_i]_{,i} \\ &= g_2^N\, E_b\, b_{i,i} + g_3^N\, E_c\, c_{i,i}. \end{aligned} \tag{8.10}$$

In obtaining the final member of (8.10), we have used the orthogonality relations $b_i B_{,i} = c_i B_{,i} = 0$.

II.9. Cases with g_1 Non-Degenerate

Ericksen [6] has shown that if the eigenvalue g_1 is non-degenerate, then the surfaces B=constant must be concentric spheres, parallel planes, or coaxial right circular cylinders, and he has determined the corresponding permissible deformations in each of these cases. We now seek to couple permissible fields to these deformations, to form controllable states.

From Section 4 we note that the invariant $I_3 = E_i E_i$ must be constant over each cylinder of a coaxial family if the field is not constant, and constant everywhere if the field is uniform. Because I_3 is a function of B, it follows that the surfaces B=constant cannot be parallel planes or concentric spheres unless the field is uniform. Furthermore, from the requirement (8.3) that the field must be perpendicular to $B_{,i}$ everywhere, we see that a uniform field is possible only if the B-surfaces are coaxial cylinders with the field in the axial direction, or if the B-surfaces are parallel planes.

These remarks imply that there are no solutions for which the B-surfaces are concentric spheres.

If the B-surfaces are parallel planes, the electric field must be a uniform field orthogonal to $B_{,i}$. Conversely, if the deformation is any one of those which Ericksen obtained in this case, and E_i is any uniform field orthogonal to $B_{,i}$, the resulting state is controllable (Section I.11).

If the B-surfaces are coaxial cylinders, the surfaces r=constant for the permissible field in Section 4 must be identified with the surfaces $B(r)$=constant in Ericksen's solutions. Conversely, every such combination of deformation and field produces a controllable state (Sections I.9 and I.13).

II.10. Cases with g_1 Degenerate. I

Now let g_1 be a repeated eigenvalue. If g_1 is triply degenerate, incompressibility implies that there is no deformation. In this case, for which $g_{ij}=\delta_{ij}$, all of the tensors (3.3) to (3.8) are symmetric, and (I.5.2) and (3.10) are satisfied, for every field E_i of the admissible form (4.3). This case of no deformation with the corresponding field (4.3) is a special case of the controllable states considered in Section I.9.

In the remainder of this section and in Section 11 we consider cases in which g_1 is doubly degenerate. In Section 5 we have adopted the convention that $g_3=g_1$ in this case, and incompressibility then yields $g_2=1/g_1^2$. Now, with $g_2 \neq g_3$, it follows from (8.10) that

$$E_b b_{i,i}=E_c c_{i,i}=0. \tag{10.1}$$

From (5.9), with (5.4) and (5.6), we obtain

$$b_{i,i}=0, \quad b_j b_{i,j}=F_b(B) B_{,i}. \tag{10.2}$$

The condition (10.1b) requires that either E_c or $c_{i,i}$ be zero. The case $E_c \neq 0$ will be considered in Section 11. In the remainder of the present section we consider the case $E_c=0$. In this instance it follows from (8.7) that the electric field is of the form

$$E_i=E_b b_i. \tag{10.3}$$

Now, E_i must be the gradient of a potential, $-V_{,i}$, and thus the field b_i is normal to the equipotential surfaces. ERICKSEN [6] has shown that in cases for which (10.2) is valid and for which the field b_i is normal to some family of surfaces, the deformation is either homogeneous or a certain inhomogeneous deformation with constant strain invariants. (He accidentally overlooked the latter case, although he obtained the relevant strain g_{ij}.)

It is convenient to leave the case of homogeneous deformations aside until Section 12. In the case involving an inhomogeneous deformation with constant strain invariants, the field b_i obtained from ERICKSEN's results yields, with (10.3), a field E_i which is the special case $E_z=0$ of the permissible fields (4.3). This combination of deformation and field yields a controllable state which has been considered in Section I.9.

II.11. Cases with g_1 Degenerate. II

We now consider the case $g_1=g_3 \neq g_2$, $E_c \neq 0$. In this case (10.1b) implies that $c_{i,i}=0$.

ERICKSEN's [6] analysis of the deformations possible when g_1 is non-degenerate is based on the relation (5.8), which states that $(a_i a_j)_{,j}$ is the gradient of some function of B. We will show that a relation of this form is satisfied in the present case even though g_1 is degenerate. With this result, the discussion in Section 9 is applicable, and we obtain no new solution.

To show that (5.8) is satisfied, we first consider the identity

$$(a_i a_j+b_i b_j+c_i c_j)_{,j}=\delta_{ij,j}=0. \tag{11.1}$$

From (10.2) we know that $(b_i b_j)_{,j}$ is the gradient of a function of B. We will show that $(c_i c_j)_{,j}$ is also the gradient of a function of B. The desired result (5.8) then follows from (11.1).

To prove that $(c_i c_j)_{,j}$ is of the required form, we use (8.6). In (8.6) we use the definition (2.1) of $h_{ij}^{(N)}$ and the requirement (3.10a). We also use the representations of g_{ij} and E_i in terms of the eigenvectors, (5.3) and (8.7) respectively, and we recall that $b_i B_{,i} = c_i B_{,i} = 0$. After some manipulation, we obtain

$$g_2^N(B) E_b(B)(b_j E_{i,j} + E_j b_{i,j}) + g_3^N(B) E_c(B)(c_j E_{i,j} + E_j c_{i,j}) = D_N(B) B_{,i}. \quad (11.2)$$

From this system of equations, with $g_2 \neq g_3$, it follows that the coefficients of g_2^N and g_3^N must be gradients of functions of B. The same is then true of the difference of these coefficients. Hence, with the representation (8.7) for E_i, we obtain a relation of the form

$$E_c^2(B) c_j c_{i,j} - E_b^2(B) b_j b_{i,j} = H(B) B_{,i}. \quad (11.3)$$

By using (10.2), $c_{i,i} = 0$, and $E_c \neq 0$, we obtain from (11.3) the desired result, that $(c_i c_j)_{,j}$ is the gradient of a function of B.

II.12. Homogeneous Deformations

If the strain components g_{ij} and field components E_i are constants, all conditions are satisfied (Section I.6). We now consider cases in which the strain is constant but the electric field is non-uniform. We show that the only homogeneous deformations which can be combined with the permissible fields (4.3) are simple extensions in the axial direction (Section I.9).

If the field (4.3) is not uniform, then $I_3(B)$ is constant over the coordinate surfaces r=constant, whence $B = B(r)$. From (5.6) it then follows that the eigenvector a_i is in the radial direction. Because g_{ij} is constant, it can have a non-constant eigenvector a_i with eigenvalue g_1 only if g_1 is degenerate. By convention, we let $g_1 = g_3$. The eigenvector c_i then lies in the azimuthal direction, and b_i lies in the axial direction. The resulting strain (5.3), with $g_2 = 1/g_1^2$ and g_1 constant, is the strain tensor for simple extension in the axial direction. The resulting state is controllable (Section I.9).

II.13. States with all Invariants Constant

In the remainder of Part II we consider cases in which all of the invariants I_μ ($\mu = 1, \ldots, 5$) are constant. In this case, the tensors (3.4), (3.5), (3.7), and (3.8) are trivially symmetric, and (3.10b) is satisfied trivially. The tensors (3.3) are symmetric if and only if there exist potentials χ_N such that

$$\overset{N}{g}_{ij,j} = \chi_{N,i}. \quad (13.1)$$

The tensors (3.6) are symmetric if and only if there exist potentials ψ_N such that

$$h_{ij,j}^{(N)} = \psi_{N,i}. \quad (13.2)$$

With (13.1), the requirement (3.10a) takes the form

$$\overset{N}{g}_{ij} E_{j,i} + \chi_{N,j} E_j = 0. \quad (13.3)$$

Because I_3 is now constant, the only permissible non-zero fields (4.3) are the uniform fields $E_i = E\delta_{3i}$, say, where E is a non-zero constant.

We use the spectral representation (5.3) of g_{ij}, in which g_1, g_2, and g_3 are now constant by virtue of the incompressibility condition $g_1 g_2 g_3 = 1$ and the constancy of I_1 and I_2. It is also convenient to represent the constant field E_i in the form

$$E_i = E\,\delta_{3i} = E_a\,a_i + E_b\,b_i + E_c\,c_i. \tag{13.4}$$

We note that in (5.3) and (13.4), the eigenvectors a_i, b_i, and c_i are not necessarily constant. The constant invariants I_3, I_4, and I_5 are

$$I_{3+\mu} = g_1^\mu\,E_a^2 + g_2^\mu\,E_b^2 + g_3^\mu\,E_c^2. \tag{13.5}$$

If the eigenvalues are all equal, we obtain a solution involving a uniform field and no deformation. For the remainder of the analysis, we assume that the eigenvalues are not all equal, and we let g_1 be the non-degenerate eigenvalue if two eigenvalues are equal. In that case, the eigenvectors b_i and c_i are arbitrary to the extent of a common rotation about the direction of a_i, and we fix c_i by the convention that $E_i\,c_i = 0$, i.e. $E_c = c_3 = 0$.

With the latter convention covering the degenerate case $g_2 = g_3$, it follows from (13.5) that because the eigenvalues and invariants are constants, then E_a, E_b, and E_c are also constants. Because $E_a = Ea_3$ where E is constant and not zero, it follows that a_3 is constant. Similarly, b_3 and c_3 are constant. From (5.3) it follows that g_{33}^N is constant. From the orthonormality conditions (5.4) it follows that each of the following two-dimensional inner products is constant:

$$a_\alpha\,a_\alpha, \quad b_\alpha\,b_\alpha, \quad c_\alpha\,c_\alpha, \quad a_\alpha\,b_\alpha, \quad b_\alpha\,c_\alpha, \quad c_\alpha\,a_\alpha. \tag{13.6}$$

Here, and in the sequel, Greek subscripts α, β, *etc.* have the range 1, 2, and we use the summation convention over this range.

With a constant field $E_i = E\delta_{3i}$, (13.3) yields $\chi_{N,3} = 0$. Then (13.1), with g_{33}^N constant, gives

$$g^N_{3\alpha,\alpha} = 0, \qquad g^N_{\alpha\beta,\beta} + g^N_{\alpha 3,3} = \chi_{N,\alpha}(x_1, x_2). \tag{13.7}$$

By using the definition (2.1) of $h_{ij}^{(N)}$ and the constancy of E_i, from (13.2) we derive

$$\psi_{N,i} = E_i(g^N_{jk,j}\,E_k) + (g^N_{ik,j}\,E_k)\,E_j = E^2\,g^N_{i3,3}, \tag{13.8}$$

where the final member was obtained by using (13.4) and (13.7a). With g_{33}^N constant, (13.8) implies that $\psi_{N,3} = 0$, and yields

$$E^2\,g^N_{\alpha 3,3} = \psi_{N,\alpha}(x_1, x_2). \tag{13.9}$$

We can integrate (13.9) to obtain

$$g^N_{\alpha 3} = x_3\,E^{-2}\,\psi_{N,\alpha}(x_1, x_2) + F_{N\alpha}(x_1, x_2). \tag{13.10}$$

From the constancy of g_1, g_2, g_3, a_3, b_3, c_3, and the quantities (13.6), it follows that $g^N_{\alpha 3}\,g^N_{\alpha 3}$ is constant. Hence, the coefficient of x_3 in (13.10) must vanish. Thus, $\psi_{N,\alpha} = 0$. By using the spectral representation of g_{ij} in (13.9), we then obtain

$$g_1^N\,a_3\,a_{\alpha,3} + g_2^N\,b_3\,b_{\alpha,3} + g_3^N\,c_3\,c_{\alpha,3} = 0. \tag{13.11}$$

By using the spectral representation of g_{ij} in (13.7), and noticing that $g^N_{\alpha 3,3}=0$ according to (13.9), with $\psi_{N,\alpha}=0$, we obtain

$$g_1^N a_3\, a_{\alpha,\alpha} + g_2^N b_3\, b_{\alpha,\alpha} + g_3^N c_3\, c_{\alpha,\alpha} = 0 \tag{13.12}$$

and

$$g_1^N (a_\alpha\, a_\beta)_{,\beta} + g_2^N (b_\alpha\, b_\beta)_{,\beta} + g_3^N (c_\alpha\, c_\beta)_{,\beta} = \chi_{N,\alpha}(x_1, x_2). \tag{13.13}$$

From (13.11), with the convention $c_3=0$ if $g_2=g_3$, we get

$$a_3\, a_{\alpha,3} = b_3\, b_{\alpha,3} = c_3\, c_{\alpha,3} = 0. \tag{13.14}$$

Similarly, from (13.12) we obtain

$$a_3\, a_{\alpha,\alpha} = b_3\, b_{\alpha,\alpha} = c_3\, c_{\alpha,\alpha} = 0. \tag{13.15}$$

From (13.13), with g_1 conventionally non-degenerate, it follows that $(a_\alpha\, a_\beta)_{,\beta}$ must be a two-dimensional gradient:

$$(a_\alpha\, a_\beta)_{,\beta} = \chi_{,\alpha}(x_1, x_2). \tag{13.16}$$

In the remainder of Part II we use (13.14) to (13.16), together with the spectral representation of g_{ij} and the compatibility conditions (3.11), to determine all controllable deformations with constant invariants which are consistent with a uniform field. As suggested by (13.14) and (13.15), the analysis will be organized according to the number of values a_3, b_3, and c_3 which are zero.

II.14. The Case $a_3\, b_3\, c_3 \neq 0$

In the present section we show that only pure homogeneous deformations are possible if a_3, b_3, and c_3 are all different from zero. In this case, it follows from the orthonormality conditions (5.4) that the two-dimensional inner products (13.6) are all different from zero.

From (13.14) we find that a_α, b_α, and c_α are independent of x_3. The problem of determining these vectors is therefore strictly two-dimensional. It follows from (13.15) that the (two-dimensional) divergence of each of these fields is zero.

With (13.15) and (13.16), the equations governing a_α are the equations for steady plane motion of a perfect incompressible fluid of unit density, with velocity a_α and pressure $-\chi$. The speed is constant since $a_\alpha\, a_\alpha$ is constant. Neményi & Prim [*24*] show, for a rather more general case of plane rotational perfect gas flows, that the streamlines for such a motion must either be parallel straight lines or concentric circles. In fact it is easy to prove as follows.

With $a_{\alpha,\alpha}=0$, (13.16) becomes

$$a_{\alpha,\beta}\, a_\beta = \chi_{,\alpha}(x_1, x_2). \tag{14.1}$$

Since $a_\alpha\, a_\alpha=$constant, it follows from (14.1) that the curves $\chi=$constant are trajectories of a_α. In view of $a_{\alpha,\alpha}=0$ and $a_\alpha\, a_\alpha=$constant, these trajectories are parallel curves. This in turn gives the magnitude of two-dimensional gradient $|\nabla\chi|$ as constant along the curve so that the curvature is constant according to (14.1). Hence the trajectories of a_α are either parallel straight lines or concentric circles.

Consider first the case of parallel straight lines. In this case a_α is constant, and from the constancy of the inner products (13.6) it follows that b_α and c_α are constant. Then, g_{ij} is also constant.

Now consider the case of concentric circles. From the constancy of the inner products (13.6) it follows that the trajectories of the field b_α must make a constant angle with the circular a_α trajectories. Therefore the radial and azimuthal components of the field b_α are constants. The radial component is not zero because the orthogonality conditions would imply $c_3=0$ in that case, contrary to hypothesis. Hence, the divergence $b_{\alpha,\alpha}$ cannot vanish, as it must for a solution to exist, and therefore we obtain no solution in this case.

II.15. The Case $c_3=0$, $a_3 b_3 \neq 0$

We now consider the case when one of the values a_3, b_3, or c_3 is zero. Since g_1 is non-degenerate according to our convention, it does not matter which of a_3, b_3, or c_3 is regarded as zero when g_2 is also non-degenerate. However, if g_2 is degenerate, then $c_3=0$ conventionally. Therefore we shall take $c_3=0$ to cover both the possibilities of degenerate and non-degenerate g_2. We show that only pure homogeneous deformations are possible.

With $a_3 \neq 0$, the equations governing a_α are the same as in Section 14, and the trajectories of the field a_α are consequently parallel straight lines or concentric circles. From the constancy of the inner products (13.6), with the fact that a_α is independent of x_3, it follows that b_α and c_α are also independent of x_3.

As in Section 14, we obtain pure homogeneous deformations in the case of parallel straight lines. In the case of concentric circles, we can introduce a system of cylindrical polar coordinates r, ϑ, z in which a_α and b_α are azimuthal vectors and c_α lies in the radial direction. In terms of the unit vectors $r_{,i}$, $r\vartheta_{,i}$, and $z_{,i}$, the vectors a_i, b_i, and c_i can be represented in the forms

$$a_i=r\vartheta_{,i}\cos\gamma+z_{,i}\sin\gamma, \qquad b_i=-r\vartheta_{,i}\sin\gamma+z_{,i}\cos\gamma, \qquad c_i=r_{,i}, \tag{15.1}$$

where γ is a constant for which $\sin\gamma\cos\gamma \neq 0$. By using (15.1) in (5.3), we obtain

$$\begin{aligned} g_{ij}=g_3\, r_{,i} r_{,j} &+(g_1\cos^2\gamma+g_2\sin^2\gamma)\, r^2\vartheta_{,i}\vartheta_{,j}+ \\ &+(g_1\sin^2\gamma+g_2\cos^2\gamma)\, z_{,i} z_{,j}+ \\ &+(g_1-g_2)\sin\gamma\cos\gamma(r\vartheta_{,i} z_{,j}+z_{,i} r\vartheta_{,j}), \end{aligned} \tag{15.2}$$

and g_{ij}^{-1} is obtained from (15.2) by substituting g_1^{-1}, g_2^{-1}, g_3^{-1} for g_1, g_2, and g_3, respectively.

From the compatibility condition (3.11) we obtain in particular $R_{i3\,i3}=0$. With $g_{ij,3}=0$ and $g_{33,i}=0$, this condition assumes the form

$$g_{\alpha\beta}(g^{-1}_{3\alpha,\gamma}-g^{-1}_{3\gamma,\alpha})(g^{-1}_{3\beta,\gamma}-g^{-1}_{3\gamma,\beta})=0. \tag{15.3}$$

Because $g_{\alpha\beta}$ is positive definite according to (15.2), (15.3) yields

$$g^{-1}_{3\alpha,\beta}-g^{-1}_{3\beta,\alpha}=0, \tag{15.4}$$

or, with (15.2),

$$(g_1^{-1}-g_2^{-1})\sin\gamma\cos\gamma\,[(r\vartheta_{,\alpha})_{,\beta}-(r\vartheta_{,\beta})_{,\alpha}]=0, \tag{15.5}$$

which is satisfied only if $g_1 = g_2$ or $\sin\gamma\cos\gamma = 0$, both contrary to hypothesis. Hence, we obtain no solutions in this case.

II.16. The Case $b_3 = 1$. Basic Equations

We now suppose, finally, that two of the three values a_3, b_3, and c_3 are zero. In this case, the third of these values is unity, and the corresponding vector is a unit vector in the x_3-direction. The conditions (13.14) and (13.15) are satisfied trivially.

Let us abandon the convention that g_1 is necessarily non-degenerate. Then, without loss of generality, we can let b_i be the constant vector in the x_3 direction. By using the relations

$$b_i = \delta_{3i} \quad \text{and} \quad c_i c_j = \delta_{ij} - a_i a_j - b_i b_j \tag{16.1}$$

in the spectral representation (5.3), we obtain

$$g_{ij}^N = (g_1^N - g_3^N)\, a_i a_j + (g_2^N - g_3^N)\, \delta_{3i}\delta_{3j} + g_3^N \delta_{ij}. \tag{16.2}$$

If $g_1 = g_3$, the deformation is a simple extension in the x_3 direction, which has been considered in Section 12.

If $g_1 \neq g_3$, by using (16.2) in (13.13) we find that $(a_\alpha a_\beta)_{,\beta}$ must be the gradient of a scalar function of x_1 and x_2 only, even if $g_1 = g_2$:

$$(a_\alpha a_\beta)_{,\beta} = \chi_{,\alpha}(x_1, x_2). \tag{16.3}$$

In order to obtain further conditions on a_α, we must use the compatibility conditions (3.11). From these conditions, we will first show that a_α is independent of x_3. We will then show that the following equation must be satisfied:

$$(a_\alpha a_\beta)_{,\alpha\beta} = 0. \tag{16.4}$$

In Section 17 we obtain a_α from (16.3) and (16.4), and in Section 18 we determine the deformation implied by (16.2) with the known field a_α.

We first consider the compatibility condition $R_{i3\,i3} = 0$. Because g_1, g_2, and g_3 are constants, and $a_3 = 0$, it follows from (16.2) that g_{3i}^{-1} and g_{ii}^{-1} are constants. The condition $R_{i3\,i3} = 0$ accordingly becomes

$$g_{\alpha\beta}\, g_{\gamma\alpha,3}^{-1}\, g_{\gamma\beta,3}^{-1} = 0. \tag{16.5}$$

Because $g_{\alpha\beta}$ is positive definite according to (16.2), it follows from (16.5) that $g_{\alpha\beta,3}^{-1} = 0$. Hence, with (16.2), and recalling that $a_\alpha a_\alpha = 1$ while $g_1 \neq g_3$, we find that $a_{\alpha,3} = 0$:

$$a_\alpha = a_\alpha(x_1, x_2). \tag{16.6}$$

With a strain tensor of the form (16.2), where $a_3 = 0$ and $a_{i,3} = 0$, the components R_{ijkl} in (3.11) vanish identically if i, j, k, or l is equal to 3. The components $R_{\alpha\beta\gamma\delta}$ all vanish if R_{1212} is zero, by virtue of the general antisymmetry conditions satisfied by R_{ijkl}. Necessary and sufficient for R_{1212} to vanish is that $R_{\alpha\beta\alpha\beta} = 0$, which, with (3.11) and (3.12), means

$$4\, g_{\alpha\beta,\alpha\beta}^{-1} = g_{\gamma\delta}(A_{\alpha\alpha\gamma} A_{\beta\beta\delta} - A_{\alpha\beta\gamma} A_{\alpha\beta\delta}), \tag{16.7}$$

where

$$A_{\alpha\beta\gamma}=A_{\beta\alpha\gamma}=g^{-1}_{\alpha\gamma,\beta}+g^{-1}_{\beta\gamma,\alpha}-g^{-1}_{\alpha\beta,\gamma}. \tag{16.8}$$

In writing (16.7) we have used the results that the invariant g^{-1}_{ii} is constant and that the components g_{ij} and g^{-1}_{ij} are functions only of x_1 and x_2.

When (16.2) is inserted in (16.7) and note is taken of $a_3=0$, $a_\alpha a_\alpha=1$, and the constancy of the eigenvalues g_1, g_2, and g_3, we find, after lengthy manipulation, that the right hand side of (16.7) vanishes identically, thus reducing the condition (16.7) to the form (16.4).

II.17. The Case $b_3=1$. Solution of Basic Equations

We now determine the fields a_α which satisfy (16.3), (16.4), (16.6), and the condition $a_\alpha a_\alpha=1$. We use the following complex variables:

$$\begin{aligned} z&=x_1+i x_2, & \bar{z}&=x_1-i x_2,\\ a&=a_1+i a_2, & \bar{a}&=a_1-i a_2. \end{aligned} \tag{17.1}$$

We note that because the behavior of a_α for complex values of x_1 and x_2 is at our disposal, we can demand that all conditions be satisfied for arbitrary, independent values of z and $\bar{z}$. Because a_1 and a_2 must be real for real values of x_1 and x_2, the formal conjugate of each equation which we obtain must also be satisfied.

From (16.3) and (16.4) it follows that the potential χ must be harmonic:

$$2\chi=\Phi(z)+\bar{\Phi}(\bar{z}). \tag{17.2}$$

Here $\bar{\Phi}$ is the function whose value at $\bar{z}$ is the complex conjugate of $\Phi(z)$. Then, from (16.3) we obtain

$$\partial a^2/\partial z+\partial(a\bar{a})/\partial\bar{z}=\bar{\Phi}'(\bar{z}). \tag{17.3}$$

The condition $a_\alpha a_\alpha=1$ yields $a\bar{a}=1$. Hence, from (17.3) we obtain

$$a^2=z\bar{\Phi}'(\bar{z})+\bar{G}(\bar{z}). \tag{17.4}$$

Then $(a\bar{a})^2=1$ implies that

$$[z\bar{\Phi}'(\bar{z})+\bar{G}(\bar{z})][\bar{z}\Phi'(z)+G(z)]=1. \tag{17.5}$$

The functions $\Phi'(z)$ and $G(z)$ must be of such forms that (17.5) is satisfied identically, for arbitrary independent values of z and $\bar{z}$.

We assume that coordinates have been chosen in such a way that $\Phi'(z)$ is not singular at $z=0$ and $G(z)$ is not singular at $z=1$. By setting $z=\bar{z}=0$ in (17.5), we find that $G(0)\,\bar{G}(0)=1$, whence

$$G(0)=e^{i\alpha}, \quad \alpha \text{ real}. \tag{17.6}$$

Let $F(z)$ be defined by

$$F(z)=\Phi'(z)+G(z). \tag{17.7}$$

Then F is also non-singular at $z=0$. In terms of F, (17.5) is

$$[z\bar{F}(\bar{z})-(z-1)\,\bar{G}(\bar{z})][\bar{z}F(z)-(\bar{z}-1)\,G(z)]=1. \tag{17.8}$$

By setting $z=\bar{z}=1$ in (17.8), we obtain $F(1)\,\bar{F}(1)=1$, whence

$$F(1)=e^{i\beta}, \qquad \beta \text{ real}. \tag{17.9}$$

By setting $\bar{z}=0$ in (17.8), we obtain

$$G(z)=[z\,\bar{F}(0)-(z-1)\,e^{-i\alpha}]^{-1}. \tag{17.10}$$

By setting $\bar{z}=1$ in (17.8), we obtain

$$F(z)=[z\,e^{-i\beta}-(z-1)\,\bar{G}(1)]^{-1}. \tag{17.11}$$

Further restrictions on the forms of F and G are now to be obtained by using (17.10) and (17.11) in (17.8), and demanding that the resulting equation be satisfied identically in z and $\bar{z}$. These restrictions can be derived by examining the relations which arise from equating coefficients of like powers of z and $\bar{z}$, once the equation has been cleared of fractions. However, to avoid this tedious process, we use a different method.

We now show that $F(z)$ is a constant multiple of $G(z)$. The first factor in (17.8), regarded as a function of z, either is a constant or has a simple zero. It is a constant only if $F(\bar{z})=G(\bar{z})$, in which case our assertion is correct. If it has a simple zero, then it also has a simple pole at infinity. To satisfy the equation in this case, the second factor, regarded as a function of z, must have a simple pole and no other singularity, and it must have a simple zero at infinity. If it has a zero at infinity, then with (17.10) and (17.11) we see that either $F(z)$ and $G(z)$ are both constant, satisfying our assertion, or each has a simple zero at infinity. In the latter case, it follows from (17.10) and (17.11) that each also has a simple pole. Recalling that the second factor in (17.8) can have only one simple pole in the case under consideration, it follows that the poles of F and G coincide, and the assertion is proved.

Now, if $F(z)$ is a constant multiple $\bar{C}+1$, say, of $G(z)$, then with (17.10), we can obtain an expression of the form

$$z\,\bar{F}(\bar{z})-(z-1)\,\bar{G}(\bar{z})=e^{-i\alpha}\frac{C\,z+1}{D\,\bar{z}+1}, \tag{17.12}$$

and it is evident that (17.8) is satisfied if and only if $D=\bar{C}$. Then, by using (17.7) and (17.12) in (17.4), we obtain

$$a^2=e^{-i\alpha}\frac{C\,z+1}{\bar{C}\,\bar{z}+1}. \tag{17.13}$$

II.18. The Case $b_3=1$. Deformations

If $C=0$ in (17.13), then a is constant, and it follows from (17.1) that a_1 and a_2 are constant. The strain g_{ij} obtained by using this result in (16.2) is then also constant. Cases of homogeneous deformation have been considered in Section 12.

If $C\neq 0$ in (17.13), we can shift the origin to the point $z=-1/C$ with no loss of generality. If we denote the new variables by z' and $\bar{z}'$, (17.13) then takes the form

$$a^2=e^{-i\alpha}(C/\bar{C})(z'/\bar{z}'). \tag{18.1}$$

Because the magnitude of $C/\bar{C}$ is unity, we can write

$$e^{-i\alpha} C/\bar{C} = e^{2i\gamma}, \quad \gamma \text{ real}. \tag{18.2}$$

Then (18.1) yields $a = e^{i\gamma} z'/r$, where the ambiguity in sign has been absorbed in the definition of γ, and $r^2 = z' \bar{z}'$. By writing $z' = re^{i\vartheta}$, we obtain

$$a = e^{i(\vartheta+\gamma)}. \tag{18.3}$$

Thus, the real vector field a_α, which is related to a by (17.1), makes a constant angle γ with the radii ϑ = constant. The physical components of the field in cylindrical polar coordinates are

$$a_r = \cos\gamma, \quad a_\vartheta = \sin\gamma, \quad a_z = 0. \tag{18.4}$$

By using (18.4) in (16.2), we find that the physical components of strain are

$$\begin{gathered} g_{rr} = g_1 \cos^2\gamma + g_3 \sin^2\gamma, \quad g_{\vartheta\vartheta} = g_1 \sin^2\gamma + g_3 \cos^2\gamma, \\ g_{zz} = g_2, \quad g_{r\vartheta} = (g_1 - g_3)\sin\gamma\cos\gamma, \quad g_{\vartheta z} = g_{rz} = 0, \end{gathered} \tag{18.5}$$

where $g_1 g_2 g_3 = 1$.

A deformation of this type is produced if the particle initially at the point R, Θ, Z moves to the point r, ϑ, z given by

$$r = AR, \quad \vartheta = B\log R + C\Theta, \quad z = Z/A^2 C, \tag{18.6}$$

where

$$A^2 = g_1 \cos^2\gamma + g_3 \sin^2\gamma, \quad B = A^{-2}(g_1 - g_3)\sin\gamma\cos\gamma, \quad C^2 = 1/A^4 g_2. \tag{18.7}$$

This deformation combines with a uniform field in the axial direction to form a controllable state (Section I.14).

This completes the investigation of controllable states involving a prescribed non-zero electric field.

Part III. States with Specified Dielectric Displacement Field

III.1. Introduction

We have asserted that every controllable state with either a prescribed electric field or a prescribed dielectric displacement field is among those described in Part I. The first part of this assertion was verified in Part II, where we determined every controllable state involving a prescribed non-zero electric field. We now show that every controllable state with a prescribed non-zero dielectric displacement field is also included in Part I.

The analysis follows an outline similar to that of Part II. In Section 3, the conditions governing controllable states are derived. From the controllability conditions which involve only the dielectric displacement field, in Section 4 we show that the field must be either uniform, or a radial field in cylindrical or spherical coordinates.

Cases in which at least one invariant is not constant are examined in Sections 5 and 6. The analysis of deformations possible in these cases is reduced to a problem which ERICKSEN [*6*] has solved, and the corresponding controllable states are then easily determined. Because states involving homogeneous deformation might be overlooked when relying on ERICKSEN'S analysis, in Section 7 we consider such states separately.

Cases with all invariants constant are considered in Section 8. The problem is reduced to a problem which we have solved in Part II. In this case, and throughout Part III, we rely heavily on results obtained in Part II to abbreviate the analysis.

III.2. Notation

We use the notation

$$f_{ij}^{(N)}=D_i(g_{jk}^N D_k)+D_j(g_{ik}^N D_k). \tag{2.1}$$

The constitutive equations (I.3.12) and (I.3.13) are respectively

$$E_i=(\Omega_0\,\delta_{ij}+\Omega_1\,g_{ij}+\Omega_2\,g_{ij}^2)\,D_j \tag{2.2}$$

and

$$S_{ij}=\Psi_1\,g_{ij}+\Psi_2\,g_{ij}^2+\Psi_3\,f_{ij}^{(0)}+\Psi_4\,f_{ij}^{(1)}+\Psi_5\,f_{ij}^{(2)}, \tag{2.3}$$

where the coefficients Ω and Ψ are functions of the invariants J_μ defined in (I.3.14),

$$J_\mu=g_{ii}^\mu\quad(\mu=1,2),\qquad J_{\mu+3}=D_i\,g_{ij}^\mu\,D_j\quad(\mu=0,1,2). \tag{2.4}$$

III.3. Controllable States

If an isochoric deformation $x_i(X_A)$ and a solenoidal field $D_i(x_j)$ are specified, then the corresponding extra stress S_{ij} and electric field strength E_i are given by the constitutive equations (2.2) and (2.3), respectively. If S_{ij} and E_i satisfy (II.3.1) and (I.5.2), respectively, then the state specified by $x_i(X_A)$ and $D_i(x_j)$ can be supported without body force or distributed charge in a material of the type considered.

It is possible for the strain g_{ij} and flux D_i to be of such forms that (II.3.1) and (I.5.2) are satisfied identically, whatever may be the forms of the coefficients Ψ and Ω in the constitutive equations. By following a procedure similar to that used in the corresponding problem in Part II, we find that in order for this to be true, it is necessary and sufficient that each of the following tensors be symmetric:

$$g_{ij}^N(J_{\mu,j}\,J_{\lambda,k}+J_{\lambda,j}\,J_{\mu,k}), \tag{3.1}$$

$$g_{ij,j}^N\,J_{\mu,k}+(g_{ij}^N\,J_{\mu,j})_{,k}, \tag{3.2}$$

$$g_{ij,jk}^N, \tag{3.3}$$

$$f_{ij}^{(N)}(J_{\mu,j}\,J_{\lambda,k}+J_{\lambda,j}\,J_{\mu,k}), \tag{3.4}$$

$$f_{ij,j}^{(N)}\,J_{\mu,k}+(f_{ij}^{(N)}\,J_{\mu,j})_{,k}, \tag{3.5}$$

$$f_{ij,jk}^{(N)}, \tag{3.6}$$

$$(g_{ij}^N\,D_j)_{,k}, \tag{3.7}$$

$$(g_{ij}^N\,D_j)\,J_{\mu,k}. \tag{3.8}$$

Here $N=0$, 1, 2 and λ, $\mu=1,\dots,5$.

Let us suppose that a symmetric, positive definite tensor field g_{ij} satisfying the compatibility conditions (II.3.11) and the incompressibility condition is given, as well as a non-zero vector field D_i satisfying (I.5.1). If the tensors (3.1) to (3.8) calculated from g_{ij} and D_i are all symmetric, then g_{ij} and D_i are the strain and dielectric displacement for a controllable state.

III.4. Permissible Fields

The dielectric displacement fields which can be involved in controllable states with D_i specified can be determined immediately, from the conditions that the divergence and curl of such a field must be zero, and that the gradient of $D_i D_i$ must be parallel to D_i. The first of these conditions are respectively (I.5.1) and the symmetry requirement on the tensor (3.7) for $N=0$:

$$D_{i,i}=D_{i,j}-D_{j,i}=0. \tag{4.1}$$

Symmetry of the tensor (3.8) with $N=0$ and $\mu=3$ yields, with the definition (2.4) of J_3,

$$D_i(D_j D_j)_{,k}=D_k(D_j D_j)_{,i}. \tag{4.2}$$

This relation is satisfied if and only if $(D_j D_j)_{,i}$ is a multiple of D_i. Thus, with (4.1),

$$D_j D_{i,j}=FD_i. \tag{4.3}$$

According to (4.1), D_i is the gradient of a harmonic potential:

$$D_i=\psi_{,i}. \tag{4.4}$$

From (4.3) it follows that the trajectories of the field D_i, the orthogonal trajectories to the surfaces $\psi=$constant, are straight lines. Consequently, the surfaces $\psi=$constant form a parallel family, and the magnitude of the gradient $\psi_{,i}$ is constant over each such surface. It then follows from the fact that ψ is harmonic, by an analysis of the type used by ERICKSEN ([6], Sec. 3) in a similar problem, that the surfaces $\psi=$constant must be parallel planes, coaxial right circular cylinders, or concentric spheres.

In the case of parallel planes, the field D_i being normal to the planes, it follows from $D_{i,i}=0$ that the field is uniform. In the case of coaxial right circular cylinders, it follows from (4.4) and (4.1a) that the physical components of the field in an appropriate system of cylindrical polar coordinates must be of the form

$$D_r=C/r, \quad D_\vartheta=D_z=0, \tag{4.5}$$

where C is a constant. Similarly, in the case of concentric spheres, the physical components in an appropriate system of spherical coordinates r, ϑ, φ are

$$D_r=C/r^2, \quad D_\vartheta=D_\varphi=0, \tag{4.6}$$

where again C is a constant.

III.5. Invariants not all Constant. General Analysis

The spectral representation of g_{ij} will be used in the analysis. Let a_i, b_i, and c_i be an orthonormal system of eigenvectors of g_{ij}, with corresponding eigenvalues g_1, g_2, and g_3. Then

$$g^N_{ij} = g^N_1 a_i a_j + g^N_2 b_i b_j + g^N_3 c_i c_j, \tag{5.1}$$

where

$$a_i a_i = b_i b_i = c_i c_i = 1, \qquad a_i b_i = b_i c_i = c_i a_i = 0. \tag{5.2}$$

The field D_i can be represented in the form

$$D_i = D_a a_i + D_b b_i + D_c c_i. \tag{5.3}$$

The invariants J_μ ($\mu = 1, \ldots, 5$) can then be written as

$$\begin{aligned} J_\mu &= g^\mu_1 + g^\mu_2 + g^\mu_3 \qquad &(\mu = 1, 2), \\ J_{3+\mu} &= g^\mu_1 D^2_a + g^\mu_2 D^2_b + g^\mu_3 D^2_c \qquad &(\mu = 0, 1, 2). \end{aligned} \tag{5.4}$$

The incompressibility condition is

$$g_1 g_2 g_3 = 1. \tag{5.5}$$

In the remainder of the present section, and in Section 6, we suppose that at least one of the invariants J_μ is not constant. We also assume that the field D_i is not identically zero.

We now consider the implications of symmetry of the tensors (3.1) to (3.8). Because a closely similar system of equations has been analyzed in Part II, we will for the most part omit detailed proofs of our statements.

Symmetry of (3.8) for $N=0$ implies that $J_{\mu,i}$ is a multiple of D_i, and thus, with (4.4), a multiple of $\psi_{,i}$. Consequently, J_μ must be a function of ψ:

$$J_\mu = J_\mu(\psi) \quad (\mu = 1, \ldots, 5). \tag{5.6}$$

With (5.4) and (5.5), it then follows that the eigenvalues of g_{ij} are also functions of ψ:

$$g_1 = g_1(\psi), \qquad g_2 = g_2(\psi), \qquad g_3 = g_3(\psi). \tag{5.7}$$

From (5.6), with the assumption that $J'_\mu(\psi) \neq 0$ for some μ, it follows that the tensors (3.1) are all symmetric if and only if $\psi_{,i}$ is an eigenvector of g_{ij}. With no loss of generality, we take the eigenvector a_i to be in the direction of $\psi_{,i}$:

$$a_i = \psi_{,i} / (\psi_{,j} \psi_{,j})^{\frac{1}{2}}. \tag{5.8}$$

Then

$$g_{ij} \psi_{,j} = g_1(\psi) \psi_{,i}. \tag{5.9}$$

From (5.6) and (5.9), with $J'_\mu(\psi) \neq 0$ for some μ, it follows that the tensors (3.2) and (3.3) are all symmetric if and only if there exist functions $C_N(\psi)$ such that

$$g^N_{ij,j} = C_N(\psi) \psi_{,i}. \tag{5.10}$$

From the results so far derived, it follows that all of the remaining tensors (3.4) to (3.8) are symmetric. Symmetry of (3.7) is shown by using (5.9) and (4.4). Symmetry of (3.8) follows from (4.4), (5.6), and (5.9). To verify the symmetry of the tensors (3.4) to (3.6), we first use (4.4) and (5.9) in the definition (2.1) of $f_{ij}^{(N)}$, to obtain

$$f_{ij}^{(N)} = 2 g_1^N(\psi)\, \psi_{,i}\, \psi_{,j}. \tag{5.11}$$

Symmetry of (3.4) now follows from (5.6) and (5.11). To prove symmetry of (3.5), we also make use of the fact that $\psi_{,i}\,\psi_{,i}$ $(=J_3(\psi))$ is a function of ψ, and that $\psi_{,ii}=0$ according to (4.1) and (4.4). Then

$$f_{ij,j}^{(N)} = [2(g_1^N)'\, J_3 + g_1^N J_3']\, \psi_{,i}, \tag{5.12}$$

where primes are used to denote differentiation with respect to ψ. Now symmetry of (3.5) follows from (5.6), (5.11), and (5.12), while symmetry of (3.6) follows from (5.6), (5.7), and (5.12).

We note that from (5.8), with $\psi_{,i}\,\psi_{,i}=J_3(\psi)$ and $\psi_{,ii}=0$, there follows

$$(a_i\, a_j)_{,j} = -(J_3'/2 J_3)\, \psi_{,i}. \tag{5.13}$$

Furthermore, by using the spectral representation (5.1) in (5.10), and taking (5.13) into account, it can be shown that

$$(b_i\, b_j)_{,j} = G(\psi)\, \psi_{,i} \quad \text{if} \quad g_2 \neq g_3. \tag{5.14}$$

Here G is a certain function whose only relevant property is that it depends only on ψ.

III.6. Solutions with Invariants not all Constant

In ERICKSEN'S [6] analysis of deformations possible in every homogeneous, isotropic, incompressible, elastic material, he obtained equations of the forms (5.13) and (5.14), with (5.13) holding only if g_1 is a non-degenerate eigenvalue, and not a generally valid relation as in the present case. The remaining conditions on the deformation in the present problem are the same as those which ERICKSEN used.

ERICKSEN found all of the deformations which are possible if an equation of the form (5.13) is valid. Consequently, the deformations which are possible in the present problem, in the case under consideration (invariants not all constant), are those which ERICKSEN obtained under the assumption that g_1 is non-degenerate. He found that the surfaces ψ=constant must be parallel planes, coaxial right circular cylinders, or concentric spheres. (He uses the symbol B rather than ψ.) We have obtained the same result in Section 4, by obtaining equations for ψ which are of the forms which ERICKSEN used. Our analysis has shown that the surfaces ψ=constant must necessarily be identified with ERICKSEN'S surfaces B=constant. Furthermore, all conditions in the present problem are satisfied if we make such an identification.

It follows that if the surfaces ψ=constant are parallel planes, then only the deformations which ERICKSEN found for this case are admissible, and the field D_i must be a uniform field normal to these planes. These controllable states have been considered in Section I.10.

In the case of coaxial cylinders, the radial field (4.5) must be associated with the deformations found by ERICKSEN for this case. These controllable states have been considered in Sections I.8 and I.12.

Finally, the deformations found by ERICKSEN for the case of concentric spheres, combined with the radial field (4.6), yield controllable states which have been considered in Section I.7.

III.7. Homogeneous Deformations

Because homogeneous deformations are easy to overlook when relying on ERICKSEN'S [6] analysis, we now give separate consideration to cases with g_{ij} constant. If the components D_i are also constant, then all controllability conditions are satisfied. The case of homogeneous deformations with uniform fields has been considered in Section I.6.

If the surfaces ψ = constant are coaxial right circular cylinders, the field D_i is the radial field (4.5). The constant matrix g_{ij} then has all radial vectors as eigenvectors, with a common eigenvalue g_1, according to (5.9). It follows that the axial direction is a third principal direction. The strain g_{ij} thus corresponds to simple extension in the axial direction. Simple extension with a radial field is a special case of the controllable states considered in Section I.8.

If the surfaces ψ = constant are concentric spheres, then every radial vector (4.6) is an eigenvector of the constant matrix g_{ij}, according to (5.9). Hence, $g_{ij}=g_1\,\delta_{ij}$, and it follows from the incompressibility condition (5.5) that $g_1=1$. This case of no deformation with a spherically symmetric radial field is a special case of the controllable states considered in Section I.7.

III.8. All Invariants Constant

We now consider cases in which all of the invariants J_μ ($\mu=1, \ldots, 5$) are constants, and the field D_i is not identically zero. From Section 4 we find that if $J_3=D_i D_i$ is constant, the corresponding field D_i must be uniform. We let the x_3 axis lie along the direction of this field, so that $D_i=D\delta_{3i}$, where D is a non-zero constant.

The general relations (5.1) to (5.5) remain valid if all invariants are constants. The eigenvalues g_1, g_2, and g_3 are now constants, but the eigenvectors a_i, b_i, and c_i need not be constant. If the three eigenvalues are equal, there is no deformation. The case of a uniform field with no deformation has been covered in Section 7.

For the remainder of the analysis, we suppose that at least one eigenvalue is non-degenerate, and with no loss of generality we let g_1 be that eigenvalue. If $g_2=g_3$, the eigenvectors b_i and c_i are undetermined to the extent of a common rotation about the direction of a_i, and we can impose the conventional restriction that $D_i\,c_i=0$, i.e. $c_3=0$.

With these conventions, and recalling that eigenvalues and invariants are now constants, from (5.4) we find that the field components D_a, D_b, and D_c in the representation (5.3) are also constants. With $D_a=D_i\,a_i=D a_3$, *etc.*, it follows that a_3, b_3, and c_3 are constants. Then, from the orthogonality conditions (5.2) we find that each of the following two-dimensional inner products is also constant:

$$a_\alpha a_\alpha, \quad b_\alpha b_\alpha, \quad c_\alpha c_\alpha, \quad a_\alpha b_\alpha, \quad b_\alpha c_\alpha, \quad c_\alpha a_\alpha. \tag{8.1}$$

Here, and in the remainder of the paper, Greek subscripts α, β, etc. have the range 1, 2, and we use the summation convention over this range. With the preceding results, it follows from (5.1) that g^N_{33} is constant.

Constancy of the invariants J_μ implies that all of the tensors (3.1) to (3.8) are zero, and thus trivially symmetric, except (3.3), (3.6), and (3.7). With $D_i = D\delta_{3i}$, symmetry of the tensors (3.7) implies that there are scalar potentials Θ_N such that

$$g^N_{i3} = \Theta_{N,i}. \tag{8.2}$$

Because g^N_{33} is constant, from (8.2) with $i=3$ we obtain

$$\Theta_N = g^N_{33}\, x_3 + \varphi_N(x_1, x_2), \tag{8.3}$$

and thus

$$g^N_{3\alpha} = \varphi_{N,\alpha}(x_1, x_2). \tag{8.4}$$

Before considering the tensors (3.6), we use $D_i = D\delta_{3i}$ in the definition (2.1) of $f^{(N)}_{ij}$ to obtain

$$f^{(N)}_{ij} = D^2(\delta_{3i}\, g^N_{j3} + \delta_{3j}\, g^N_{i3}). \tag{8.5}$$

Then, with g^N_{33} constant and $g^N_{3\alpha}$ independent of x_3, we obtain

$$f^{(N)}_{ij,j} = D^2\, \delta_{3i}\, g^N_{3\alpha,\alpha}. \tag{8.6}$$

The tensors (3.6) are symmetric if and only if $f^{(N)}_{ij,j}$ is the gradient of a scalar. This gradient is in the x_3 direction, according to (8.6), and it follows that $g^N_{3\alpha,\alpha}$ in (8.6) must be a function of x_3 only. Hence, with (8.4),

$$\varphi_{N,\alpha\alpha\beta}(x_1, x_2) = 0. \tag{8.7}$$

By using the spectral representation (5.1) in (8.4), we obtain a system of linear equations ($N=0, 1, 2$) for $a_3\, a_\alpha$, $b_3\, b_\alpha$, and $c_3\, c_\alpha$. Recalling our conventions for cases with degenerate eigenvalues, we find that the solution of these equations is of the form

$$a_3\, a_\alpha = \varphi_{a,\alpha}(x_1, x_2), \quad b_3\, b_\alpha = \varphi_{b,\alpha}(x_1, x_2), \quad c_3\, c_\alpha = \varphi_{c,\alpha}(x_1, x_2), \tag{8.8}$$

where φ_a, φ_b, and φ_c are linear combinations of the potentials φ_N in (8.4). From (8.7) we then obtain

$$\varphi_{a,\alpha\alpha\beta} = \varphi_{b,\alpha\alpha\beta} = \varphi_{c,\alpha\alpha\beta} = 0. \tag{8.9}$$

Now, recalling that a_3 and $a_\alpha\, a_\alpha$ are constants, from (8.8) and (8.9) we find

$$0 = (a_3^2\, a_\alpha\, a_\alpha)_{,\beta\beta} = 2\varphi_{a,\alpha\beta}\, \varphi_{a,\alpha\beta}. \tag{8.10}$$

Hence, $\varphi_{a,\alpha\beta}$ is zero, and $\varphi_{a,\alpha}$ is thus constant. From (8.8) it then follows that $a_3\, a_\alpha$ is constant. We similarly find that each of the following quantities is constant:

$$a_3\, a_\alpha, \quad b_3\, b_\alpha, \quad c_3\, c_\alpha. \tag{8.11}$$

We can now draw the conclusion that if some eigenvector is oblique to the x_3 direction, then a_i, b_i, and c_i are all constant. To show this, suppose for example that $a_3 \neq 0$ and $a_\alpha\, a_\alpha \neq 0$. With $a_3 \neq 0$, it follows from the constancy of a_3 and $a_3\, a_\alpha$ that a_α is constant. With a_α constant and not the zero vector, constancy of the two-dimensional inner products (8.1) implies that b_α and c_α are also constants. Hence a_i, b_i, and c_i are all constant. In this case the strain g_{ij} is constant, and we have a case of homogeneous deformation with uniform field, covered in Section 7.

It remains to consider cases in which no eigenvector is oblique to the field $D_i = D\delta_{3i}$. In such cases, one eigenvector must be parallel to the x_3 direction, and the other two must be perpendicular to it. If we abandon the convention that g_1 is necessarily non-degenerate, we can let b_i be the eigenvector in the x_3 direction with no loss of generality. In the spectral representation (5.1) we use

$$b_i = \delta_{3i} \quad \text{and} \quad c_i c_j = \delta_{ij} - a_i a_j - b_i b_j, \tag{8.12}$$

to obtain

$$g_{ij}^N = (g_1^N - g_3^N)\, a_i a_j + (g_2^N - g_3^N)\, \delta_{3i} \delta_{3j} + g_3^N \delta_{ij}, \tag{8.13}$$

where g_1, g_2, and g_3 are constants, with $g_1 g_2 g_3 = 1$, and $a_3 = 0$.

If $g_1 = g_3$, the deformation corresponding to (8.13) is a simple extension in the x_3 direction, which has already been considered. We now suppose that $g_1 \neq g_3$. Symmetry of the tensors (3.3), which requires that $g_{ij,j}^N$ be the gradient of a scalar, has not yet been used. With g_{ij}^N given by (8.13), where $g_1 \neq g_3$, this condition implies that there is a potential χ such that

$$(a_i a_j)_{,j} = \chi_{,i}. \tag{8.14}$$

Because $a_3 = 0$, then $\chi_{,3} = 0$, and we obtain

$$(a_\alpha a_\beta)_{,\beta} = \chi_{,\alpha}(x_1, x_2). \tag{8.15}$$

The problem which remains is to find all of the fields a_α which satisfy (8.15) and the requirement $a_\alpha a_\alpha = 1$, for which the resulting strain tensor (8.13) satisfies the compatibility conditions. We have solved this problem in Sections II.16 to II.18, and have obtained the corresponding deformations. The controllable states obtained by combining these deformations with a constant field D_i in the x_3 direction have been considered in Section I.14.

This concludes the proof that all controllable states with specified deformation and non-zero dielectric displacement field have been included among the solutions described in Part I.

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Department of Engineering Mechanics
North Carolina State University
and
Division of Applied Mathematics
Brown University

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