On the Exterior Stationary Problem for the Navier-Stokes Equations, and Associated Perturbation Problems

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Contents

1. Introduction

The exterior stationary problem for the Navier-Stokes equations

$$-\frac{\partial \boldsymbol{w}}{\partial t} + \nu \Delta \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{w} - \nabla p = -\boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{w} = 0 \tag{1}$$

consists in finding a time independent solution pair $\boldsymbol{w}(x)$, $p(x)$ of (1) in a domain $\mathscr{E}$ exterior to a closed surface Σ, such that $\boldsymbol{w}(x) \to \boldsymbol{w}_0$, a prescribed constant vector, as $x \to \infty$, and $\boldsymbol{w}(x)$ assumes prescribed data $\boldsymbol{w}^*$ on Σ.

The equations (1) correspond to the motion of an incompressible viscous fluid. The quantities which appear have the following physical significance:

$\boldsymbol{w}$: flow velocity vector;
t: time;
ν: kinematic viscosity;
p: pressure;
$\boldsymbol{f}$: external force, assumed prescribed;
x: position in space.

If $\boldsymbol{w}^* \equiv 0$, $\boldsymbol{w}_0 \neq 0$, the problem amounts to the determination, in a coordinate frame attached to Σ, of the flow velocities in a steady motion of Σ through the fluid with velocity $-\boldsymbol{w}_0$, under the assumption that the fluid adheres at the surface (the no-slip condition). This problem is the principal motivation for the present paper. One of the results (§ 4) will be a proof (constructive) that if the data are sufficiently small, there exists exactly one solution whose behavior corresponds to physical experience. This solution exhibits a "wake region" of fluid behind Σ which tends to follow Σ along the direction $-\boldsymbol{w}_0$, and the work required to move that mass of fluid results in a "drag force" on Σ in the opposite direction. More precisely, the velocity is shown to tend to its limit at infinity to the order $|x|^{-1}$ within the wake, and at the more rapid rate, $|x|^{-2}$, in any direction other than that of $\boldsymbol{w}_0$. The wake is asymptotically paraboloidal. The hypothesis $\oint_\Sigma \boldsymbol{w}^* \cdot d\boldsymbol{\sigma} = 0$ (the zero outflux condition), which has appeared in earlier literature, is not required in this paper.

Other physical problems are also accessible to the method. For example, one may consider a rotating sphere in a fluid at rest or in uniform motion at infinity (the baseball problem), or a situation in which Σ has porous walls through which fluid is being forced at prescribed rates. The results of this paper apply to these and to many other cases of interest; the existence of physically reasonable solutions is, however, demonstrated only in the case in which the data on Σ are close to the prescribed vector $\boldsymbol{w}_0$.

The extent to which the requirement of small data reflects the actual behavior of the solutions of (1) is not clear. Both physical experience and a recent result of VELTE [*1*] strongly suggest that uniqueness will fail if the data are large. On the other hand, it has long been known (*cf.* LERAY [*2*], FINN [*3, 4*]) that smooth solutions exist for very general data, in a class for which the energy dissipation is finite. No regularity properties at infinity of these solutions have been demonstrated, however, beyond continuity, and no information is available on uniqueness. It is not even known whether there are multiple solutions in this class which vanish on Σ and at infinity.

It appears that solutions in three dimensions, $x = (x_1, x_2, x_3)$, are more easily studied than in two or higher dimensions, and the three-dimensional case is the only one considered in this paper. Some partial results for two-dimensional flows have been obtained recently by D. SMITH and will appear elsewhere.

It will be assumed that the applied force is time independent, so that $\boldsymbol{f} = \boldsymbol{f}(x)$. Since the solutions to be constructed also have this property, (1) may be written in the form

$$\begin{aligned} \nu \Delta \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{w} - \nabla p &= -\boldsymbol{f}(x), \\ \nabla \cdot \boldsymbol{w} &= 0. \end{aligned} \tag{2}$$

Solutions are sought as perturbations of the particular solution $\boldsymbol{w}(x) \equiv \boldsymbol{w}_0 = \text{const}$. Writing $\boldsymbol{w}(x; \lambda) = \boldsymbol{w}_0 + \lambda \boldsymbol{u}(x; \lambda)$, (2) becomes, after relabeling of $\boldsymbol{f}(x)$ and $p(x)$, the system

$$\begin{aligned} \nu \Delta \boldsymbol{u} - \boldsymbol{w}_0 \cdot \nabla \boldsymbol{u} - \nabla p &= -\boldsymbol{f}(x) + \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned} \tag{3}$$

for $\boldsymbol{u}(x; \lambda)$. The requirement of small data is conveniently expressed by prescribing fixed data for $\boldsymbol{u}(x; \lambda)$ and letting λ be small. The solution of the stated problem is then constructed as a perturbation series in powers of λ, using the solution $U(x; w_0)$ of the linearized system

$$\nu \Delta u - w_0 \cdot \nabla u - \nabla p = -f(x), \qquad \nabla \cdot u = 0 \tag{4}$$

which satisfies the same data, as initial term. It is shown that for given data u^* the series converges in an interval $|\lambda| < \Lambda$, with Λ independent of w_0. Consequently, if w_0 is sufficiently small, there is necessarily a solution $w(x; \lambda)$ which vanishes on Σ.

It is necessary to prove the existence of the "parametrix" solution of (4) and to obtain for it certain *a priori* estimates. This step is carried out in § 2. The existence proof is effected by *Galerkin's method* and follows closely the approach used by FUJITA [5] in his study of (2). The method is constructive. Using the fundamental solution tensor $E(x-y; w_0) = (E_{ij})$ of (4), which has been given explicitly by OSEEN [6], one then obtains a Green's tensor $G(x, y; w_0)$, which has a fundamental singularity and whose components vanish on Σ and at infinity.

The core of the existence proof for the nonlinear equations lies in the estimate

$$|x| \int_{\mathscr{E}} |y|^{-2} |\nabla_y G(x, y; w_0)| \, dy < H < \infty \tag{5}$$

uniformly for all $x \in \mathscr{E}$ and all $|w_0|$ in any finite interval. The demonstration of this result presents technical difficulties, chiefly owing to the absence of spatial homogeneity in the equation when $w_0 \neq 0$. It is proved in § 3 by exploiting the invariance of (4) under transformations which preserve the "Reynolds number", and by adapting potentialtheoretic methods to the geometry imposed by the equation.

The *a priori* estimate (5) leads easily to the construction (§ 4) of solutions of the nonlinear system (2) for small data, which are such that $|w(x) - w_0| < C|x|^{-1}$ as $x \to \infty$. The proof that these solutions are unique in a sensible class, and that they are the physically reasonable ones which are sought, is carried out in § 5. The idea of this section is taken from my earlier paper [7], and the results are very similar. The material is here developed in a more systematic way than in [7], resulting in an improved estimate which is essential in what follows.

Since the construction of the strict solution of (2) requires many steps, it is both natural and important to examine the sense in which these solutions can be approximated by solutions of the linearized system (4). It turns out that there are four possibilities, leading to different results according to whether or not $w_0 = 0$, and whether the flow is perturbed at infinity. It is also of interest to examine the same problem for the solutions of LERAY mentioned above, which presumably have weaker regularity properties at infinity. Except in one case (the one on which the existence theorem of § 4 is based) the perturbation appears to be singular, owing to the fact that $\mathscr{E}$ is an infinite region. If $w_0 = 0$ and the perturbation is nonvanishing at infinity, this phenomenon finds its expression in two dimensions as the Stokes paradox [8, 9, 10] and in three dimensions as Whitehead's paradox [6, p. 163]. In § 6 the perturbation and its derivatives are estimated in all eight cases. It is seen that the estimates are weakened either by perturbing the flow at infinity or by choosing as base flow the identically vanishing solution $\boldsymbol{w}_0 \equiv 0$. Thus the attempt by STOKES [8] to approximate a solution with nonvanishing limiting velocity by a solution of the system

$$\nu \Delta \boldsymbol{u} - \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0 \tag{6}$$

led to singular behavior at infinity, presumably induced by two causes. In two dimensions, this behavior results in the nonexistence of solutions of (6) with the given data. In the three-dimensional case considered here, the solution exists, and the approximation to the strict solution of (2) is even uniform throughout $\mathscr{E}$ (Theorems 6.5, 6.6), but the approximation is nevertheless weaker than in any of the other cases considered (see Table).

In most of the cases studied in § 6, the estimates obtained are considerably more precise than the ones stated in my preliminary announcement of results [10]. However, in the one case $\boldsymbol{w}_0 \neq 0$, $\boldsymbol{u}_0 = 0$ of Theorem 6.5 the logarithmic expression in the definition of $\tau_0(x;\lambda)$ does not appear in the corresponding Theorem 3.4 of [10]. I regret that I have misplaced the notes on which the statement of that theorem was based, and I am now unable to reproduce that estimate.

It will be assumed throughout that the bounding surface Σ is of class $\mathscr{C}^{(1+\alpha)}$ in appropriate non-singular parameters. It was originally my intention to develop the material for the class of surfaces studied recently by EDWARDS [11], who proved the existence of flows in the class D (see below) for surfaces admitting isolated singular curves and conical points — including, in particular, surfaces such as a circular disk embedded in three dimensions. It is my opinion that such a result is accessible to the methods of this paper; however, it would require a considerable technical effort, and it seemed best in this initial contribution to concentrate on those features of the underlying ideas which are qualitatively new.

In addition to the notation already introduced, the following symbols will be used:

$\mathscr{C}^k(\mathscr{E})$: vector functions whose derivatives up to k^{th} order are continuous in $\mathscr{E}$.

$\mathscr{C}^{k+\alpha}(\mathscr{E})$: functions of class $\mathscr{C}^k(\mathscr{E})$, whose k^{th} order derivatives are Hölder-continuous with exponent α.

$\mathscr{C}_0^{k+\alpha}(\mathscr{E})$: functions of class $\mathscr{C}^{k+\alpha}(\mathscr{E})$ which have compact support in $\mathscr{E}$.

$\mathscr{C}_\sigma^{k+\alpha}(\mathscr{E})$: functions φ of class $\mathscr{C}^{k+\alpha}(\mathscr{E})$ which are solenoidal, *i.e.*, such that $\nabla \cdot \varphi = 0$ in $\mathscr{E}$.

$\mathscr{C}_{0,\sigma}^{k+\alpha}(\mathscr{E})$: functions of class $\mathscr{C}_\sigma^{k+\alpha}(\mathscr{E})$ which have compact support in $\mathscr{E}$.

$(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int \boldsymbol{\varphi}(x)\, \boldsymbol{\psi}(x)\, dx$ over some given region.

$((\boldsymbol{\varphi})) = \left(\int |\boldsymbol{\varphi}(x)|^2 dx\right)^{\frac{1}{2}}$.

$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = \int \nabla \boldsymbol{\varphi} \cdot \nabla \boldsymbol{\psi}\, dx$.

$\|\boldsymbol{\varphi}\| = \left(\int |\nabla \boldsymbol{\varphi}|^2 dx\right)^{\frac{1}{2}}$.

$\{\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\omega}\} = \int \boldsymbol{\varphi}(x) \cdot \boldsymbol{\psi}(x) \cdot \nabla \boldsymbol{\omega}(x)\, dx$.

$\mathcal{H}_0(\mathcal{E})$: Hilbert space obtained by completion of $\mathcal{C}_0^\infty(\mathcal{E})$ in the norm $\|\varphi\|$.

$\mathcal{H}_{0,\sigma}(\mathcal{E})$: Hilbert space obtained by completion of $\mathcal{C}_{0,\sigma}^\infty(\mathcal{E})$ in the norm $\|\varphi\|$.

D: class of solutions of (2) in $\mathcal{E}$, having finite Dirichlet integral, see Definition 5.2.

PR: class of solution of (2) in $\mathcal{E}$, which are physically reasonable at infinity, see Definition 5.1.

C: a quantity which is constant in some expression. Its value may, however, change within a demonstration.

$\boldsymbol{E}$: fundamental solution tensor associated with systems (4) or (6); defined in § 2.3.

$\boldsymbol{E}^\gamma$: truncated tensor, see § 2.3.

$\boldsymbol{G}$: Green's tensor for (4).

$\boldsymbol{G}_0$: Green's tensor for (6).

$T\boldsymbol{w}$: stress tensor, $(T\boldsymbol{w})_{ij} = -p\,\delta_{ij} + \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$.

$\operatorname{def}\boldsymbol{w}$: deformation tensor, $(\operatorname{def}\boldsymbol{w})_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$.

$(\ )_0$: an equation in which the forcing term does not appear; thus $(4)_0$ means equation (4) with $f \equiv 0$.

In most of the material of this paper the parameter ν is conveniently eliminated by a coordinate transformation, and I shall assume that this has been done. However, in the uniqueness theorem (5.10), an explicit criterion is given depending on the Reynold's number — or equivalently for fixed Σ — on ν and $\boldsymbol{w}_0$, and that result is therefore formulated in terms of these parameters.

As indicated above, Σ may consist of a finite number of smooth disjoint surfaces. However, in the interest of simplicity I shall carry out the demonstrations under the supposition that Σ is a single surface. The extension to the more general case requires only formal changes.

2. The Linearized Equations

2.1. Preliminary lemmas

Lemma 2.1. *Let $\boldsymbol{w}^* \in \mathcal{C}^{2+\alpha}$ with respect to nonsingular local parameters on Σ, and suppose $\oint_\Sigma \boldsymbol{w}^* \cdot d\boldsymbol{\sigma} = 0$. Then there is a solenoidal field $\boldsymbol{\zeta}(x) \in \mathcal{C}^2$ in $\mathcal{E}$ such that $\boldsymbol{\zeta}(x)$ vanishes outside a prescribed neighborhood of Σ, and $\boldsymbol{\zeta}(x) = \boldsymbol{w}^*$ on Σ.*

A proof of this result appears as Lemma 2.1 in [*13*].

Lemma 2.2. *Let $\mathcal{E}$ be the entire space, and let $\boldsymbol{\varphi}(x)$ be a vector valued function such that $\boldsymbol{\varphi}(x) \in \mathcal{C}^0(\mathcal{E})$. Suppose $\boldsymbol{\varphi}(x)$ has first derivatives almost everywhere in $\mathcal{E}$, which are square integrable over $\mathcal{E}$. Then there is a vector $\boldsymbol{\varphi}_0$ such that*

$$\int_{\mathcal{E}} \frac{|\boldsymbol{\varphi}(y) - \boldsymbol{\varphi}_0|^2}{r_{xy}^2}\, dy \leq 4 \int_{\mathcal{E}} |\nabla \boldsymbol{\varphi}(y)|^2\, dy \tag{7}$$

for any choice of the point x.

The proof is based on a lemma of PAYNE & WEINBERGER [12], which states that if $\boldsymbol{\varphi}(x)\in\mathscr{C}^1(\mathscr{E})$, and if $\mathscr{E}_R$ is the exterior of a sphere Σ_R of radius R, then there is a constant vector $\boldsymbol{\varphi}_0$, such that

$$\frac{1}{R}\oint_{\Sigma_R}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_0)^2\,d\sigma\leqq\int_{\mathscr{E}_R}|\nabla\boldsymbol{\varphi}|^2\,dy. \tag{8}$$

In proving (7) we may clearly assume that $\int_{\mathscr{E}}|\nabla\boldsymbol{\varphi}|^2dy<\infty$, for otherwise no proof would be needed. Also we may assume at first that $\boldsymbol{\varphi}(x)\in\mathscr{C}^1(\mathscr{E})$. Consider the identity, for arbitrary $\boldsymbol{u}(x)\in\mathscr{C}^1(\mathscr{E})$,

$$\int_{\mathscr{I}_R}\boldsymbol{u}^2\Delta\log r\,dy=\int_{\mathscr{I}_R}\frac{\boldsymbol{u}^2}{r^2}\,dy=-2\int_{\mathscr{I}_R}\frac{1}{r}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla r\,dy+\oint_{\Sigma_R}\boldsymbol{u}^2\frac{\partial\log r}{\partial n}\,d\sigma \tag{9}$$

where $\mathscr{I}_R$ is the interior of a sphere Σ_R about the origin. From (9) follows immediately

$$\int_{\mathscr{I}_R}\frac{\boldsymbol{u}^2}{r^2}\,dy\leqq\frac{1}{2}\int_{\mathscr{I}_R}\frac{\boldsymbol{u}^2}{r^2}\,dy+2\int_{\mathscr{I}_R}|\nabla\boldsymbol{u}|^2dy+\frac{1}{R}\oint_{\Sigma_R}\boldsymbol{u}^2\,d\sigma.$$

The stated result for $\boldsymbol{\varphi}(x)\in\mathscr{C}^1(\mathscr{E})$ is obtained by choosing $\boldsymbol{\varphi}_0$ from (8), setting $\boldsymbol{u}=\boldsymbol{\varphi}-\boldsymbol{\varphi}_0$, choosing the origin of coordinates at x, and letting $R\to\infty$. The more general assertion of the lemma then follows by a standard approximation procedure.

Corollary 2.2a. *Let $\boldsymbol{\varphi}(x)\in\mathscr{C}^0(\mathscr{E}_R)$ and have generalized first derivatives in the exterior $\mathscr{E}_R$ of the sphere Σ_R about the origin. Then there is a vector $\boldsymbol{\varphi}_0$ such that, if either $x=0$ or else $|x|\geqq 2R$,*

$$\int_{\mathscr{E}_R}\frac{|\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}_0|^2}{r_{xy}^2}\,dy\leqq K\int_{\mathscr{E}_R}|\nabla\boldsymbol{\varphi}(y)|^2dy \tag{10}$$

where we may choose $K=3+2\sqrt{2}<6$.

Proof. In the case considered, the counterpart of (9) is

$$\begin{aligned}\int_{\mathscr{E}_R}\frac{\boldsymbol{u}^2}{r_{xy}^2}\,dy&=-2\int_{\mathscr{E}_R}\frac{1}{r_{xy}}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla r_{xy}\,dy+\oint_{\Sigma_R}\boldsymbol{u}^2\frac{\partial\log r_{xy}}{\partial n}\,d\sigma\\&\leqq\lambda\int_{\mathscr{E}_R}\frac{\boldsymbol{u}^2}{r_{xy}^2}\,dy+\frac{1}{\lambda}\int_{\mathscr{E}_R}|\nabla\boldsymbol{u}|^2dy+\frac{1}{R}\oint_{\Sigma_R}\boldsymbol{u}^2\,d\sigma\end{aligned}$$

for any $\lambda>0$. The choice $\lambda=\sqrt{2}-1$ yields the stated result.

Corollary 2.2b. *Let $\mathscr{E}$ and $\boldsymbol{\varphi}(x)$ be as in Lemma 2.2, and suppose $\|\boldsymbol{\varphi}\|_{\mathscr{E}}<\infty$. Then there is a vector $\boldsymbol{\varphi}_0$ such that $\boldsymbol{\varphi}-\boldsymbol{\varphi}_0\in\mathscr{H}_0(\mathscr{E})$.*

Proof. Choose $\varepsilon>0$, and choose R so that $\|\boldsymbol{\varphi}\|^2_{\mathscr{E}_R}<\varepsilon$. Define $\boldsymbol{\psi}(x)$ by the conditions:

i) $\boldsymbol{\psi}(x)=\boldsymbol{\varphi}(x)-\boldsymbol{\varphi}_0$ if $|x|\leqq R$,

ii) $\boldsymbol{\psi}(x)=\dfrac{2R-r}{r}\,(\boldsymbol{\varphi}(x)-\boldsymbol{\varphi}_0)$ if $R\leqq|x|\leqq 2R$,

iii) $\boldsymbol{\psi}(x)=0$ if $|x|\geqq 2R$.

Evidently $\psi(x) \in \mathscr{H}_0(\mathscr{E})$. We have

$$\|\psi\|^2_{\mathscr{E}_R} \leqq 2\|\varphi\|^2_{\mathscr{E}_R} + 8\left(\left(\frac{\varphi-\varphi_0}{r}\right)\right)^2.$$

Hence by Corollary 2.2a,

$$\|\psi\|^2_{\mathscr{E}_R} \leqq 50\|\varphi\|^2_{\mathscr{E}_R} \leqq 50\varepsilon,$$

so that

$$\|\psi - (\varphi - \varphi_0)\|^2_{\mathscr{E}_R} \leqq 2\|\psi\|^2_{\mathscr{E}_R} + 2\|\varphi\|^2_{\mathscr{E}_R} \leqq 102\varepsilon.$$

Since ε is arbitrary, the result follows.

Corollary 2.2c. *The conclusion of Lemma* 2.2 *holds with* $\varphi_0 = 0$, *for all* $\varphi(x) \in \mathscr{H}_0(\mathscr{E})$.

The proof is immediate, since functions in $\mathscr{H}_0(\mathscr{E})$ can be approximated in norm by continuous functions. Similarly, one has

Corollary 2.2d. *The conclusion of Corollary* 2.2a *holds with* $\varphi_0 = 0$, *for all* $\varphi(x) \in \mathscr{H}_0(\mathscr{E})$.

2.2. Generalized solutions; the Dirichlet bound

Let $\boldsymbol{w}^*$ be prescribed data on Σ, subject to the restrictions of the preceding section. It will not, however, be assumed that $\oint_\Sigma \boldsymbol{w}^* \cdot d\boldsymbol{\sigma} = 0$. We may suppose the origin of coordinates to lie interior to Σ and set $\boldsymbol{\gamma}(x) = \gamma_0 \nabla\left(\frac{1}{r}\right)$ where γ_0 is a real constant so chosen that $\oint_\Sigma (\boldsymbol{w}^* + \boldsymbol{\gamma}) \cdot d\boldsymbol{\sigma} = 0$ (note that $\nabla \cdot \boldsymbol{\gamma} = 0$). We may then apply Lemma 2.1 to obtain a solenoidal field $\boldsymbol{\zeta}(x)$ such that $\boldsymbol{\zeta}(x) = \boldsymbol{w}^* + \boldsymbol{\gamma}$ on Σ and $\boldsymbol{\zeta}(x)$ vanishes outside a neighborhood of Σ.

In terms of $\boldsymbol{v}(x) = \boldsymbol{u}(x) - \boldsymbol{\zeta}(x) + \boldsymbol{\gamma}(x)$, equations (4) take the form

$$\begin{aligned} \Delta \boldsymbol{v} - \boldsymbol{w}_0 \cdot \nabla \boldsymbol{v} - \nabla p &= -\boldsymbol{f} - \Delta \boldsymbol{\zeta} + \boldsymbol{w}_0 \cdot \nabla(\boldsymbol{\zeta} - \boldsymbol{\gamma}), \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned} \tag{11}$$

since $\Delta \boldsymbol{\gamma} = 0$. A solution of (11) will be sought which satisfies homogeneous boundary conditions on Σ and at infinity.

Definition 2.1. *A field* $\boldsymbol{v}(x)$ *over* $\mathscr{E}$ *will be said to be a generalized solution of* (11), *and the corresponding* $\boldsymbol{u}(x) = \boldsymbol{v}(x) + \boldsymbol{\zeta}(x) - \boldsymbol{\gamma}(x)$ *a generalized solution of* (4), *whenever* i) $\boldsymbol{v}(x) \in \mathscr{H}_{0,\sigma}(\mathscr{E})$, *and* ii) *the relation*

$$[\boldsymbol{v}, \boldsymbol{\varphi}] - \{\boldsymbol{v}, \boldsymbol{w}_0, \boldsymbol{\varphi}\} = (\boldsymbol{f}, \boldsymbol{\varphi}) - [\boldsymbol{\zeta}, \boldsymbol{\varphi}] + \{\boldsymbol{\zeta} - \boldsymbol{\gamma}, \boldsymbol{w}_0, \boldsymbol{\varphi}\} \tag{12}$$

holds for all $\boldsymbol{\varphi}(x) \in \mathscr{C}^\infty_{0,\sigma}(\mathscr{E})$.

Evidently, if $\boldsymbol{u}(x)$ is a generalized solution of (4) and if $\boldsymbol{u}(x)$ is smooth, then $\boldsymbol{u}(x)$ is a strict solution corresponding to a suitable pressure $p(x)$, and $\boldsymbol{u}(x) \to \boldsymbol{w}^*$ on Σ. By Lemma 2.2, $\boldsymbol{u}(x) \to 0$ at infinity in the sense that $\int_{\mathscr{E}} \frac{\boldsymbol{u}^2(x)}{r^2} dx < \infty$. It will be shown later that in fact $\boldsymbol{u}(x) = O\left(\frac{1}{r}\right)$ at infinity, and still more precise information will be obtained.

Let $\{\boldsymbol{\varphi}^{(n)}(x)\}$ be a complete set of functions in $\mathscr{H}_{0,\sigma}(\mathscr{E})$. It may (and shall) be assumed that each $\boldsymbol{\varphi}^{(n)}(x)\in\mathscr{C}^{\infty}_{\sigma,0}(\mathscr{E})$, that

$$(13) \qquad [\boldsymbol{\varphi}^{(n)}(x), \boldsymbol{\varphi}^{(m)}(x)] = \delta^n_m,$$

and that the $\{\boldsymbol{\varphi}^{(n)}(x)\}$ are complete in the uniform topology, that is, any $\boldsymbol{\varphi}(x)\in\mathscr{C}^0_{0,\sigma}(\mathscr{E})$ can be uniformly approximated in $\mathscr{E}$ by a linear combination of the $\{\boldsymbol{\varphi}^{(n)}(x)\}$.

Definition 2.2. *A field $\boldsymbol{v}(x)$ over $\mathscr{E}$ will be said to be an approximating solution of* (12) *of order N, and the corresponding field $\boldsymbol{u}(x)$ an approximating solution of* (4) *of order N, whenever* i) *$\boldsymbol{v}(x)$ can be expressed as a linear combination of the functions $\boldsymbol{\varphi}^{(1)}(x), \dots, \boldsymbol{\varphi}^{(N)}(x)$, and* ii) *$\boldsymbol{v}(x)$ satisfies the generalized equation* (12) *for each of the functions $\boldsymbol{\varphi}^{(1)}(x), \dots, \boldsymbol{\varphi}^{(N)}(x)$.*

Lemma 2.3. *For each integer $N=1, 2, 3, \dots$, there exists a unique approximating solution $\boldsymbol{v}(x)$ of* (12).

Proof. Writing

$$(14) \qquad \boldsymbol{v}(x) = \sum_1^N \xi_i \boldsymbol{\varphi}^{(i)}(x)$$

and choosing $\boldsymbol{\varphi}=\boldsymbol{\varphi}^{(k)}(x)$ in (12) leads, because of (13), to the linear system

$$(15) \qquad \sum_1^N (\delta^i_k \alpha^i_k \xi_i) = f_k - Z_k - \zeta_k + \gamma_k$$

where

$$\alpha^i_k = \{\boldsymbol{\varphi}^{(i)}, \boldsymbol{w}_0, \boldsymbol{\varphi}^{(k)}\},$$

$$f_k = (\boldsymbol{f}, \boldsymbol{\varphi}), \qquad Z_k = [\boldsymbol{\zeta}, \boldsymbol{\varphi}],$$

$$\zeta_k = \{\boldsymbol{\zeta}, \boldsymbol{w}_0, \boldsymbol{\varphi}^{(k)}\}, \quad \gamma_k = \{\boldsymbol{\gamma}, \boldsymbol{w}_0, \boldsymbol{\varphi}^{(k)}\}.$$

Since $\alpha^i_k = -\alpha^k_i$, we conclude easily that the determinant of the system (15) is positive, hence for any given $\boldsymbol{\zeta}(x)$, $\boldsymbol{\gamma}(x)$ there is a unique solution $(\xi_1, \dots, \xi_N)=\xi$ of (15).

Lemma 2.4. *Suppose $|x|\,\boldsymbol{f}(x)\in L_2(\mathscr{E})$. Then there exists a constant K, not depending on N, such that for each approximating solution $\boldsymbol{v}(x)$ there holds $\|\boldsymbol{v}\| = [\boldsymbol{v}(x), \boldsymbol{v}(x)]^{\frac{1}{2}} \leqq K$.*

Proof. If $\boldsymbol{v}(x)$ is an approximating solution, we may choose $\boldsymbol{\varphi}(x)\equiv\boldsymbol{v}(x)$ in (12), obtaining

$$\|\boldsymbol{v}\|^2 \leqq ((r\boldsymbol{f}))\left(\left(\frac{\boldsymbol{v}}{r}\right)\right) + \|\boldsymbol{\zeta}\|\,\|\boldsymbol{v}\| + |\boldsymbol{w}_0|\,\|\boldsymbol{\zeta}\|\,\|\boldsymbol{v}\| + |\boldsymbol{w}_0|\,((\boldsymbol{\gamma}))\,\|\boldsymbol{v}\|$$

by Schwarz' inequality, since $\{\boldsymbol{v}, \boldsymbol{w}_0, \boldsymbol{v}\}=0$. The result then follows from Corollary 2.2c, since $\boldsymbol{\zeta}(x)$ has compact support and $|\boldsymbol{\gamma}| = |\gamma_0|\, r^{-2}$ is square integrable over $\mathscr{E}$.

Corollary 2.4. *Under the above hypothesis there is a constant K_1, not depending on N, such that $\|\boldsymbol{u}\|\leqq K_1$ for all approximating solutions $\boldsymbol{u}(x)$.*

This result is now evident from the definition $\boldsymbol{u}(x)=\boldsymbol{v}(x)+\boldsymbol{\zeta}(x)-\boldsymbol{\gamma}(x)$ and from Minkowski's inequality.

Theorem 2.5. *Suppose $|x|\,\boldsymbol{f}(x)\in L_2(\mathscr{E})$. Then there is a generalized solution $\boldsymbol{u}(x)$ of (4), corresponding to arbitrary prescribed data $\boldsymbol{w}^*$ on Σ. The solution $\boldsymbol{u}(x)$ tends to zero at infinity in the sense that $\int\limits_{\mathscr{E}} \frac{\boldsymbol{u}^2}{r^2}\,dx<\infty$.*

Since $\|\boldsymbol{v}\|<K$ for each N, the Rellich choice theorem [*14*] yields the existence of a subsequence of approximating solution of (12) which converges weakly in $\mathscr{H}_{0,\sigma}(\mathscr{E})$ and strongly in L_2 on any compact subset of $\mathscr{E}+\Sigma$. Diagonalization yields a limit function in $\mathscr{H}_{0,\sigma}(\mathscr{E})$ which satisfies (12). Since an approximating solution corresponding to the index N is an approximating solution for any index $M<N$, one sees that for the limit function, (12) holds for any of the $\{\boldsymbol{\varphi}^{(j)}(x)\}$, hence for any $\boldsymbol{\varphi}(x)\in\mathscr{C}^{\infty}_{0,\sigma}$.

Remark. Note that it is not required that $\oint\limits_{\Sigma}\boldsymbol{w}^*\cdot d\boldsymbol{\sigma}=0$.

2.3. Fundamental solutions; the local representation

A fundamental solution tensor $\boldsymbol{E}(x-y)$ for (4) has been determined explicitly in a particularly elegant form by Oseen [*6*]. His construction yields the components $E_{ij}(x-y)$ and associated "pressure vector" $\boldsymbol{e}=\{e_j(x-y)\}$ in the form

$$
\begin{aligned}
E_{ij}&=\delta_{ij}\,\Delta\Phi-\frac{\partial^2\Phi}{\partial x_i\,\partial x_j},\\
e_j&=-\frac{\partial}{\partial x_j}\left(\Delta\Phi-\boldsymbol{w}_0\cdot\nabla\Phi\right),\\
\Phi&=-\frac{1}{8\pi\sigma}\int\limits_0^{\sigma s}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha,\\
\sigma&=\frac{|\boldsymbol{w}_0|}{2},\qquad s=|x-y|-\frac{\boldsymbol{w}_0\cdot(x-y)}{|\boldsymbol{w}_0|}.
\end{aligned}
\tag{16}
$$

Corresponding to any (smooth) vector field $\boldsymbol{w}(x)$ and scalar $p(x)$ we may define the *stress tensor* $T\boldsymbol{w}$ by the relation

$$(T\boldsymbol{w})_{ij}=-p\,\delta_{ij}+\left(\frac{\partial w_i}{\partial x_j}+\frac{\partial w_j}{\partial x_i}\right).$$

If $\boldsymbol{u}(x)$ is a solution of (4) in a closed region $\mathscr{G}$ with boundary Σ, there holds the representation

$$\boldsymbol{u}(x)=\int\limits_{\mathscr{G}}\boldsymbol{E}\cdot\boldsymbol{f}\,dy+\oint\limits_{\Sigma}\{\boldsymbol{u}\cdot T\boldsymbol{E}\cdot-\boldsymbol{E}\cdot T\boldsymbol{u}\cdot+(\boldsymbol{E}\cdot\boldsymbol{u})\,\boldsymbol{w}_0\cdot\}\,d\boldsymbol{\sigma} \tag{17}$$

where $d\boldsymbol{\sigma}$ is understood as a directed surface element on Σ, and $T\boldsymbol{E}$ is formed by interpreting the components $\{e_j\}$ as pressures. Similarly,

$$p(x)=\int\limits_{\mathscr{G}}\boldsymbol{e}\cdot\boldsymbol{f}\,dy+\oint\limits_{\Sigma}\{\boldsymbol{u}\cdot T\boldsymbol{e}-\boldsymbol{e}\cdot T\boldsymbol{u}+(\boldsymbol{e}\cdot\boldsymbol{u})\,\boldsymbol{w}_0\cdot\}\,d\boldsymbol{\sigma} \tag{18}$$

where the "pressure" in the term $T\boldsymbol{e}$ is defined to be $e^*=\boldsymbol{w}_0\cdot\nabla\left(\frac{1}{|x-y|}\right)$.

We shall need also a *truncated fundamental tensor* (FUJITA [5], p. 96). Let $\eta(t)$ be a function of class $\mathscr{C}^\infty(t)$, such that

$$\eta(t) = \begin{cases} 1, & 0 \leqq t \leqq 1, \\ 0, & 2 \leqq t, \end{cases}$$

and define $\eta^\gamma(t) = \eta(t\gamma^{-1})$. Let $\Phi^\gamma(x-y) = \eta^\gamma \Phi(x-y)$. The truncated tensor $\boldsymbol{E}^\gamma = \{E_{ij}^\gamma\}$ is then defined by the relations (16) simply by inserting Φ^γ in place of Φ in these relations. The utility of this tensor consists in the fact that it leads to a representation for the solutions of (4), in which boundary integrals do not appear. In fact, one has, for any solution $u(x)$,

$$\boldsymbol{u}(x) = \int \boldsymbol{E}^\gamma \cdot \boldsymbol{f}\, dy + \int \boldsymbol{H}^\gamma \cdot \boldsymbol{u}\, dy \tag{19}$$

the integrations being extended over all space. Here the tensor $\boldsymbol{H}^\gamma(x-y)$, defined by

$$H_{ij}^\gamma(x) = \begin{cases} -\delta_{ij} \Delta(\Delta - \boldsymbol{w}_0 \cdot \nabla)\, \Phi^\gamma(x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is of class $\mathscr{C}^\infty$ in all space and vanishes (as does $\boldsymbol{E}^\gamma(x)$) if $|x| \geqq 2\gamma$.

It will be necessary to have a representation of the form (19) for the generalized solutions constructed in the preceding section. This will be done first with the somewhat simpler tensors $\boldsymbol{E}_0^\gamma$, $\boldsymbol{H}_0^\gamma$ arising in the case $\boldsymbol{w}_0 = 0$. The ensuing discussion follows FUJITA [5].

For any tensor $\boldsymbol{A}(x, y)$ and vector $\boldsymbol{\psi}(x)$, set

$$\boldsymbol{A}\boldsymbol{\psi} \equiv \int \boldsymbol{A}(x, y) \cdot \boldsymbol{\psi}(y)\, dy, \qquad \boldsymbol{A}^*\boldsymbol{\psi} \equiv \int \boldsymbol{A}(y, x) \cdot \boldsymbol{\psi}(y)\, dy.$$

Then, if $\boldsymbol{\psi}(x)$ is smooth, one has $\boldsymbol{E}_0\boldsymbol{\psi} \equiv \boldsymbol{E}_0^*\boldsymbol{\psi}$, and

$$\begin{aligned} &\Delta \boldsymbol{E}_0\boldsymbol{\psi} - \nabla \boldsymbol{e}_0\boldsymbol{\psi} = \boldsymbol{\psi}, && \nabla \cdot \boldsymbol{E}_0\boldsymbol{\psi} = 0, \\ &\Delta \boldsymbol{E}_0^\gamma\boldsymbol{\psi} - \nabla \boldsymbol{e}_0^\gamma\boldsymbol{\psi} = \boldsymbol{\psi} - H_0^\gamma\boldsymbol{\psi}, && \nabla \cdot \boldsymbol{E}_0^\gamma\boldsymbol{\psi} = 0. \end{aligned} \tag{20}$$

Let V^γ denote the sphere $|x| \leqq 2\gamma$, and let $\boldsymbol{\psi}(x) \in \mathscr{C}_0^\infty(V^\gamma)$. Let $\boldsymbol{\varphi}(x) = \boldsymbol{E}_0^\gamma\boldsymbol{\psi}$, $\pi = \boldsymbol{e}_0^\gamma\boldsymbol{\psi}$. Then $\boldsymbol{\varphi}(x) \in \mathscr{C}_{0,\sigma}^\infty(\mathscr{E})$, and $\Delta\boldsymbol{\varphi} - \nabla\pi = \boldsymbol{\psi} - \boldsymbol{H}_0^\gamma\boldsymbol{\psi}$. Inserting $\boldsymbol{\varphi}(x)$ as a test function in the generalized equation (12), which we may write in the form

$$[\boldsymbol{u}, \boldsymbol{\varphi}] - \{\boldsymbol{u}, \boldsymbol{w}_0, \boldsymbol{\varphi}\} = (\boldsymbol{f}, \boldsymbol{\varphi}), \tag{21}$$

leads to

$$(\boldsymbol{u}, \boldsymbol{\psi} - \boldsymbol{H}_0^\gamma\boldsymbol{\psi}) + \{\boldsymbol{E}_0^\gamma\boldsymbol{\psi}, \boldsymbol{w}_0, \boldsymbol{u}\} = (\boldsymbol{f}, \boldsymbol{E}_0^\gamma\boldsymbol{\psi}),$$

and an interchange of order of integration yields

$$(\boldsymbol{\psi}, \boldsymbol{u} - \boldsymbol{H}_0^\gamma\boldsymbol{u}) + (\boldsymbol{\psi}, \boldsymbol{E}_0^\gamma\boldsymbol{w}_0 \cdot \nabla\boldsymbol{u}) = (\boldsymbol{\psi}, \boldsymbol{E}_0^\gamma\boldsymbol{f}).$$

Since $\boldsymbol{\psi}$ is arbitrary in $\mathscr{C}_0^\infty(V^\gamma)$, there must hold

$$\boldsymbol{u}(x) = \int \boldsymbol{E}_0^\gamma \cdot (\boldsymbol{f} - \boldsymbol{w}_0 \cdot \nabla\boldsymbol{u})\, dy + \int \boldsymbol{H}_0^\gamma \cdot \boldsymbol{u}\, dy \tag{22}$$

for almost every x whose distance from Σ exceeds 2γ, whenever $\boldsymbol{u}(x)$ is a generalized solution of (4).

2.4. Local Regularity of Generalized Solutions

From (22) follows immediately the interior regularity of the generalized solution $\boldsymbol{u}(x)$, the smoothness of the solution depending only on the smoothness of $\boldsymbol{f}(x)$.

Theorem 2.6. *Let $\boldsymbol{f}(x)\in\mathscr{C}^{r+\alpha}(\mathscr{E})$, $r\geqq 0$. Then any generalized solution $\boldsymbol{u}(x)\in\mathscr{C}_\sigma^{r+2+\alpha}(\mathscr{E})$, and $p(x)\in\mathscr{C}^{r+1+\alpha}(\mathscr{E})$. The system* (4) *is satisfied strictly throughout $\mathscr{E}$. If also $\boldsymbol{f}(x)\to 0$ as $x\to\infty$, then so does $\boldsymbol{u}(x)$ together with its derivatives of order up to $r+2$.*

Proof. The order of singularity of $\boldsymbol{E}_0^\gamma$ at $x=y$ can be calculated from the explicit formula, valid for $|x-y|\leqq\gamma$,

$$\boldsymbol{E}_0^\gamma(x-y)\equiv\boldsymbol{E}_0(x-y)=\left\{\frac{\delta_{ij}}{r}-\frac{(x_i-y_i)\,(x_j-y_j)}{r^3}\right\}.$$

Schwarz' inequality and Corollary 2.2c, applied to (22), yield immediately a bound on $|\boldsymbol{u}(x)|$, uniform at all points of distance exceeding 2γ from Σ. Writing (22) in the form

$$\boldsymbol{u}(x)=\int\boldsymbol{E}_0^\gamma\cdot\boldsymbol{f}\,dy+\int\boldsymbol{u}\cdot\boldsymbol{w}_0\cdot\nabla\boldsymbol{E}_0^\gamma\,dy+\int\boldsymbol{H}_0^\gamma\cdot\boldsymbol{u}\,dy, \tag{23}$$

and using the bound on $|\boldsymbol{u}(x)|$, one finds that $\boldsymbol{u}(x)\in\mathscr{C}^{0+\beta}(\mathscr{E})$ for any $\beta<1$. Placing this result again into (23) permits successive improvement of the estimate, until the (prescribed) smoothness of $\boldsymbol{f}(x)$ has been fully exploited. The result is the assertion of Theorem 2.6 with regard to $\boldsymbol{u}(x)$. The corresponding property of $p(x)$ can then be obtained directly from the strict equation (4). The fact that $\boldsymbol{u}(x)\to 0$ at infinity is obtained by applying (19) for large $|x|$ and using Corollary 2.2d. The vanishing of the derivatives of $\boldsymbol{u}(x)$ at infinity then follows by repeated use of (19), as above.

We have also the following general result:

Corollary 2.6. *Let $\boldsymbol{f}(x)\in\mathscr{C}^{r+\alpha}$, and let H_f denote a Hölder constant for the r^{th} derivatives of $\boldsymbol{f}(x)$, with exponent α. Suppose $\boldsymbol{u}(x)$ satisfies the generalized equation*

$$[\boldsymbol{u},\boldsymbol{\Phi}]-\{\boldsymbol{u},\boldsymbol{w}_0,\boldsymbol{\Phi}\}=(\boldsymbol{f},\boldsymbol{\Phi})$$

in a sphere $V^\gamma\subset\mathscr{E}$ of radius γ, for all test functions $\boldsymbol{\Phi}(x)\in\mathscr{C}_{0,\sigma}^\infty(V^\gamma)$. Suppose $\boldsymbol{u}(x)$ can be extended to a field in $\mathscr{H}_\sigma(V^\gamma)\cap\mathscr{H}_0(\mathscr{E})$, with $\|\boldsymbol{u}\|\leqq M$. Then there is a constant C, depending only on $\boldsymbol{w}_0$, on γ, and on q, such that at the center of V^γ there holds for all q^{th} derivatives $D_q\boldsymbol{u}$, $0\leqq q\leqq r+2$,

$$|D_q\boldsymbol{u}|\leqq C\big(M+H_f+\max_{V^\gamma}|\boldsymbol{f}(x)|\big). \tag{24}$$

The proof follows by adjoining Corollary 2.2c to the above discussion.

Theorem 2.7. *Suppose Σ is of class $\mathscr{C}^{2+\alpha}$ and w^* of class $\mathscr{C}^{1+\alpha}$ on Σ, and suppose $\boldsymbol{f}$ is bounded near Σ. Then $\nabla\boldsymbol{u}(x)$, $p(x)\in\mathscr{C}^{0+\alpha}$ in a closed (outer) neighborhood of Σ, and $\boldsymbol{u}(x)\to\boldsymbol{w}^*$ on Σ.*

The proof can be obtained from the ODQVIST estimates [*15*] for the Green's tensor for (6) in an annular region bounded by Σ and by a sphere Σ_0 containing Σ. The reasoning follows the lines of classical potential theory, and I shall omit details (*cf.* [*2, 5, 16*]).

2.5. Representation in the Large

Let $\boldsymbol{u}(x)$ be a generalized solution of (4) in $\mathscr{E}$, in the sense of Definition 2.1, and suppose $\boldsymbol{f}(x) \in \mathscr{C}^{0+\alpha}(\mathscr{E})$. By Theorem 2.6, $\boldsymbol{u}(x)$ is a strict solution in $\mathscr{E}$, and we may apply the representation (17) to $\boldsymbol{u}(x)$ in the annular region $\mathscr{E}_R$ bounded by Σ and by a sphere Σ_R of large radius R, obtaining

$$\boldsymbol{u}(x) = \int_{\mathscr{E}_R} \boldsymbol{E} \cdot \boldsymbol{f}\, dy + \oint_{\Sigma + \Sigma_R} \{\boldsymbol{u} \cdot T\boldsymbol{E} \cdot - \boldsymbol{E} \cdot T\boldsymbol{u} \cdot + (\boldsymbol{E} \cdot \boldsymbol{u})\, \boldsymbol{w}_0 \cdot\}\, d\boldsymbol{\sigma} \tag{25}$$

for $x \in \mathscr{E}_R$.

Let us assume that $\boldsymbol{f}(x)$ has one of the following properties: either i) $\boldsymbol{f}(x) \equiv \boldsymbol{g}(x) \cdot \nabla \boldsymbol{g}(x)$, with $\boldsymbol{g}(x) \in \mathscr{H}_0(\mathscr{E} + \Sigma)$, or ii) $|x|^\beta \boldsymbol{f}(x) \in L_2(\mathscr{E})$, for some $\beta > \frac{1}{2}$. Then formal estimation of the volume integral in (25) shows that it can be extended to an integral over all of $\mathscr{E}$ which converges absolutely and approaches zero as $x \to \infty$. The former property implies that for each fixed x the outer surface integral tends to a finite limit as $R \to \infty$. Thus,

$$\boldsymbol{F}(x) = \lim_{R \to \infty} \oint_{\Sigma_R} \{\boldsymbol{u} \cdot T\boldsymbol{E} \cdot - \boldsymbol{E} \cdot T\boldsymbol{u} \cdot + (\boldsymbol{E} \cdot \boldsymbol{u})\, \boldsymbol{w}_0 \cdot\}\, d\boldsymbol{\sigma} \tag{26}$$

exists and is finite for each x.

In the integrand of (26), all terms involving $\boldsymbol{u}(x)$ and its derivatives vanish at infinity, by Theorem 2.6. The same theorem shows that $\nabla p \to 0$, since (4) equates ∇p to a sum of terms which tend to zero. Hence $p(x) = o(r)$ as $x \to \infty$. On the other hand, the defining relations (16) imply that the successive derivatives of $\boldsymbol{E}(x)$ and of $\boldsymbol{e}(x)$ in arbitrary directions, if of sufficiently high order, will tend to zero more rapidly than any prescribed negative power of r. It is not obvious that (26) can be differentiated under the sign, but it is possible to interchange the limit operation with the formation of difference quotients, which decay asymptotically with the same order as the corresponding derivatives. Thus, letting $\delta^{(N)} \boldsymbol{F}(x)$ denote the result of taking N successive differences in arbitrary directions, there will hold

$$\begin{aligned} \delta^{(N)} \boldsymbol{F}(x) &= \lim_{R \to \infty} \int_{\Sigma_R} \{\boldsymbol{u} \cdot T\delta^{(N)}\boldsymbol{E} \cdot - \delta^{(N)}\boldsymbol{E} \cdot T\boldsymbol{u} + (\delta^{(N)}\boldsymbol{E} \cdot \boldsymbol{u})\, \boldsymbol{w}_0 \cdot\}\, d\boldsymbol{\sigma} \\ &= 0 \end{aligned}$$

identically in x whenever N is sufficiently large. It follows that $\boldsymbol{F}(x)$ is a polynomial in the components of x, $\boldsymbol{F}(x) \equiv \boldsymbol{P}_N(x)$ where $\boldsymbol{P}_N(x)$ has degree at most $N-1$. Hence (25) implies

$$\boldsymbol{u}(x) = \boldsymbol{u}_{\boldsymbol{f}}(x) + \boldsymbol{u}_\Sigma(x) + \boldsymbol{P}_N(x)$$

where $\boldsymbol{u}_{\boldsymbol{f}}(x)$ and $\boldsymbol{u}_\Sigma(x) \to 0$ as $x \to \infty$. But by Theorem 2.6, $\boldsymbol{u}(x) \to 0$, hence $\boldsymbol{P}_N(x) \to 0$, and we conclude $\boldsymbol{F}(x) \equiv \boldsymbol{P}_N(x) \equiv 0$ in x.

The above reasoning can be repeated with little change when there is an inhomogeneous term of the form $\boldsymbol{f}(x)=\boldsymbol{h}(x)\cdot\nabla\boldsymbol{h}(x)$, for which it is only known that $\nabla\cdot\boldsymbol{h}(x)\equiv 0$, that the integral

$$I(x)=\int_{\mathscr{E}}\boldsymbol{h}(y)\cdot\boldsymbol{h}(y)\cdot\nabla\boldsymbol{E}(x-y)\,dy \tag{27}$$

converges absolutely, and that $I(x)\to 0$ as $x\to\infty$. For then

$$\int_{\mathscr{E}_R}\boldsymbol{E}\cdot\boldsymbol{h}\cdot\nabla\boldsymbol{h}\,dy=-\int_{\mathscr{E}_R}\boldsymbol{h}\cdot\boldsymbol{h}\cdot\nabla\boldsymbol{E}\,dy+\oint_{\Sigma+\Sigma_R}(\boldsymbol{E}\cdot\boldsymbol{h})\,\boldsymbol{h}\,d\sigma, \tag{28}$$

and the reasoning can be repeated with the additional term $\lim_{R\to\infty}\oint_{\Sigma_R}(\boldsymbol{E}\cdot\boldsymbol{h})\,\boldsymbol{h}\,d\sigma$ added to $\boldsymbol{F}(x)$. In order that $I(x)\to 0$, it suffices to have $|\boldsymbol{h}(x)|=o(r^{-\frac{3}{4}})$ as $x\to\infty$, although weaker conditions would do.

A similar discussion can be given for the pressure $p(x)$, starting with the representation (18).

Summarizing, we have the following result:

Theorem 2.8. *Suppose* $\boldsymbol{f}(x)=\boldsymbol{f}_1(x)+\boldsymbol{f}_2(x)+\boldsymbol{f}_3(x)$, *where* $\boldsymbol{f}_1(x),\boldsymbol{f}_2(x),\boldsymbol{f}_3(x)\in C^{0+\alpha}(\mathscr{E})$, *and suppose* $\boldsymbol{f}_1(x)\equiv\boldsymbol{g}(x)\cdot\nabla\boldsymbol{g}(x)$ *with* $\boldsymbol{g}(x)\in\mathscr{H}_0(\mathscr{E}+\Sigma)$, *while* $|x|^\beta\boldsymbol{f}_2(x)\in L_2(\mathscr{E})$ *for some* $\beta>\frac{1}{2}$, *and* $\boldsymbol{f}_3(x)\equiv\boldsymbol{h}(x)\cdot\nabla\boldsymbol{h}(x)$, *where* $\boldsymbol{h}(x)$ *is as above* (e.g., $\nabla\cdot\boldsymbol{h}\equiv 0$ *and* $|\boldsymbol{h}(x)|=o(r^{-\frac{3}{4}})$ *at infinity*). *Let* $\boldsymbol{u}(x)$ *be a generalized solution of* (4) *in* $\mathscr{E}$ *in the sense of Definition* 2.1, *and suppose that* $\nabla\boldsymbol{u}(x)$, $p(x)$ *are continuous up to* Σ. *Then*

$$\begin{aligned}\boldsymbol{u}(x)&=\int_{\mathscr{E}}\boldsymbol{E}(x-y)\cdot\boldsymbol{f}(y)\,dy+\oint_{\Sigma}\{\boldsymbol{u}\cdot T\boldsymbol{E}-\boldsymbol{E}\cdot T\boldsymbol{u}+(\boldsymbol{E}\cdot\boldsymbol{u})\,\boldsymbol{w}_0\}\,d\boldsymbol{\sigma},\\ p(x)&=p_0+\int_{\mathscr{E}}\boldsymbol{e}(x-y)\cdot\boldsymbol{f}(y)\,dy+\oint_{\Sigma}\{\boldsymbol{u}\cdot T\boldsymbol{e}-\boldsymbol{e}\cdot T\boldsymbol{u}+(\boldsymbol{e}\cdot\boldsymbol{u})\,\boldsymbol{w}_0\}\,d\boldsymbol{\sigma}\end{aligned} \tag{28}$$

in $\mathscr{E}$, *where* p_0 *is an arbitrary constant.*

An unpleasant but formal computation, starting with the definition (16), yields the following estimates for the upper bounds of all components of $\boldsymbol{E}$, $\nabla\boldsymbol{E}$, when $|x|=r$ is large:[1]

$$\begin{aligned}|\boldsymbol{E}|&<C\,\frac{1}{r}\,\frac{1-e^{-\sigma s}}{\sigma s},\\ |\nabla\boldsymbol{E}|&<C\left[\frac{\sigma^{\frac{1}{2}}}{r^{\frac{3}{2}}}\,\frac{1-e^{-\sigma s}-\sigma s\,e^{-\sigma s}}{(\sigma s)^{\frac{3}{2}}}+\frac{1}{r^2}\,\frac{1-e^{-\sigma s}}{\sigma s}\right]\end{aligned} \tag{29}$$

for some constant C. Similar estimates hold for the higher derivatives of $\boldsymbol{E}(x)$. Also,

$$\boldsymbol{e}(x)=\frac{1}{4\pi}\,\nabla\left(\frac{1}{r}\right). \tag{30}$$

In terms of these estimates, the mean value theorem applied to (28) yields:

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[1] In [7, p. 392] and in [13, p. 204], the second term in the estimate of $|\nabla\boldsymbol{E}|$ is omitted, apparently due to an error in copying. This omission does not affect the content of those papers.

Corollary 2.8. *Suppose $\boldsymbol{f}(x)$ has compact support in $\mathscr{E}+\Sigma$. Then the behavior of the generalized solution $\boldsymbol{u}(x)$, $p(x)$ at infinity is controlled by that of the fundamental tensor $\boldsymbol{E}(x)$, $\boldsymbol{e}(x)$, in the sense*

$$\begin{aligned} \boldsymbol{u}(x) &= \boldsymbol{a}\cdot\boldsymbol{E}(x) + b\,\boldsymbol{e}(x) + \boldsymbol{\sigma}(x) \\ p(x) &= p_0 + \boldsymbol{\beta}\cdot\boldsymbol{e}(x) + \tau(x) \end{aligned} \tag{31}$$

for constant vectors $\boldsymbol{a}$, $\boldsymbol{\beta}$ and scalar b. Here $|\boldsymbol{\sigma}(x)|$ may be chosen in the form of the right side of (29), *while $|\tau(x)| < C r^{-3}$. Both expressions* (31) *may be differentiated any number of times, with corresponding improved rate of decay of $\boldsymbol{\sigma}(x)$, $\tau(x)$.*

A particular consequence of (31) is that the solution $\boldsymbol{u}(x)$ admits a paraboloidal "wake region" in the direction of $\boldsymbol{w}_0$, interior to which $|\boldsymbol{u}(x)| < C r^{-1}$. Exterior to this region, the decay of $\boldsymbol{u}(x)$ is progressively more rapid until, exterior to any circular cone with axis in direction $\boldsymbol{w}_0$, one has $|\boldsymbol{u}(x)| < C r^{-2}$. This behavior will be discussed in further detail in § 5.

Note that (31) completely characterizes the qualitative asymptotic structure of any generalized solution; the method of construction of these solutions yielded no information *a priori* beyond the fact that $\boldsymbol{u}(x)\in\mathscr{H}_0(\mathscr{E})$.

2.6. The Green's Tensor

Theorem 2.6 has as particular consequence the existence in $\mathscr{E}$ of a tensor $\mathsf{A}_{ij}(x, y)$, whose row and column vectors satisfy as function of y the adjoint system

$$\begin{aligned} \Delta\boldsymbol{u} + \boldsymbol{w}_0\cdot\nabla\boldsymbol{u} - \nabla p &= 0, \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned} \tag{32}$$

and which has the properties i) if $y\in\Sigma$, then $\mathsf{A}_{ij}(x, y) = -\mathsf{E}_{ij}(x, y)$, ii) $\mathsf{A}_{ij}(x, y)\to 0$ as $y\to\infty$, any fixed $x\in\mathscr{E}$. Thus, there exists a *Green's tensor*

$$\mathsf{G}(x, y) = \mathsf{E}(x, y) + \boldsymbol{A}(x, y)$$

for the system (4), whose components vanish, as functions of y, on Σ and at infinity.

The qualitative behavior of $\boldsymbol{G}(x, y)$ as $y\to\infty$ is known from Corollary 2.8. Let $\boldsymbol{G}^*(x, y)$ be the Green's tensor for the adjoint system (32), so that its row and column vectors satisfy the original system $(4)_0$. *The representation formula* (17), *together with Corollary* 2.8, *yields the symmetry property*

$$\mathsf{G}_{ij}(x, y) = \mathsf{G}^*_{ij}(y, x). \tag{33}$$

From (33) we conclude:

i) *the row and column vectors of $\boldsymbol{G}(x, y)$ satisfy $(4)_0$ as functions of x;*

ii) $\lim_{x\to\Sigma}\boldsymbol{G}(x, y) = 0$, *uniformly for all $y\in\mathscr{E}$ which exceed a fixed distance from Σ.*

The following lemmas characterize quantitatively the asymptotic behavior of $\boldsymbol{G}(x, y)$ at infinity, and its local behavior near Σ.

Lemma 2.9. *Let $|x| = R$ be fixed. There exists an R_0 and a constant $C(R_0)$ such that, for all (fixed) $R > R_0$, $\|\boldsymbol{A}\| < C R^{-1}$. The estimate is uniform in $\boldsymbol{w}_0$, in any finite interval of this parameter.*

It suffices to choose R_0 so that all points of Σ lie interior to a sphere of radius R_0.

Proof. Because of the estimates (29), there is a constant C for which $|\boldsymbol{E}| < CR^{-1}$, $|\nabla \boldsymbol{E}| < CR^{-1}$ on Σ. It is evident from the definition that a similar estimate holds for the derivatives of $\boldsymbol{E}(x, y)$ of arbitrary order on Σ. By Lemma 2.1, there is a solenoidal field $\boldsymbol{Z}(x; y)$ in $\mathscr{E}$, having compact support in a prescribed neighborhood $\mathscr{E}_\delta$ of Σ, such that $\boldsymbol{Z}(x; y) = \boldsymbol{E}(x, y)$ if $y \in \Sigma$, and such that $|\boldsymbol{Z}(x; y)| < CR^{-1}$, $|\nabla \boldsymbol{Z}(x; y)| < CR^{-1}$ throughout $\mathscr{E}_\delta$. (One verifies easily that $\oint_\Sigma \boldsymbol{E}\, d\boldsymbol{\sigma} = 0$, thus assuring the existence of $\boldsymbol{Z}(x; y)$.)

In terms of $\boldsymbol{\Xi} = \boldsymbol{A} - \boldsymbol{Z}$, equation (32) becomes

$$\Delta \boldsymbol{\Xi} + \boldsymbol{w}_0 \cdot \nabla \boldsymbol{\Xi} - \nabla \boldsymbol{\alpha} = -\Delta \boldsymbol{Z} - \boldsymbol{w}_0 \cdot \nabla \boldsymbol{Z}$$

where $\boldsymbol{\alpha}$ is the "pressure" term associated with $\boldsymbol{A}(x, y)$. Multiply by $\boldsymbol{\Xi}$, integrate over an annular region $\mathscr{E}_R$ of outer radius R, and let $R \to \infty$, obtaining

$$\|\boldsymbol{\Xi}\|^2 = -[\boldsymbol{\Xi}, \boldsymbol{Z}] + \{\boldsymbol{Z}, \boldsymbol{w}_0, \boldsymbol{\Xi}\}$$

the outer surface integral vanishing in the limit by Corollary 2.8. Hence

$$\|\boldsymbol{\Xi}\|^2 \leqq \frac{1}{2} \|\boldsymbol{\Xi}\|^2 + \frac{1}{2} \|\boldsymbol{Z}\|^2 + \frac{|\boldsymbol{w}_0|}{2\lambda} ((\boldsymbol{Z})) + \frac{\lambda}{2} |\boldsymbol{w}_0| \, \|\boldsymbol{\Xi}\|$$

for any positive λ. The result now follows, after suitable choice of λ, by Minkowski's inequality.

Lemma 2.10. *Let Σ_0: $|x| = R_0$ be an arbitrary but fixed spherical surface surrounding Σ. Then at all points $y \in \Sigma_0$ there holds, if $|x| = R > 2R_0$, $|\boldsymbol{A}(x, y)| < CR^{-1}$, $|\nabla \boldsymbol{A}(x, y)| < CR^{-1}$, $|\nabla \boldsymbol{\alpha}(x, y)| < CR^{-1}$. The estimates are uniform on any finite interval of the parameter $\boldsymbol{w}_0$.*

The proof is obtained immediately from Lemma 2.9 and Corollary 2.6.

Lemma 2.11. *Let Σ_0 be as above. Then for y outside Σ_0 and $|x| = R \geqq 2R_0$, there holds, uniformly in any finite interval of the parameter $\boldsymbol{w}_0$,*

$$\begin{aligned} &|\boldsymbol{A}(x, y)| < CR^{-1} |\boldsymbol{a} \cdot \boldsymbol{E}(-y) + b\,\boldsymbol{e}(-y) + \boldsymbol{\sigma}(-y)|, \\ &|\boldsymbol{\alpha}(x, y)| < CR^{-1} |\boldsymbol{\beta} \cdot \boldsymbol{e}(-y) + \tau(-y)| \end{aligned} \tag{34}$$

after normalization of $\boldsymbol{\alpha}(x, y)$ by a suitable additive constant. Here $\boldsymbol{\sigma}(y)$, $\tau(y)$, $\boldsymbol{a}$, b, $\boldsymbol{\beta}$ have the same meaning as in Corollary 2.8. These inequalities may be differentiated formally with respect to y on both sides.

Proof. For fixed x exterior to Σ_0, we may apply Theorem 2.8 to obtain, after normalizing $\alpha(x, y)$,

$$\boldsymbol{A}(x, y) = \oint_{\Sigma_0} \{\boldsymbol{A} \cdot T\boldsymbol{E}^* - \boldsymbol{E}^* \cdot T\boldsymbol{A} + (\boldsymbol{E}^* \cdot \boldsymbol{A})\, \boldsymbol{w}_0\}\, d\boldsymbol{\sigma},$$

$$\boldsymbol{\alpha}(x, y) = \oint_{\Sigma_0} \{\boldsymbol{\alpha} \cdot T\boldsymbol{e}^* - \boldsymbol{e}^* \cdot T\boldsymbol{\alpha} + (\boldsymbol{e}^* \cdot \boldsymbol{u})\, \boldsymbol{w}_0\}\, d\boldsymbol{\sigma}$$

where $\boldsymbol{E}^*$, $\boldsymbol{e}^*$ are the fundamental tensor and pressure for the adjoint system (32), whenever y is exterior to Σ_0. The stated result then follows by using the symmetry property $E^*(x-y)=E(y-x)$, placing the estimates of Lemma 2.10 — and the corresponding estimates for the higher derivatives of E — into the integrand and repeating the reasoning that led to Corollary 2.8.

Lemma 2.12. *Let Σ_0 be as above. In any closed neighborhood $\mathcal{N}_y$ of Σ which does not contain Σ_0 there holds, uniformly for w_0 in any finite interval,*

$$|A(x,y)| < C R^{-1}, \qquad |\nabla A(x,y)| < C R^{-1},$$
$$|\alpha(x,y)| < C R^{-1}$$

whenever $|x| = R > 2R_0$.

Proof. In the annular region bounded by Σ and by Σ_0, $A(x,y)$, $\alpha(x,y)$ can be represented, as functions of y, by the Green's tensor for the system (32). The result then follows from Lemma 2.9, Lemma 2.10, and from the estimates of ODQVIST on the Green's tensor for (32) in a finite region. The procedure is identical to the method of proof of Theorem 2.7, and details can be found in the reference cited there.

3. The Underlying Estimate

The existence theorem for physically reasonable solutions of (2) will be made to depend on the following estimate for the Green's tensor $G(x,y;w_0)$ of $(4)_0$ in $\mathcal{E}$:

Theorem 3.1. *There is a constant H, uniform in any finite interval of w_0, such that*

$$|x| \int_{\mathcal{E}} |y|^{-2} |\nabla_y G(x,y;w_0)|\, dy < H \tag{35}$$

for all $x \in \mathcal{E}$.

Several auxiliary lemmas will be needed to prove this result.

Lemma 3.2. *Let x be interior to a sphere Σ_R of radius R, and suppose the distance from x to Σ_R exceeds $R/2$. There exists a constant C, not depending on R or on w_0, such that*

$$\oint_{\Sigma_R} |\nabla E(x-y;w_0)|\, d\sigma < C. \tag{36}$$

Proof. If $w_0=0$, then $|\nabla E| < C|x-y|^{-2}$, from which the lemma follows immediately. Otherwise, the dependence on this parameter can be eliminated by observing that $E(x;\lambda w_0) \equiv \lambda E(\lambda x; w_0)$. Thus, setting $\xi = \lambda x$, $\eta = \lambda y$, one has

$$\oint_{\Sigma_R} |\nabla E(x-y);\lambda w_0)|\, d\sigma = \oint_{\Sigma_{\lambda R}} |\nabla E(\xi-\eta;w_0)|\, d\sigma.$$

It follows that if the lemma is established for a single value $w_0 \neq 0$ and arbitrary R, it holds also for all other values $w_0 \neq 0$. But for given w_0, an estimate (36) is obtained for large R by an easy calculation, using the estimates (29). For small R, (36) is a consequence of the local estimate $|\nabla E| < C|x-y|^2$, as $|x-y| \to 0$.

Lemma 3.3. *There is a constant H such that*

$$|x| \int_{\mathcal{E}} |y|^{-2} |\nabla_y A(x,y;w_0)|\, dy < \tfrac{1}{2} H$$

uniformly in any finite interval of w_0, for all sufficiently large $|x|$.

This result follows from Lemmas 2.11, 2.12, and 3.2. As a consequence, the proof of (35) in the case of large $|x|$ is reduced to the study of the corresponding expression, in which G is replaced by the fundamental tensor $\boldsymbol{E}(x-y;\boldsymbol{w}_0)$.

Lemma 3.4. *Let $|x|=R$ and suppose x lies exterior to a sphere Σ_r about the origin and of radius $r\leqq\frac{1}{2}R$. Let $\boldsymbol{w}_0$ be arbitrary but fixed, with $\boldsymbol{w}_0\neq 0$. There is a constant C, depending on $\boldsymbol{w}_0$ but not on R, such that for all $R>1$ there holds*

$$\oint_{\Sigma_r}|\nabla\boldsymbol{E}(x-y;\boldsymbol{w}_0)|\,d\sigma_y< C\,r^2 R^{-\frac{3}{2}}. \tag{37}$$

Proof. The estimate (29) yields in particular $|\nabla\boldsymbol{E}(x;\boldsymbol{w}_0)|<C R^{-\frac{3}{2}}$ if $R>1$, and from this the result follows.

Lemma 3.5. *Under the above assumptions, suppose $R^{\frac{1}{2}}\leqq 2r\leqq R$. There is a constant C, depending on $\boldsymbol{w}_0$ but not on R, such that for all $R>1$ there holds*

$$\oint_{\Sigma_r}|\nabla\boldsymbol{E}(x-y;\boldsymbol{w}_0)|\,d\sigma_y< C\,r^{\frac{5}{7}} R^{-\frac{6}{7}}. \tag{38}$$

Proof. Consider the contribution to the integral of that part of Σ_r which lies in a cylinder Z of radius $r^{\frac{3}{7}}R^{\frac{2}{7}}$, whose axis contains x and has the direction $\pm\boldsymbol{w}_0$. Replacing $|\nabla\boldsymbol{E}|$ by the estimate (29), and the quantity $|x-y|$ in (38) by its minimum, $R-r$, one sees that this contribution will be maximized when Z subtends the maximum possible area on Σ_r. This situation in turn will occur when Z lies interior to a parallel cylinder Z_0 of radius $2r^{\frac{3}{7}}R^{\frac{2}{7}}$, whose axis is tangent to Σ_r. The area $\mathscr{A}_0$ of that part of Σ_r which lies interior to Z_0 is easily estimated and has the order of magnitude $r^{\frac{8}{7}}R^{\frac{3}{7}}$.

It does not suffice to multiply by $\mathscr{A}_0$ the maximum of $|\nabla\boldsymbol{E}|$ on Σ_r in Z_0, as the resulting estimate would be too weak. It is necessary to split the integration into two parts, as follows:

i) a concentric subcylinder Z_0' in Z_0, of radius $r^{\frac{1}{7}}R^{\frac{3}{7}}$ (smaller than $2r^{\frac{3}{7}}R^{\frac{2}{7}}$, since $2r>R^{\frac{1}{2}}$). The area subtended by Z_0' is of order $r^{\frac{5}{7}}R^{\frac{9}{14}}$. In Z_0', replace $|\nabla\boldsymbol{E}|$ by the uniform estimate of order $R^{-\frac{3}{2}}$, obtaining as a bound for the net contribution of the integral over this part of Σ_r an order of magnitude

$$r^{\frac{5}{7}}R^{-\frac{6}{7}};$$

ii) the remainder of Z_0. The integral here can be estimated by multiplying the full area of Σ_r in Z_0, of order $r^{\frac{8}{7}}R^{\frac{3}{7}}$, by the maximum of the estimate (29) for $|\nabla\boldsymbol{E}|$ outside Z_0' on Σ_r. We obtain an order

$$r^{\frac{8}{7}}R^{\frac{3}{7}}\cdot r^{-\frac{3}{7}}R^{-\frac{9}{7}}=r^{\frac{5}{7}}R^{-\frac{6}{7}}.$$

Finally, we estimate the contribution from the part of Σ_r exterior to Z_0. To do so, it suffices to multiply the area of the full sphere Σ_r by the maximum of the estimate (29) for $|\nabla\boldsymbol{E}|$ outside Z_0. We find an order,

$$r^2\cdot r^{-\frac{9}{7}}R^{-\frac{6}{7}}=r^{\frac{5}{7}}R^{-\frac{6}{7}}.$$

Thus, all three estimates have the same order and yield the stated result.

Lemma 3.6. *Under the hypotheses of Theorem 3.1, there is a constant H, not depending on $\boldsymbol{w}_0$, such that*

$$|x| \int |y|^{-2} |\nabla_y \boldsymbol{E}(x-y;\boldsymbol{w}_0)|\, dy < \tfrac{1}{2} H \tag{39}$$

the integration being extended over all space.

If $\boldsymbol{w}_0 = 0$, the uniform estimate $|\nabla_y \boldsymbol{E}(x-y;0)| < C|x-y|^{-2}$ reduces the lemma to a classical potentialtheoretic result. If $\boldsymbol{w}_0 \neq 0$, the relation $\boldsymbol{E}(x;\lambda\boldsymbol{w}_0) = \lambda \boldsymbol{E}(\lambda x;\boldsymbol{w}_0)$ shows that the left side of (39) is independent of $\boldsymbol{w}_0$, and it therefore suffices to prove the result for any particular $\boldsymbol{w}_0 \neq 0$. To do so, write $|x| = R$ and split the region of integration into three parts, as follows:

I. The exterior of a sphere Σ_{2R} about the origin. The desired estimate follows from Lemma 3.2.

II. The annular region between $\Sigma_{R/2}$ and Σ_{2R}. In this region, $R < 2|y| < 4R$, hence Lemma 3.2 can again be applied to obtain the correct estimate.

III. The interior of $\Sigma_{R/2}$. If $R \leqq 1$, we have the uniform estimate $|\nabla \boldsymbol{E}(x;\boldsymbol{w}_0)| < C|x|^{-2}$ for small $|x|$, and the desired estimate is immediate. If $R > 1$, we split this integral again into two parts:

IIIa. The interior of a concentric sphere of radius $\frac{1}{2}R^{\frac{1}{2}}$. The uniform estimate $|\nabla \boldsymbol{E}| < CR^{-\frac{3}{2}}$ over this sphere leads to an evaluation of the form

$$CR \cdot R^{-\frac{3}{2}} \cdot \int_0^{R^{\frac{1}{2}}} dr = C$$

for the integral in question.

IIIb. The remaining part Q of the interior of $\Sigma_{R/2}$. By Lemma 3.5, if $R^{\frac{1}{2}} \leqq 2r \leqq R$, then $\oint_{\Sigma_r} |\nabla \boldsymbol{E}(x-y;\boldsymbol{w}_0)|\, d\sigma < Cr^{\frac{5}{7}} R^{-\frac{6}{7}}$. Denote this integral by $\mathscr{I}_r$. Then

$$R \int_Q |y|^{-2} |\nabla_y \boldsymbol{E}(x-y;\boldsymbol{w}_0)\, dy = R \int_{\frac{1}{2}R^{\frac{1}{2}}}^{\frac{1}{2}R} r^{-2} \mathscr{I}_r\, dr <$$

$$< CR^{\frac{1}{7}} \int_{\frac{1}{2}R^{\frac{1}{2}}}^{\frac{1}{2}R} r^{-\frac{9}{7}}\, dr < C$$

which completes the proof of the lemma.

Lemma 3.7. *Suppose Σ lies interior to a sphere Σ_{R_0}, and let $\mathscr{A}_0$ be the annular region between the two surfaces. Let y lie on the concentric surface Σ_{2R_0}. Then on any finite interval of values $\boldsymbol{w}_0$, $|\boldsymbol{G}(x,y;\boldsymbol{w}_0)|$ is bounded, uniformly in $\boldsymbol{w}_0$ and for all $x \in \mathscr{A}_0$. The same assertion holds for all derivatives of $\boldsymbol{G}(x,y;\boldsymbol{w}_0)$, and all bounds tend to zero uniformly as $x \to \Sigma$.*

This assertion follows from the definition of $\boldsymbol{G}(x,y;\boldsymbol{w}_0)$, from the symmetry property (33), from Lemma 2.9, and from Corollary 2.6.

Applying Corollary 2.8 to the function $\boldsymbol{u}(y) = \boldsymbol{G}(x,y;\boldsymbol{w}_0)$ exterior to Σ_{2R_0}, we find

Lemma 3.8. *Under the hypotheses of the preceding lemma, if $x \in \mathscr{A}_0$ and if y is exterior to Σ_{3R_0}, then*

$$|\nabla \boldsymbol{G}(x,y;\boldsymbol{w}_0)| < A(|\nabla \boldsymbol{E}(-y)| + |\nabla \boldsymbol{e}(-y)| + \text{lower order terms})$$

where $A \to 0$ as $x \to \Sigma$.

Finally, estimates on $G(x, y; w_0)$ will be needed in the case that both $x, y \in \mathscr{A}_0$. These estimates will be obtained by comparing $G(x, y; w_0)$ with the simpler tensor $G_0(x, y)$ corresponding to the Stokes system (6), in the (finite) region $\mathscr{B}_0$ bounded by Σ and by Σ_{2R_0}.

Let $W(x, y; w_0) = G(x, y; w_0) - G_0(x, y)$.

As function of y for fixed $x \in \mathscr{A}_0$, W satisfies the system

$$\nabla W + w_0 \cdot \nabla W + w_0 \cdot \nabla G_0 - \nabla P = 0, \qquad \nabla \cdot W = 0 \tag{40}$$

and W, ∇W are finite in $\mathscr{B}_0$. On Σ, $W=0$. On Σ_{2R_0}, $W = G(x, y; w_0)$. By Lemma 2.1 there is a solenoidal field $\zeta(x; y)$ in $\mathscr{B}_0$, such that $\zeta(x; y) \equiv 0$ for $y \in \mathscr{A}_0$ and such that $\zeta(x; y) = W(x, y; w_0)$ for $y \in \Sigma_{2R_0}$. Further, using Lemma 3.7, it is clear that ζ can be chosen so that $|\zeta|$, $|\nabla_y \zeta|$ are uniformly bounded when $x \in \mathscr{A}_0$. Let $\Omega = W - \zeta$. Then $\Omega = 0$ on Σ, Σ_{2R_0}, and $\nabla \cdot \Omega = 0$ in $\mathscr{B}_0$. Multiply (40) by Ω and integrate over $\mathscr{B}_0$. We find, for arbitrary fixed $x \in \mathscr{A}_0$,

$$\int_{\mathscr{B}_0} |\nabla \Omega|^2 dy = -\int_{\mathscr{B}_0} \zeta \cdot w_0 \cdot \nabla \Omega \, dy - \int_{\mathscr{B}_0} (\nabla \Omega \cdot \nabla \zeta) \, dy - \int_{\mathscr{B}_0} G_0 \cdot w_0 \cdot \nabla \Omega \, dy$$

from which, for any $\lambda > 0$,

$$\int_{\mathscr{B}_0} |\nabla \Omega|^2 dy \leqq \frac{3\lambda}{2} \int_{\mathscr{B}_0} |\nabla \Omega|^2 dy + \frac{1}{2\lambda} \int_{\mathscr{B}_0} [\zeta^2 w_0^2 + |\nabla \zeta|^2 + G_0^2 w_0^2] \, dy. \tag{41}$$

The estimates of ODQVIST [15] imply in particular $|G_0(x, y)| < C|x-y|^{-1}$, uniformly for $x, y \in \mathscr{B}_0$. Hence (41) implies:

Lemma 3.9. *If $G_0(x, y)$ is the Green's tensor for* (6) *in $\mathscr{B}_0$ and $G(x, y; w_0)$ the Green's tensor for* (4) *in $\mathscr{E}$, then $W = G - G_0$ satisfies $\|W\| \mathscr{B}_0 < C$ as function of y, uniformly for $x \in \mathscr{A}_0$ and for w_0 in any finite interval.*

Let us represent the tensor field W in $\mathscr{B}_0$ with the aid of the fundamental tensor G_0 and the auxiliary field $\zeta(x, y)$. We find

$$\begin{aligned} W(x, y; w_0) = \zeta(x, y) &+ \int_{\mathscr{B}_0} G_0(y, z) \cdot w_0 \cdot \nabla W(x, z) \, dz - \\ &- \int_{\mathscr{B}_0} G_0(y, z) \cdot w_0 \cdot \nabla G_0(x, z) \, dz - \int_{\mathscr{B}_0} G_0(y, z) \cdot \Delta \zeta(x, z) \, dz. \end{aligned}$$

If $y \neq x$, we may differentiate under the sign with respect to y. Using the Odqvist estimate $|\nabla G_0(x, y)| < C|x-y|^{-2}$, we have following:

$$|\nabla_y W(x, y)| \leqq C \int_{\mathscr{B}_0} r_{yz}^{-2} |\nabla_z W(x, z)| \, dz + C \int_{\mathscr{B}_0} r_{yz}^{-2} r_{xz}^{-2} \, dz + C.$$

Setting $|y| = r_{0y}$, multiplying by r_{0y}^{-2} and integrating over $\mathscr{B}_0$, we obtain

$$\begin{aligned} \int_{\mathscr{B}_0} r_{0y}^{-2} |\nabla_y W(x, y)| \, dy &\leqq C \int_{\mathscr{B}_0} r_{0z}^{-1} |\nabla_z W(x, z)| \, dz + C \int_{\mathscr{B}_0} r_{0y}^{-2} r_{xy}^{-1} \, dy + C \\ &\leqq C \int_{\mathscr{B}_0} r_{0z}^{-2} \, dz + C \int_{\mathscr{B}_0} |\nabla W|^2 \, dz + C |\log r_{0x}| \end{aligned}$$

for a suitable constant C. We have proved:

Lemma 3.10. *Under the hypotheses of the preceding lemma there holds for* $x \in \mathscr{A}_0$,

$$|x| \int_{\mathscr{B}_0} |y|^{-2} |\nabla G(x, y; \boldsymbol{w}_0)| \, dy < \frac{H}{2}$$

for some constant H, *uniformly on any finite set of values* $\boldsymbol{w}_0$.

This lemma has completed the proof of Theorem 3.1, which now follows from Lemmas 3.3 and 3.6 if $|x|$ is large, and from Lemmas 3.8 and 3.10 if x is near Σ.

4. The Nonlinear Problem; Existence

In terms of the variable $\boldsymbol{u}(x) = \lambda^{-1}(\boldsymbol{w} - \boldsymbol{w}_0)$, the Navier-Stokes equations (2) take the form

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{w}_0 \cdot \nabla \boldsymbol{u} - \nabla p &= \boldsymbol{f}(x) + \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{3}$$

Let $\boldsymbol{u}^*$ be prescribed data on Σ, and suppose $\boldsymbol{f}(x)$ is locally smooth and decays suitably at infinity.

It is proposed to solve (3) for a function $\boldsymbol{u}(x)$ satisfying $|\boldsymbol{u}(x)| < C r^{-1}$ at infinity, by an approximation procedure. To do so, we begin by representing an arbitrary solution $\boldsymbol{u}(x)$ in $\mathscr{E}$ by the Green's tensor $\boldsymbol{G}(x, y; \boldsymbol{w}_0)$ introduced in the preceding section. For an annular region $\mathscr{A}_R$ bounded by Σ and by a sphere Σ_R of (large) radius R, we find

$$\begin{aligned} \boldsymbol{u}(x) = \oint_{\Sigma} \boldsymbol{u}^* \cdot T\boldsymbol{G} \, d\sigma + \int_{\mathscr{A}_R} \boldsymbol{G} \cdot \boldsymbol{f} \, dy + \lambda \int_{\mathscr{A}_R} \boldsymbol{G} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{u} \, dy + \\ + \oint_{\Sigma_R} [\boldsymbol{u} \cdot T\boldsymbol{G} - \boldsymbol{G} \cdot T\boldsymbol{u} + (\boldsymbol{G} \cdot \boldsymbol{u}) \boldsymbol{w}_0] \, d\sigma. \end{aligned}$$

We may also write, since $\boldsymbol{G}(x, y) = 0$ if $y \in \Sigma$,

$$\int_{\mathscr{A}_R} \boldsymbol{G} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{u} \, dy = -\int_{\mathscr{A}_R} \boldsymbol{u} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{G} \, dy + \oint_{\Sigma_R} (\boldsymbol{G} \cdot \boldsymbol{u}) \, \boldsymbol{u} \, d\sigma.$$

If we assume for the moment the results of § 5, then Corollary 2.8, applied to $G(x, y)$ for $y \to \infty$, shows that the outer surface integral vanishes in the limit, and we obtain the representation, valid whenever $|x|^{\beta} \boldsymbol{f}(x) \in L_2(\mathscr{E})$, $\beta > \frac{1}{2}$,

$$\boldsymbol{u}(x) = \oint_{\Sigma} \boldsymbol{u}^* \cdot T\boldsymbol{G} \, d\sigma + \int_{\mathscr{E}} \boldsymbol{G} \cdot \boldsymbol{f} \, dy - \lambda \int_{\mathscr{E}} \boldsymbol{u} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{G} \, dy. \tag{42}$$

The first two terms on the right represent the solution $\boldsymbol{U}(x)$ of the inhomogeneous linear equation (4), such that $\boldsymbol{U}(x) = \boldsymbol{u}^*$ on Σ. Thus, any solution $\boldsymbol{u}(x)$ of (3) with the specified decay at infinity admits the representation

$$\boldsymbol{u}(x) = \boldsymbol{U}(x) - \lambda \int_{\mathscr{E}} \boldsymbol{u} \cdot \boldsymbol{u} \cdot \nabla \boldsymbol{G} \, dy \tag{43}$$

where $\boldsymbol{G} = \boldsymbol{G}(x, y; \boldsymbol{w}_0)$ in the Green's tensor for (4) in $\mathscr{E}$.

Let us suppose $\boldsymbol{f}(x)$ so chosen that $|\boldsymbol{U}(x)| < C r^{-1}$ in $\mathscr{E}$. In particular, this will be the case if $\boldsymbol{f}(x)$ has compact support in $\mathscr{E}$. We seek to write $\boldsymbol{u}(x)$ as an expansion

$$\boldsymbol{u}(x) = \boldsymbol{U}(x) + \sum_{1}^{\infty} \boldsymbol{u}_j(x) \, \lambda^j \tag{44}$$

in the parameter λ, such that $|\boldsymbol{u}_j(x)| < C r^{-1}$ for each j. Assuming (44), set $\boldsymbol{v}_j(x) = r\,\boldsymbol{u}_j(x)$, $\boldsymbol{v}_0(x) = r\,\boldsymbol{U}(x)$, obtaining

(45) $$\boldsymbol{v}(x) = r\,\boldsymbol{u}(x) = \sum_0^\infty \boldsymbol{v}_j(x)\,\lambda^j.$$

The representation (43) leads to the recursion relation

(46) $$\begin{aligned}\boldsymbol{v}_{n+1}(x) = |x| \int_{\mathscr{E}} |y|^{-2} [\boldsymbol{v}_0(y)\cdot\boldsymbol{v}_n(y) + \boldsymbol{v}_1(y)\cdot\boldsymbol{v}_{n-1}(y) + \cdots +\\ + \boldsymbol{v}_n(y)\cdot\boldsymbol{v}_0(y)]\cdot \nabla \boldsymbol{G}(x, y; \boldsymbol{w}_0)\,dy.\end{aligned}$$

By Theorem 3.1 there holds

$$|x| \int_{\mathscr{E}} |y|^{-2} |\nabla \boldsymbol{G}(x, y; \boldsymbol{w}_0)|\,dy < H$$

uniformly for $x \in \mathscr{E}$. Thus the series with constant coefficients

(47) $$V = \sum_0^\infty V_j \lambda^j$$

will be a dominant series for (45), provided

$$|\boldsymbol{v}_0(x)| \leqq V_0, \qquad \text{all } x \in \mathscr{E},$$
$$V_{n+1} = (V_0 V_n + V_1 V_{n-1} + \cdots + V_n V_0)\,H.$$

The convergence of (47) then implies

$$V = V_0 + \lambda H V^2.$$

The solution of this equation has a branch which is analytic in λ in a circle about the origin, whose radius is determined by the vanishing of the discriminant. *Thus, the series* (47), *and hence also the series* (44), *converges whenever* $\lambda < (4 H V_0)^{-1}$, *and provides a solution of* (3) *in* $\mathscr{E}$ *for such* λ.

We may state further: *the solution* $\boldsymbol{u}(x) = \boldsymbol{u}_0(x) + \sum_1^\infty \boldsymbol{u}_j(x)\lambda^j$ *is unique among* all *solutions* $\bar{\boldsymbol{u}}(x)$ *satisfying the same boundary conditions and admitting corresponding expansions, such that* $|\bar{\boldsymbol{u}}_j| = O(r^{-1})$ *at infinity.* For then the recursion relations, applied to the difference $\boldsymbol{u}(x) - \bar{\boldsymbol{u}}(x)$, lead successively to the identities $\boldsymbol{u}_j(x) - \bar{\boldsymbol{u}}_j(x) \equiv 0$ in $\mathscr{E}$, for each j.

One consequence of the above result for the original Navier-Stokes equations (2) is as follows. Suppose data $\boldsymbol{w}^*$ are prescribed on Σ. Set $\boldsymbol{u}^* = \boldsymbol{w}^* - \boldsymbol{w}_0$, and let $\boldsymbol{U}(x)$ be the solution of the linearized system (4) with data $\boldsymbol{u}^*$. Suppose $\boldsymbol{u}^*$ and $\boldsymbol{f}$ are sufficiently small that $4H \sup_{\mathscr{E}} |x|\,|\boldsymbol{U}(x)| < 1$. Then the series (44) converges for the value $\lambda = 1$. We have proved:

Theorem 4.1. *If the data* $\boldsymbol{w}^*$ *are sufficiently close to* $\boldsymbol{w}_0$, *and the external force* $\boldsymbol{f}(x)$ *sufficiently small, in the sense indicated above, then there is a solution* $\boldsymbol{w}(x)$ *of the Navier-Stokes equations* (2) *in* $\mathscr{E}$, *such that* $\boldsymbol{w}(x) = \boldsymbol{w}^*$ *on* Σ, $\boldsymbol{w}(x) \to \boldsymbol{w}_0$ *at infinity and* $|\boldsymbol{w}(x) - \boldsymbol{w}_0| < C r^{-1}$ *in* $\mathscr{E}$. *The solution can be obtained explicitly by a successive approximation procedure, requiring only the solution of linear inhomogeneous equations, with vanishing boundary conditions.*

The estimates leading to Theorem 4.1 are uniform on any finite interval of values $\boldsymbol{w}_0$. Hence, we obtain the following special case of the above result:

Corollary 4.1. *If $\boldsymbol{w}_0$ and $\boldsymbol{f}(x)$ are sufficiently small, then there is a solution $\boldsymbol{w}(x)$ of (2) in $\mathscr{E}$, such that $\boldsymbol{w}(x)=0$ on Σ, $\boldsymbol{w}(x)\to\boldsymbol{w}_0$ at infinity, and $|\boldsymbol{w}(x)-\boldsymbol{w}_0| < C r^{-1}$ in $\mathscr{E}$.*

The solutions just constructed are unique in a broad class for small data, and they are physically reasonable in the sense that they exhibit a paraboloidal "wake region" in the direction of the flow at infinity. The demonstrations of these facts require, however, some effort; they are a consequence of the material in the following section.

5. Asymptotic Structure of the Solutions; Uniqueness

The material to follow is based on the representation

$$\boldsymbol{u}(x) = -\int_{\mathscr{E}} \boldsymbol{u}\cdot\boldsymbol{u}\cdot\nabla\boldsymbol{E}\,dy + \oint_{\Sigma}[\boldsymbol{u}\cdot T\boldsymbol{E} - \boldsymbol{E}\cdot T\boldsymbol{u} + (\boldsymbol{E}\cdot\boldsymbol{u})(\boldsymbol{u}+\boldsymbol{w}_0)]\,d\boldsymbol{\sigma} \tag{48}$$

with $\boldsymbol{u}(x)=\boldsymbol{w}(x)-\boldsymbol{w}_0$, which, like (42), is valid for any of the solutions of $(2)_0$ in $\mathscr{E}$ to be considered. Under suitable assumptions, we shall estimate $\boldsymbol{u}(x)$ at infinity.

5.1. The case $\boldsymbol{w}_0 \neq 0$

If $\boldsymbol{w}_0\neq 0$, the solutions constructed in the preceding section have at infinity a structure which is identical, up to terms of small order, with that of the fundamental tensor $\boldsymbol{E}(x)$, $\boldsymbol{e}(x)$, in the sense

$$\boldsymbol{w}(x)=\boldsymbol{w}_0+\boldsymbol{a}\cdot\boldsymbol{E}(x)+b\,\boldsymbol{e}(x)+\boldsymbol{\sigma}(x)$$

where $\boldsymbol{a}$, b, $\boldsymbol{\sigma}(x)$ have the significance indicated in Corollary 2.8. To show this, it suffices to show that the nonlinear operator

$$N[\boldsymbol{u}] = \int_{\mathscr{E}} |\boldsymbol{u}(y)|^2\,|\nabla\boldsymbol{E}(x,y)|\,dy$$

has the order $\boldsymbol{\sigma}(x)$ at infinity whenever $\boldsymbol{u}(x)$ is a solution of (3) satisfying $|\boldsymbol{u}(x)|=O(r^{-1})$. The proof of this can be obtained as a consequence of the following general result.

Theorem 5.1. *Suppose $\boldsymbol{w}_0\neq 0$. Let τ denote distance along an axis in the direction $\boldsymbol{w}_0$, and let ϱ be distance orthogonal to the axis. Suppose $\boldsymbol{u}(x)$ satisfies in $\mathscr{E}$ the inequalities*

$$|\boldsymbol{u}(x)| < \begin{cases} |x|^{-\alpha} & \textit{throughout } \mathscr{E}, \\ |x|^{-\alpha+\frac{1}{2}\beta}\varrho^{-\beta} & \textit{if } \ \varrho \geqq |\tau|^{\frac{1}{2}}, \\ |x|^{-\alpha-\frac{1}{2}\beta} & \textit{if } \ \tau<0, \\ 1 & \textit{if } \ |x|\leqq 1, \end{cases} \tag{49}$$

where $\frac{1}{2}<\alpha<\frac{3}{2}$ and $\beta\geqq 0$. Then $N[\boldsymbol{u}]$ satisfies, for some constant C independent of the particular choice of $\boldsymbol{u}(x)$,

$$N[\boldsymbol{u}] < \begin{cases} C|x|^{-\bar\alpha} & \textit{throughout } \mathscr{E}, \\ C|x|^{-\bar\alpha+\frac{1}{2}\bar\beta}\varrho^{-\bar\beta} & \textit{if } \ \varrho \geqq |\tau|^{\frac{1}{2}}, \\ C|x|^{-\bar\alpha-\frac{1}{2}\bar\beta} & \textit{if } \ \tau<0, \\ C & \textit{if } \ |x|\leqq 1 \end{cases} \tag{50}$$

where $\bar{\alpha}$ *is any quantity such that*

$$\text{(51)}\qquad \bar{\alpha} < \begin{cases} \min\left[\alpha + \frac{\varepsilon}{2},\ \alpha + \frac{\beta}{2}\right] & \text{if } \beta \leqq \frac{\varepsilon}{2}, \\ \min\left[\alpha + \varepsilon,\ \alpha + \frac{\beta^*}{2}\right] & \text{if } \beta > \frac{\varepsilon}{2} \end{cases}$$

with $\varepsilon = \alpha - \frac{1}{2}$, $\beta^* = \min[\beta, 1]$, *and* β *is replaced by*

$$\text{(52)}\qquad \bar{\beta} = \begin{cases} \varepsilon & \text{if } \beta \leqq \frac{\varepsilon}{2}, \\ \min[2\varepsilon + \beta, 2\alpha + \beta^*, 2\beta, \beta + 2, 3], & \text{if } \beta \geqq \frac{\varepsilon}{2}. \end{cases}$$

Remark 1. The hypothesis (49) implies the existence of a paraboloidal "wake region" in the direction $\boldsymbol{w}_0$, in which $\boldsymbol{u}(x) = O(|x|^{-\alpha})$. Outside any cone whose axis coincides with that of the paraboloid, there holds $\boldsymbol{u}(x) = O\left(|x|^{-\alpha-\frac{\beta}{2}}\right)$.

Remark 2. The reason for excluding equality in (51) is the possible occurrence of logarithmic terms. Such terms can occur in the estimate for $N[\boldsymbol{u}]$ in the three cases $\beta = 1$, $1 + \beta - 2\alpha = 0$, or $3 - 2\alpha - \beta^* = 0$. In all other cases the inequality in (51) can be replaced by equality.

Theorem 5.1 will be proved by splitting the region of integration into a number of subregions, in each of which one of the two factors occurring in the integrand of the expression for $N[\boldsymbol{u}]$ is constant on each of a suitable family of surfaces of integration. The decomposition is indicated in Fig. 1 (overleaf), the various symmetries in the figure being indicated in the end view. The origin of coordinates is designated by 0, x is the point of evaluation, $|x| = R$, and x is (at first) assumed at distance $CR^{\frac{1}{2}+\sigma}$ from the paraboloidal axis through 0, where $0 \leqq \sigma \leqq \frac{1}{2}$. $\sigma = 0$ corresponds to a paraboloidal "wake" surface, while $\sigma = \frac{1}{2}$ yields a cone containing all such paraboloids at infinity. For purposes of definiteness the value $C = 2$ is chosen in Fig. 1. The points 0 and x are each enclosed in cylinders of radius $\frac{1}{2}R^{\frac{1}{2}+\sigma}$, and both these cylinders are enclosed in a larger one of radius $\frac{5}{2}R^{\frac{1}{2}+\sigma}$.

In the estimates that follow, multiplicative constants having no relevance to order of magnitude are omitted; thus a term R^α is to be understood as a term bounded for large R by CR^α for some constant C. The Roman numerals correspond to the estimates for $\int \boldsymbol{u}^2 |\nabla \boldsymbol{E}|\, dy$ over the regions indicated in Fig. 1. The symbol λ denotes the exponent of r which appears in each estimate.

I. In this region, $|\boldsymbol{u}|^2 < R^{(-2\alpha-\beta)(\frac{1}{2}+\sigma)}$, $|\nabla \boldsymbol{E}| < R^{-3(\frac{1}{2}+\sigma)}$. Hence

$$\lambda_{\mathrm{I}} \leqq (-2\alpha - \beta - 3)(\tfrac{1}{2} + \sigma) + 3(\tfrac{1}{2} + \sigma)$$
$$= -(2\alpha + \beta)(\tfrac{1}{2} + \sigma).$$

II. Here $|\boldsymbol{u}|^2 < \tau^{-2\alpha+\beta} R^{-2\beta(\frac{1}{2}+\sigma)}$, while $|\nabla \boldsymbol{E}| < R^{-\frac{3}{2}-3\sigma}$. Hence

$$\mathrm{II} < R^{-(2\beta+3)(\frac{1}{2}+\sigma)} \int_{R^{\frac{1}{2}+\sigma}}^{R} \tau^{-2\alpha+\beta}\, d\tau,$$

so that

$$\lambda_{\mathrm{II}} \leqq \max\left\{-\alpha - \varepsilon - 2\beta\sigma - \sigma,\ -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma\right\}$$

unless $1 + \beta - 2\alpha = 0$, in which case a logarithmic term appears.

III. A formal calculation yields the estimate for the surface integral of $u^2(y)$ over a sphere V_r of radius $r \geqq 1$ about the origin:

$$\oint_{\Sigma_r} u^2 d\sigma < r^{2-2\alpha-\beta^*}.$$

Hence

$$\int_{V_r} u^2 dy < \max\{r^{3-2\alpha-\beta^*}, 1\}$$

unless $3-2\alpha-\beta=0$, in which case a logarithmic factor enters.

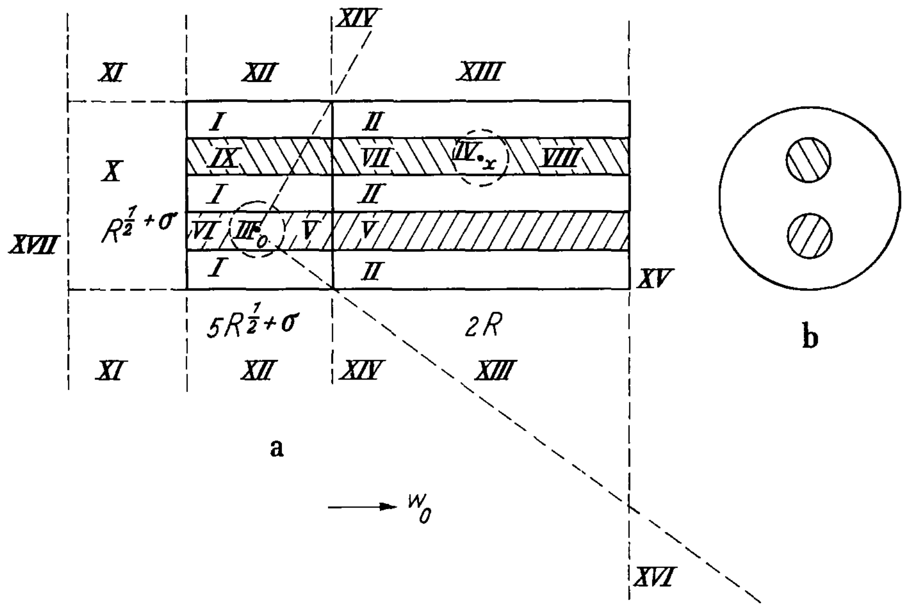

Fig. 1

The radius of the sphere III is $r=R^{\frac{1}{2}+\sigma}$. In this sphere $|\nabla E|<R^{-3(\frac{1}{2}+\sigma)}$. Thus if $3-2\alpha-\beta \neq 0$, then

$$\lambda_{\mathrm{III}} < \max\{-(2\alpha+\beta)(\tfrac{1}{2}+\sigma), -3(\tfrac{1}{2}+\sigma)\},$$

a logarithmic factor appearing in the other case.

IV. One has

$$\int_{V_r} |\nabla E|\, dy < r^{\frac{1}{2}}.$$

Again $r=R^{\frac{1}{2}+\sigma}$, while $|u|^2<R^{-2(\alpha+\beta\sigma)}$ in IV. Thus

$$\begin{aligned}\lambda_{\mathrm{IV}} &< -\alpha-\varepsilon-2\beta\sigma-\frac{1}{4}+\frac{\sigma}{2}\\ &< -\alpha-\varepsilon-2\beta\sigma\end{aligned}$$

since $\sigma<\frac{1}{2}$.

V. There holds uniformly $|\nabla E|<R^{-3(\frac{1}{2}+\sigma)}$. Consider first the subregion Va defined by setting $\varrho^2<\tau$. In Va one has $|u|^2<\tau^{-2\alpha}$. Therefore

$$\mathrm{Va} < R^{-3(\frac{1}{2}+\sigma)} \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau\, \tau^{-2\alpha} \int_0^{\tau^{\frac{1}{2}}} \varrho\, d\varrho.$$

Since $\alpha > \frac{1}{2}$,

$$\lambda_{\mathrm{Va}} < -(2\alpha+1)(\tfrac{1}{2}+\sigma).$$

In the remaining part Vb of V there holds $|\boldsymbol{u}|^2 < \tau^{-2\alpha+\beta}\varrho^{-2\beta}$, so that

$$\begin{aligned}\mathrm{Vb} &\leqq R^{-3(\frac{1}{2}+\sigma)} \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau\, \tau^{-2\alpha+\beta} \int_{\tau^{\frac{1}{2}}}^{R^{\frac{1}{2}+\sigma}} \varrho^{-2\beta+1}\, d\varrho \\ &= R^{-3(\frac{1}{2}+\sigma)} \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau\, \tau^{-2\alpha+\beta} \begin{cases} R^{-2(\beta-1)(\frac{1}{2}+\sigma)}, & \beta < 1, \\ \tau^{-(\beta-1)}, & \beta > 1. \end{cases}\end{aligned}$$

Thus,

$$\lambda_{\mathrm{Vb}} \leqq \max \begin{cases} -\alpha - \varepsilon - 2\beta\sigma - \sigma, \\ -\alpha - \dfrac{\beta}{2} - 2\alpha\sigma - \beta\sigma, \\ -\alpha - \varepsilon - 3\sigma, \\ -\alpha - \dfrac{1}{2} - 2\alpha\sigma \end{cases}$$

unless $\beta = 1$ or $1 + \beta - 2\alpha = 0$, in which cases logarithmic factors appear.

VI. In this region all quantities are of not larger magnitude than in V, hence the above estimate dominates.

VII. The distance from 0 to x along the τ-axis has the order R, and there is no loss of generality in setting it equal to R. One then has $\tau = R - s$, where s is distance from x along the τ-axis in the direction $-\boldsymbol{w}_0$. Let Γ_s denote a section of the cylinder containing x, at distance $s \geqq R^{\frac{1}{2}+\sigma}$ from x. One computes from (29)

$$\int_{\Gamma_s} |\nabla \boldsymbol{E}|\, d\sigma < s^{-\frac{1}{2}} = (R-\tau)^{-\frac{1}{2}}.$$

In VII, $|\boldsymbol{u}|^2 < \tau^{-2\alpha+\beta} R^{-2\beta(\frac{1}{2}+\sigma)}$. Thus,

$$\mathrm{VII} \leqq R^{-2\beta(\frac{1}{2}+\sigma)} \int_{R^{\frac{1}{2}+\sigma}}^{R} \tau^{-2\alpha+\beta} (R-\tau)^{-\frac{1}{2}}\, d\tau.$$

Set $\tau = Rt$. Then

$$\begin{aligned}\mathrm{VII} &\leqq R^{-2\beta(\frac{1}{2}+\sigma)} R^{-2\alpha+\beta+\frac{1}{2}} \int_{R^{-\frac{1}{2}+\sigma}}^{1} t^{-2\alpha+\beta}(1-t)^{-\frac{1}{2}}\, dt \\ &\leqq R^{-2\alpha-2\beta\sigma+\frac{1}{2}} \left[1 + R^{(-2\alpha+\beta+1)(-\frac{1}{2}+\sigma)}\right]\end{aligned}$$

or a logarithm replacing the bracketed expression if $1 + \beta - 2\alpha = 0$. Therefore

$$\lambda_{\mathrm{VII}} \leqq \max\left\{-\alpha - \varepsilon - 2\beta\sigma,\ -\alpha - \frac{\beta}{2} - \beta\sigma - 2\varepsilon\sigma\right\}$$

a logarithm appearing in the exceptional case $1 + \beta - 2\alpha = 0$.

VIII. Here

$$|\boldsymbol{u}|^2 < R^{-2\alpha+\beta} R^{-2\beta(\frac{1}{2}+\sigma)},$$

$$\int_{\Gamma_{-s}} |\nabla \boldsymbol{E}|\, d\sigma < |s|^{-3} R^{1+2\sigma},$$

so that

$$\mathrm{VIII} \leqq R^{-2\alpha-2\beta\sigma} R^{1+2\sigma} R^{-2(\frac{1}{2}+\sigma)}$$

or

$$\lambda_{\mathrm{VIII}} \leqq -\alpha - \frac{1}{2} - \frac{\varepsilon}{2} - (2\beta+\varepsilon)\,\sigma.$$

IX. In this region, $|\boldsymbol{u}|^2 < R^{(-2\alpha-\beta)(\frac{1}{2}+\sigma)}$, $\int_{\Gamma_t} |\nabla \boldsymbol{E}|\, d\sigma < R^{-\frac{1}{2}}$. Thus

$$\mathrm{IX} \leqq R^{(-2\alpha-\beta)(\frac{1}{2}+\sigma)} R^{-\frac{1}{2}} R^{\frac{1}{2}+\sigma},$$

so that

$$\lambda_{\mathrm{IX}} \leqq -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma + \sigma$$
$$= -\alpha - \frac{\beta}{2} - 2\varepsilon\sigma - \beta\sigma.$$

X. There holds here $\int_{\Gamma_t} |\nabla \boldsymbol{E}|\, d\sigma < R^{-\frac{1}{2}}$, $|\boldsymbol{u}|^2 < \tau^{-2\alpha-\beta}$. Hence

$$\mathrm{X} \leqq R^{-\frac{1}{2}} \int_{R^{\frac{1}{2}+\sigma}}^{R} \tau^{-2\alpha-\beta}\, d\tau$$

from which

$$\lambda_{\mathrm{X}} \leqq -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma + \sigma$$
$$\leqq -\alpha - \frac{\beta}{2} - 2\varepsilon\sigma - \beta\sigma$$

since $\alpha = \frac{1}{2} + \varepsilon$.

XI. We have $|\boldsymbol{u}|^2 < \tau^{-2\alpha-\beta}$, $|\nabla \boldsymbol{E}| < \varrho^{-3}$, so that

$$\mathrm{XI} < \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau\, \tau^{-2\alpha-\beta} \int_{R^{\frac{1}{2}+\sigma}}^{\infty} \varrho^{-2}\, d\varrho$$

from which

$$\lambda_{\mathrm{XI}} < \max\left\{-\alpha - \varepsilon - \beta - \sigma,\ -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma\right\}$$

unless $1 + \beta - 2\alpha = 0$, in which case a logarithm appears.

XII. Here $|\boldsymbol{u}|^2 < \varrho^{-2\alpha-\beta}$, $|\nabla \boldsymbol{E}| < \varrho^{-3}$, hence

$$\mathrm{XII} \leqq R^{\frac{1}{2}+\sigma} \int_{R^{\frac{1}{2}+\sigma}}^{\infty} \varrho^{-2\alpha-\beta} \varrho^{-2}\, d\varrho$$

and

$$\lambda_{\mathrm{XII}} \leqq -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma.$$

XIII. In this region, $|\boldsymbol{u}|^2 < \tau^{-2\alpha+\beta} \varrho^{-2\beta}$, $|\nabla \boldsymbol{E}| < \varrho^{-3}$, so that

$$\mathrm{XIII} < \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau\, \tau^{-2\alpha+\beta} \int_{R^{\frac{1}{2}+\sigma}}^{R} \varrho^{-2\beta-2}\, d\varrho$$

and there follows

$$\lambda_{\mathrm{XIII}} \leqq \max\left\{-\alpha - \varepsilon - 2\beta\sigma - \sigma,\ -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma\right\}$$

with the proviso that a logarithmic factor enters if $1 + \beta - 2\alpha = 0$.

XIV. Here $|\boldsymbol{u}|^2 < \varrho^{-2\alpha-\beta}$, $|\nabla \boldsymbol{E}| < \varrho^{-3}$,

$$\mathrm{XIV} \leqq \int_{R^{\frac{1}{2}+\sigma}}^{R} d\tau \int_{\tau}^{\infty} \varrho^{-2\alpha-\beta-2}\, d\varrho$$

so that

$$\lambda_{\mathrm{XIV}} \leqq -\alpha - \frac{\beta}{2} - 2\alpha\sigma - \beta\sigma = -(2\alpha + \beta)\left(\frac{1}{2} + \sigma\right).$$

XV. We integrate over concentric spherical surfaces S_r of radius $r>2R$ centered at 0. On these surfaces, $|\nabla E|<r^{-3}$, while $\int_{S_r} u^2 d\sigma$ is less than the corresponding integral over the entire sphere of that radius,

$$\int_{S_r} u^2 d\sigma < \oint_{\Sigma_r} u^2 d\sigma < r^{2-2\alpha-\beta^*},$$

a result obtained by an easy calculation, using the hypothesis (49). Thus,

$$\mathrm{XV} < \int_R^\infty r^{-1-2\alpha-\beta^*} dr,$$

and

$$\lambda_{\mathrm{XV}} < -2\alpha-\beta^*.$$

XVI. In this case $|u|^2<r^{-2\alpha-\beta}$, $|\nabla E|<r^{-3}$. There certainly holds

$$\mathrm{XVI} < \int_R^\infty r^{-3} r^{-2\alpha-\beta} r^2 dr$$

so that

$$\lambda_{\mathrm{XVI}} \leqq -2\alpha-\beta.$$

XVII. Again we integrate over concentric spherical surfaces about 0, noting this time that

$$\oint_{\Sigma_r} |\nabla E|\, d\sigma < r^{-\frac{1}{2}}.$$

Since $|u|^2<r^{-2\alpha-\beta}$ in XVII, we obtain

$$\lambda_{\mathrm{XVII}} \leqq -\alpha-\beta-\varepsilon.$$

This completes the estimates in the situation considered, for which $\varrho=CR^{\frac{1}{2}+\sigma}$, $0\leqq\sigma\leqq\frac{1}{2}$. There are two cases not covered by this situation, namely the one in which x lies exterior to a cone $\varrho=CR$ which opens in the direction w_0, and the one in which $\varrho<R^{\frac{1}{2}}$. In the former case, all the above estimates apply with $\sigma=\frac{1}{2}$ (in fact, they simplify greatly). The latter case requires a special discussion.

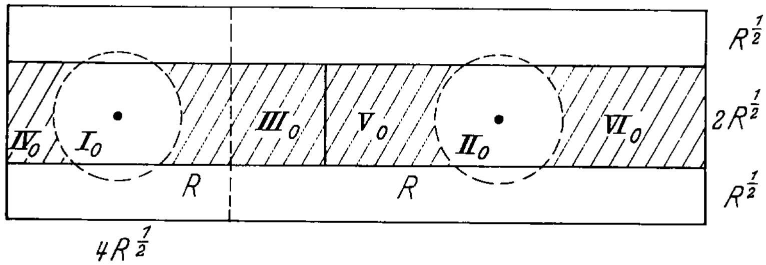

Fig. 2

The situation is illustrated in Fig. 2. We may assume that 0 lies on the axis of a cylinder Z of a radius $2R^{\frac{1}{2}}$ and that x is of distance $<R^{\frac{1}{2}}$ from this axis. This cylinder is now enclosed in another concentric one of radius $3R^{\frac{1}{2}}$. One sees immediately that all estimates corresponding to regions outside Z proceed as before, simply by setting $\sigma=0$. In Z, the estimate is now most easily established, using the decomposition indicated in Fig. 2.

I_0. Here $|\nabla E|<R^{-\frac{3}{2}}$, and

$$\oint_{\Sigma_r} |u|^2 d\sigma < r^{2-2\alpha-\beta^*}.$$

Thus,

$$\lambda_{I_0} = \max\left\{-\frac{3}{2}, -\alpha - \frac{\beta^*}{2}\right\}$$

unless $3-2\alpha-\beta^*=0$, in which event a logarithm appears.

II_0. We have $|u|^2 < R^{-2\alpha}$, $\oint_{\Sigma_r} |\nabla E|\, d\sigma < r^{-\frac{1}{2}}$. We compute

$$\lambda_{II_0} < -2\alpha + \tfrac{1}{4} = -\alpha - \tfrac{1}{4} - \varepsilon.$$

III_0. There holds $|\nabla E| < R^{-\frac{3}{2}}$, and on a cylindrical section Γ_τ,

$$\int_{\Gamma_\tau} |u|^2 d\sigma = \tau^{-2\alpha} \int_0^{\tau^{\frac{1}{2}}} \varrho\, d\varrho + \tau^{-2\alpha+\beta} \int_{\tau^{\frac{1}{2}}}^{R^{\frac{1}{2}}} \varrho^{-2\beta} \varrho\, d\varrho$$

$$= \tau^{1-2\alpha} + \tau^{-2\alpha+\beta} \begin{cases} \tau^{1-2\alpha}, & \beta > 1 \\ \tau^{-2\alpha+\beta} R^{-(\beta-1)}, & \beta < 1. \end{cases}$$

Integrating with respect to τ, we find

$$\mathrm{III}_0 < R^{-\frac{3}{2}} \int_{R^{\frac{1}{2}}}^{R} |u|^2 d\sigma$$

$$< R^{-\frac{3}{2}} \begin{cases} R^{1-\alpha}, & \beta > 1 \\ R^{-\alpha+\frac{3}{2}-\frac{\beta}{2}}, & \beta < 1, \end{cases}$$

so that

$$\lambda_{III_0} \leqq -\alpha - \frac{\beta^*}{2}.$$

IV_0. Here all quantities are smaller than in III_0, so the estimate for that region prevails.

V_0. Throughout this region, $|u|^2 < R^{-2\alpha}$. On a section Γ_s, $\int_{\Gamma_s} |\nabla E|\, d\sigma < s^{-\frac{1}{2}}$. Integrating in s, we find

$$\lambda_{V_0} < -2\alpha + \tfrac{1}{2} = -\alpha - \varepsilon.$$

VI_0. All quantities entering in the estimations are smaller than in the case just considered, so the estimate for V_0 prevails.

The demonstration of Theorem 5.1 is now completed by collecting all the above estimates and examining the finite number of possibilities that can occur.

A case of special interest is that in which $u(x)$ decays asymptotically as the fundamental tensor $E(x-y)$. Then $\alpha=1$, $\beta=2$. This is one of the exceptional cases in which a logarithm appears (*cf.* Remark 2 following Theorem 5.1). All estimates remain unchanged, however, except III and III_0. We are led to the result:

Corollary 5.1. *Under the hypotheses of Theorem* 5.1, *if* $\alpha=1$, $\beta=2$, *then* $N[u]$ *satisfies:*

$$(53) \qquad N[u] < \begin{cases} C|x|^{-\frac{3}{2}} \log(|x|+1) & \text{throughout } \mathscr{E}, \\ C\varrho^{-3} \log(|x|+1) & \text{if } \varrho \geqq |\tau|^{\frac{1}{2}}, \\ C|x|^{-3} \log(|x|+1) & \text{if } \tau < 0, \\ C & \text{if } |x| \leqq 1 \end{cases}$$

where the constant C *does not depend on* $u(x)$.

We may apply the above results to the solutions of $(2)_0$ in $\mathscr{E}$. Let $\boldsymbol{w}(x)$ be such a solution, such that $\boldsymbol{w}(x)\to\boldsymbol{w}_0$ at infinity. Set $\boldsymbol{u}(x)=\boldsymbol{w}-\boldsymbol{w}_0$, and suppose $|\boldsymbol{u}(x)|<C|x|^{-\alpha}$ with $\alpha>\frac{1}{2}$. We may write

$$\boldsymbol{u}(x)=\boldsymbol{u}_\Sigma(x)+\boldsymbol{u}_{\mathscr{E}}(x)$$

corresponding to the surface and volume integrals in the representation (48). $\boldsymbol{u}_\Sigma(x)$ satisfies (50) with $\alpha=1$, $\beta=2$, so that $N[\boldsymbol{u}_\Sigma]$ satisfies (53). On the other hand, $\boldsymbol{u}(x)$ satisfies (50) with the given α and with $\beta=0$. According to Theorem 5.1, $N[\boldsymbol{u}]$ satisfies again (50) with any $\bar{\alpha}<\alpha$ and with some $\bar{\beta}>0$ (in this case we may choose $\bar{\alpha}=\alpha$). Since $|\boldsymbol{u}(x)|=|\boldsymbol{u}_\Sigma(x)+\boldsymbol{u}_{\mathscr{E}}(x)|\leqq|\boldsymbol{u}_\Sigma(x)|+N[\boldsymbol{u}]$, it follows that the original solution $\boldsymbol{u}(x)$ satisfies this improved estimate, with the proviso that $\bar{\alpha}\leqq 1$. Inserting this result into $N[\boldsymbol{u}]$ now yields larger values for α and for β in the estimate (50). A finite number of iterations then leads to the result:

Theorem 5.2. *Let $\boldsymbol{w}(x)$ be smooth in $\mathscr{E}+\Sigma$ and satisfy $(2)_0$ in $\mathscr{E}$. Suppose $|\boldsymbol{w}(x)-\boldsymbol{w}_0|<C|x|^{-\alpha}$ as $x\to\infty$ for some $\alpha>\frac{1}{2}$. Let $\boldsymbol{u}(x)=\boldsymbol{w}(x)-\boldsymbol{w}_0$. Then*

$$\boldsymbol{u}(x)=\boldsymbol{u}_\Sigma(x)+\boldsymbol{N}[\boldsymbol{u}] \tag{54}$$

where $\boldsymbol{u}_\Sigma(x)$ is the surface integral in (48) and $\boldsymbol{N}[\boldsymbol{u}]$, the volume integral in (48), satisfies (50) with $\bar{\beta}=3$ and any $\bar{\alpha}$ such that $\bar{\alpha}<\frac{3}{2}$. In particular, $\boldsymbol{w}(x)$ exhibits a paraboloidal "wake region" at infinity. The behavior of $\boldsymbol{u}(x)$ along any path tending to infinity is, up to terms of order $N[\boldsymbol{u}]$, identical to that of the fundamental tensor $E(x;\boldsymbol{w}_0)$ considered as function of x.

Let us consider the derivatives of the solution $\boldsymbol{w}(x)$. The representation (48) cannot be differentiated under the sign, but the following device is effective. Consider that part of the volume integral which is extended over a unit ball V centered at x. We may write, since $\nabla\cdot\boldsymbol{w}=\nabla\cdot\boldsymbol{u}=0$,

$$\int_V \boldsymbol{u}\cdot\boldsymbol{u}\cdot\nabla E\,dy=\oint_S(\boldsymbol{E}\cdot\boldsymbol{u})\,\boldsymbol{u}\,d\boldsymbol{\sigma}-\int_V \boldsymbol{E}\cdot\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,dy \tag{55}$$

where S is the bounding surface.

At any point z having distance $>2\gamma$ from Σ, we may represent $\boldsymbol{u}(x)$ with the aid of the truncated tensor $\boldsymbol{E}^\gamma(x-z;\boldsymbol{w}_0)$ (*cf.* § 2.3) obtaining

$$\begin{aligned}\boldsymbol{u}(z)&=\int_{V^\gamma}\boldsymbol{E}^\gamma\cdot\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,dy+\int_{V^\gamma}\boldsymbol{H}^\gamma\cdot\boldsymbol{u}\,dy\\&=-\int_{V^\gamma}\boldsymbol{u}\cdot\boldsymbol{u}\cdot\nabla\boldsymbol{E}^\gamma dy+\int_{V^\gamma}\boldsymbol{H}^\gamma\cdot\boldsymbol{u}\,dy.\end{aligned}$$

The latter relation leads to a Hölder bound on $\boldsymbol{u}(z)$ for any exponent $\alpha<1$, with constant proportional (for fixed α) to $\max_{V^\gamma}|\boldsymbol{u}(x)|$. But we may also write

$$\boldsymbol{u}(z)=-\int_{V^\gamma}[\boldsymbol{u}(y)\cdot\boldsymbol{u}(y)-\boldsymbol{u}(z)\cdot\boldsymbol{u}(z)]\cdot\nabla\boldsymbol{E}^\gamma dy+\int_{V^\gamma}\boldsymbol{H}^\gamma\cdot\boldsymbol{u}\,dy. \tag{56}$$

Because of the Hölder continuity of $\boldsymbol{u}(x)$, (56) can be differentiated under the sign, leading to the estimate

$$|\nabla\boldsymbol{u}(z)|<C\max_{V^\gamma}|\boldsymbol{u}(y)|,$$

C depending only on w_0 and on γ. Thus, $|\nabla \boldsymbol{u}|$ satisfies at infinity the same estimates which were shown in Theorem 5.2 to hold for $|\boldsymbol{u}(x)|$. Placing this result in (55) shows that the quantity $\nabla\{\int_V \boldsymbol{u}(y)\cdot\boldsymbol{u}(y)\cdot\nabla \boldsymbol{E}(x-y)\,dy\}$ satisfies an estimate (50) with $\bar{\alpha}=2$, $\bar{\beta}=4$.

The remaining terms in the representation (48) can now be estimated by differentiating under the sign and repeating the reasoning which led to Theorem 5.2. The procedure is formally unchanged, and details are omitted. We find

Theorem 5.3. *Under the hypotheses of Theorem* 5.2, *we may write*

$$\nabla \boldsymbol{u}(x)=\nabla \boldsymbol{u}_\Sigma(x)+\boldsymbol{N}'[\boldsymbol{u}]$$

where $|\boldsymbol{N}'[\boldsymbol{u}]|$ *satisfies* (50) *with* $\bar{\alpha}=2$, $\bar{\beta}=4$.

Similarly, one may discuss the pressure term, starting from the representation (see (28))

$$(57)\quad p(x)=p_0+\int_{\mathscr{E}} \boldsymbol{e}(x-y)\cdot\boldsymbol{u}\cdot\nabla \boldsymbol{u}\,dy+\oint_\Sigma\{\boldsymbol{u}\cdot T\boldsymbol{e}-\boldsymbol{e}\cdot T\boldsymbol{u}+(\boldsymbol{e}\cdot\boldsymbol{u})\,\boldsymbol{w}_0\}\,d\boldsymbol{\sigma}$$

and using the fact that the kernel $\boldsymbol{e}(x-y)$ can be expressed as a gradient. We obtain

Theorem 5.4. *Under the hypotheses of Theorem* 5.2, *there holds, analogous to* (54),

$$p(x)=p_\Sigma(x)+P[\boldsymbol{u}]$$

where $P[\boldsymbol{u}]$ *satisfies* (50) *with* $\bar{\alpha}=2$, $\bar{\beta}=2$.

5.2. *The case* $\boldsymbol{w}_0=0$

If $\boldsymbol{w}_0=0$, the nonlinear term in (48) does not appear in general to decay at infinity more rapidly than the surface integral. Nevertheless, this term admits a qualitative estimate for all functions $\boldsymbol{w}(x)$ such that $|\boldsymbol{w}(x)|=O(r^{-1})$. The estimate can be obtained by subdividing the region of integration into a sphere of radius $\frac{1}{2}|x|$ about the origin, the exterior of a concentric sphere of radius $2|x|$, and the remaining annular region. We find easily $N[\boldsymbol{w}]<C|x|^{-1}$. If $\boldsymbol{w}(x)$ is a solution in $\mathscr{E}$, then $\nabla\boldsymbol{w}$ and $p(x)$ can be estimated as above. Thus:

Theorem 5.5. *Suppose* $\boldsymbol{w}_0=0$, *and* $|\boldsymbol{w}(x)|<|x|^{-1}$ *as* $x\to\infty$. *Then* $N[\boldsymbol{w}]<C|x|^{-1}$, *where* C *does not depend on the choice of* $\boldsymbol{w}(x)$. *If* $\boldsymbol{w}(x)$ *is a solution of* $(2)_0$ *in* $\mathscr{E}$, *then* $N'[\boldsymbol{w}]<C|x|^{-2}\log|x|$ *as* $x\to\infty$, *and* $P[\boldsymbol{w}]<C|x|^{-2}$.

Theorem 5.5 exhibits the compactness property of the operator $N[\boldsymbol{w}]$, which is the underlying reason that the existence theorem of the preceding section could be obtained with vanishing data on Σ and nonvanishing data at infinity. The class of functions which decay as $|x|^{-1}$ is mapped compactly into itself, uniformly on an interval of $\boldsymbol{w}_0$ which extends to $\boldsymbol{w}_0=0$. This is evidently not the case for any family of functions which decay as $|x|^{-\alpha}$, if $\alpha<1$, whereas the method requires such an estimate to hold for some $\alpha\leqq 1$.

In the material that follows, the following definitions will be adopted:

Definition 5.1. *A solution* $\boldsymbol{w}(x)$ *of* (2) *in* $\mathscr{E}$ *will be said to be in the class* PR *(physically reasonable) if it satisfies the hypotheses of Theorem* 5.2 *or of Theorem* 5.5.

The justification for the definition lies in the conclusions of those theorems. Although those results were proved under the assumption $\boldsymbol{f} \equiv 0$, they evidently hold whenever the volume integrals containing $\boldsymbol{f}(x)$ are sufficiently well behaved at infinity. This is the case, for example, whenever $\boldsymbol{f}(x)$ has compact support in $\mathscr{E}+\Sigma$.

Definition 5.2. *A solution $\boldsymbol{w}(x)$ of* (2) *in $\mathscr{E}$ will be said to be in the class* D *if it has finite Dirichlet integral over $\mathscr{E}$*, i.e., *if*

$$\int_{\mathscr{E}} |\nabla \boldsymbol{w}|^2 \, dx < \infty .$$

Solutions of class D were first constructed by LERAY [*2*], see also [*5, 13, 16, 17, 4*]. Such solutions are known to be continuous at infinity, but no further asymptotic properties have been demonstrated (see, however, the discussion in § 6 of [*13*]). On the other hand, it follows from Theorems 5.3 and 5.5 that *every solution of class* PR *is also in the class* D.

5.3. Consequences

Let $\mathscr{F}$ be the force exerted on Σ by the fluid,

$$\mathscr{F} = -\oint_{\Sigma} T\boldsymbol{w}\, d\boldsymbol{\sigma},$$

and define the deformation tensor by $(\operatorname{def} \boldsymbol{w})_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$. The following results are easy formal consequences of the above estimates. (For further details, see [*7*].)

Theorem 5.6. *Let $\boldsymbol{w}(x) \in PR$, $\boldsymbol{f} \equiv 0$. Then the term $\boldsymbol{N}[\boldsymbol{u}]$ in the representation* (48) *yields no contribution to the force on Σ, that is, $\mathscr{F}$ is determined by the solution of the linear system* $(4)_0$ *defined by the surface integral in* (48).

The hypothesis $\boldsymbol{f} \equiv 0$ can, of course, be weakened.

Theorem 5.7. *Let $\boldsymbol{w}(x) \in PR$ in $\mathscr{E}$, $\boldsymbol{f} \equiv 0$, and suppose $\boldsymbol{w}(x) \to \boldsymbol{w}^* \equiv$* constant *on Σ. Then*

$$\mathscr{F} \cdot (\boldsymbol{w}_0 - \boldsymbol{w}^*) = 2 \int_{\mathscr{E}} (\operatorname{def} \boldsymbol{w})^2 \, dx . \tag{58}$$

Corollary 5.7a. *Under the "physical" boundary condition $\boldsymbol{w}^* = 0$, there holds $\mathscr{F} \cdot \boldsymbol{w}_0 > 0$, that is, there is a "drag" force in the direction $\boldsymbol{w}_0$.*

Corollary 5.7b. *If $\boldsymbol{w}(x) \in PR$ in $\mathscr{E}$, $\boldsymbol{f} \equiv 0$ and $\boldsymbol{w}(x) = \boldsymbol{w}_0$ on Σ, then $\boldsymbol{w}(x)$ is the uniform flow $\boldsymbol{w}(x) \equiv \boldsymbol{w}_0$ in $\mathscr{E}$.*

Theorem 5.8. *Let $\boldsymbol{w}(x) \in PR$ in $\mathscr{E}$, $\boldsymbol{f} \equiv 0$, and set*

$$\boldsymbol{a} = \oint_{\Sigma} \{T\boldsymbol{w} - \boldsymbol{w}\boldsymbol{w} - \boldsymbol{w}_0 \boldsymbol{w}\}\, d\boldsymbol{\sigma}.$$

Then $|\boldsymbol{w}(x) - \boldsymbol{w}_0| = o(|x|^{-1})$ at infinity if and only if $\boldsymbol{a} = 0$. In particular, if the flux and momentum flux across Σ of the (prescribed) data both vanish, then $|\boldsymbol{w} - \boldsymbol{w}_0| = o(|x|^{-1})$ if and only if the net force exerted by the fluid on Σ is zero.

Corollary 5.8a (*cf.* BERKER [*18*]). *Let $\boldsymbol{w}(x)$ satisfy $(2)_0$ in $\mathscr{E}$, $\boldsymbol{w} \to \boldsymbol{w}^* \equiv$* const. *on Σ, and suppose $|\boldsymbol{w}(x) - \boldsymbol{w}_0| = o(|x|^{-1})$ at infinity. Then $\boldsymbol{w}(x) \equiv \boldsymbol{w}_0$ in $\mathscr{E}$.*

Theorem 5.9. *Let $\boldsymbol{w}(x)$ be a solution of $(2)_0$ throughout Euclidean three-space, and suppose $\boldsymbol{w}(x) \in PR$. Then $\boldsymbol{w}(x) \equiv \boldsymbol{w}_0$.*

Finally we establish the uniqueness of solutions $\boldsymbol{w}(x) \in PR$, which are sufficiently close to a uniform flow.

Theorem 5.10. *Let $\boldsymbol{w}(x)$ be a solution of $(2)_0$, such that $\boldsymbol{w}(x) = \boldsymbol{w}^*$ on Σ. Suppose $|\boldsymbol{w}(x) - \boldsymbol{w}_0| < \frac{\nu}{2}|x|^{-1}$ throughout $\mathscr{E}$. Let $\boldsymbol{v}(x)$ be a solution of $(2)_0$ such that $\boldsymbol{v}(x) \in PR$, and suppose $\boldsymbol{v}(x) = \boldsymbol{w}^*$ on Σ. Then $\boldsymbol{v}(x) \equiv \boldsymbol{w}(x)$ throughout $\mathscr{E}$.*

Proof. Let $\boldsymbol{\eta} = \boldsymbol{w} - \boldsymbol{v}$, and let $q(x)$ be the difference of pressures. Then

$$\nu \Delta \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} - \nabla q = (\boldsymbol{\eta} \cdot \nabla \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{\eta}).$$

Multiply by $\boldsymbol{\eta}$ and integrate over the annular region $\mathscr{E}_R$ bounded between Σ and a large sphere Σ_R. We obtain

$$(59) \qquad \begin{aligned} -\nu \int_{\mathscr{E}_R} |\nabla \boldsymbol{\eta}|^2 dx = \int_{\mathscr{E}_R} \boldsymbol{\eta} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{w}\, dx - \frac{1}{2} \oint_{\Sigma_R} |\boldsymbol{\eta}|^2 \boldsymbol{w}\, d\boldsymbol{\sigma} + \\ + \oint_{\Sigma_R} |\boldsymbol{\eta}|^2 \boldsymbol{\eta}\, d\boldsymbol{\sigma} - \frac{\nu}{2} \oint_{\Sigma_R} \nabla |\boldsymbol{\eta}|^2 d\boldsymbol{\sigma} + \oint_{\Sigma_R} q \boldsymbol{\eta}\, d\boldsymbol{\sigma}. \end{aligned}$$

By Theorems 5.2 to 5.5 all boundary terms vanish in the limit. Further,

$$\begin{aligned} \int_{\mathscr{E}_R} \boldsymbol{\eta} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{w}\, dx &= \int_{\mathscr{E}_R} \boldsymbol{\eta} \cdot \boldsymbol{\eta} \cdot \nabla (\boldsymbol{w} - \boldsymbol{w}_0)\, dx \\ &= -\int_{\mathscr{E}_R} (\boldsymbol{w} - \boldsymbol{w}_0) \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}\, dx + \oint_{\Sigma_R} \boldsymbol{\eta} \cdot (\boldsymbol{w} - \boldsymbol{w}_0)\, \boldsymbol{\eta}\, d\boldsymbol{\sigma}. \end{aligned}$$

Again the surface integral tends to zero. But by hypothesis, $|\boldsymbol{w} - \boldsymbol{w}_0| < \frac{\nu}{2}|x|^{-1}$, hence if $\nabla \boldsymbol{\eta} \not\equiv 0$, there holds

$$\begin{aligned} \left| \int_{\mathscr{E}_R} (\boldsymbol{w} - \boldsymbol{w}_0) \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}\, dx \right|^2 &< \left| \frac{\nu}{2} \int_{\mathscr{E}_R} r^{-1} \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}\, dx \right|^2 \\ &< \left(\frac{\nu}{2}\right)^2 \int_{\mathscr{E}_R} r^{-2} |\boldsymbol{\eta}|^2 dx \int_{\mathscr{E}_R} |\nabla \boldsymbol{\eta}|^2 dx \end{aligned}$$

by the Schwarz inequality. Since $\boldsymbol{\eta} = 0$ on Σ, we may extend $\boldsymbol{\eta}$ to a continuous field in all space by setting $\boldsymbol{\eta} \equiv 0$ in the complement of $\mathscr{E} + \Sigma$. Applying Lemma 2.2, we find

$$\int_{\mathscr{E}} r^{-2} |\boldsymbol{\eta}|^2 dx \leqq 4 \int_{\mathscr{E}} |\nabla \boldsymbol{\eta}|^2 dx.$$

Inserting this result into (59) leads to a contradiction unless $\nabla \boldsymbol{\eta} \equiv 0$ in $\mathscr{E}$. But $\boldsymbol{\eta} = 0$ on Σ, hence $\boldsymbol{\eta} \equiv 0$ in $\mathscr{E}$, which was to be proved.

Remark. Note that the hypothesis $\boldsymbol{f} \equiv 0$ could be weakened considerably. It would suffice, for example, that $(\boldsymbol{E}, \boldsymbol{f})_{\mathscr{E}}$, $(\nabla \boldsymbol{E}, \boldsymbol{f})_{\mathscr{E}}$, $(\boldsymbol{e}, \boldsymbol{f})_{\mathscr{E}} = o(r^{-1})$ at infinity.

6. Perturbation to Zero Reynolds' Number

Let $\boldsymbol{w}(x)$ be a solution of the exterior problem for (2) in $\mathscr{E}$, such that $\boldsymbol{w}(x) = \boldsymbol{w}^*$ on Σ and $\boldsymbol{w}(x) \to \boldsymbol{w}_0$ at infinity. If $\boldsymbol{w}^* - \boldsymbol{w}_0$ is small, can the corresponding solution $\boldsymbol{u}(x)$ of the linear system (4) or (6) serve as an approximation to $\boldsymbol{w}(x)$ in $\mathscr{E}$, and if so, in what sense? The answer to this question appears to depend heavily on the asymptotic structure of the given solution $\boldsymbol{w}(x)$, and also on the way in which the approximation is constructed. The question is conveniently rephrased as a perturbation problem. A given solution $\boldsymbol{w}(x)$ of (2) in $\mathscr{E}$ will be supposed embedded, for small λ, in a one-parameter family of solutions $\boldsymbol{w}(x;\lambda)=\boldsymbol{w}_0+\lambda\,\boldsymbol{u}(x;\lambda)$, where $\boldsymbol{u}(x;\lambda)$, appearing as a solution of (3), is to achieve fixed boundary data $\boldsymbol{u}^*$, $\boldsymbol{u}_0$ on Σ and at infinity. Here $\boldsymbol{w}_0$ is an arbitrary constant vector, which may or may not be velocity at infinity. Let $\boldsymbol{U}(x)$ be a solution of the corresponding linear equations (4), and satisfying the same conditions on Σ and at infinity. One seeks to estimate $[\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)]$ uniformly in $\mathscr{E}$ for small λ.

The various possibilities available to us may be illustrated by the following considerations. Suppose for simplicity $\boldsymbol{w}(x)=\boldsymbol{w}(x;1)$ and $\boldsymbol{w}(x)\to\boldsymbol{w}_0^*$ at infinity. Then it may be convenient to set $\boldsymbol{w}_0=\boldsymbol{w}_0^*$ and prescribe $\boldsymbol{u}(x;\lambda)\to 0$ at infinity for each λ, that is, $\boldsymbol{u}_0=0$. It will be seen shortly that this case leads to a perturbation which is uniformly well behaved throughout $\mathscr{E}$. On the other hand, the equation satisfied by $\boldsymbol{U}(x)$ is still relatively complicated, because of the factor $\boldsymbol{w}_0$. Alternatively, one may set $\boldsymbol{w}_0=0$, $\boldsymbol{u}_0=\boldsymbol{w}_0^*$, thus perturbing the data at infinity. Then $\boldsymbol{U}(x)$ satisfies the simpler system (6), but the perturbation turns out to be singular at infinity. It leads in the three-dimensional case considered here to Whitehead's paradox [*6*, p. 163], and in two dimensions to the more striking Stokes paradox, which arises from the fact that, in general, solutions $\boldsymbol{U}(x)$ corresponding to the prescribed data $\boldsymbol{u}^*$, $\boldsymbol{u}_0$ do not exist [*8*, *9*, *10*][2].

A further difficulty arises according to the asymptotic behavior assumed of $\boldsymbol{w}(x)$ at infinity. If the solutions are assumed of the type constructed in § 4, *i.e.*, if $\boldsymbol{w}(x)\in PR$ (§ 5.2), then for small λ (or, alternatively, for given λ and small $\boldsymbol{u}^*$, $\boldsymbol{u}_0$) the solutions are unique (Theorem 5.10) and the perturbation completely determined by the expansion (44). However, solutions exist which satisfy the stated conditions on Σ and at infinity and which are only known to be of class D. The relation between these solutions and those of class PR is not clear; nevertheless, one may again examine the perturbations for the presumably broader class D in both cases, obtaining uniform estimates in $\mathscr{E}$.

There are thus four cases to consider, and we shall take them in turn. In each of these cases it will be necessary to distinguish the subcase $\boldsymbol{w}_0=0$, as the order of approximation appears generally to be weaker in this situation. Throughout the ensuring discussion it will be supposed that $\boldsymbol{f}(x)\equiv 0$, although this assumption could be replaced by suitable hypotheses on $\boldsymbol{f}(x)$.

6.1. Perturbations Vanishing at Infinity

Case 1, $\boldsymbol{w}\in PR$. We consider a family of solutions $\boldsymbol{w}(x;\lambda)$ of $(2)_0$ in $\mathscr{E}$, corresponding to data $\boldsymbol{w}^*=\boldsymbol{w}_0+\lambda\,\boldsymbol{u}^*$ on Σ, with $\boldsymbol{u}^*$ prescribed, and we suppose $|\boldsymbol{w}(x;\lambda)-\boldsymbol{w}_0|<\lambda C(\lambda)\,|x|^{-\alpha}$ at infinity for some $\alpha>\frac{1}{2}$, for each λ. $C(\lambda)$ is not supposed uniform in λ. If λ is sufficiently small, a solution $\boldsymbol{w}(x;\lambda)$ can be obtained as an expansion in powers of λ (§ 4), and for small λ this is the only solution in PR (Theorem 5.10). Thus, uniformly in $\mathscr{E}$, there holds

$$\boldsymbol{w}(x;\lambda)-\boldsymbol{w}_0=\lambda\left[\boldsymbol{U}(x)+\sum_{1}^{\infty}\boldsymbol{u}_j(x)\,\lambda^j\right] \tag{60}$$

[2] The former perturbation ($u_0=0$) is similar to the "outer expansion" of Kaplun, or of Proudman & Pearson (see, *e.g.*, [*19*, *20*]). If $u_0\neq 0$, one obtains a perturbation analogous to their "inner expansion". See, however, Footnote 7 in [*10*].

where $U(x)$ is the solution of the linear system $(4)_0$ which assumes data u^* on Σ and vanishes at infinity. This relation may be differentiated term by term as many times as the smoothness of the data u^* will permit. Note that in (60) the functions $U(x)$, $u_j(x)$ do not depend on λ. This shows, in particular, that $C(\lambda)$ can be chosen uniformly for all λ sufficiently small.

As a particular consequence we obtain:

Theorem 6.1. *Let $w(x;\lambda)$ be a solution of $(2)_0$ in PR for each λ and set $u(x;\lambda)=\lambda^{-1}[w(x;\lambda)-w_0]$. Suppose $u(x;\lambda)=\begin{Bmatrix} u^* \text{ on } \Sigma \\ 0 \text{ at infinity}\end{Bmatrix}$, and let $U(x)$ be the solution of $(4)_0$ in $\mathscr{E}$ with the same data. Then if $w_0 \neq 0$, there holds uniformly in $\mathscr{E}$ as $\lambda \to 0$,*

$$(61) \qquad |u(x;\lambda)-U(x)| < C\lambda\tau(x)$$

where $\tau(x)$ is bounded in $\mathscr{E}$ and satisfies (50) with $\bar\beta=2$ and $\bar\alpha=1$. If $w_0=0$, then

$$(62) \qquad |u(x;\lambda)-U(x)| < C\lambda|x|^{-1}$$

in $\mathscr{E}$.

That is, any solution $w(x)$ with data on Σ sufficiently close to w_0 can be approximated in $\mathscr{E}$ by the solution of the linearized equations, to the indicated order.

The proof of (61) may be obtained from (43), noting that $\nabla G(x,y;w_0)=\nabla E+\nabla A$. Thus, $\int_{\mathscr{E}} u\cdot u\cdot\nabla G\,dy$ splits into two terms, the first of which can be estimated by Theorem 5.1. As to the second, we observe that $A(x,y;w_0)$ is a solution in $\mathscr{E}$ of the adjoint of $(4)_0$, and, if $w_0\neq 0$, assumes boundary data of the form $B(x)\tau(x)$, where $B(x)$ is bounded with all its tangential derivatives on Σ. The methods of § 2.6 then yield estimates for $A(x,y;w_0)$ as function of y, from which the result follows.

The derivatives admit a corresponding estimate when $w_0\neq 0$, with $\bar\beta=3$ and any $\bar\alpha<\frac{3}{2}$. For the pressure one obtains $\bar\beta=2$ and any $\bar\alpha<2$. If $w_0=0$, then for the derivatives and pressure, $|x|^{-1}$ should be replaced by $|x|^{-2}\log|x|$, $|x|^{-2}$, respectively.

Case 2, Perturbation of Class D. A family of solutions $u(x;\lambda)=\lambda^{-1}[w(x;\lambda)-w_0]$ of $(3)_0$ is considered, such that $u(x;\lambda)=u^*$ on Σ, $u(x;\lambda)\to 0$ at infinity, and $\|u\|_{\mathscr{E}}<C<\infty$, uniformly for λ sufficiently small[3]. Again let $U(x)$ be the solution of the linearized system $(4)_0$ with the same data. Then

$$(63) \qquad u(x;\lambda)-U(x)=\lambda\int_{\mathscr{E}} G(x,y;w_0)\cdot u(y;\lambda)\cdot\nabla u(y;\lambda)\,dy.$$

Now $G(x,y;w_0)=E(x-y;w_0)+A(x,y;w_0)$. Because of (29), there holds in particular $|E(x-y;w_0)|<C|x-y|^{-1}$. The estimates of § 2.6 show that for large x, $|A(x,y;w_0)|<C|x|^{-1}|y|^{-1}$. We may therefore write

$$\Big(\int_{\mathscr{E}} G\cdot u\cdot\nabla u\,dy\Big)^2 \leqq \int_{\mathscr{E}}|u|^2(|x-y|^{-2}+|y|^{-2})\,dy\int_{\mathscr{E}}|\nabla u|^2 dy \leqq C$$

[3] The existence of such solutions can be proved by a procedure due to LERAY [*2*]; *cf.* the remarks in [*13*, p. 237].

for some uniform constant C, by Schwarz' inequality and Lemma 2.2. Thus, setting $\boldsymbol{\eta}(x)=\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)$,

$$|\boldsymbol{\eta}(x)|=|\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)|<C\lambda \tag{64}$$

as $\lambda\to 0$, for all x bounded from Σ. The representation (*cf.* § 2.3)

$$\boldsymbol{\eta}(x)=\lambda\int \boldsymbol{E}^{\gamma}\cdot\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,dy+\int \boldsymbol{H}^{\gamma}\cdot\boldsymbol{\eta}\,dy \tag{65}$$

then yields a corresponding estimate for the derivatives of $\boldsymbol{\eta}$ up to second order. Thus, in particular, a bound of the form (64) holds for $\boldsymbol{\eta}(x)$ and its derivatives on the surface of a sphere Σ_0 containing Σ and bounding, with Σ, an annular region $\mathscr{A}_0$.

Let $\boldsymbol{\xi}(x)$ be a solenoidal field in $\mathscr{A}_0$ such that $\boldsymbol{\xi}(x)=\boldsymbol{u}^*$ on Σ and $\boldsymbol{\xi}(x)\equiv 0$ outside Σ_0. Let $\boldsymbol{v}(x;\lambda)=\boldsymbol{u}(x;\lambda)-\boldsymbol{\xi}(x)$. Let Σ_1 be a concentric sphere containing Σ_0, let $\mathscr{A}_1$ be the region between Σ and Σ_1. We may represent $\boldsymbol{v}(x;\lambda)$ in $\mathscr{A}_1$ by the Green's tensor $\boldsymbol{G}_0(x,y)$ of (6):

$$v(x;\lambda)=\{\boldsymbol{G}_0,\boldsymbol{w}_0,v\}+\lambda\{\boldsymbol{G}_0,\boldsymbol{u},\boldsymbol{u}\}+\{\boldsymbol{G}_0,\boldsymbol{w}_0,\boldsymbol{\xi}\}+[\boldsymbol{G}_0,\boldsymbol{\xi}]-(\boldsymbol{G}_0,\boldsymbol{f})+ \\ +\oint_{\Sigma_1}\boldsymbol{u}\cdot T\,\boldsymbol{G}_0\,d\boldsymbol{\sigma}.$$

Because of (64), $\boldsymbol{u}(x;\lambda)$ is bounded on Σ_1. By assumption, $\|\boldsymbol{u}(x;\lambda)\|<C$, hence $\|\boldsymbol{v}(x;\lambda)\|<C$. Schwarz' inequality and the Odqvist estimates $|\boldsymbol{G}_0(x,y)|<C|x-y|^{-1}$, $|\nabla\boldsymbol{G}_0|<C|x-y|^{-2}$ then show that $|\boldsymbol{v}(x;\lambda)|<C$ in $\mathscr{A}_0$; hence also $|\boldsymbol{u}(x;\lambda)|<C$ in $\mathscr{A}_0$.

Now let $\boldsymbol{\zeta}(x;\lambda)$ be a solenoidal field in $\mathscr{A}_0$, equal to $\boldsymbol{\eta}$ on Σ_0 and vanishing outside a neighborhood of Σ_0 which does not meet Σ. Because of what has been proved, $\boldsymbol{\zeta}(x)$ can be constructed so that

$$\|\boldsymbol{\zeta}\|_{\mathscr{A}_0}<C\lambda.$$

Set $\boldsymbol{\gamma}=\boldsymbol{\eta}-\boldsymbol{\zeta}$. Since $\boldsymbol{\gamma}=0$ on Σ, Σ_0, we have in $\mathscr{A}_0$,

$$\begin{aligned}\|\boldsymbol{\gamma}\|^2&=-\lambda\{\boldsymbol{\gamma},\boldsymbol{u},\boldsymbol{u}\}-[\boldsymbol{\gamma},\boldsymbol{\zeta}]-\{\boldsymbol{\gamma},\boldsymbol{w}_0,\boldsymbol{\zeta}\}\\&=\lambda\{\boldsymbol{u},\boldsymbol{u},\boldsymbol{\gamma}\}-[\boldsymbol{\gamma},\boldsymbol{\zeta}]-\{\boldsymbol{\gamma},\boldsymbol{w}_0,\boldsymbol{\zeta}\}\\&\leqq \tfrac{1}{2}\|\boldsymbol{\gamma}\|^2+C\lambda^2+C\|\boldsymbol{\zeta}\|^2\end{aligned}$$

for a suitable C, since $\boldsymbol{u}(x;\lambda)$ is bounded in $\mathscr{A}_0$. Hence $\|\boldsymbol{\gamma}\|_{\mathscr{A}_0}<C\lambda$, and also

$$\|\boldsymbol{\eta}\|_{\mathscr{A}_0}<C\lambda. \tag{66}$$

Finally, we represent $\boldsymbol{\eta}$ in $\mathscr{A}_0$ by the fundamental tensor $\boldsymbol{G}_0$, obtaining

$$\boldsymbol{\eta}(x;\lambda)=\{\boldsymbol{G}_0,\boldsymbol{w}_0,\boldsymbol{\eta}\}+\lambda\{\boldsymbol{G}_0,\boldsymbol{u},\boldsymbol{u}\}+\oint_{\Sigma_0}\boldsymbol{\eta}\cdot T\,\boldsymbol{G}_0\,d\boldsymbol{\sigma}.$$

Since on Σ_0, $|\boldsymbol{\eta}|<C\lambda$ by (64), (66) implies that (64) holds up to Σ for a suitable C. Similarly, one may estimate the derivatives and the pressure terms. We have proved

Theorem 6.2. *Let $\boldsymbol{w}(x;\lambda)$ be a family of solutions of $(2)_0$, such that if $\boldsymbol{u}(x;\lambda)=\lambda^{-1}[\boldsymbol{w}(x;\lambda)-\boldsymbol{w}_0]$, then $\boldsymbol{u}(x;\lambda)\to 0$ at infinity and $\boldsymbol{u}(x;\lambda)=\boldsymbol{u}^*$ on Σ. Suppose further that $\|\boldsymbol{u}(x;\lambda)\|<C$, uniformly for small λ. Then there holds uniformly in*

$\mathscr{E}$, as $\lambda\to 0$,

$$|u(x;\lambda)-U(x)|<C\lambda \tag{67}$$

for the solution $U(x)$ of $(4)_0$ in $\mathscr{E}$ with corresponding data.

Note that in this case the method does not lead to improved estimates when $w_0 \neq 0$.

For the derivatives one obtains a stronger estimate:

Theorem 6.2a. *Under the hypotheses of Theorem 6.2, there holds for x bounded from Σ,*

$$|\nabla u(x;\lambda)-\nabla U(x)|<C[\lambda^2+\lambda\,\tau_1(x)]$$

where $\tau_1(x)$ satisfies (50) with $\bar\beta=3$ and any $\bar\alpha<\frac{3}{2}$ when $w_0\neq 0$; if $w_0=0$, the same inequality holds if $\tau_1(x)$ is replaced by $|x|^{-2}\log|x|$. Uniformly up to Σ, one has

$$|\nabla u(x;\lambda)-\nabla U(x)|<C\lambda.$$

Proof. From

$$\eta(x)=\lambda\{G,u,u\}$$

we find

$$\begin{aligned}\eta(x)&=\lambda\{G,\eta,\eta\}+\lambda\{G,\eta,U\}+\lambda\{G,U,\eta\}+\lambda\{G,U,U\}\\&=\lambda(\eta_1+\eta_2+\eta_3+\eta_4).\end{aligned}$$

Let V be a unit ball centered at x, S its surface. Then for any derivative $\eta'(x)$,

$$\begin{aligned}\eta_1'&=\{G',\eta,\eta\}_V+\{G',\eta,\eta\}_{\mathscr{E}-V},\\|\eta_1'|&\leqq C\lambda^2+\lambda\|G'\|_{\mathscr{E}-V}\|\eta\|_{\mathscr{E}-V}\\&\leqq C\lambda\end{aligned}$$

for small λ, by Theorem 6.2, Schwarz' inequality, (65), and the hypothesis $\|u\|<C$. η_2' is estimated similarly. Next, integrating by parts,

$$\begin{aligned}\eta_3'&=\{G',U,\eta\}_V+\oint_S(G'\cdot U)\,\eta\,d\sigma-\{\eta,U,G'\}_{\mathscr{E}-V},\\|\eta_3'|&\leqq\lambda|x|^{-1}+\lambda|x|^{-1}+\lambda[U,G'']_{\mathscr{E}-V}\\&\leqq C\lambda\end{aligned}$$

for small λ. Finally, we may write

$$\eta_4=-\{U,U,E\}-\{U,U,A\}.$$

To the first term, we may apply the method of proof of Theorem 5.3 or Theorem 5.5. In the second, we observe that $A_x(x,y;w_0)$ is a solution in $\mathscr{E}$ of the adjoint of $(4)_0$, and, if $w_0\neq 0$, assumes boundary data of the form $B(x)\,\tau_1(x)$, where $\tau_1(x)$ satisfies (50) with $\bar\beta=3$, any $\bar\alpha<\frac{3}{2}$, and $B(x)$ is bounded with all its tangential derivatives on Σ. The result then follows from the estimates of § 2.6. If $w_0=0$, one uses the estimate $|A_x|<C|x|^{-2}$ on Σ. This establishes the theorem when x is bounded away from Σ. For x near Σ, the result is obtained by a procedure analogous to the discussion of this case for Theorem 6.2.

6.2. A Lemma on Green's Tensor

We shall need information on the variation of the Green's tensor, as function of the parameter $\boldsymbol{w}_0$.

Lemma 6.3. *Uniformly for* $y \in \Sigma$, *there holds as* $x \to \infty$, *if* $\boldsymbol{w}_0 \neq 0$, $\overline{\boldsymbol{w}}_0 = (1-\lambda)\boldsymbol{w}_0$,

(68) $$|T\boldsymbol{G}(x,y;\boldsymbol{w}_0) - T\boldsymbol{G}(x,y;\overline{\boldsymbol{w}}_0)| < C\lambda\tau(x)$$

where $\tau(x)$ *satisfies* (50) *with* $\alpha = 1$, $\beta = 2$. *Also,*

(69) $$|T\nabla_x\boldsymbol{G}(x,y;\boldsymbol{w}_0) - T\nabla_x\boldsymbol{G}(x,y;\overline{\boldsymbol{w}}_0)| < C\lambda\tau_1(x)$$

where $\tau_1(x)$ *satisfies* (50) *with any* $\alpha < \frac{3}{2}$ *and* $\beta = 3$.

Proof. We have $\boldsymbol{G}(x,y;\boldsymbol{w}_0) = \boldsymbol{E}(x-y;\boldsymbol{w}_0) + \boldsymbol{A}(x,y;\boldsymbol{w}_0)$. Using the relation $\boldsymbol{E}(x;t\boldsymbol{w}_0) = t\boldsymbol{E}(tx;\boldsymbol{w}_0)$ with $t = 1-\lambda$ leads to

(70) $$\boldsymbol{E}(x;t\boldsymbol{w}_0) - \boldsymbol{E}(x;\boldsymbol{w}_0) = \boldsymbol{E}(tx;\boldsymbol{w}_0) - \boldsymbol{E}(x;\boldsymbol{w}_0) - \lambda\boldsymbol{E}(tx;\boldsymbol{w}_0),$$

so that, applying the mean value theorem to the first two terms on the right,

(71) $$|\boldsymbol{E}(x;t\boldsymbol{w}_0) - \boldsymbol{E}(x;\boldsymbol{w}_0)| < C\lambda\tau(x)$$

where $\tau(x)$ has the properties indicated above.

Differentiating (70) in x,

(72) $$\begin{aligned}&\nabla\boldsymbol{E}(x;t\boldsymbol{w}_0) - \nabla\boldsymbol{E}(x;\boldsymbol{w}_0)\\ &= \nabla\boldsymbol{E}(tx;\boldsymbol{w}_0) - \nabla\boldsymbol{E}(x;\boldsymbol{w}_0) - \lambda(2-\lambda)\nabla\boldsymbol{E}(tx;\boldsymbol{w}_0).\end{aligned}$$

Again using the mean value theorem,

(73) $$|\nabla\boldsymbol{E}(x;t\boldsymbol{w}_0) - \nabla\boldsymbol{E}(x;\boldsymbol{w}_0)| < C\lambda\tau_1(x)$$

where τ_1 satisfies (50) with $\alpha = \frac{3}{2}$, $\beta = 3$. The pressure term, $\boldsymbol{e}(x-y)$ does not depend on $\boldsymbol{w}_0$. Hence, (68) is verified for the singular part of $\boldsymbol{G}(x,y;\boldsymbol{w}_0)$.

The regular part $\boldsymbol{A}(x,y;\boldsymbol{w}_0)$ satisfies the system

(74) $$\begin{aligned}\Delta\boldsymbol{u} + \boldsymbol{w}_0\cdot\nabla\boldsymbol{u} - \nabla p &= 0,\\ \nabla\cdot\boldsymbol{u} &= 0\end{aligned}$$

as function of y, and $\boldsymbol{A} = \boldsymbol{A}^* = \boldsymbol{E}(x,y;\boldsymbol{w}_0)$ for y on Σ. We may write $\boldsymbol{A}^* = \boldsymbol{B}^*\tau(x)$, where $\boldsymbol{B}^*$ is bounded and has bounded derivatives on Σ, uniformly as $x \to \infty$.

The field $\overline{\boldsymbol{A}} = \boldsymbol{A}(x,y;t\boldsymbol{w}_0)$ satisfies

(75) $$\begin{aligned}\Delta\boldsymbol{u} + \boldsymbol{w}_0\cdot\nabla\boldsymbol{u} - \nabla p &= -\lambda\boldsymbol{w}_0\cdot\nabla\boldsymbol{u},\\ \nabla\cdot\boldsymbol{u} &= 0.\end{aligned}$$

We may write, because of the linearity of (75),

$$\overline{\boldsymbol{A}} = \overline{\boldsymbol{A}}_1 + \overline{\boldsymbol{A}}_2$$

where $\overline{\boldsymbol{A}}_1 = \boldsymbol{A}^*$ on Σ, $\overline{\boldsymbol{A}}_2 = \boldsymbol{E}(x,y;\boldsymbol{w}_0) - \boldsymbol{E}(x,y;t\boldsymbol{w}_0)$ on Σ, and $\overline{\boldsymbol{A}}_1$, $\overline{\boldsymbol{A}}_2$ each satisfy (75) in y.

We may write $\boldsymbol{A} - \overline{\boldsymbol{A}} = \boldsymbol{\eta} - \overline{\boldsymbol{A}}_2$, where $\boldsymbol{\eta} = \boldsymbol{A} - \overline{\boldsymbol{A}}_1$. $\boldsymbol{\eta}$ and $\overline{\boldsymbol{A}}_2$ will be estimated in turn.

Using the estimates of Theorem 2.8 and its Corollary on the solutions of the linear systems, we find

$$\bar{A}_1(x, y; \lambda) = B_1(x, y; \lambda)\,\tau(x)$$

where $B_1(x, y; \lambda)$ satisfies (75) in y, is bounded together with its derivatives up to Σ, and admits estimates of the form (31) in y uniformly in x for all sufficiently small λ.

The field $-\eta(x, y)$ satisfies

$$\Delta\eta - w_0 \cdot \nabla\eta - \nabla p = \lambda\, w_0 \cdot \nabla \bar{A}_1, \qquad \nabla \cdot \eta = 0 \tag{76}$$

in $\mathscr{E}$, and $\eta = 0$ for $y \in \Sigma$. Hence

$$\eta(x, y) = -\lambda\{G(x, y; t\,w_0), w_0, \bar{A}\} = -\lambda\,\tau(x)\{G(x, y; t\,w_0), w_0, B_1\}.$$

For large x, $G(x, y; t\,w_0)$ is in $L_p(\mathscr{E})$ whenever $2 < p < 3$, uniformly for $|t-1|$ sufficiently small. Also, $|\nabla B|$ is in $L_q(\mathscr{E})$ whenever $q > \frac{4}{3}$. Choosing p and q in these intervals, such that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$|\eta(x, y)| \leqq \lambda\,\tau(x)\,|w_0| \Big(\int_{\mathscr{E}} |G|^p\Big)^{\frac{1}{p}} \Big(\int_{\mathscr{E}} |\nabla B|^q\Big)^{\frac{1}{q}} < C\,\lambda\,\tau(x) \tag{77}$$

uniformly for small λ and large x.

Suppose $y \in \Sigma_0$, a spherical surface containing Σ. We represent $\eta(x, y)$ by the truncated tensor (*cf.* § 2.3), obtaining

$$\eta(x, y) = -\lambda\{E^\gamma(y, z; t\,w_0), w_0, \bar{A}_1\}_{V^\gamma} + (H^\gamma, \eta)_{V^\gamma}. \tag{78}$$

From (78) we conclude

$$|\nabla_y \eta(x, y)| < C\,\lambda\,\tau(x)$$

for $y \in \Sigma$, and corresponding estimates for all higher derivatives of η.

Hence we may construct, for each x and t, a solenoidal field $\zeta(y)$ in the region $\mathscr{A}_0$ between Σ and Σ_0, vanishing outside a given neighborhood of Σ_0, such that $\zeta = \eta$ on Σ_0, and such that

$$\|\zeta\|_{\mathscr{A}_0} < C\,\lambda\,\tau(x).$$

Set $v = \eta - \zeta$. We find, since $v = 0$ on Σ, Σ_0,

$$\|v\|^2 = \lambda\{v, w_0, \bar{A}_1\} - [v, \zeta] - \{v, w_0, \zeta\}$$

from which we conclude

$$\|v\|_{\mathscr{A}_0} < C\,\lambda\,\tau(x)$$

and hence

$$\|\eta\|_{\mathscr{A}_0} < C\,\lambda\,\tau(x).$$

Now represent η in $\mathscr{A}_0$ by the Odqvist tensor $G_0(x, y)$ for (6):

$$\eta(x, y) = -\{G_0, w_0, \eta\} - \lambda\{G_0, w_0, \bar{A}_1\} - \oint_{\Sigma_0} \eta \cdot T\,G_0\,d\sigma.$$

The above estimates of $\boldsymbol{\eta}$ on Σ_0, of $\|\boldsymbol{\eta}\|_{\mathscr{A}_0}$, and of $\bar{\boldsymbol{A}}_1$, and the Odqvist estimates for $\boldsymbol{G}_0(x, y)$ in $\mathscr{A}_0$, yield the results

$$|\boldsymbol{\eta}(x, y)|, |T\boldsymbol{\eta}(x, y)| < C\lambda\tau(x) \tag{79}$$

up to Σ.

It remains to estimate $\bar{\boldsymbol{A}}_2$. On Σ, $|\bar{\boldsymbol{A}}_2| < C\lambda\tau(x)$ with a corresponding estimate for its tangential derivatives of all orders. Hence we may write

$$\bar{\boldsymbol{A}}_2(x, y; \lambda) = \lambda\tau(x)\boldsymbol{B}_2(x, y; \lambda)$$

where $\boldsymbol{B}_2(x, y; \lambda)$ has the same properties as were shown for $\boldsymbol{B}_1(x, y; \lambda)$. Thus, $|T\boldsymbol{B}_2|$ is bounded up to Σ, and therefore

$$|T\bar{\boldsymbol{A}}_2| < C\lambda\tau(x)$$

up to Σ. This completes the proof of the first part of the lemma. The remaining estimate (69) is obtained by the same procedure, using the appropriate estimates for the derivatives of the fundamental tensor when $y \in \Sigma$ and $|x|$ is large.

Lemma 6.4. *Suppose* $\boldsymbol{w}_0 = 0$, $|\bar{\boldsymbol{w}}_0| < \lambda$. *Then there holds for* $y \in \Sigma$, *as* $x \to \infty$,

$$|T\boldsymbol{G}(x, y; 0) - T\boldsymbol{G}(x, y; \bar{\boldsymbol{w}}_0)| < C\,\tau_0(x; \lambda)$$

where

$$\tau_0(x; \lambda) = \min\left(\lambda \log \lambda^{-1}, |x|^{-1} \log |x|\right).$$

Also,

$$|T\nabla_x \boldsymbol{G}(x, y; 0) - T\nabla_x \boldsymbol{G}(x, y; \bar{\boldsymbol{w}}_0)| < C\,\tau_0'(x, y)$$

where $\tau_0' = \sqrt{\lambda}\,|x|^{-1}$.

Proof. Observe first that the change of $\boldsymbol{E}$ can not be estimated as above, and it is necessary to obtain this estimate directly from the defining relations (16). A formal calculation yields

$$\begin{aligned} &|\boldsymbol{E}(x; \boldsymbol{w}_0) - \boldsymbol{E}(x; 0)| < C \min\left(\lambda, |x|^{-1}\right), \\ &|\nabla\boldsymbol{E}(x; \boldsymbol{w}_0) - \nabla\boldsymbol{E}(x; 0)| < C\lambda^{\frac{1}{2}}|x|^{-1}. \end{aligned} \tag{80}$$

For the regular part $\boldsymbol{A}(x, y)$ we write as before $\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}_1 + \bar{\boldsymbol{A}}_2$, with $\bar{\boldsymbol{A}}_1 = \boldsymbol{A}$ on Σ, and set $\boldsymbol{\eta} = \boldsymbol{A} - \bar{\boldsymbol{A}}_1$. $\boldsymbol{\eta}(x, y)$ satisfies

$$\begin{aligned} \Delta\boldsymbol{\eta} - \nabla p &= -\bar{\boldsymbol{w}}_0 \cdot \nabla\bar{\boldsymbol{A}}_1, \\ \nabla \cdot \boldsymbol{\eta} &= 0 \end{aligned} \tag{81}$$

as function of y, and $\boldsymbol{\eta} = 0$ on Σ. We represent $\boldsymbol{\eta}$ over the region $\mathscr{E}_R$ between Σ and a sphere of large radius Σ_R, using the Green's tensor $\boldsymbol{G}(x, y)$ for (6) in $\mathscr{E}$:

$$\boldsymbol{\eta}(x, y) = -\int_{\mathscr{E}_R} \boldsymbol{G}(y, z) \cdot \bar{\boldsymbol{w}}_0 \cdot \nabla\bar{\boldsymbol{A}}_1(z)\, dz + \oint_{\Sigma_R} (\boldsymbol{\eta} \cdot T\boldsymbol{G} - \boldsymbol{G} \cdot T\boldsymbol{\eta})\, d\boldsymbol{\sigma}. \tag{82}$$

Using (29) and the results of § 2.5, we find $\bar{\boldsymbol{A}}_1 = \boldsymbol{B}_1(x, y)\,|x|^{-1}$ where $\boldsymbol{B}_1(x, y)$ satisfies (81) and remains bounded as $x \to \infty$, $\lambda \to 0$. Consequently, on a surface Σ_r, there holds

$$\oint_{\Sigma_R} |\nabla\boldsymbol{B}_1|\, d\sigma < C \tag{83}$$

uniformly for large x, as $\lambda \to 0$.

Consider (82) for a point y on a fixed spherical surface Σ_0 containing Σ but inside Σ_R. One has

$$|\boldsymbol{G}(y,z)| < C[r_z^{-1} r_y^{-1} + r_{yz}^{-1}];$$

hence using (83),

$$\left|\int_{\mathscr{E}_R} \boldsymbol{G}(y,z) \cdot \overline{\boldsymbol{w}}_0 \cdot \nabla \boldsymbol{B}(z)\, dz\right| < C\lambda \log R$$

for $y \in \Sigma_0$.

On Σ_R, $|\boldsymbol{\eta}| < CR^{-1}$, $|\boldsymbol{G}| < CR^{-1}$, $|T\boldsymbol{G}| < CR^{-2}$, and $\oint_{\Sigma_R} |T\boldsymbol{\eta}|\, d\sigma < C$, if $y \in \Sigma_0$. Hence

$$\left|\oint_{\Sigma_R} (\boldsymbol{\eta} \cdot T\boldsymbol{G} - \boldsymbol{G} \cdot T\boldsymbol{\eta})\, d\boldsymbol{\sigma}\right| < CR^{-1}$$

for large R and $y \in \Sigma_0$. Thus

$$|\boldsymbol{\eta}(x,y)| < C[R^{-1} + \lambda |x|^{-1} \log R].$$

The choice $R = \lambda^{-1}|x|$ yields the estimate

$$|\boldsymbol{\eta}(x,y)| < C\lambda |x|^{-1} \log \lambda^{-1}|x|$$

for large x and small λ, and $y \in \Sigma_0$.

Now we may represent $\boldsymbol{\eta}(x,y)$ in $\mathscr{A}_0$ by means of the Green's tensor $\boldsymbol{G}_0(x,y)$ for $\mathscr{A}_0$. The procedure is identical to the one used in proving the previous lemma. We find

$$|T\boldsymbol{\eta}| < C\lambda |x|^{-1} \log \lambda^{-1}|x|$$

for large x, small λ, and $y \in \Sigma$.

To estimate $\overline{\boldsymbol{A}}_2$, observe that on Σ we have $|\overline{\boldsymbol{A}}_2| < C \min(\lambda, |x|^{-1})$ with stronger estimates for its tangential derivatives, and $\overline{\boldsymbol{A}}_2$ satisfies

$$\Delta \overline{\boldsymbol{A}}_2 - \nabla p = \overline{\boldsymbol{w}}_0 \cdot \nabla \overline{\boldsymbol{A}}_2,$$

$$\nabla \cdot \overline{\boldsymbol{A}}_2 = 0$$

in $\mathscr{E}$. The results of § 2.5 show that

$$|T\overline{\boldsymbol{A}}_2| < C \min(\lambda, |x|^{-1})$$

on Σ.

Collecting the above estimates, one obtains easily the first assertion of the lemma. The second assertion follows from the same procedure, using the estimate for the variation of $\nabla \boldsymbol{E}$. In this case, the volume integral is convergent.

6.3. General Perturbation of Class PR

It will be assumed throughout that the direction of the velocity field is unvaried at infinity, that is, $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ are parallel if $\boldsymbol{w}_0 \neq 0$. If $\boldsymbol{w}_0 = 0$, no restriction is made on $\boldsymbol{u}_0$.

Theorem 6.5. *Let $\boldsymbol{w}(x;\lambda)$ be a family of solutions of $(2)_0$ of class* PR, *such that if $\boldsymbol{u}(x;\lambda) = \lambda^{-1}[\boldsymbol{w}(x;\lambda) - \boldsymbol{w}_0]$, then $\boldsymbol{u}(x;\lambda) = \boldsymbol{u}^*$ on Σ and $\boldsymbol{u}(x;\lambda) \to \boldsymbol{u}_0$ as $x \to \infty$. Let $\boldsymbol{U}(x)$ be the solution of the linearized system $(4)_0$ with corresponding data. Then if $\boldsymbol{w}_0 \neq 0$ there holds uniformly in $\mathscr{E}$, as $\lambda \to 0$,*

$$|\boldsymbol{u}(x;\lambda) - \boldsymbol{U}(x)| < C\lambda\, \tau(x)$$

where $\tau(x)$ *satisfies* (50) *with* $\alpha=1$, $\beta=2$. *If* $\boldsymbol{w}_0=0$, *then*

$$|\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)|<C\,\tau_0(x;\lambda)$$

where $\tau_0(x;\lambda)=\min(\lambda\log\lambda^{-1}, |x|^{-1}\log|x|)$.

Proof. For small λ, solutions of the indicated class exist (§ 4), and they are unique (Theorem 5.10). Therefore the solutions $\boldsymbol{w}(x;\lambda)$ can be represented by an expansion (44) based on the equations

$$\begin{aligned}\Delta\boldsymbol{u}-(\boldsymbol{w}_0+\lambda\boldsymbol{u}_0)\cdot\nabla\boldsymbol{u}-\nabla p&=\lambda(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\nabla\boldsymbol{u},\\ \nabla\cdot\boldsymbol{w}&=0\end{aligned}\tag{84}$$

for each λ, from which one sees from the method of construction that the perturbations are uniformly in PR, *i.e.*, $|\boldsymbol{u}-\boldsymbol{u}_0|<C|x|^{-1}$, with C independent of λ if λ is small.

We have

$$\begin{aligned}\boldsymbol{u}(x;\lambda)-\boldsymbol{u}_0&=\lambda\{\overline{\boldsymbol{G}},\boldsymbol{u}-\boldsymbol{u}_0,\boldsymbol{u}-\boldsymbol{u}_0\}+\oint_{\Sigma}(\boldsymbol{u}^*-\boldsymbol{u}_0)\cdot T\overline{\boldsymbol{G}}\,d\boldsymbol{\sigma}\\ &=-\lambda\{\boldsymbol{u}-\boldsymbol{u}_0,\boldsymbol{u}-\boldsymbol{u}_0,\overline{\boldsymbol{G}}\}+\oint_{\Sigma}(\boldsymbol{u}^*-\boldsymbol{u}_0)\cdot T\overline{\boldsymbol{G}}\,d\boldsymbol{\sigma}\end{aligned}$$

where $\overline{\boldsymbol{G}}=\boldsymbol{G}(x,y;\boldsymbol{w}_0+\lambda\boldsymbol{u}_0)$, while

$$\boldsymbol{U}(x)-\boldsymbol{u}_0=\oint_{\Sigma}(\boldsymbol{u}^*-\boldsymbol{u}_0)\cdot T\boldsymbol{G}\,d\boldsymbol{\sigma}$$

with $\boldsymbol{G}=\boldsymbol{G}(x,y;\boldsymbol{w}_0)$. Thus

$$\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)=-\lambda\{\boldsymbol{u}-\boldsymbol{u}_0,\boldsymbol{u}-\boldsymbol{u}_0,\overline{\boldsymbol{G}}\}+\oint_{\Sigma}(\boldsymbol{u}^*-\boldsymbol{u}_0)(T\overline{\boldsymbol{G}}-T\boldsymbol{G})\,d\boldsymbol{\sigma}.\tag{85}$$

Suppose first $\boldsymbol{w}_0\neq 0$. Since $\boldsymbol{u}-\boldsymbol{u}_0\in PR$, it satisfies (50) with $\alpha=1$, $\beta=2$. Therefore, as in the proof of Theorem 6.1, the volume integral on the right satisfies (50) with any $\bar\alpha=1$, $\bar\beta=2$. To the surface integral we apply Lemma 6.3. This yields the asserted theorem if $|x|$ is large. If x is near Σ one may proceed in a way analogous to the corresponding estimate near Σ in the proof of Theorem 6.2, and there should be no need to reiterate the details.

If $\boldsymbol{w}_0=0$, the method is the same, only Lemma 6.3 must be replaced by Lemma 6.4.

Theorem 6.5a. *Under the hypotheses of Theorem* 6.5, *if* $\boldsymbol{w}_0\neq 0$, *there holds*

$$|\nabla\boldsymbol{u}(x;\lambda)-\nabla\boldsymbol{U}(x)|<C\lambda\tau_1(x)\tag{86}$$

where $\tau_1(x)$ *is as in Lemma* 6.3. *If* $\boldsymbol{w}_0=0$, *then*

$$|\nabla\boldsymbol{u}(x;\lambda)-\nabla\boldsymbol{U}(x)|<C\sqrt{\lambda}\,|x|^{-1}.\tag{87}$$

Proof. The volume integral in (85) can be differentiated and estimated as in the proof of Theorem 6.2a. To the surface integral we apply the second part of Lemma 6.3 or 6.4.

6.4. General Perturbation of Class D

Theorem 6.6. *Let* $\boldsymbol{w}(x;\lambda)$ *be a family of solutions of* $(2)_0$, *such that if* $\boldsymbol{u}(x;\lambda)=\lambda^{-1}[\boldsymbol{w}(x;\lambda)-\boldsymbol{w}_0]$, *then* $\boldsymbol{u}(x;\lambda)\to\boldsymbol{u}_0$ *at infinity and* $\boldsymbol{u}(x;\lambda)=\boldsymbol{u}^*$ *on* Σ. *Suppose further that* $\|\boldsymbol{u}(x;\lambda)\|<C$, *uniformly for small* λ. *Then if* $\boldsymbol{w}_0\neq 0$, *there holds*

uniformly in $\mathscr{E}$, as $\lambda \to 0$,

$$|\boldsymbol{u}(x;\lambda) - \boldsymbol{U}(x)| < C\lambda$$

for the solution $\boldsymbol{U}(x)$ of $(4)_0$ in $\mathscr{E}$ with corresponding data. If $\boldsymbol{w}_0 = 0$, one has

$$(88) \qquad |\boldsymbol{u}(x;\lambda) - \boldsymbol{U}(x)| < C\big(\tau_0(x;\lambda) + \lambda\big)$$

where $\tau_0(x;\lambda) = \min\big(\lambda \log \lambda^{-1}, |x|^{-1} \log |x|\big)$.

Proof. Suppose first $\boldsymbol{w}_0 \neq 0$. The functions $\boldsymbol{u}(x;\lambda) - \boldsymbol{U}(x) = \boldsymbol{\eta}(x;\lambda)$ admit the representation

$$\boldsymbol{\eta}(x;\lambda) = -\lambda\{\overline{\boldsymbol{G}}, \boldsymbol{u} - \boldsymbol{u}_0, \boldsymbol{u} - \boldsymbol{u}_0\} + \oint_{\Sigma} (\boldsymbol{u}^* - \boldsymbol{u}_0)(T\overline{\boldsymbol{G}} - T\boldsymbol{G})\, d\boldsymbol{\sigma}$$

with $\overline{\boldsymbol{G}} = \boldsymbol{G}(x, y; \boldsymbol{w}_0 + \lambda \boldsymbol{u}_0)$, $\boldsymbol{G} = \boldsymbol{G}(x, y; \boldsymbol{w}_0)$. Thus, applying the Schwarz inequality and Lemma 2.2 to the volume integral, and Lemma 6.3 to the surface integral, we obtain $|\boldsymbol{\eta}| < C\lambda$ uniformly over $\mathscr{E}$, as $\lambda \to 0$.

If $\boldsymbol{w}_0 = 0$, the same reasoning, with Lemma 6.3 replaced by Lemma 6.4, yields $|\boldsymbol{\eta}| < C[\min(\lambda \log \lambda^{-1}, |x|^{-1} \log |x|) + \lambda]$ the stated result.

For the derivatives one may obtain an improved result. Again suppose first $\boldsymbol{w}_0 \neq 0$. We have, for a derivative $\boldsymbol{\eta}'$,

$$\boldsymbol{\eta}'(x;\lambda) = -\lambda\{\overline{\boldsymbol{G}}', \boldsymbol{u} - \boldsymbol{u}_0, \boldsymbol{u} - \boldsymbol{u}_0\} + \oint_{\Sigma} (\boldsymbol{u}^* - \boldsymbol{u}_0)(T\overline{\boldsymbol{G}}' - T\boldsymbol{G}')\, d\boldsymbol{\sigma}.$$

The volume integral is amenable to the method of proof of Theorem 6.2a, yielding again an estimate

$$|\{\overline{\boldsymbol{G}}', \boldsymbol{u} - \boldsymbol{u}_0, \boldsymbol{u} - \boldsymbol{u}_0\}| < C\big(\lambda^2 + \lambda\,\tau_1(x)\big)$$

where $\tau_1(x)$ is as in Theorem 6.2a. To the surface term we apply Lemma 6.3. In case $\boldsymbol{w}_0 = 0$, we repeat the reasoning, using the second part of Theorem 6.2a and Lemma 6.4.

These estimates hold if x is bounded from Σ. Estimates up to Σ follow from the method used in the proof of Theorem 6.2. Thus we have

Theorem 6.6a. *Under the hypotheses of Theorem 6.6, if $\boldsymbol{w}_0 \neq 0$, there holds*

$$(89) \qquad |\nabla \boldsymbol{u}(x;\lambda) - \nabla \boldsymbol{U}(x)| < C\lambda$$

uniformly in $\mathscr{E}$, and

$$(90) \qquad |\nabla \boldsymbol{u}(x;\lambda) - \nabla \boldsymbol{U}(x)| < C\big(\lambda^2 + \lambda\,\tau_1(x)\big)$$

for x bounded from Σ, where $\tau(x)$ satisfies (50) with $\alpha = 1$, $\beta = 2$. If $\boldsymbol{w}_0 = 0$, (89) and (90) are replaced by

$$(91) \qquad |\nabla \boldsymbol{u}(x;\lambda) - \nabla \boldsymbol{U}(x)| < C\lambda \log \lambda^{-1}$$

uniformly in $\mathscr{E}$ for small λ, and

$$(92) \qquad |\nabla \boldsymbol{u}(x;\lambda) - \nabla \boldsymbol{U}(x)| < C\big(\lambda^2 + \sqrt{\lambda}\,|x|^{-1}\big)$$

for x bounded from Σ.

The Table summarizes the results of this section. In this table the following notation is used:

$\boldsymbol{\eta}(x;\lambda)=\boldsymbol{u}(x;\lambda)-\boldsymbol{U}(x)$, where $\boldsymbol{u}(x;\lambda)$, $\boldsymbol{U}(x)$ are solutions of $(3)_0$, $(4)_0$, respectively, with the same data.

$\tau_0(x;\lambda)=\min\left(\lambda\log\lambda^{-1}, |x|^{-1}\log|x|\right)$.

$\tau(x)=$ function satisfying (50), with $\alpha=1$, $\beta=2$.

$\tau_1(x)=$ function satisfying (50), with $\beta=3$, any $\alpha<\frac{3}{2}$.

$\tau_2(x)=$ function satisfying (50), with $\alpha=2$, $\beta=4$.

Table. *Perturbation estimates for small Reynolds' number*

The symbols $\boldsymbol{\eta}$, τ_j are defined at the end of § 6.4.

Flow Class	Limiting Velocity	Limiting Perturbation	Uniform Estimate in $\mathscr{E}$	Estimate away from Σ
PR	$\boldsymbol{w}_0\neq 0$	$\boldsymbol{u}_0=0$	$\boldsymbol{\eta}(x;\lambda)=\sum_1^\infty u_j(x)\,\lambda^j$ $\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\tau(x)$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\tau(x)$	same
		$\boldsymbol{u}_0\neq 0$	$\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\tau(x)$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\tau_1(x)$	same
	$\boldsymbol{w}_0=0$	$\boldsymbol{u}_0=0$	$\boldsymbol{\eta}(x;\lambda)=\sum_1^\infty u_j(x)\,\lambda^j$ $\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\lvert x\rvert^{-1}$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\,\lvert x\rvert^{-2}\log\lvert x\rvert$	same
		$\boldsymbol{u}_0\neq 0$	$\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\tau_0(x;\lambda)$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\sqrt{\lambda}\,\lvert x\rvert^{-1}$	same
D	$\boldsymbol{w}_0\neq 0$	$\boldsymbol{u}_0=0$	$\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$	$\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C[\lambda^2+\lambda\,\tau_1(x)]$
		$\boldsymbol{u}_0\neq 0$	$\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$	$\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C[\lambda^2+\lambda\,\tau_1(x)]$
	$\boldsymbol{w}_0=0$	$\boldsymbol{u}_0=0$	$\lvert\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda$	$\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<$ $<C[\lambda^2+\lvert x\rvert^{-2}\log\lvert x\rvert]$
		$\boldsymbol{u}_0\neq 0$	$\lvert\boldsymbol{\eta}(x;\lambda)<C(\tau_0(x;\lambda)+\lambda)$ $\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C\,\lambda\log\lambda^{-1}$	$\lvert\nabla\boldsymbol{\eta}(x;\lambda)\rvert<C[\lambda^2+\sqrt{\lambda}\,\lvert x\rvert^{-1}]$

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