Isotropic Integrity Bases for Vectors and Second-Order Tensors

Part I

A. J. M. SPENCER & R. S. RIVLIN

1. Introduction

In previous papers [2, 3] it has been shown how an arbitrary matrix polynomial in any number of symmetric 3×3 matrices may be expressed in a canonical form. From these results an integrity basis under the orthogonal transformation group for an arbitrary number of symmetric 3×3 matrices has been derived. This consists of traces of products formed from the matrices which have total degree six or less in the matrices. In deriving these results a number of theorems were obtained which enabled us to express a product formed from any number of 3×3 matrices, whether symmetric or non-symmetric, as a sum of products of particular types formed from these matrices, with coefficients which are polynomials in traces of products formed from the matrices.

In the present paper it is shown how these results may be used to obtain finite integrity bases under the full and proper orthogonal transformation groups for an arbitrary number of three-dimensional vectors and symmetric 3×3 matrices. This is done by replacing the vectors by skew-symmetric 3×3 matrices. The integrity bases derived consist of elements which involve the symmetric matrices alone and elements which involve both the vectors and matrices. The former elements are the same for both the full and proper orthogonal groups and form the integrity basis for 3×3 matrices derived in the previous papers. The integrity bases derived in this paper for the full and proper orthogonal groups differ in the elements which involve both vectors and matrices. In neither case is the integrity basis irreducible. It is intended to pursue the further reduction of the integrity bases in a later paper.

The results obtained in the present paper are applied in §7 to the problem of the formulation of constitutive equations for isotropic materials, which are applicable to physical phenomena described by the relation between the value of a tensor of arbitrary order at some instant and the values of the displacement gradients and a number of vectors at that instant and at times preceding that instant.

2. Proper orthogonal transformation group

In a previous paper [1] it has been shown that an integrity basis, under the proper orthogonal group, for a single symmetric tensor G_{ij} and ν vectors $V_i^{(\alpha)}$

($\alpha=1, 2, \ldots, \nu$) in three dimensions, is given, with the notation $G=\|G_{ij}\|$, by

$$
\begin{aligned}
&V_i^{(\alpha)}(G^N)_{ij}V_j^{(\beta)}, \qquad \operatorname{tr} G^{N+1}, \\
&e_{ijk}(G^M)_{ip}V_p^{(\alpha)}(G^N)_{jq}V_q^{(\beta)}(G^P)_{kr}V_r^{(\gamma)}, \\
&e_{ijk}(G^M)_{ip}V_p^{(\alpha)}(G^N)_{jk},
\end{aligned}
\tag{2.1}
$$

where $M, N, P=0, 1, 2, \ldots$ and $\alpha, \beta, \gamma=1, 2, \ldots, \nu$ and $e_{ijk}=1$ or -1 accordingly as i, j, k is an even or an odd permutation of $1, 2, 3$ and otherwise $e_{ijk}=0$. Invariants of the last type vanish identically since G_{ij} is symmetric.

The first two invariants in (2.1) are invariant under both the proper and the full orthogonal groups. The remaining invariants in (2.1) are invariant under the proper orthogonal group but not under the full orthogonal group.

In a similar manner it can be shown that an integrity basis, under the proper orthogonal group for μ symmetric second-order tensors $G_{ij}^{(\alpha)}$ ($\alpha=1, 2, \ldots, \mu$) and ν vectors $V_i^{(\beta)}$ ($\beta=1, 2, \ldots, \nu$) in three dimensions is given by

$$
V_i^{(\alpha)}(\mathbf{\Pi}_L)_{ij}V_j^{(\beta)}, \qquad \operatorname{tr}\mathbf{\Pi}_K, \tag{2.2}
$$

$$
\begin{aligned}
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}_N)_{kr}V_r^{(\gamma)}, \\
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)}, \\
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{qj}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)}, \\
&e_{ijk}(\mathbf{\Pi}_L)_{pi}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{qj}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)},
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jk}, \\
&e_{ijk}(\mathbf{\Pi}_L)_{pi}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jk},
\end{aligned}
\tag{2.4}
$$

where $\mathbf{\Pi}_K$, $\mathbf{\Pi}_L$, $\mathbf{\Pi}_M$ and $\mathbf{\Pi}_N$ are any matrix products formed from the μ symmetric matrices $G_\alpha (=\|G_{ij}^{(\alpha)}\|)$. $\mathbf{\Pi}_L, \mathbf{\Pi}_M$ and $\mathbf{\Pi}_N$, but not $\mathbf{\Pi}_K$, may as particular cases be the unit matrix. The invariants (2.2) are invariant under both the proper and the full orthogonal groups. The invariants (2.3) and (2.4) are invariant under the proper orthogonal group but not under the full orthogonal group. Now,

$$
(\mathbf{\Pi}_L)_{pi}=(\mathbf{\Pi}'_L)_{ip}, \qquad (\mathbf{\Pi}_M)_{qj}=(\mathbf{\Pi}'_M)_{jq}, \qquad (\mathbf{\Pi}_N)_{rk}=(\mathbf{\Pi}'_N)_{kr}, \tag{2.5}
$$

where $\mathbf{\Pi}'_L, \mathbf{\Pi}'_M$ and $\mathbf{\Pi}'_N$ denote the transposes of $\mathbf{\Pi}_L, \mathbf{\Pi}_M$ and $\mathbf{\Pi}_N$ respectively. Introducing (2.5) into (2.3), we have

$$
\begin{aligned}
e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)} &= e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}'_N)_{kr}V_r^{(\gamma)}, \\
e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{qj}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)} &= e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}'_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}'_N)_{kr}V_r^{(\gamma)}, \\
e_{ijk}(\mathbf{\Pi}_L)_{pi}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{qj}V_q^{(\beta)}(\mathbf{\Pi}_N)_{rk}V_r^{(\gamma)} &= e_{ijk}(\mathbf{\Pi}'_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}'_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}'_N)_{kr}V_r^{(\gamma)}.
\end{aligned}
\tag{2.6}
$$

Since $\mathbf{\Pi}_L$ is a matrix product formed from the symmetric matrices G_α, $\mathbf{\Pi}'_L$ is the matrix product formed by writing down the factors in $\mathbf{\Pi}_L$ in reverse order. With similar considerations applied to $\mathbf{\Pi}'_M$ and $\mathbf{\Pi}'_N$, we see that the three invariants on the right hand sides of (2.6) are each of the form of the first of the invariants (2.3). Hence we may omit from the integrity basis the last three of the invariants (2.3). Similar considerations enable us to omit from the integrity basis the last of the invariants (2.4). We thus see that an integrity basis for μ symmetric tensors $G_{ij}^{(\alpha)}$ and ν vectors $V_i^{(\alpha)}$ under the proper orthogonal group is formed by

$$
\begin{aligned}
&V_i^{(\alpha)}(\mathbf{\Pi}_L)_{ij}V_j^{(\beta)}, \qquad \operatorname{tr}\mathbf{\Pi}_K,\\
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}_N)_{kr}V_r^{(\gamma)},\\
&e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jk},
\end{aligned}
\tag{2.7}
$$

where the first two invariants in (2.7) are invariant under both the proper and the full orthogonal groups, and the last two invariants in (2.7) are invariant under the proper orthogonal group, but not under the full orthogonal group.

We now define the matrices $\boldsymbol{v}_\alpha(=\|v_{ij}^{(\alpha)}\|)$ by

$$v_{ij}^{(\alpha)} = e_{ijk}V_k^{(\alpha)}. \tag{2.8}$$

Employing the relation

$$e_{ijp}e_{ijk} = 2\delta_{pk}, \tag{2.9}$$

we readily obtain the inverse relation

$$V_p^{(\alpha)} = \tfrac{1}{2}e_{ijp}v_{ij}^{(\alpha)}. \tag{2.10}$$

Introducing (2.10) into the first of the invariants (2.7), we obtain

$$V_i^{(\alpha)}(\mathbf{\Pi}_L)_{ij}V_j^{(\beta)} = \tfrac{1}{4}e_{kli}v_{kl}^{(\alpha)}(\mathbf{\Pi}_L)_{ij}e_{mnj}v_{mn}^{(\beta)}. \tag{2.11}$$

Introducing into (2.11) the relation

$$
\begin{aligned}
e_{kli}e_{mnj} &= \begin{vmatrix} \delta_{km}, & \delta_{kn}, & \delta_{kj}\\ \delta_{lm}, & \delta_{ln}, & \delta_{lj}\\ \delta_{im}, & \delta_{in}, & \delta_{ij}\end{vmatrix}\\
&= [\delta_{km}\delta_{ln}\delta_{ij} - \delta_{km}\delta_{lj}\delta_{in} + \delta_{kn}\delta_{lj}\delta_{im} -\\
&\quad - \delta_{kn}\delta_{lm}\delta_{ij} + \delta_{kj}\delta_{lm}\delta_{in} - \delta_{kj}\delta_{ln}\delta_{im}],
\end{aligned}
\tag{2.12}
$$

we obtain

$$
\begin{aligned}
V_i^{(\alpha)}(\mathbf{\Pi}_L)_{ij}V_j^{(\beta)} &= -\tfrac{1}{2}(\mathbf{\Pi}_L)_{ii}v_{jk}^{(\alpha)}v_{kj}^{(\beta)} + v_{ij}^{(\alpha)}v_{jk}^{(\beta)}(\mathbf{\Pi}_L)_{ki}\\
&= -\tfrac{1}{2}\operatorname{tr}\mathbf{\Pi}_L\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta + \operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\mathbf{\Pi}_L.
\end{aligned}
\tag{2.13}
$$

In deriving this result, we use the relation

$$v_{ij}^{(\alpha)} = -v_{ji}^{(\alpha)}, \tag{2.14}$$

obtained directly from (2.8).

Now, introducing (2.10) into the last of the invariants (2.7), we obtain, with (2.12),

$$
\begin{aligned}
e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jk} &= \tfrac{1}{2}e_{ijk}e_{mnp}(\mathbf{\Pi}_L)_{ip}v_{mn}^{(\alpha)}(\mathbf{\Pi}_M)_{jk}\\
&= \tfrac{1}{2}[\delta_{im}\delta_{jn}\delta_{kp} - \delta_{im}\delta_{jp}\delta_{kn} + \delta_{in}\delta_{jp}\delta_{km} -\\
&\quad - \delta_{in}\delta_{jm}\delta_{kp} + \delta_{ip}\delta_{jm}\delta_{kn} - \delta_{ip}\delta_{jn}\delta_{km}]\times\\
&\quad \times(\mathbf{\Pi}_L)_{ip}v_{mn}^{(\alpha)}(\mathbf{\Pi}_M)_{jk}.
\end{aligned}
\tag{2.15}
$$

With (2.5) and (2.14), equation (2.15) yields

$$e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jk} = -\operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}'_M\boldsymbol{v}_\alpha + \operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}_M\boldsymbol{v}_\alpha - \operatorname{tr}\mathbf{\Pi}_L\operatorname{tr}\boldsymbol{v}_\alpha\mathbf{\Pi}_M. \tag{2.16}$$

Again, introducing (2.10) into the third of the invariants (2.7), we obtain, with (2.5), (2.12) and (2.14),

$$\begin{aligned} & e_{ijk}(\mathbf{\Pi}_L)_{ip}V_p^{(\alpha)}(\mathbf{\Pi}_M)_{jq}V_q^{(\beta)}(\mathbf{\Pi}_N)_{kr}V_r^{(\gamma)} \\ &= \tfrac{1}{8}e_{ijk}e_{pmn}e_{qst}e_{ruw}(\mathbf{\Pi}_L)_{ip}(\mathbf{\Pi}_M)_{jq}(\mathbf{\Pi}_N)_{kr}v_{mn}^{(\alpha)}v_{st}^{(\beta)}v_{uw}^{(\gamma)} \\ &= \tfrac{1}{8}[\delta_{ip}\delta_{jm}\delta_{kn} - \delta_{ip}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kp} - \delta_{im}\delta_{jp}\delta_{kn} + \\ &\quad + \delta_{in}\delta_{jp}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kp}][\delta_{qr}\delta_{su}\delta_{tw} - \delta_{qr}\delta_{sw}\delta_{tu} + \\ &\quad + \delta_{qu}\delta_{sw}\delta_{tr} - \delta_{qu}\delta_{sr}\delta_{tw} + \delta_{qw}\delta_{sr}\delta_{tu} - \delta_{qw}\delta_{su}\delta_{tr}] \times \\ &\quad \times (\mathbf{\Pi}_L)_{ip}(\mathbf{\Pi}_M)_{jq}(\mathbf{\Pi}_N)_{kr}v_{mn}^{(\alpha)}v_{st}^{(\beta)}v_{uw}^{(\gamma)} \\ &= \tfrac{1}{8}[-2\operatorname{tr}\mathbf{\Pi}_L\operatorname{tr}\mathbf{\Pi}'_M\boldsymbol{v}_\alpha\mathbf{\Pi}_N\operatorname{tr}\boldsymbol{v}_\beta\boldsymbol{v}_\gamma - 8\operatorname{tr}\mathbf{\Pi}_L\operatorname{tr}\mathbf{\Pi}_M\boldsymbol{v}_\gamma\boldsymbol{v}_\beta\mathbf{\Pi}'_N\boldsymbol{v}_\alpha + \\ &\quad + 2\operatorname{tr}\mathbf{\Pi}_L\operatorname{tr}\mathbf{\Pi}_M\mathbf{\Pi}'_N\boldsymbol{v}_\alpha\operatorname{tr}\boldsymbol{v}_\beta\boldsymbol{v}_\gamma + 4\operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}_N\mathbf{\Pi}'_M\boldsymbol{v}_\alpha\operatorname{tr}\boldsymbol{v}_\beta\boldsymbol{v}_\gamma - \\ &\quad - 8\operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}_N\boldsymbol{v}_\beta\boldsymbol{v}_\gamma\mathbf{\Pi}'_M\boldsymbol{v}_\alpha - 4\operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}_M\mathbf{\Pi}'_N\boldsymbol{v}_\alpha\operatorname{tr}\boldsymbol{v}_\beta\boldsymbol{v}_\gamma + \\ &\quad + 8\operatorname{tr}\mathbf{\Pi}_L\mathbf{\Pi}_M\boldsymbol{v}_\gamma\boldsymbol{v}_\beta\mathbf{\Pi}'_N\boldsymbol{v}_\alpha]. \end{aligned} \tag{2.17}$$

We note, from (2.13), (2.16) and (2.17), that each of the invariants (2.7) may be written as a polynomial in traces of products formed the matrices $\boldsymbol{G}_\alpha$ ($\alpha=1, 2, \ldots, \mu$) and $\boldsymbol{v}_\beta$ ($\beta=1, 2, \ldots, \nu$) which are of degree three or less in the matrices $\boldsymbol{v}_\beta$ ($\beta=1, 2, \ldots, \nu$). In the next section we shall show how the trace of any of these products may be expressed as a polynomial in traces of a limited number of products of the same type.

3. Derivation of a finite integrity basis

As a particular case of a theorem proven in a previous paper [*2*, Theorem 1'], we have

Theorem 1. *Any matrix product* $\mathbf{\Pi}$ *in* R, 3×3 *matrices* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) *can be expressed as a matrix polynomial in which*

(i) *each matrix product is either the unit matrix* $\boldsymbol{I}$ *or is formed from one or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) *and at most two of the factors* $\boldsymbol{a}_P^2$ ($P=1, 2, \ldots, R$);

(ii) *no two factors in a single matrix product are the same;*

(iii) *each matrix product is of lower or equal partial degree in each of the matrices* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) *than the matrix product* $\mathbf{\Pi}$;

(iv) *matrix products containing two of the factors* $\boldsymbol{a}_P^2$ ($P=1, 2, \ldots, R$) *contain them consecutively;*

(v) *no matrix product containing both of the factors* $\boldsymbol{a}_K^2$ *and* $\boldsymbol{a}_L^2$ *contains either of the factors* $\boldsymbol{a}_K$ *or* $\boldsymbol{a}_L$ *unless it is of the form* $\boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_K^2$;

(vi) *each matrix product which contains both* $\boldsymbol{a}_K$ *and* $\boldsymbol{a}_K^2$ *as factors has* $\boldsymbol{a}_K$ *as the first factor and* $\boldsymbol{a}_K^2$ *as the last factor;*

(vii) *no matrix product has total degree greater than five in the matrices* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$;

(viii) *the coefficients in the matrix polynomial are polynomials in traces of matrix products formed from the matrices* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$.

It follows immediately from this result that any matrix product in the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ may be expressed as a matrix polynomial in which the matrix terms are

$$\begin{aligned}
&\boldsymbol{I};\ \boldsymbol{a}_K,\ \boldsymbol{a}_K^2;\ \boldsymbol{a}_K\boldsymbol{a}_L,\ \boldsymbol{a}_K\boldsymbol{a}_L^2,\ \boldsymbol{a}_K^2\boldsymbol{a}_L,\ \boldsymbol{a}_K^2\boldsymbol{a}_L^2,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_K^2,\ \boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_K^2;\\
&\boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M,\ \boldsymbol{a}_K^2\boldsymbol{a}_L\boldsymbol{a}_M,\ \boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_M,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M^2,\ \boldsymbol{a}_K^2\boldsymbol{a}_L^2\boldsymbol{a}_M,\ \boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_M^2,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_K^2;\\
&\boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N,\ \boldsymbol{a}_K^2\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N,\ \boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_M\boldsymbol{a}_N,\\
&\boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M^2\boldsymbol{a}_N,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N^2,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_S;
\end{aligned} \tag{3.1}$$

where K, L, M, N, S are integers, which are all different and are chosen in all possible ways from the integers $1, 2, \ldots, R$.

We may now readily obtain*

Lemma 1. *The trace of any matrix product* $\boldsymbol{\Pi}$ *formed from the matrices* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ *may be expressed as a polynomial in expressions of the form* $\operatorname{tr}\boldsymbol{a}_T\boldsymbol{\Pi}_\alpha$, *where* $\boldsymbol{\Pi}_\alpha$ $(\alpha=1, 2, \ldots)$ *are the matrix products* (3.1) *and* T *may be any of the integers* $(1, 2, \ldots, R)$.

We employ the following theorem which has been proven in a previous paper [*3*, Theorem 2]:

Theorem 2. *The trace of any matrix product formed from the* R, 3×3 *matrices* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ *may be expressed as a polynomial in traces of matrix products of lower or equal partial degrees in each of the matrices and of lower or equal extensions, having the forms*

$$\begin{aligned}
&\operatorname{tr}\boldsymbol{y}\boldsymbol{a}_K^2\boldsymbol{a}_L^2 \qquad (K\neq L),\\
&\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2,\\
&\operatorname{tr}\boldsymbol{u},\\
&\operatorname{tr}\boldsymbol{a}_M\boldsymbol{a}_L\boldsymbol{a}_K^2\boldsymbol{a}_L^2 \qquad (K\neq L,\ M\neq L),\\
\textit{and}\quad &\operatorname{tr}\boldsymbol{a}_K^3,
\end{aligned} \tag{3.2}$$

where $\boldsymbol{y}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;\ P\neq K, L)$, *no two factors in* $\boldsymbol{y}$ *being the same*; $\boldsymbol{v}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, *such that* $\boldsymbol{a}_K$ *is not the first or last factor in* $\boldsymbol{v}$ *and no two factors in* $\boldsymbol{v}$ *are the same*; $\boldsymbol{u}$ *is a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, *no two factors in which are the same.*

Combining this result with Lemma 1, we see that in Theorem 2 we may take $\boldsymbol{y}$, $\boldsymbol{v}$ and $\boldsymbol{u}$ to have total degrees in the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ not greater than two, four and six respectively.

* See [*3*, § 5] for an argument similar to that used in obtaining this result.

We thus obtain

Theorem 3. *The trace of any matrix product in the* 3×3 *matrices* $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$ *may be expressed as a polynomial in traces of the products*

$$\begin{aligned}
&\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_K^2\boldsymbol{a}_L^2,\ \boldsymbol{a}_M\boldsymbol{a}_K^2\boldsymbol{a}_L^2,\ \boldsymbol{a}_K^2\boldsymbol{a}_L^2;\\
&\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_P\boldsymbol{a}_K^2,\ \boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_K^2,\ \boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_K^2,\ \boldsymbol{a}_L\boldsymbol{a}_K^2,\ \boldsymbol{a}_K^2;\\
&\boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_P\boldsymbol{a}_Q,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N\boldsymbol{a}_P,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_N,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_M,\ \boldsymbol{a}_K\boldsymbol{a}_L,\ \boldsymbol{a}_K;\\
&\boldsymbol{a}_L\boldsymbol{a}_K\boldsymbol{a}_N\boldsymbol{a}_P\boldsymbol{a}_K^2,\ \boldsymbol{a}_L\boldsymbol{a}_M\boldsymbol{a}_K\boldsymbol{a}_P\boldsymbol{a}_K^2,\ \boldsymbol{a}_L\boldsymbol{a}_K\boldsymbol{a}_N\boldsymbol{a}_K^2;\\
&\boldsymbol{a}_M\boldsymbol{a}_L\boldsymbol{a}_K^2\boldsymbol{a}_L^2,\ \boldsymbol{a}_K\boldsymbol{a}_L\boldsymbol{a}_K^2\boldsymbol{a}_L^2;\\
&\boldsymbol{a}_K^3;
\end{aligned}\tag{3.3}$$

in which K, L, M, N, P *and* Q *are integers, all different, chosen from* $1, 2, \ldots, R$.

We shall now consider that the set of matrices $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$ consists of the μ symmetric matrices $\boldsymbol{G}_P$ $(P=1,2,\ldots,\mu)$* and the ν skew-symmetric matrices $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$. We note that

$$\operatorname{tr}\boldsymbol{v}_\alpha=\operatorname{tr}\boldsymbol{v}_\alpha^3=\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta^2=\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{G}_P=\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{G}_P^2=0.\tag{3.4}$$

We note also

Lemma 2. *If* $\boldsymbol{\Pi}_1$ *is a matrix product formed from the symmetric matrices* $\boldsymbol{G}_P$ $(P=1,2,\ldots,\mu)$ *and the skew-symmetric matrices* $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$ *and* $\boldsymbol{\Pi}_2$ *is the matrix product formed by writing the factors of* $\boldsymbol{\Pi}_1$ *in inverse order, then* $\operatorname{tr}\boldsymbol{\Pi}_1=\operatorname{tr}\boldsymbol{\Pi}_2$, *if* $\boldsymbol{\Pi}_1$ *is of even degree in the matrices* $\boldsymbol{v}_\alpha$, *and* $\operatorname{tr}\boldsymbol{\Pi}_1=-\operatorname{tr}\boldsymbol{\Pi}_2$, *if* $\boldsymbol{\Pi}_1$ *is of odd degree in the matrices* $\boldsymbol{v}_\alpha$.

We shall also use

Lemma 3. *If* $\boldsymbol{\Pi}_1$ *and* $\boldsymbol{\Pi}_2$ *are two matrices (which may be matrix products), then* $\operatorname{tr}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2=\operatorname{tr}\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1$.

We substitute the set of matrices $\boldsymbol{G}_P$ $(P=1,2,\ldots,\mu)$ and $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$ for $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$ in (3.3) and recall, from §2, that any polynomial invariant of the tensors $G_{ij}^{(P)}$ and the vectors $V_i^{(\alpha)}$ may be expressed as a polynomial in traces of matrix products of the matrices $\boldsymbol{G}_P$ $(P=1,2,\ldots,\mu)$ and $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$, which are of degree three or less in the matrices $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$. Then employing the relations (3.4) and Lemmas 2 and 3, we obtain from Theorem 3,

Lemma 4. *The trace of any matrix product in the* μ *symmetric* 3×3 *matrices* $\boldsymbol{G}_P$ $(P=1,2,\ldots,\mu)$ *and the* ν *skew-symmetric* 3×3 *matrices* $\boldsymbol{v}_\alpha$ $(\alpha=1,2,\ldots,\nu)$ *which is of degree three or less in the matrices* $\boldsymbol{v}_\alpha$, *may be expressed as a polynomial in traces of matrix products which do not involve the matrices* $\boldsymbol{v}_\alpha$, *together with traces of the following matrix products:*

$$\begin{aligned}
&\boldsymbol{v}_\alpha\boldsymbol{G}_N\boldsymbol{G}_K^2\boldsymbol{G}_L^2,\ \boldsymbol{v}_\alpha\boldsymbol{G}_K^2\boldsymbol{G}_L^2;\\
&\boldsymbol{v}_\alpha\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{G}_L\boldsymbol{v}_\alpha\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_K^2,\ \boldsymbol{G}_L\boldsymbol{v}_\alpha\boldsymbol{G}_N\boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha\boldsymbol{G}_M\boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha\boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_Q,\ \boldsymbol{v}_\alpha\boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P,\ \boldsymbol{v}_\alpha\boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N,\ \boldsymbol{v}_\alpha\boldsymbol{G}_L\boldsymbol{G}_M;\\
&\boldsymbol{v}_\alpha\boldsymbol{G}_K\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{G}_L\boldsymbol{G}_K\boldsymbol{v}_\alpha\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha\boldsymbol{G}_M\boldsymbol{G}_K\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha\boldsymbol{G}_K\boldsymbol{G}_N\boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha\boldsymbol{G}_L\boldsymbol{G}_K^2\boldsymbol{G}_L^2;
\end{aligned}\tag{3.5}$$

* We shall use roman capitals as subscripts to the G's in order to avoid confusion with the subscripts on the $\boldsymbol{v}$'s.

$$
\begin{aligned}
&G_M G_N G_K^2 v_\alpha^2,\ G_M G_K^2 v_\alpha^2,\ G_K^2 v_\alpha^2;\\
&G_L G_M G_N G_P v_\alpha^2,\ G_L G_M G_N v_\alpha^2,\ G_L G_M v_\alpha^2,\ G_L v_\alpha^2,\ v_\alpha^2;\\
&G_M G_L v_\alpha^2 G_L^2;\\
&v_\alpha v_\beta G_K^2 G_L^2;\\
&v_\alpha v_\beta G_N G_P G_K^2,\ G_L v_\alpha v_\beta G_P G_K^2,\ v_\alpha G_M G_N v_\beta G_K^2,\\
&v_\alpha G_M v_\beta G_P G_K^2,\ v_\alpha v_\beta G_N G_K^2,\ v_\alpha G_M v_\beta G_K^2,\ v_\alpha v_\beta G_K^2;\\
&v_\alpha v_\beta G_M G_N G_P G_Q,\ v_\alpha G_L v_\beta G_N G_P G_Q,\ v_\alpha G_L G_M v_\beta G_P G_Q,\ v_\alpha v_\beta G_M G_N G_P,\\
&v_\alpha G_L v_\beta G_N G_P,\ v_\alpha v_\beta G_M G_N,\ v_\alpha G_L v_\beta G_N,\ v_\alpha v_\beta G_M,\ v_\alpha v_\beta;\\
&v_\alpha G_K v_\beta G_P G_K^2,\ v_\alpha G_K G_N v_\beta G_K^2,\ G_L G_K v_\alpha v_\beta G_K^2;\\
&v_\alpha G_K v_\beta G_K^2;
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&G_L v_\alpha G_N G_P v_\alpha^2,\ G_L v_\alpha G_N v_\alpha^2,\ G_M v_\alpha G_K^2 v_\alpha^2,\ v_\alpha G_L v_\alpha^2 G_L^2;\\
&v_\alpha G_N v_\beta^2 G_L^2,\ v_\alpha G_N G_K^2 v_\beta^2,\ v_\alpha G_K^2 v_\beta^2;\\
&v_\alpha G_M G_N G_P v_\beta^2,\ G_L v_\alpha G_N G_P v_\beta^2,\ v_\alpha G_M G_N v_\beta^2,\ G_L v_\alpha G_N v_\beta^2,\ v_\alpha G_M v_\beta^2;\\
&v_\alpha G_L v_\beta^2 G_L^2;\\
&v_\alpha v_\beta v_\gamma G_P G_K^2,\ v_\alpha v_\beta G_N v_\gamma G_K^2,\ v_\alpha v_\beta v_\gamma G_K^2;\\
&v_\alpha v_\beta v_\gamma G_N G_P G_Q,\ v_\alpha v_\beta G_M v_\gamma G_P G_Q,\ v_\alpha G_L v_\beta G_N v_\gamma G_Q,\ v_\alpha v_\beta v_\gamma G_N G_P,\\
&v_\alpha v_\beta G_M v_\gamma G_P,\ v_\alpha v_\beta v_\gamma G_N,\ v_\alpha v_\beta v_\gamma,\ v_\alpha G_K v_\beta v_\gamma G_K^2,
\end{aligned}
\tag{3.7}
$$

where K, L, M, N, P, Q are integers, all different, chosen from $1, 2, \ldots, \mu$ and α, β, γ are integers, all different, chosen from $1, 2, \ldots, \nu$.

4. Further reductions of the traces of matrix products

In this section it will be shown that the traces of certain of the matrix products listed in (3.5), (3.6) and (3.7) can be expressed in terms of traces of other of these products. As in previous papers we use the notation

$$\operatorname{tr} \boldsymbol{P} \equiv 0 \tag{4.1}$$

to express the fact that the trace of some matrix polynomial $\boldsymbol{P}$ of degree n in a set of matrices $\boldsymbol{a}_P$ ($P = 1, 2, \ldots, R$) can be expressed as a polynomial in traces of matrix products $\boldsymbol{\Pi}_i$ in the matrices $\boldsymbol{a}_P$, in which each matrix product $\boldsymbol{\Pi}_i$ has degree less than n in the matrices $\boldsymbol{a}_P$. We shall use the following results, which were proven in [2] and [3]:

$$\operatorname{tr} \boldsymbol{x}\, \boldsymbol{a}_K\, \boldsymbol{y}\, \boldsymbol{a}_K^2 + \operatorname{tr} \boldsymbol{x}\, \boldsymbol{a}_K^2\, \boldsymbol{y}\, \boldsymbol{a}_K \equiv 0, \tag{4.2}$$

$$\operatorname{tr} \boldsymbol{w} (\boldsymbol{a}_K \boldsymbol{a}_L + \boldsymbol{a}_L \boldsymbol{a}_K)\, \boldsymbol{z}\, \boldsymbol{a}_M^2 \equiv 0, \tag{4.3}$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}$ and $\boldsymbol{z}$ are matrix products formed from the matrices $\boldsymbol{a}_P$ and are not equal to the unit matrix $\boldsymbol{I}$.

$\underline{\operatorname{tr} G_L G_K v_\alpha G_P G_K^2 \text{ and } \operatorname{tr} v_\alpha G_M G_K G_P G_K^2}$. Replacing $\boldsymbol{w}, \boldsymbol{a}_K, \boldsymbol{a}_L, \boldsymbol{z}$ and $\boldsymbol{a}_M$ in (4.3) by G_L, G_K, v_α, G_P and G_K respectively, and applying Lemmas 2 and 3, we obtain

$$
\begin{aligned}
\operatorname{tr} G_L G_K v_\alpha G_P G_K^2 &\equiv -\operatorname{tr} G_L v_\alpha G_K G_P G_K^2\\
&= \operatorname{tr} v_\alpha G_L G_K^2 G_P G_K.
\end{aligned}
\tag{4.4}
$$

By a relation of the type (4.2)

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_K^2 \boldsymbol{G}_P \boldsymbol{G}_K \equiv - \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_K \boldsymbol{G}_P \boldsymbol{G}_K^2, \tag{4.5}$$

and by a further relation of the type (4.3)

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_K \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv - \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_L \boldsymbol{G}_P \boldsymbol{G}_K^2. \tag{4.6}$$

Hence, combining (4.4), (4.5) and (4.6)

$$\operatorname{tr} \boldsymbol{G}_L \boldsymbol{G}_K \boldsymbol{v}_\alpha \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_L \boldsymbol{G}_P \boldsymbol{G}_K^2. \tag{4.7}$$

Further, by a relation of the type (4.3)

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_K \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv - \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_M \boldsymbol{G}_P \boldsymbol{G}_K^2. \tag{4.8}$$

Thus $\operatorname{tr} \boldsymbol{G}_L \boldsymbol{G}_K \boldsymbol{v}_\alpha \boldsymbol{G}_P \boldsymbol{G}_K^2$ and $\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_K \boldsymbol{G}_P \boldsymbol{G}_K^2$ can be expressed in terms of traces of matrix products of the form $\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2$, and traces of matrix products of lower degree, and so they may be omitted from the set (3.5).

$\underline{\operatorname{tr} \boldsymbol{G}_M \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2}$. By a relation of the type (4.2) and Lemmas 2 and 3

$$\begin{aligned}\operatorname{tr} \boldsymbol{G}_M \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2 &\equiv - \operatorname{tr} \boldsymbol{G}_M \boldsymbol{G}_L^2 \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L \\ &\equiv - \operatorname{tr} \boldsymbol{G}_M \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2.\end{aligned}$$

Hence

$$\operatorname{tr} \boldsymbol{G}_M \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2 \equiv 0, \tag{4.9}$$

and $\boldsymbol{G}_M \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2$ may be omitted from (3.6).

We next derive a further relation for matrix products containing two or more skew-symmetric matrices. Let $v_{ij}^{(\alpha)}$ be the ij^{th} component of a skew-symmetric matrix $\boldsymbol{v}_\alpha$. Then, using the relation

$$e_{ijr} e_{pqr} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}, \tag{4.10}$$

we have

$$\begin{aligned} v_{ij}^{(\alpha)} &= \tfrac{1}{2} (v_{ij}^{(\alpha)} - v_{ji}^{(\alpha)}) \\ &= \tfrac{1}{2} (\delta_{ip} \delta_{jq} - \delta_{jp} \delta_{iq}) v_{pq}^{(\alpha)} \\ &= \tfrac{1}{2} e_{ijr} e_{pqr} v_{pq}^{(\alpha)}. \end{aligned} \tag{4.11}$$

Now let $\boldsymbol{v}_\alpha$ $(= \|v_{ij}^{(\alpha)}\|)$ and $\boldsymbol{v}_\beta$ $(= \|v_{ij}^{(\beta)}\|)$ be skew-symmetric matrices, and let $\boldsymbol{\Pi}$ $(= \|\Pi_{ij}\|)$ be an arbitrary matrix product. Then, using relations of the type (4.11)

$$\begin{aligned} (\boldsymbol{v}_\alpha \boldsymbol{\Pi} \boldsymbol{v}_\beta)_{il} &= v_{ij}^{(\alpha)} \Pi_{jk} v_{kl}^{(\beta)} \\ &= \tfrac{1}{4} e_{ijp} e_{rsp} v_{rs}^{(\alpha)} \Pi_{jk} e_{klq} e_{mnq} v_{mn}^{(\beta)} \\ &= \tfrac{1}{4} e_{ijp} e_{klq} e_{rsp} e_{mnq} v_{rs}^{(\alpha)} v_{mn}^{(\beta)} \Pi_{jk}. \end{aligned} \tag{4.12}$$

Introducing relations of the type (2.12) into (4.12) we obtain

$$\begin{aligned} (\boldsymbol{v}_\alpha \boldsymbol{\Pi} \boldsymbol{v}_\beta)_{il} = {} & \tfrac{1}{4} [\delta_{ik} \delta_{jl} \delta_{pq} - \delta_{ik} \delta_{jq} \delta_{pl} + \delta_{il} \delta_{jq} \delta_{pk} - \delta_{il} \delta_{jk} \delta_{pq} + \\ & + \delta_{iq} \delta_{jk} \delta_{pl} - \delta_{iq} \delta_{jl} \delta_{pk}] [\delta_{rm} \delta_{sn} \delta_{pq} - \delta_{rm} \delta_{sq} \delta_{pn} + \\ & + \delta_{rn} \delta_{sq} \delta_{pm} - \delta_{rn} \delta_{sm} \delta_{pq} + \delta_{rq} \delta_{sm} \delta_{pn} - \delta_{rq} \delta_{sn} \delta_{pm}] v_{rs}^{(\alpha)} v_{mn}^{(\beta)} \Pi_{jk} \\ = {} & - v_{jk}^{(\alpha)} v_{kl}^{(\beta)} \Pi_{ji} - v_{ij}^{(\alpha)} v_{jk}^{(\beta)} \Pi_{lk} + v_{ij}^{(\alpha)} v_{jl}^{(\beta)} \Pi_{kk} + \\ & + \tfrac{1}{2} v_{jk}^{(\alpha)} v_{kj}^{(\beta)} \Pi_{li} + \delta_{il} (v_{jk}^{(\alpha)} v_{kp}^{(\beta)} \Pi_{jp} - \tfrac{1}{2} v_{jk}^{(\alpha)} v_{kj}^{(\beta)} \Pi_{pp}). \end{aligned}$$

This relation may be written

$$\begin{aligned} & \boldsymbol{v}_\alpha \boldsymbol{\Pi} \boldsymbol{v}_\beta + \boldsymbol{\Pi}' \boldsymbol{v}_\alpha \boldsymbol{v}_\beta + \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}' \\ & \qquad = \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \operatorname{tr} \boldsymbol{\Pi} + \tfrac{1}{2} \boldsymbol{\Pi}' \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta + \boldsymbol{I} (\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}' - \tfrac{1}{2} \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \operatorname{tr} \boldsymbol{\Pi}), \end{aligned} \tag{4.13}$$

where $\boldsymbol{\Pi}'$ denotes the transpose of $\boldsymbol{\Pi}$. Unlike other relations which have been used in the reduction of matrix polynomials, (4.13) does not appear to be a consequence of the Hamilton-Cayley theorem, although it reduces to the Hamilton-Cayley theorem for skew-symmetric 3×3 matrices when $\boldsymbol{v}_\alpha = \boldsymbol{v}_\beta = \boldsymbol{\Pi}$.

Replacing $\boldsymbol{\Pi}$ in (4.13) by $\boldsymbol{\Pi}_1$, multiplying the relation so obtained on the right by a matrix product $\boldsymbol{\Pi}_2$, and taking the trace of each side of the resulting equation, we obtain

$$\begin{aligned} \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{\Pi}_1 \boldsymbol{v}_\beta \boldsymbol{\Pi}_2 = & - \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}_2 \boldsymbol{\Pi}_1' - \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}_1' \boldsymbol{\Pi}_2 + \\ & + \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}_2 \operatorname{tr} \boldsymbol{\Pi}_1 + \tfrac{1}{2} \operatorname{tr} \boldsymbol{\Pi}_1' \boldsymbol{\Pi}_2 \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta + \\ & + \operatorname{tr} \boldsymbol{\Pi}_2 (\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}_1' - \tfrac{1}{2} \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \operatorname{tr} \boldsymbol{\Pi}_1), \end{aligned} \tag{4.14}$$

if $\boldsymbol{\Pi}_2 \neq \boldsymbol{I}$. If $\boldsymbol{\Pi}_2 = \boldsymbol{I}$, by Lemmas 2 and 3

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{\Pi}_1 \boldsymbol{v}_\beta = \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{\Pi}_1'. \tag{4.15}$$

Thus we have

Lemma 5. *The trace of any matrix product of 3×3 matrices which contains two skew-symmetric matrices $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ as factors may be expressed as a polynomial in traces of matrix products which either*

a) contain both $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ as consecutive factors

or

b) contain neither $\boldsymbol{v}_\alpha$ nor $\boldsymbol{v}_\beta$ as factors.

From Lemma 5, it follows that the traces of all of the matrix products in (3.6) in which $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ do not occur consecutively may be expressed in terms of traces of matrix products in which $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ do occur consecutively, or do not occur at all. Hence all of the matrix products in which $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ do not occur consecutively may be omitted from (3.6). Similarly, the matrix product $\boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{v}_\beta \boldsymbol{G}_N \boldsymbol{v}_\gamma \boldsymbol{G}_Q$ may be omitted from (3.7).

$\underline{\operatorname{tr} \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_P \boldsymbol{G}_K^2}$. Replacing $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ in (4.14) by $\boldsymbol{G}_L$ and $\boldsymbol{G}_P \boldsymbol{G}_K^2$ respectively, we obtain

$$\operatorname{tr} (\boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{v}_\beta + \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{v}_\beta + \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_L) \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv 0. \tag{4.16}$$

Also, from a relation of the type (4.3)

$$\operatorname{tr} \boldsymbol{v}_\alpha (\boldsymbol{G}_L \boldsymbol{v}_\beta + \boldsymbol{v}_\beta \boldsymbol{G}_L) \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv 0. \tag{4.17}$$

Subracting (4.17) from (4.16), it follows that

$$\operatorname{tr} \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv 0, \tag{4.18}$$

and so $\boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_P \boldsymbol{G}_K^2$ may be omitted from (3.6).

$\underline{\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_P \boldsymbol{G}_K^2 \text{ and } \operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_K^2 \boldsymbol{v}_\beta^2}$. Replacing $\boldsymbol{v}_\alpha, \boldsymbol{v}_\beta, \boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ in (4.14) by $\boldsymbol{v}_\beta, \boldsymbol{v}_\gamma, \boldsymbol{G}_P$ and $\boldsymbol{G}_K^2 \boldsymbol{v}_\alpha$ respectively, and using Lemma 3, we have the relation

$$\operatorname{tr} \boldsymbol{v}_\alpha (\boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_P + \boldsymbol{v}_\beta \boldsymbol{G}_P \boldsymbol{v}_\gamma + \boldsymbol{G}_P \boldsymbol{v}_\beta \boldsymbol{v}_\gamma) \boldsymbol{G}_K^2 \equiv 0. \tag{4.19}$$

Also, by a relation of the type (4.3),

$$\operatorname{tr} \boldsymbol{v}_\alpha (\boldsymbol{v}_\beta \boldsymbol{G}_P + \boldsymbol{G}_P \boldsymbol{v}_\beta) \boldsymbol{v}_\gamma \boldsymbol{G}_K^2 \equiv 0. \tag{4.20}$$

Hence from (4.19) and (4.20)

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_P \boldsymbol{G}_K^2 \equiv 0. \tag{4.21}$$

Further, replacing $\boldsymbol{v}_\alpha, \boldsymbol{v}_\gamma$ and $\boldsymbol{G}_P$ in (4.21) by $\boldsymbol{v}_\beta, \boldsymbol{v}_\alpha$ and $\boldsymbol{G}_N$ respectively, and using Lemma 3, we have

$$\operatorname{tr} \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_K^2 \boldsymbol{v}_\beta^2 \equiv 0, \tag{4.22}$$

and so $\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_P \boldsymbol{G}_K^2$ and $\boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_K^2 \boldsymbol{v}_\beta^2$ may be omitted from (3.7).

Employing the relations (4.7), (4.8), (4.9), (4.18), (4.21) and (4.22), and Lemma 5, we obtain from Lemma 4

Theorem 4. *The trace of any matrix product in the μ symmetric 3×3 matrices $\boldsymbol{G}_P$ $(P=1, 2, \dots, \mu)$ and the ν skew-symmetric 3×3 matrices $\boldsymbol{v}_\alpha$ $(\alpha=1, 2, \dots, \nu)$ which is of degree three or less in the matrices $\boldsymbol{v}_\alpha$, may be expressed as a polynomial in traces of matrix products which do not involve the matrices $\boldsymbol{v}_\alpha$, together with traces of the following matrix products:*

$$\begin{aligned}
&\boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_K^2 \boldsymbol{G}_L^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_K^2 \boldsymbol{G}_L^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2,\ \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_K^2,\ \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_Q,\ \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P,\ \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{G}_N,\ \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_M;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_K \boldsymbol{G}_N \boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{G}_K^2 \boldsymbol{G}_L^2;
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
&\boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_K^2 \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_M \boldsymbol{G}_K^2 \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_K^2 \boldsymbol{v}_\alpha^2;\\
&\boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_L \boldsymbol{G}_M \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_L \boldsymbol{v}_\alpha^2,\ \boldsymbol{v}_\alpha^2;
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_K^2 \boldsymbol{G}_L^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_N \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_Q,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M \boldsymbol{G}_N,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta;\\
&\boldsymbol{G}_L \boldsymbol{G}_K \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_K^2;
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{v}_\alpha^2,\ \boldsymbol{G}_M \boldsymbol{v}_\alpha \boldsymbol{G}_K^2 \boldsymbol{v}_\alpha^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{v}_\alpha^2 \boldsymbol{G}_L^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{v}_\beta^2 \boldsymbol{G}_L^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_K^2 \boldsymbol{v}_\beta^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{v}_\beta^2,\ \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{v}_\beta^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{v}_\beta^2;\ \boldsymbol{G}_L \boldsymbol{v}_\alpha \boldsymbol{G}_N \boldsymbol{v}_\beta^2,\ \boldsymbol{v}_\alpha \boldsymbol{G}_M \boldsymbol{v}_\beta^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{G}_L \boldsymbol{v}_\beta^2 \boldsymbol{G}_L^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_N \boldsymbol{v}_\gamma \boldsymbol{G}_K^2,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_K^2;\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_Q,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M \boldsymbol{v}_\gamma \boldsymbol{G}_P \boldsymbol{G}_Q,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_N \boldsymbol{G}_P,\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_M \boldsymbol{v}_\gamma \boldsymbol{G}_P,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma \boldsymbol{G}_N,\ \boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{v}_\gamma;\\
&\boldsymbol{v}_\alpha \boldsymbol{v}_\beta \boldsymbol{G}_K \boldsymbol{v}_\gamma \boldsymbol{G}_K^2,
\end{aligned} \tag{4.25}$$

where K, L, M, N, P, Q are integers, all different, chosen from $1, 2, \dots, \mu$, and α, β, γ are integers, all different, chosen from $1, 2, \dots, \nu$.

5. Expression of the results in terms of vectors

In this section we shall show how the traces of the matrix products (4.23), (4.24) and (4.25) may be expressed in terms of the components of the vectors $V_i^{(\alpha)}$, which are related to the matrices $\boldsymbol{v}_\alpha$ $(=\|v_{ij}^{(\alpha)}\|)$ by (2.8) and (2.10).

We note, with Lemma 3, that the traces of each of the matrix products (4.23) may be expressed in the form $\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{\Pi}$, where $\boldsymbol{\Pi}$ is a matrix product formed from the matrices $\boldsymbol{G}_P$ $(P=1, 2, \ldots, \mu)$. With (2.8), we have

$$\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{\Pi} = v_{ij}^{(\alpha)}(\boldsymbol{\Pi})_{ji} = e_{ijk}V_k^{(\alpha)}(\boldsymbol{\Pi})_{ji}. \tag{5.1}$$

Thus the traces of the matrix products (4.23) may be written in the form (5.1) where $\boldsymbol{\Pi}$ is given by

$$\begin{aligned}
&\boldsymbol{G}_N\boldsymbol{G}_K^2\boldsymbol{G}_L^2,\ \boldsymbol{G}_K^2\boldsymbol{G}_L^2;\\
&\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2\boldsymbol{G}_L,\ \boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_K^2,\ \boldsymbol{G}_N\boldsymbol{G}_K^2\boldsymbol{G}_L,\ \boldsymbol{G}_M\boldsymbol{G}_K^2;\\
&\boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_Q,\ \boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P,\ \boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N,\ \boldsymbol{G}_L\boldsymbol{G}_M;\\
&\boldsymbol{G}_K\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{G}_K\boldsymbol{G}_N\boldsymbol{G}_K^2;\\
&\boldsymbol{G}_L\boldsymbol{G}_K^2\boldsymbol{G}_L^2.
\end{aligned} \tag{5.2}$$

Again, with Lemma 3, we see that the traces of each of the matrix products (4.24) may be expressed either in the form $\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi}$ or in the form $\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi}$, where $\boldsymbol{\Pi}$ is a matrix product formed from the matrices $\boldsymbol{G}_P$ $(P=1, 2, \ldots, \mu)$. With (2.8) and the relation (4.10), we have

$$\begin{aligned}
\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi} &= v_{ij}^{(\alpha)}v_{jk}^{(\beta)}(\boldsymbol{\Pi})_{ki}\\
&= e_{ijp}V_p^{(\alpha)}e_{jkq}V_q^{(\beta)}(\boldsymbol{\Pi})_{ki}\\
&= [\delta_{pk}\delta_{iq}-\delta_{pq}\delta_{ki}]V_p^{(\alpha)}V_q^{(\beta)}(\boldsymbol{\Pi})_{ki}\\
&= V_p^{(\alpha)}(\boldsymbol{\Pi})_{pq}V_q^{(\beta)} - V_p^{(\alpha)}V_p^{(\beta)}\operatorname{tr}\boldsymbol{\Pi}.
\end{aligned} \tag{5.3}$$

Taking $\boldsymbol{v}_\alpha=\boldsymbol{v}_\beta$, and correspondingly $V_i^{(\alpha)}=V_i^{(\beta)}$, we obtain

$$\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi} = V_p^{(\alpha)}(\boldsymbol{\Pi})_{pq}V_q^{(\alpha)} - V_p^{(\alpha)}V_p^{(\alpha)}\operatorname{tr}\boldsymbol{\Pi}. \tag{5.4}$$

We see that each of the expressions of the form $\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi}$ listed in (4.24) can be expressed as a polynomial in $V_p^{(\alpha)}V_p^{(\alpha)}$, traces of matrix products formed from $\boldsymbol{G}_P$ $(P=1, 2, \ldots, \mu)$, and expressions of the form $V_p^{(\alpha)}(\boldsymbol{\Pi})_{pq}V_q^{(\alpha)}$ where $\boldsymbol{\Pi}$ has the values

$$\begin{aligned}
&\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_K^2,\ \boldsymbol{G}_M\boldsymbol{G}_K^2,\ \boldsymbol{G}_K^2;\\
&\boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P,\ \boldsymbol{G}_L\boldsymbol{G}_M\boldsymbol{G}_N,\ \boldsymbol{G}_L\boldsymbol{G}_M,\ \boldsymbol{G}_L.
\end{aligned} \tag{5.5}$$

Also, each of the expressions of the form $\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi}$ listed in (4.24) can be expressed as a polynomial in $V_p^{(\alpha)}V_p^{(\beta)}$, traces of matrix products formed from $\boldsymbol{G}_P$ $(P=1, 2, \ldots, \mu)$, and expressions of the form $V_p^{(\alpha)}(\boldsymbol{\Pi})_{pq}V_q^{(\beta)}$, where $\boldsymbol{\Pi}$ has the values

$$\begin{aligned}
&\boldsymbol{G}_K^2\boldsymbol{G}_L^2;\\
&\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_K^2,\ \boldsymbol{G}_N\boldsymbol{G}_K^2,\ \boldsymbol{G}_K^2;\\
&\boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_Q,\ \boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P,\ \boldsymbol{G}_M\boldsymbol{G}_N,\ \boldsymbol{G}_M;\\
&\boldsymbol{G}_K^2\boldsymbol{G}_L\boldsymbol{G}_K.
\end{aligned} \tag{5.6}$$

Finally, with Lemma 3, we see that the traces of each of the matrix products (4.25) may be expressed either in the form $\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi}_1\boldsymbol{v}_\gamma\boldsymbol{\Pi}_2$, or $\operatorname{tr}\boldsymbol{v}_\beta^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2$, or $\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2$, where $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ are matrix products formed from the matrices $\boldsymbol{G}_P$ $(P=1, 2, \ldots, \mu)$ which, as particular cases, may be the unit matrix. With (2.8) and the relation (4.10), we have

$$\begin{aligned}\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi}_1\boldsymbol{v}_\gamma\boldsymbol{\Pi}_2 &= v_{ij}^{(\alpha)} v_{jk}^{(\beta)} (\boldsymbol{\Pi}_1)_{kl} v_{lm}^{(\gamma)} (\boldsymbol{\Pi}_2)_{mi} \\ &= e_{ijp} V_p^{(\alpha)} e_{jkq} V_q^{(\beta)} (\boldsymbol{\Pi}_1)_{kl} e_{lmr} V_r^{(\gamma)} (\boldsymbol{\Pi}_2)_{mi} \\ &= [\delta_{pk}\delta_{iq} - \delta_{ik}\delta_{pq}]\, e_{lmr} V_p^{(\alpha)} V_q^{(\beta)} (\boldsymbol{\Pi}_1)_{kl} V_r^{(\gamma)} (\boldsymbol{\Pi}_2)_{mi} \\ &= e_{lmr} (\boldsymbol{\Pi}_1')_{lp} V_p^{(\alpha)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\beta)} V_r^{(\gamma)} - e_{lmr} (\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\gamma)} \,.\, V_p^{(\alpha)} V_p^{(\beta)}.\end{aligned} \tag{5.7}$$

Taking $\boldsymbol{v}_\alpha = \boldsymbol{v}_\beta$, and replacing $\boldsymbol{v}_\gamma$ by $\boldsymbol{v}_\alpha$ in (5.7),

$$\operatorname{tr}\boldsymbol{v}_\beta^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2 = e_{lmr} (\boldsymbol{\Pi}_1')_{lp} V_p^{(\beta)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\beta)} V_r^{(\alpha)} - e_{lmr} (\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\alpha)} \,.\, V_p^{(\beta)} V_p^{(\beta)}. \tag{5.8}$$

Taking $\boldsymbol{v}_\beta = \boldsymbol{v}_\alpha$ in (5.8),

$$\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2 = e_{lmr} (\boldsymbol{\Pi}_1')_{lp} V_p^{(\alpha)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\alpha)} V_r^{(\alpha)} - e_{lmr} (\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\alpha)} \,.\, V_p^{(\alpha)} V_p^{(\alpha)}. \tag{5.9}$$

From (5.7), we see that each of the expressions of the form $\operatorname{tr}\boldsymbol{v}_\alpha\boldsymbol{v}_\beta\boldsymbol{\Pi}_1\boldsymbol{v}_\gamma\boldsymbol{\Pi}_2$ listed in (4.25) can be expressed as a polynomial in expressions of the forms $V_p^{(\alpha)} V_p^{(\beta)}$, $e_{lmr}(\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\gamma)}$ and $e_{lmr}(\boldsymbol{\Pi}_1')_{lp} V_p^{(\alpha)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\beta)} V_r^{(\gamma)}$, where $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ take the sets of values

$$\begin{aligned}&\left.\begin{matrix}\boldsymbol{G}_N\\ \boldsymbol{G}_K^2\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_K^2\end{matrix}\right\};\\ &\left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_N\boldsymbol{G}_P\boldsymbol{G}_Q\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{G}_M\\ \boldsymbol{G}_P\boldsymbol{G}_Q\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_N\boldsymbol{G}_P\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{G}_M\\ \boldsymbol{G}_P\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_N\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{I}\end{matrix}\right\};\\ &\left.\begin{matrix}\boldsymbol{G}_K\\ \boldsymbol{G}_K^2\end{matrix}\right\}.\end{aligned} \tag{5.10}$$

From (5.8), we see similarly that each of the expressions of the form $\operatorname{tr}\boldsymbol{v}_\beta^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2$ listed in (4.25) can be expressed as a polynomial in expressions of the forms $V_p^{(\beta)} V_p^{(\beta)}$, $e_{lmr}(\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\alpha)}$ and $e_{lmr}(\boldsymbol{\Pi}_1')_{lp} V_p^{(\beta)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\beta)} V_r^{(\alpha)}$, where $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ take the sets of values

$$\begin{aligned}&\left.\begin{matrix}\boldsymbol{G}_L^2\\ \boldsymbol{G}_N\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_K^2\end{matrix}\right\};\\ &\left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_M\boldsymbol{G}_N\boldsymbol{G}_P\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{G}_L\\ \boldsymbol{G}_N\boldsymbol{G}_P\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_M\boldsymbol{G}_N\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{G}_L\\ \boldsymbol{G}_N\end{matrix}\right\},\ \left.\begin{matrix}\boldsymbol{I}\\ \boldsymbol{G}_M\end{matrix}\right\};\\ &\left.\begin{matrix}\boldsymbol{G}_L^2\\ \boldsymbol{G}_L\end{matrix}\right\}.\end{aligned} \tag{5.11}$$

Also, we see from (5.9) that each of the expressions of the form $\operatorname{tr}\boldsymbol{v}_\alpha^2\boldsymbol{\Pi}_1\boldsymbol{v}_\alpha\boldsymbol{\Pi}_2$ listed in (4.25) can be expressed as a polynomial in expressions of the forms $V_p^{(\alpha)} V_p^{(\alpha)}$, $e_{lmr}(\boldsymbol{\Pi}_1'\boldsymbol{\Pi}_2')_{lm} V_r^{(\alpha)}$ and $e_{lmr}(\boldsymbol{\Pi}_1')_{lp} V_p^{(\alpha)} (\boldsymbol{\Pi}_2)_{mq} V_q^{(\alpha)} V_r^{(\alpha)}$, where $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$

take the sets of values
$$\left.\begin{matrix} G_L \\ G_N G_P \end{matrix}\right\}, \quad \left.\begin{matrix} G_L \\ G_N \end{matrix}\right\}, \quad \left.\begin{matrix} G_M \\ G_K^2 \end{matrix}\right\}, \quad \left.\begin{matrix} G_L^2 \\ G_L \end{matrix}\right\}. \tag{5.12}$$

The results of this section show that the traces of the matrix products (4.23), (4.24) and (4.25) can be expressed as polynomials in expressions of the forms

$$\begin{aligned} &\operatorname{tr} \mathbf{\Pi}, \\ &e_{ijk} V_k^{(\alpha)} (\mathbf{\Pi})_{ji}, \\ &V_p^{(\alpha)} V_p^{(\beta)}, \\ &V_p^{(\alpha)} (\mathbf{\Pi})_{pq} V_q^{(\beta)}, \\ \text{and} \quad &e_{lmr} (\mathbf{\Pi}_1')_{lp} V_p^{(\alpha)} (\mathbf{\Pi}_2)_{mq} V_q^{(\beta)} V_r^{(\gamma)}, \end{aligned} \tag{5.13}$$

where α, β and γ are not necessarily all different, and $\mathbf{\Pi}$, $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are matrix products formed from the matrices G_P $(P=1, 2, \ldots, \mu)$, whose possible values have been given above. From (2.2), (2.3) and (2.4), we see that each of the expressions (5.13) is invariant under the proper orthogonal group, and consequently the trace of each of the matrix products listed in (4.23), (4.24) and (4.25) is an invariant under this group*. Since it was also shown in Sections 2 to 4 that any invariant under the proper orthogonal groups of the tensors $G_{ij}^{(\alpha)}$ $(\alpha=1, 2, \ldots, \mu)$ and the vectors $V_i^{(\beta)}$ $(\beta=1, 2, \ldots, \nu)$ can be expressed as a polynomial in traces of the matrix products (4.23), (4.24) and (4.25), and matrix products formed from the matrices G_P $(P=1, 2, \ldots, \mu)$, it follows that the traces of the matrix products listed in (4.23), (4.24) and (4.25), together with the traces of the matrix products which can be formed from the matrices G_P, form an integrity basis, under the *proper* orthogonal group, for the μ symmetric second-order tensors $G_{ij}^{(\alpha)}$ $(\alpha=1, 2, \ldots, \mu)$ and the ν vectors $V_i^{(\beta)}$ $(\beta=1, 2, \ldots, \nu)$ in three dimensions.

By a similar argument, the results of Sections 2 to 5 show that the traces of the matrix products listed in (4.24), together with the traces of matrix products formed from the matrices G_P $(P=1, 2, \ldots, \mu)$ form an integrity basis under the *full* orthogonal group for the tensors $G_{ij}^{(\alpha)}$ $(\alpha=1, 2, \ldots, \mu)$ and the vectors $V_i^{(\beta)}$ $(\beta=1, 2, \ldots, \nu)$.

The integrity bases derived above are finite. They are, however, not irreducible since there are further relations between the invariants of which no account has been taken. They do have the possible advantage of symmetry with respect to interchanges among the symmetric matrices G_P and among the skew-symmetric matrices $\boldsymbol{v}_\alpha$.

6. Products of a vector and symmetric matrices

In this section we consider products of the forms $\boldsymbol{V}_\alpha \mathbf{\Pi}$ and $\mathbf{\Pi} \boldsymbol{V}_\alpha$, where $\boldsymbol{V}_\alpha$ is a vector in three dimensions, with components $V_i^{(\alpha)}$, and $\mathbf{\Pi}$ is a matrix product formed from 3×3 symmetric matrices G_β $(\beta=1, 2, \ldots, \mu)$. We note that

$$(\boldsymbol{V}_\alpha \mathbf{\Pi})_i = (\mathbf{\Pi}' \boldsymbol{V}_\alpha)_i. \tag{6.1}$$

* This result can also be derived directly, by applying (2.8) and the transformation laws for vectors and tensors to the traces of the matrix products formed from the matrices G_P and $\boldsymbol{v}_\alpha$ and listed in (4.23), (4.24) and (4.25).

Let $\boldsymbol{X}$ be an arbitrary vector with components X_i and let Π_{ij} denote the ij^{th} component of $\boldsymbol{\Pi}$. Then

$$\begin{aligned}(\boldsymbol{\Pi}\,\boldsymbol{V}_\alpha)_i &= \Pi_{ij}\,V_j^{(\alpha)}\\ &= \frac{\partial}{\partial X_i}\,(X_k\,\Pi_{kj}\,V_j^{(\alpha)}).\end{aligned} \tag{6.2}$$

Now define a skew-symmetric matrix $\boldsymbol{v}_\alpha$ as in (2.8), and a skew-symmetric matrix $\boldsymbol{x}$ in an analogous manner. Then, from (2.13)

$$X_k\,\Pi_{kj}\,V_j^{(\alpha)} = -\tfrac{1}{2}\operatorname{tr}\boldsymbol{\Pi}\operatorname{tr}\boldsymbol{x}\,\boldsymbol{v}_\alpha + \operatorname{tr}\boldsymbol{x}\,\boldsymbol{v}_\alpha\,\boldsymbol{\Pi}. \tag{6.3}$$

From Theorem 4, $\operatorname{tr}\boldsymbol{x}\boldsymbol{v}_\alpha\boldsymbol{\Pi}$ can be expressed as a polynomial in traces of matrix products, which are either products which do not involve $\boldsymbol{x}$ or $\boldsymbol{v}_\alpha$, or are products of the forms listed below the line in (4.24), with $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ replaced by $\boldsymbol{x}$ and $\boldsymbol{v}_\alpha$ respectively or $\boldsymbol{v}_\alpha$ and $\boldsymbol{x}$ respectively. Hence (6.3) may be written in the form

$$X_k\,\Pi_{kj}\,V_j^{(\alpha)} = \sum_R\,(\psi_R\operatorname{tr}\boldsymbol{x}\,\boldsymbol{v}_\alpha\,\boldsymbol{\Pi}_R + \psi'_R\operatorname{tr}\boldsymbol{v}_\alpha\,\boldsymbol{x}\,\boldsymbol{\Pi}_R), \tag{6.4}$$

where ψ_R and ψ'_R are polynomials in traces of matrix products formed from the matrices $\boldsymbol{G}_\beta$ $(\beta=1, 2, \ldots, \mu)$, and $\boldsymbol{x}\boldsymbol{v}_\alpha\boldsymbol{\Pi}_R$ and $\boldsymbol{v}_\alpha\boldsymbol{x}\boldsymbol{\Pi}_R$ are matrix products of the forms given below the line in (4.24), with $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ replaced in the first case by $\boldsymbol{x}$ and $\boldsymbol{v}_\alpha$ respectively, and in the second case by $\boldsymbol{v}_\alpha$ and $\boldsymbol{x}$ respectively.

Next, from (5.3)

$$\begin{aligned}\operatorname{tr}\boldsymbol{x}\,\boldsymbol{v}_\alpha\,\boldsymbol{\Pi}_R &= X_k\,\Pi_{kj}^{(R)}\,V_j^{(\alpha)} - X_k\,V_k^{(\alpha)}\operatorname{tr}\boldsymbol{\Pi}_R,\\ \operatorname{tr}\boldsymbol{v}_\alpha\,\boldsymbol{x}\,\boldsymbol{\Pi}_R &= V_k^{(\alpha)}\,\Pi_{kj}^{(R)}\,X_j - X_k\,V_k^{(\alpha)}\operatorname{tr}\boldsymbol{\Pi}_R,\end{aligned}$$

and hence with (6.4)

$$X_k\,\Pi_{kj}\,V_j^{(\alpha)} = \sum_R\,(\chi_R\,X_k\,\Pi_{kj}^{(R)}\,V_j^{(\alpha)} + \chi'_R\,V_k^{(\alpha)}\,\Pi_{kj}^{(R)}\,X_j), \tag{6.5}$$

where χ_R and χ'_R are polynomials in traces of matrix products formed from the matrices $\boldsymbol{G}_\beta$ $(\beta=1, 2, \ldots, \mu)$ and $\Pi_{ij}^{(R)}$ are the ij^{th} components of the matrix products $\boldsymbol{\Pi}_R$, which are matrix products formed from the matrices $\boldsymbol{G}_\beta$ $(\beta= 1, 2, \ldots, \mu)$ of the forms listed in (5.6), with the addition of the unit matrix.

From (6.2) and (6.5), it follows that

$$\boldsymbol{\Pi}\,\boldsymbol{V}_\alpha = \sum_R\,(\chi_R\,\boldsymbol{\Pi}_R\,\boldsymbol{V}_\alpha + \chi'_R\,\boldsymbol{V}_\alpha\,\boldsymbol{\Pi}_R). \tag{6.6}$$

Finally, from (6.1) we see that $\boldsymbol{V}_\alpha\boldsymbol{\Pi}$ is also expressible in the form (6.6) and, also from (6.1), that $\boldsymbol{\Pi}\,\boldsymbol{V}_\alpha$ (or $\boldsymbol{V}_\alpha\boldsymbol{\Pi}$) may be expressed in the forms

$$\boldsymbol{\Pi}\,\boldsymbol{V}_\alpha = \sum_R\,(\chi_R\,\boldsymbol{\Pi}_R\,\boldsymbol{V}_\alpha + \chi'_R\,\boldsymbol{\Pi}'_R\,\boldsymbol{V}_\alpha) = \sum_R\,(\chi_R\,\boldsymbol{V}_\alpha\,\boldsymbol{\Pi}'_R + \chi'_R\,\boldsymbol{V}_\alpha\,\boldsymbol{\Pi}_R). \tag{6.7}$$

Theorem 5. *Any product of the form $\boldsymbol{\Pi}\,\boldsymbol{V}_\alpha$ or $\boldsymbol{V}_\alpha\boldsymbol{\Pi}$ of a three-dimensional vector $\boldsymbol{V}_\alpha$ and a matrix product $\boldsymbol{\Pi}$, where $\boldsymbol{\Pi}$ is formed from 3×3 symmetric matrices $\boldsymbol{G}_\beta$ $(\beta=1, 2, \ldots, \mu)$, can be expressed as a sum, with coefficients which are polynomials in traces of matrix products formed from the matrices $\boldsymbol{G}_\beta$ $(\beta=1, 2, \ldots, \mu)$, of products* *of the forms* $\mathbf{\Pi}_R \mathbf{V}_\alpha$ $(= \mathbf{V}_\alpha \mathbf{\Pi}_R')$ *and* $\mathbf{V}_\alpha \mathbf{\Pi}_R$ $(= \mathbf{\Pi}_R' \mathbf{V}_\alpha)$, *where* $\mathbf{\Pi}_R$ *are the matrix products*

$$\begin{aligned}
&\boldsymbol{G}_K^2 \boldsymbol{G}_L^2;\\
&\boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_K^2,\ \boldsymbol{G}_N \boldsymbol{G}_K^2,\ \boldsymbol{G}_K^2;\\
&\boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P \boldsymbol{G}_Q,\ \boldsymbol{G}_M \boldsymbol{G}_N \boldsymbol{G}_P,\ \boldsymbol{G}_M \boldsymbol{G}_N,\ \boldsymbol{G}_M;\\
&\boldsymbol{G}_K^2 \boldsymbol{G}_L \boldsymbol{G}_K;\\
&\boldsymbol{I};
\end{aligned} \tag{6.8}$$

and K, L, M, N, P, Q *are integers, all different, chosen from* $1, 2, \ldots, \mu$.

7. Application to isotropic materials possessing a centre of symmetry

In [5], RIVLIN has discussed the form taken by a tensor $u_{i_1 i_2 \ldots i_\mu}$ of order μ whose components describe some physical property of a body which in its initial undeformed state is isotropic and possesses a centre of symmetry. The components of $u_{i_1 i_2 \ldots i_\mu}$ are assumed to have continuous functional dependence on the deformation gradients and on the components of ν vectors $v_p^{(\alpha)}(\tau)$ $(\alpha = 1, 2, \ldots, \nu)$ at all times τ up to t, with particular dependence on their values at time t. It is further assumed that $v_p^{(\alpha)} = 0$ $(\alpha = 1, 2, \ldots, \nu)$ in the initial state at $\tau = 0$. The deformation is described by the dependence of the coordinates $x_i(\tau)$ in a rectangular Cartesian coordinate system x of a generic particle of the body on the time τ and the coordinates X_i of the particle in the coordinate system x at time $\tau = 0$. Thus it is assumed that $u_{i_1 i_2 \ldots i_\mu}$ is a continuous functional of $\partial x_p(\tau)/\partial X_q$ and $v_p^{(\alpha)}(\tau)$ $(\alpha = 1, 2, \ldots, \nu)$ over the compact aggregate of these functions which are continuous over the range $0 \leqq \tau \leqq t$, with particular dependence on their values at time t.

The following notation is used:

$$\begin{gathered}
G_{rs} = \frac{\partial x_p(\tau)}{\partial X_r} \frac{\partial x_p(\tau)}{\partial X_s},\\
\boldsymbol{G}(\tau) = \|G_{rs}(\tau)\|, \qquad \boldsymbol{G} = \boldsymbol{G}(t),\\
V^{(\alpha)}(\tau) = \frac{\partial x_p(\tau)}{\partial X_r} v_p^{(\alpha)}(\tau), \qquad V_r^{(\alpha)} = V_r^{(\alpha)}(t),
\end{gathered} \tag{7.1}$$

and $\boldsymbol{V}_\alpha(\tau)$ and $\boldsymbol{V}_\alpha$ denote the vectors whose components are $V_r^{(\alpha)}(\tau)$ and $V_r^{(\alpha)}$ respectively.

Then it was shown in [5] that, to any required degree of approximation, $u_{i_1 i_2 \ldots i_\mu}$ can be expressed in the form

$$u_{i_1 i_2 \ldots i_\mu} = \frac{\partial x_{i_1}}{\partial X_{j_1}} \frac{\partial x_{i_2}}{\partial X_{j_2}} \cdots \frac{\partial x_{i_\mu}}{\partial X_{j_\mu}} \Phi_{j_1 j_2 \ldots j_\mu}, \tag{7.2}$$

where $\Phi_{j_1 j_2 \ldots j_\mu}$ is a sum of outer products of order μ formed from matrices of the type

$$\int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_1, \tau_2, \ldots, \tau_R) \mathbf{\Pi}^*[\boldsymbol{G}, \boldsymbol{G}(\tau_1), \boldsymbol{G}(\tau_2), \ldots, \boldsymbol{G}(\tau_R)]\, d\tau_1\, d\tau_2 \ldots d\tau_R, \tag{7.3}$$

and vectors of the types

$$\int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_1, \tau_2, \ldots, \tau_R)\, \boldsymbol{V}_\alpha \mathbf{\Pi}^*[\boldsymbol{G}, \boldsymbol{G}(\tau_1), \boldsymbol{G}(\tau_2), \ldots, \boldsymbol{G}(\tau_R)]\, d\tau_1\, d\tau_2 \ldots d\tau_R, \tag{7.4}$$

and

$$\int_0^t\int_0^t\cdots\int_0^t \chi(t,\tau_0,\tau_1,\ldots,\tau_R)\,\boldsymbol{V}_\alpha(\tau_0)\,\boldsymbol{\Pi}^*[\boldsymbol{G},\boldsymbol{G}(\tau_1),\boldsymbol{G}(\tau_2),\ldots,\boldsymbol{G}(\tau_R)]\,d\tau_0\,d\tau_1\ldots d\tau_R, \quad (7.5)$$

the coefficients of these outer products being polynomials in expressions of the types

$$\int_0^t\int_0^t\cdots\int_0^t \chi(t,\tau_1,\tau_2,\ldots,\tau_R)\,\mathrm{tr}\,\boldsymbol{\Pi}^*[\boldsymbol{G},\boldsymbol{G}(\tau_1),\boldsymbol{G}(\tau_2),\ldots,\boldsymbol{G}(\tau_R)]\,d\tau_1\,d\tau_2,\ldots d\tau_R, \quad (7.6)$$

$$\int_0^t\int_0^t\cdots\int_0^t \chi(t,\tau_1,\tau_2,\ldots,\tau_R)\,\boldsymbol{V}_\alpha\boldsymbol{\Pi}^*[\boldsymbol{G},\boldsymbol{G}(\tau_1),\boldsymbol{G}(\tau_2),\ldots,\boldsymbol{G}(\tau_R)]\,\boldsymbol{V}_\beta\,d\tau_1\,d\tau_2\ldots d\tau_R, \quad (7.7)$$

$$\begin{aligned}\int_0^t\int_0^t\cdots\int_0^t \chi(t,\tau_0,\tau_1,\ldots,\tau_R,\tau_{R+1})\,\boldsymbol{V}_\alpha(\tau_0)\times&\\ \times\boldsymbol{\Pi}^*[\boldsymbol{G},\boldsymbol{G}(\tau_1),\boldsymbol{G}(\tau_2),\ldots,\boldsymbol{G}(\tau_R)]\,\boldsymbol{V}_\beta(\tau_{R+1})\,d\tau_0\,d\tau_1\ldots d\tau_R\,d\tau_{R+1},&\end{aligned} \quad (7.8)$$

$$\begin{aligned}\int_0^t\int_0^t\cdots\int_0^t \chi(t,\tau_1,\tau_2,\ldots,\tau_R,\tau_{R+1})\,\boldsymbol{V}_\alpha\boldsymbol{\Pi}^*[\boldsymbol{G},\boldsymbol{G}(\tau_1),\boldsymbol{G}(\tau_2),&\\ \ldots,\boldsymbol{G}(\tau_R)]\,\boldsymbol{V}_\beta(\tau_{R+1})\,d\tau_1\,d\tau_2\ldots d\tau_R\,d\tau_{R+1}.&\end{aligned} \quad (7.9)$$

In (7.3) to (7.9), the kernels χ are continuous functions of their arguments, and the matrices $\boldsymbol{\Pi}^*$ are matrix products formed from the matrices $\boldsymbol{G}$, $\boldsymbol{G}(\tau_1)$, $\boldsymbol{G}(\tau_2)$, $\ldots$, $\boldsymbol{G}(\tau_R)$ which are linear in each of the matrices $\boldsymbol{G}(\tau_1)$, $\boldsymbol{G}(\tau_2)$, $\ldots$, $\boldsymbol{G}(\tau_R)$.

We consider now further restrictions which can be placed upon the matrix products $\boldsymbol{\Pi}^*$. From (3.1), it follows that, in (7.3), $\boldsymbol{\Pi}^*$ can be expressed as a matrix polynomial in which the coefficients are polynomials in traces of matrix products formed from $\boldsymbol{G}$, $\boldsymbol{G}(\tau_1)$, $\boldsymbol{G}(\tau_2)$, $\ldots$, $\boldsymbol{G}(\tau_R)$, and the matrix terms are of the forms

$$\begin{aligned}
&G\,G(\tau_{K_1})\,G^2,\ \ G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G^2;\\
&G^2,\ \ G^2\,G(\tau_{K_1}),\ \ G(\tau_{K_1})\,G^2,\ \ G^2\,G(\tau_{K_1})\,G(\tau_{K_2}),\ \ G(\tau_{K_1})\,G^2\,G(\tau_{K_2}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G^2,\ \ G^2\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\ \ G(\tau_{K_1})\,G^2\,G(\tau_{K_2})\,G(\tau_{K_3}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G^2\,G(\tau_{K_3}),\ \ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G^2;\\
&G,\ \ G\,G(\tau_{K_1}),\ \ G(\tau_{K_1})\,G,\ \ G\,G(\tau_{K_1})\,G(\tau_{K_2}),\ \ G(\tau_{K_1})\,G\,G(\tau_{K_2}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G,\ \ G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\ \ G(\tau_{K_1})\,G\,G(\tau_{K_2})\,G(\tau_{K_3}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G\,G(\tau_{K_3}),\ \ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G,\\
&G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),\ \ G(\tau_{K_1})\,G\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G\,G(\tau_{K_3})\,G(\tau_{K_4}),\ \ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G\,G(\tau_{K_4}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4})\,G;\\
&I,\ \ G(\tau_{K_1}),\ \ G(\tau_{K_1})\,G(\tau_{K_2}),\ \ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),\ \ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4})\,G(\tau_{K_5}),
\end{aligned} \quad (7.10)$$

where K_1, K_2, K_3, K_4, K_5 are integers, all different, chosen from $1, 2, \ldots, R$.

From (6.7) and (6.8), it follows that in (7.4) and (7.5), Π^* can be expressed as a matrix polynomial in which the coefficients are polynomials in traces of matrix products formed from G, $G(\tau_1)$, $G(\tau_2)$, ..., $G(\tau_R)$, and the matrix terms are of the forms

$$\begin{aligned}
&G^2 G(\tau_{K_1}) G;\\
&G^2,\ G(\tau_{K_1}) G^2,\ G^2 G(\tau_{K_1}),\ G(\tau_{K_1}) G(\tau_{K_2}) G^2,\ G^2 G(\tau_{K_1}) G(\tau_{K_2});\\
&G,\ G G(\tau_{K_1}),\ G(\tau_{K_1}) G,\ G G(\tau_{K_1}) G(\tau_{K_2}),\ G(\tau_{K_1}) G G(\tau_{K_2}),\\
&G(\tau_{K_1}) G(\tau_{K_2}) G,\ G G(\tau_{K_1}) G(\tau_{K_2}) G(\tau_{K_3}),\ G(\tau_{K_1}) G G(\tau_{K_2}) G(\tau_{K_3}),\\
&G(\tau_{K_1}) G(\tau_{K_2}) G G(\tau_{K_3}),\ G(\tau_{K_1}) G(\tau_{K_2}) G(\tau_{K_3}) G;\\
&I,\ G(\tau_{K_1}),\ G(\tau_{K_1}) G(\tau_{K_2}),\ G(\tau_{K_1}) G(\tau_{K_2}) G(\tau_{K_3}),\ G(\tau_{K_1}) G(\tau_{K_2}) G(\tau_{K_3}) G(\tau_{K_4}),
\end{aligned} \tag{7.11}$$

where K_1, K_2, K_3 and K_4 are integers, all different, chosen from $1, 2, \ldots, R$. The matrix product $G G(\tau_{K_1}) G^2$ has been omitted from (7.11) because it can be expressed in terms of $G^2 G(\tau_{K_1}) G$ and matrix products of lower degree, by means of a relation proven in [4].

From these results, it follows that $\Phi_{j_1 j_2 \ldots j_\mu}$ can be expressed as a sum of terms of the form

$$\int_0^t \int_0^t \cdots \int_0^t \vartheta(t, \tau_1, \tau_2, \ldots, \tau_R)\, \Theta_{j_1 j_2 \ldots j_\mu}(t, \tau_1, \tau_2, \ldots, \tau_R)\, d\tau_1\, d\tau_2 \ldots d\tau_R, \tag{7.12}$$

in which $\Theta_{j_1 j_2 \ldots j_\mu}$ is an outer product of order μ formed from matrices Π_1^* and vectors $V_\alpha \Pi_2^*$ and $V_\alpha(\tau_M)\Pi_2^*$, where Π_1^* and Π_2^* are matrices of the forms listed in (7.10) and (7.11) respectively, such that $\Theta_{j_1 j_2 \ldots j_\mu}(t, \tau_1, \tau_2, \ldots, \tau_R)$ is linear in each of the matrices $G(\tau_{K_\varrho})$ and in each of the vectors $V_\alpha(\tau_{M_\sigma})$ which occurs as factors in $\Theta_{j_1 j_2 \ldots j_\mu}(t, \tau_1, \tau_2, \ldots, \tau_R)$, and the integers K_ϱ and M_σ are a permutation of the integers $1, 2, \ldots, R$. If $\Theta_{j_1 j_2 \ldots j_\mu}(t, \tau_1, \tau_2, \ldots, \tau_R)$ is an outer product of say γ matrices Π_1^* and δ vectors $V_\alpha \Pi_2^*$ and $V_\alpha(\tau_M)\Pi_2^*$, then

$$\mu = 2\gamma + \delta. \tag{7.13}$$

Since each matrix Π_1^* involves at most five distinct arguments τ_{K_ϱ}, and each vector $V_\alpha(\tau_M)\Pi_2^*$ also involves at most five distinct arguments τ_{M_σ} and τ_{K_ϱ}, it follows that $\Theta_{j_1 j_2 \ldots j_\mu}(t, \tau_1, \tau_2, \ldots, \tau_R)$ involves at most 5μ distinct arguments, apart from t. Thus in (7.12) we have $R \leq 5\mu$. As a special case we may have $R=0$, in which case $\Theta_{j_1 j_2 \ldots j_\mu}$ is a function of t only, that is, it depends only on G and V_α.

The coefficient $\vartheta(t, \tau_1, \tau_2, \ldots, \tau_R)$ in (7.12) is a continuous function of its arguments and a polynomial in expressions of the types (7.6), (7.7), (7.8) and (7.9). Expressions of the type (7.6) were considered in a previous paper [2], where it was shown that $\operatorname{tr}\Pi^*$ in (7.6) can be expressed as a polynomial in traces of matrix products of the following forms:

$$\begin{aligned}
&G,\ G^2,\ G^3;\\
&G\,G(\tau_{K_1}),\ G^2\,G(\tau_{K_1});\\
&G\,G(\tau_{K_1})\,G(\tau_{K_2}),\ G^2\,G(\tau_{K_1})\,G(\tau_{K_2}),\ G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\\
&G^2\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G\,G(\tau_{K_3})\,G^2;\\
&G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),\ G^2\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4});\\
&G\,G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4})\,G(\tau_{K_5});\\
&G(\tau_{K_1}),\ G(\tau_{K_1})\,G(\tau_{K_2}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4})\,G(\tau_{K_5}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4})\,G(\tau_{K_5})\,G(\tau_{K_6}),
\end{aligned}\tag{7.14}$$

where $K_1, K_2, \ldots, K_6$ are integers, all different, chosen from $1, 2, \ldots, R$. Also, from (2.13) and (5.6), we see that $V_\alpha \Pi^* V_\beta$ in (7.7), $V_\alpha(\tau_0)\Pi^* V_\beta(\tau_{R+1})$ in (7.8), and $V_\alpha \Pi^* V_\beta(\tau_{R+1})$ in (7.9) can be replaced by $V_\alpha P_1 V_\beta + V_\beta P_2 V_\alpha$, $V_\alpha(\tau_0) P_1 V_\beta(\tau_{R+1}) + V_\beta(\tau_{R+1}) P_2 V_\alpha(\tau_0)$, and $V_\alpha P_1 V_\beta(\tau_{R+1}) + V_\beta(\tau_{R+1}) P_2 V_\alpha$ respectively, where P_1 and P_2 are matrix polynomials, with coefficients which are traces of matrix products listed in (7.14), and matrix terms of the forms

$$\begin{aligned}
&G^2\,G(\tau_{K_1})\,G,\\
&G^2,\ G(\tau_{K_1})\,G^2,\ G(\tau_{K_1})\,G(\tau_{K_2})\,G^2,\\
&G,\ G(\tau_{K_1})\,G,\ G(\tau_{K_1})\,G\,G(\tau_{K_2}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G,\\
&G(\tau_{K_1})\,G(\tau_{K_2})\,G\,G(\tau_{K_3}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G;\\
&I,\ G(\tau_{K_1}),\ G(\tau_{K_1})\,G(\tau_{K_2}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3}),\ G(\tau_{K_1})\,G(\tau_{K_2})\,G(\tau_{K_3})\,G(\tau_{K_4}),
\end{aligned}\tag{7.15}$$

where K_1, K_2, K_3, K_4 are integers, all different, chosen from $1, 2, \ldots, R$. It follows, by an argument similar to that used in [6], that the coefficients $\vartheta(t, \tau_1, \tau_2, \ldots, \tau_R)$ $(R \leqq 5\mu)$ can be expressed as continuous functions of their arguments and polynomials in expressions of the types

$$\begin{aligned}
&\int_0^t\int_0^t\cdots\int_0^t \psi(t, \tau_1, \tau_2, \ldots, \tau_S)\,\operatorname{tr}\Pi_3^*[G, G(\tau_1), G(\tau_2), \ldots, G(\tau_S)]\,d\tau_1\,d\tau_2\ldots d\tau_S,\\
&\int_0^t\int_0^t\cdots\int_0^t \psi(t, \tau_1, \tau_2, \ldots, \tau_T)\,V_\alpha\Pi_4^*[G, G(\tau_1), G(\tau_2), \ldots, G(\tau_T)]\,V_\beta\,d\tau_1\,d\tau_2\ldots d\tau_T,\\
&\int_0^t\int_0^t\cdots\int_0^t \psi(t, \tau_0, \tau_1, \ldots, \tau_T, \tau_{T+1})\,V_\alpha(\tau_0)\,\Pi_4^*[G, G(\tau_1), G(\tau_2), \ldots, G(\tau_T)]\times\\
&\qquad\qquad\times V_\beta(\tau_{T+1})\,d\tau_0\,d\tau_1\ldots d\tau_T\,d\tau_{T+1},\\
&\int_0^t\int_0^t\cdots\int_0^t \psi(t, \tau_1, \tau_2, \ldots, \tau_T, \tau_{T+1})\,V_\alpha\Pi_4^*[G, G(\tau_1), G(\tau_2), \ldots, G(\tau_T)]\times\\
&\qquad\qquad\times V_\beta(\tau_{T+1})\,d\tau_1\,d\tau_2\ldots d\tau_T\,d\tau_{T+1},\\
&\int_0^t\int_0^t\cdots\int_0^t \psi(t, \tau_0, \tau_1, \ldots, \tau_T)\,V_\alpha(\tau_0)\,\Pi_4^*[G, G(\tau_1), G(\tau_2), \ldots, G(\tau_T)]\times\\
&\qquad\qquad\times V_\beta\,d\tau_0\,d\tau_1\ldots d\tau_T,
\end{aligned}\tag{7.16}$$

where the kernels ψ are continuous functions of their arguments, Π_3^* are matrix products listed in (7.14), with $K_1, K_2, \ldots, K_6$ chosen from the integers $1, 2, \ldots, S$, and Π_4^* are matrix products listed in (7.15), with K_1, K_2, K_3, K_4 chosen from the integers $1, 2, \ldots, T$. Since the matrix products in (7.14) involve at most 6 distinct arguments, apart from t, and the matrix products in (7.15) involve at most 4 distinct arguments, apart from t, we have $S \leqq 6$, $T \leqq 4$.

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University of Nottingham
and
Brown University
Providence, Rhode Island

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