On the Operator of Symmetric Differentiation on a Compact Riemannian Manifold with Boundary

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Abstract

We state a particular case of one of the theorems which we shall prove. Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary and let $\sigma=(\sigma_{ij})$ be a symmetric second-order tensor with components $\sigma_{ij}\in H^k(\Omega)$ for some (positive or negative) integer k; H^k are Sobolev spaces on Ω. Then we have $\sigma_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}$ for some $u_i\in H^{k+1}(\Omega)$, $i=1,\dots,n$, if and only if $\sum_{i,j}\int_\Omega \sigma_{ij}\,\omega^{ij}\,dx=0$ (if $k<0$, the integral is in fact a duality) for any symmetric tensor ω with components $\omega^{ij}\in H_0^\infty(\Omega) = \bigcap_{m\geqq 0} H_0^m(\Omega)$ and such that $\sum_j \frac{\partial \omega^{ij}}{\partial x_j}=0$. Some applications in the theory of elasticity are also given.

1. Introduction

The aim of this paper is to prove an analogue of DE RHAM's theorem ([13] §22, th. 17′) for the space of symmetric second-order tensors having (tempered) distributions as components, the operator of exterior derivation being replaced by the operator of "symmetric differentiation". We work on a compact Riemannian manifold with boundary Γ. If $\Gamma=\emptyset$, my results coincide with those of BERGER & EBIN [1] (their first decomposition theorem can easily be extended, by their methods, to Sobolev spaces of any order if $\Gamma=\emptyset$). If $\Gamma\neq\emptyset$, my theorem is a generalization of TING's results [14] in two directions: first, we show that the condition (3.4) of [14] can be replaced by $S=0$ on $\Gamma(\equiv\partial M)$ (it is very difficult, in my opinion, to do this using TING's method) and second, the symmetric second-order tensors are allowed to have arbitrary (tempered) distributions as components, not only square integrable functions.

In order to prove their results, BERGER & EBIN and TING followed essentially the same idea, namely, they reduced the problem to the proof of the existence and regularity of the solution for some elliptic system of equations (this is SPENCER's method; see KOHN & NIRENBERG [8]). My proof of the decomposition theorem (Corollary 4.2) is much easier being based only on a regularity theorem for the gradient operator and some abstract lemmas. I use an elliptic type argument only to show that this decomposition is "regular" (Theorem 4.9.).

Let us now say some words about the organization of this paper. In the rest of the introduction, the necessary definitions and notations will be given. Section 2 is devoted to some abstract considerations. Then in Section 3 we study some properties of the gradient operator. Probably all the results of this section are well known, but I have not found any reference for the essential theorem which I shall use, namely Theorem 3.1. All results are easy consequences of this theorem. In Section 3 I give also a very simple proof of a particular case of the de Rham theorem, a decomposition very useful in hydrodynamics. The decomposition theorems related to the operator of symmetric differentiation are given in Section 4. Section 5 has a special character, being devoted to a somewhat different subject. My interest in the problems treated in this paper was aroused by reading [14]. In the introduction of [14] St. Venant's compatibility conditions for "linear strain tensors" are criticised. The reproaches are the following: first "the differentiability requirements are unnecessarily strong" and second "the full set of compatibility conditions is still insufficient even for a multiply connected domain in Euclidean space". The aim of Section 5 is to remedy these defects. When the manifold is Euclidean St. Venant's compatibility conditions are augmented by a finite number of linear conditions (number equal to zero if and only if the manifold is simply connected) and we prove the conditions obtained to be necessary and sufficient even in a multiply connected domain; we do not use any requirements of differentiability beyond those necessary. I have not treated a general Riemannian manifold, for which difficulties of a different nature arise (see [15] sect. 84).

We shall always work on a C^∞, n-dimensional, Riemannian, compact, orientable and oriented manifold $\bar{\Omega}$, with a C^∞ boundary Γ. We denote by $\Omega = \bar{\Omega} \setminus \Gamma$ the interior of $\bar{\Omega}$ and $*1_\Omega$ (respectively $*1_\Gamma$) the canonical volume form on $\bar{\Omega}$ (respectively Γ; on Γ the induced Riemann structure and orientation are considered). We identify covariant and contravariant tensors using the Riemannian structure (we speak about covariant $t_{ij\ldots}$, or contravariant $t^{ij\ldots}$, *etc.*, components of a tensor t in some local coordinates). If u, v are first order tensors on $\bar{\Omega}$ we define the function $(u,v)\colon \bar{\Omega} \to \mathbb{C}$ by $(u,v)(x) = \overline{u_i(x)}\, v^i(x)$, $x \in \bar{\Omega}$ (components in local coordinates). Similarly, if ω, w are symmetric second-order tensors, then $(\omega, w)(x) = \frac{1}{2}\overline{\omega_{ij}(x)}\, w^{ij}(x)$ is a function on $\bar{\Omega}$.

Let $H(\Omega)$ (respectively $\vec{H}(\Omega)$, $\vec{\vec{H}}(\Omega)$) be the Hilbert space of square-integrable functions (respectively first-order tensors, symmetric second-order tensors) on Ω, with scalar product $(f,g)_{0,\Omega} = \int_\Omega \bar{f} g * 1_\Omega$ (if f, g are first order, or symmetric second-order tensors, then $\bar{f}g$ is replaced by (f,g)). For any $s \in \mathbb{R}$, $H^s(\Omega)$, $\vec{H}^s(\Omega)$,

$\overset{\leftrightarrow}{H}^s(\Omega)$ are the corresponding Sobolev spaces (see [9]). We recall that they are Hilbertizable topological vector spaces such that $H(\Omega)=H^0(\Omega)$, *etc.*, as topological vector spaces. If $s\geqq 0$, then $H^{-s}(\Omega)$ is identified with the strong antidual of $H_0^s(\Omega)$, so that if $s_1\leqq s_2$, then $H^{s_2}(\Omega)\subset H^{s_1}(\Omega)$ linearly. Similarly for $\vec{H}^s$ and $\overset{\leftrightarrow}{H}^s$. The antiduality $\langle\cdot,\cdot\rangle$ between H_0^s and H^{-s} ($s\geqq 0$) is defined so that if $f\in H_0^s$ and $g\in H^0\subset H^{-s}$, then $\langle f,g\rangle=(f,g)_{0,\Omega}$. Similarly for the other cases ($\vec{H}$ and $\overset{\leftrightarrow}{H}$). $H^\infty(\Omega)=\bigcap_{s\geqq 0} H^s(\Omega)$, $H_0^\infty(\Omega)=\bigcap_{s\geqq 0} H_0^s(\Omega)$ (projective limits) are the spaces of all C^∞ functions on $\bar{\Omega}$, respectively of all C^∞ functions on $\bar{\Omega}$ which are zero, together with all their derivatives (in local coordinates), on Γ. The (projective limit) topology of $H^\infty(\Omega)$ coincides with the topology of the uniform convergence on $\bar{\Omega}$ of the function and of all its derivatives (in local charts). $H_0^\infty(\Omega)$ is a closed subspace of $H^\infty(\Omega)$, its (projective limit) topology being identical with that induced by $H^\infty(\Omega)$. Then $H^{-\infty}(\Omega)=\bigcup_{s\geqq 0} H^{-s}(\Omega)$ (inductive limit), the strong antidual of $H_0^\infty(\Omega)$, is the space of all "tempered distributions" on Ω. Similarly, $\vec{H}^{-\infty}(\Omega)$, $\overset{\leftrightarrow}{H}^{-\infty}(\Omega)$ will be the spaces of first order (respectively symmetric second order) tensors having as components tempered distributions on Ω.

We denote by v the first-order tensor on $\bar{\Omega}$ defined as follows, only for points on Γ: if $x\in\Gamma$, then $v(x)$ is the outward unit normal to Γ at x. Let T be a C^1, p^{th}-order tensor on $\bar{\Omega}$, ∇T its covariant derivative (a $(p+1)^{\text{st}}$-order tensor on $\bar{\Omega}$ with covariant components $\nabla_{i_1} T_{i_2\ldots i_{p+1}}$) and S a C^1, $(p+1)^{\text{st}}$-order tensor on $\bar{\Omega}$. Then Stokes' theorem is equivalent to

$$\int_\Omega \nabla_{i_1} T_{i_2\ldots i_{p+1}}\cdot S^{i_1\ldots i_{p+1}} * 1_\Omega+\int_\Omega T_{i_1\ldots i_p}\cdot \nabla_j S^{j i_1\ldots i_p} * 1_\Omega$$
$$=\int_\Gamma T_{i_1\ldots i_p} v_j S^{j i_1\ldots i_p} * 1_\Gamma$$

(the expressions under the integrals have an invariant meaning). If f is a C^1 function on $\bar{\Omega}$, we define $\operatorname{grad} f=\nabla f$. Then the *symmetric derivative* of a C^1 first-order tensor u on $\bar{\Omega}$ will be the symmetric second order tensor $\varepsilon(u)$ on $\bar{\Omega}$ with covariant components $\varepsilon(u)_{ij}=\nabla_i u_j+\nabla_j u_i$. The divergence of u is $\operatorname{div} u=\nabla^i u_i$, and the divergence of a symmetric C^1 second-order tensor ω on $\bar{\Omega}$ will be the first order tensor $(\operatorname{Div}\omega)_i=\nabla^j\omega_{ij}$. Clearly

(1) $$(\operatorname{grad} f, u)_{0,\Omega}+(f,\operatorname{div} u)_{0,\Omega}=\int_\Gamma \bar{f} v^i u_i * 1_\Gamma,$$

(2) $$(\varepsilon(u),\omega)_{0,\Omega}+(u,\operatorname{Div}\omega)_{0,\Omega}=\int_\Gamma \bar{u}_i v_j \omega^{ij} * 1_\Gamma.$$

Using these formulas, we define the continuous operators

$$\operatorname{grad}: H^{-\infty}(\Omega)\to\vec{H}^{-\infty}(\Omega),\qquad \varepsilon: \vec{H}^{-\infty}(\Omega)\to\overset{\leftrightarrow}{H}^{-\infty}(\Omega),$$
$$\operatorname{div}: \vec{H}^{-\infty}(\Omega)\to H^{-\infty}(\Omega),\qquad \operatorname{Div}: \overset{\leftrightarrow}{H}^{-\infty}(\Omega)\to\vec{H}^{-\infty}(\Omega)$$

in the usual way. If $s\neq\frac{1}{2}$, then their restrictions to a space with index s is a continuous operator into the space with index $s-1$ (see [9], Chapter 1, §12.8).

For example, if $s \neq \frac{1}{2}$ then ε: $\vec{H}^s(\Omega) \to \vec{H}^{s-1}(\Omega)$ is continuous. The object of this paper is, essentially, to show that it has a closed range, too.

2. Abstract Preliminaries

The following lemma is due to PEETRE; see [9] Chapter 2, §5.2 for a proof.

Lemma 2.1. *Let X_1, X_2, Y be Banach spaces, i: $X_1 \to Y$ linear and compact, T: $X_1 \to X_2$ linear and continuous. Then the following conditions are equivalent:*

1) *The kernel of T is finite-dimensional, the image of T is closed and if $Tx=0$, $ix=0$, then $x=0$.*
2) *There is a finite constant c such that for any $x \in X_1$:*

$$\|x\| \leqq c(\|Tx\| + \|ix\|).$$

Remark. The same proof easily gives the following assertion: Let X_1, X_2 be Banach spaces and let T: $X_1 \to X_2$ be linear and continuous, having finite dimensional kernel and closed image. Suppose $|\cdot|_1$ is a continuous seminorm on X_1 such that $x \in \operatorname{Ker} T$, $x \neq 0 \Rightarrow |x|_1 \neq 0$. Then there is a finite constant c such that for any $x \in X_1$

$$\|x\|_{X_1} \leqq c(\|Tx\|_{X_2} + |x|_1).$$

Suppose now that X_i, Y_i $(i=1,2)$ are reflexive Banach spaces such that $X_i \subset Y_i$ continuously and densely. If we denote X_i^*, Y_i^* their strong antiduals (spaces of continuous antilinear forms), we will also have $Y_i^* \subset X_i^*$ continuously and densely. Let T: $X_1 \to X_2$ be linear, continuous and having a continuous extension $\tilde{T}$: $Y_1 \to Y_2$. It follows that the adjoint T^*: $X_2^* \to X_1^*$ of T is an extension of the adjoint $\tilde{T}^*$: $Y_2^* \to Y_1^*$ of $\tilde{T}$. In particular $\operatorname{Ker} \tilde{T}^* \subset \operatorname{Ker} T^*$, and we ask under what conditions is the first subspace dense in the second one.

Lemma 2.2. *If T and $\tilde{T}$ have closed images, then* $\operatorname{Ker} \tilde{T}^*$ *is dense in* $\operatorname{Ker} T^*$ *if and only if* $\operatorname{Im} T = \operatorname{Im} \tilde{T} \cap X_2$. *If, moreover,* $\operatorname{Ker} \tilde{T} \subset X_1$, *then this is clearly equivalent to:*

$$y \in Y_1 \quad \text{and} \quad \tilde{T} y \in X_2 \Rightarrow y \in X_1.$$

Proof. Since X_2 is reflexive, $\operatorname{Ker} \tilde{T}^*$ is dense in $\operatorname{Ker} T^*$ if and only if $\operatorname{Ker} \tilde{T}^*$ and $\operatorname{Ker} T^*$ have the same polar set in X_2. But the polar set of $\operatorname{Ker} T^*$ is $\overline{\operatorname{Im} T}$, the closure of the image of T. It is easily seen that the polar set of $\operatorname{Ker} \tilde{T}^*$ is the intersection of X_2 with the closure of the image of $\tilde{T}$ (in the topology of Y_2). Thus we obtain $\overline{\operatorname{Im} T} = X_2 \cap \overline{\operatorname{Im} \tilde{T}}$ as a necessary and sufficient condition. Q.E.D.

Lemma 2.3. *Suppose the inclusion $X_1 \subset Y_1$ compact, and suppose the condition: $\{y \in Y_1$ and $\tilde{T} y \in X_2 \Rightarrow y \in X_1\}$ verified. Then there is a finite constant c such that for any $x \in X_1$:*

$$(*) \qquad \|x\|_{X_1} \leqq c(\|Tx\|_{X_2} + \|x\|_{Y_1}).$$

In particular, $\operatorname{Ker} T = \operatorname{Ker} \tilde{T}$ *is finite-dimensional, and the image of T is closed.*

Proof. Let $D=\{y\in Y_1 \mid \tilde{T}y\in X_2\}$ with the graph topology. It is clearly a Banach space topology, $X_1\subset D$ continuously and $X_1=D$ as vector spaces. We obtain $X_1=D$ as topological vector spaces using a theorem of Banach. In particular (*) is true. Then we apply PEETRE's lemma. Q.E.D.

For any positive integer $k\geqq 0$ let $\mathscr{H}^k$ be a Hilbertizable topological vector space, and let $\mathscr{H}_0^k$ be a closed subspace of $\mathscr{H}^k$. We suppose

(I) if $k\geqq 0$, then $\mathscr{H}^{k+1}\subset\mathscr{H}^k$, the inclusion being compact and with dense image.

(II) for any $k\geqq 0$, $\mathscr{H}_0^{k+1}$ is a dense subspace of $\mathscr{H}_0^k$ ($\mathscr{H}_0^{k+1}$ is embedded in $\mathscr{H}^k$, by (I)) and $\mathscr{H}_0^0=\mathscr{H}^0$.

(III) on $\mathscr{H}^0$ is given a scalar product $(\cdot,\cdot)$, antilinear in the first variable, which defines its topology. We denote by $\mathscr{H}$ the corresponding Hilbert space.

If $k\geqq 0$, we define $\mathscr{H}^{-k}$ as the strong antidual of $\mathscr{H}_0^k$. We identify $\mathscr{H}\equiv\mathscr{H}^0\equiv\mathscr{H}^{-0}$ using the Riesz theorem. Then by transposition in $\mathscr{H}_0^{k+1}\subset\mathscr{H}_0^k\subset\mathscr{H}$ we get $\mathscr{H}\subset\mathscr{H}^{-k}\subset\mathscr{H}^{-k-1}$, the injections being compact and with dense images. In conclusion, we obtain $\mathscr{H}^k\subset\mathscr{H}^m$, the inclusion being compact and with dense image, for any integers $k\geqq m$.

Let $\mathscr{H}_0^\infty=\bigcap_{k\geqq 0}\mathscr{H}_0^k$, $\mathscr{H}^\infty=\bigcap_{k\geqq 0}\mathscr{H}^k$ with the projective limit topologies ($\mathscr{H}_0^\infty$ is a closed subspace of $\mathscr{H}^\infty$ and its topology coincides with the induced one). They are reflexive Fréchet spaces, the strong antidual of $\mathscr{H}_0^\infty$ being $\mathscr{H}^{-\infty}=\bigcup_{k\geqq 0}\mathscr{H}^{-k}$ (inductive limit topology). $\mathscr{H}^\infty$ is dense in any $\mathscr{H}^k$, $k\in\mathbb{Z}$ (=set of integers). We denote $\langle\cdot,\cdot\rangle$: $\mathscr{H}_0^\infty\times\mathscr{H}^{-\infty}\to\mathbb{C}$ the canonical sesquilinear form, antilinear in the first variable; its restriction to $\mathscr{H}_0^\infty\times\mathscr{H}^{-k}$ (some $k\geqq 0$) coincides with the restriction to the same set of the antiduality $\mathscr{H}_0^k\times\mathscr{H}^{-k}\to\mathbb{C}$, which will also be denoted $\langle\cdot,\cdot\rangle$. In particular, if $k\geqq 0$, $x\in\mathscr{H}_0^k$, $y\in\mathscr{H}\subset\mathscr{H}^{-k}$, then $\langle x,y\rangle=(x,y)$ =scalar product in $\mathscr{H}$.

Suppose $\{\mathscr{K}_0^k,\mathscr{K}^k\}$ is another set of Hilbertizable topological vector spaces with properties similar to $\{\mathscr{H}_0^k,\mathscr{H}^k\}$. Let S_0: $\mathscr{H}_0^\infty\to\mathscr{K}_0^\infty$, T_0: $\mathscr{K}_0^\infty\to\mathscr{H}_0^\infty$ be two linear continuous operators such that $(S_0x,y)=(x,T_0y)$ for any $x\in\mathscr{H}_0^\infty$, $y\in\mathscr{K}_0^\infty$. Then S_0 (respectively T_0) has a unique extension to a continuous application S: $\mathscr{H}^{-\infty}\to\mathscr{K}^{-\infty}$ (respectively T: $\mathscr{K}^{-\infty}\to\mathscr{H}^{-\infty}$) equal to the adjoint of T_0 (respectively S_0), i.e. $\langle T_0x,y\rangle=\langle x,Sy\rangle$ (respectively $\langle S_0x,y\rangle=\langle x,Ty\rangle$) if $x\in\mathscr{K}_0^\infty$, $y\in\mathscr{H}^{-\infty}$ (respectively $x\in\mathscr{H}_0^\infty$, $y\in\mathscr{K}^{-\infty}$).

Theorem 2.4. *Suppose that, for any integer k, T maps $\mathscr{K}^{k+1}$ into $\mathscr{H}^k$ and enjoys the following regularity: if $u\in\mathscr{K}^{-\infty}$ and $Tu\in\mathscr{H}^k$, then $u\in\mathscr{K}^{k+1}$. Then the following assertions are true:*

1) *T: $\mathscr{K}^{-\infty}\to\mathscr{H}^{-\infty}$ has a closed image, equal to the polar set of* $\operatorname{Ker}S_0$, *i.e.*

$$\operatorname{Im}T=\{u\in\mathscr{H}^{-\infty}\mid\langle x,u\rangle=0 \text{ if } x\in\mathscr{H}_0^\infty \text{ and } S_0x=0\}.$$

2) $\operatorname{Ker}T$ *is a finite-dimensional subspace of $\mathscr{K}^\infty$.*

3) *For any integer k the restriction $T^{(k)}=T\mid\mathscr{K}^{k+1}:\mathscr{K}^{k+1}\to\mathscr{H}^k$ is continuous and has a closed image equal to*

$$\{u\in\mathscr{H}^k\mid\langle x,u\rangle=0 \text{ if } x\in\mathscr{H}_0^\infty \text{ and } S_0x=0\}.$$

4) *Let* $\|\cdot\|_k$ *be a norm on* $\mathscr{K}^k$ *(or* $\mathscr{H}^k$*) defining its topology and* $|\cdot|_{k+1}$ *a continuous seminorm on* $\mathscr{K}^{k+1}$ *such that* $x \neq 0$, $Tx=0 \Rightarrow |x|_{k+1} \neq 0$. *Then for any integer k there is a finite constant c such that if* $u \in \mathscr{K}^{k+1}$,

$$\|u\|_{k+1} \leqq c(\|Tu\|_k + |u|_{k+1}).$$

5) *Let* E_0 *be the closure in* $\mathscr{H}$ *of* $\operatorname{Ker} S_0 = \{x \in \mathscr{H}_0^\infty \mid S_0 x = 0\}$ *and* $F = T(\mathscr{K}^1) \equiv \operatorname{Im} T^{(0)}$. *Then* $\mathscr{H} = E_0 \oplus F$ *(Hilbert direct sum).*

Proof. $T^{(k)}$: $\mathscr{K}^{k+1} \to \mathscr{H}^k$ is continuous, since it has a closed graph. By Lemma 2.3, $\operatorname{Ker} T^{(k)}$ (which is clearly equal to $\operatorname{Ker} T$, and $\operatorname{Ker} T \subset \mathscr{K}^\infty$) is finite-dimensional and $\operatorname{Im} T^{(k)}$ is closed, for any integer k. Then 4) is a consequence of the remark following Lemma 2.1. If $k \geqq 0$, it is easily seen that the adjoint of $T^{(-k-1)}$: $\mathscr{K}^{-k} \to \mathscr{K}^{-k-1}$ is the restriction $S_0^{(k)}$: $\mathscr{H}_0^{k+1} \to \mathscr{K}_0^k$ of S to $\mathscr{H}_0^{k+1}$. By Lemma 2.2, $E_0^{(k+2)} = \operatorname{Ker} S_0^{(k+1)}$ is a dense subspace of $E_0^{(k+1)} = \operatorname{Ker} S_0^{(k)}$ $(k \geqq 0)$. We prove that $E_0^{(\infty)} = \operatorname{Ker} S_0 = \bigcap_{k \geqq 1} E_0^{(k)}$ is dense in any $E_0^{(k)}$. Let $x_0 \in E_0^{(k)}$ and $\varepsilon > 0$. For $k \geqq 0$ let $\|\cdot\|_k$ be a norm on $\mathscr{H}_0^k$ defining its topology and such that $\|\cdot\|_k \leqq \|\cdot\|_{k+1}$ on $\mathscr{H}_0^{k+1}$. For any $i \geqq 0$ let $x_{i+1} \in E_0^{(k+i+1)}$ such that $\|x_i - x_{i+1}\|_{k+i} \leqq \varepsilon 2^{-i-1}$. Then the sequence $\{x_i\}$ is convergent in any $\mathscr{H}_0^m$, since, if $j \geqq i \geqq m$,

$$\|x_i - x_j\|_m \leqq \sum_{p=i}^{j-1} \|x_p - x_{p+1}\|_m \leqq \sum_{p=i}^{j-1} \|x_p - x_{p+1}\|_{k+p} \leqq \frac{\varepsilon}{2^i}$$

so that $x = \lim x_i \in \mathscr{H}_0^\infty$, $S_0 x = 0$ and

$$\|x_0 - x\|_k \leqq \sum_{i=0}^{\infty} \|x_i - x_{i+1}\|_{k+i} \leqq \varepsilon,$$

i.e. the assertion regarding density is proved. If now we use the closed range theorem, we obtain Assertion 3) of the theorem for $k = -m-1$ and $m \geqq 0$ (since $\operatorname{Im} T^{(-m-1)} =$ polar in $\mathscr{H}^{-m-1}$ of $\operatorname{Ker} S_0^{(m)}$).

Let us prove Assertion 1). The closed range theorem states that $\overline{\operatorname{Im} T} = (\operatorname{Ker} S_0)^0$. Let $u \in \mathscr{H}^{-\infty}$ with $u \in (\operatorname{Ker} S_0)^0$. There is a $k \geqq 0$ such that $u \in \mathscr{H}^{-k-1}$. By the part of Assertion 3) just proved, $u \in \operatorname{Im} T^{(-k-1)}$; in particular $u \in \operatorname{Im} T$, which proves Assertion 1). Then Assertion 3) for any k is easily obtained.

We know $F = T(\mathscr{K}^1) = \operatorname{Im} T^{(0)}$ is closed in $\mathscr{H}$ and $F = \{u \in \mathscr{H} \mid \langle x, u\rangle \equiv (x, u) = 0$ for any $x \in \operatorname{Ker} S_0\}$, *i.e.* $\mathscr{H} = E_0 \oplus F$. Q.E.D.

3. Regularity for the Gradient Operator and Consequences

The main technical point of this paper is the theorem which we now prove. While the result must be known, I have not found it stated in this generality. Theorem 3.2, Chapter 3 of DUVAUT & LIONS [4] asserts a result very near to the following one. Also, the theorem which we shall prove is a generalization of Theorem 9.7 in Chapter 1 of LIONS & MAGENES [9] for the case $s \leqq 0$ (this is essential for us). In fact, my proof is almost the same as LIONS & MAGENES', the only difference being in the choice of the extension operator P.

Theorem 3.1. *Let* $s \in \mathbb{R}$, $s \neq -(k+\frac{1}{2})$ *if* k *is a non-negative integer. If* $f \in H^{-\infty}(\Omega)$ *and* $\operatorname{grad} f \in \vec{H}^s(\Omega)$, *then* $f \in H^{s+1}(\Omega)$.

Proof. We may suppose that $f \in H^s(\Omega)$. In fact, if the assertion is proved in this case (for any s), we obtain the general case in the following way: Clearly there is $t \leqq s$ with $f \in H^t(\Omega)$; since $\operatorname{grad} f \in \vec{H}^s(\Omega) \subset \vec{H}^t(\Omega)$, we have $f \in H^{t+1}(\Omega)$. We repeat the argument until we obtain $f \in H^{t+k}(\Omega)$, $s \leqq t+k < s+1$; a new step gives $f \in H^{s+1}(\Omega)$, which finishes the proof. If $f \in H^s$, $\operatorname{grad} f \in \vec{H}^s$ and $\theta \in H^\infty(\Omega)$, then $\theta f \in H^s$, $\operatorname{grad}(\theta f) \in \vec{H}^s$. Using a partition of unity, we reduce the problem to the case in which f has as support a compact subset of a domain of chart. Then using local coordinates, we reduce the problem still further to the case in which $f \in H^s(\mathbb{R}^n_-)$, $\partial_j f \in H^s(\mathbb{R}^n_-)$ $(j=1, \ldots, n)$, where $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x^1 < 0\}$, $\partial_j \equiv \frac{\partial}{\partial x^j}$. We must show $f \in H^{s+1}(\mathbb{R}^n_-)$.

Let m be an integer, $|s| \leqq m-2$. If u is a function on $\mathbb{R}^n_-$, let Pu be the function on $\mathbb{R}^n$ defined by:

$$(Pu)(x) = \begin{cases} u(x) & \text{if } x^1 < 0 \\ \sum_{k=1}^{2m} \alpha_k u(a_k x^1, x'') & \text{if } x^1 > 0 \end{cases}$$

where $a_k < 0$ and $\sum_{k=1}^{2m} \alpha_k (a_k)^j = 1$ if $-m \leqq j \leqq m-1$ (see [9] Lemma 12.2, Chapter 1). We have set $x = (x^1, x'')$, $x'' = (x^2, \ldots, x^n)$. Then P: $H^m(\mathbb{R}^n_-) \to H^m(\mathbb{R}^n)$ is continuous and has an extension to a continuous operator $H^{-m}(\mathbb{R}^n_-) \to H^{-m}(\mathbb{R}^n)$, which also we denote by P; $Pu \mid \mathbb{R}^n_- = u$ for any $u \in H^{-m}(\mathbb{R}^n_-)$ (see Lions & Magenes, *loc. cit.*). By interpolation P: $H^t(\mathbb{R}^n_-) \to H^t(\mathbb{R}^n)$ is continuous if $|t| \leqq m$, $t \neq -k-\frac{1}{2}$, k being a non-negative integer ([9] Theorem 12.4 Chapter 1). Let P' be the operator obtained when we replace α_k by $\alpha_k a_k$. Clearly P' has exactly the same properties with respect to $m-1$ as P does with respect to m, *i.e.* P': $H^t(\mathbb{R}^n_-) \to H^t(\mathbb{R}^n)$ is continuous if $|t| \leqq m-1$, $t \neq -k-\frac{1}{2}$, k being a non-negative integer.

Let $|t| \leqq m-2$, $t \neq -k-\frac{1}{2}$, k being a non-negative integer, and $g \in H^t(\mathbb{R}^n_-)$. Then $Pg \in H^t(\mathbb{R}^n)$ and $P\partial_i g = \partial_i Pg$ if $i \geqq 2$, $P'\partial_1 g = \partial_1 Pg$. Thus for $t=s$, $g=f$: $Pf \in H^s(\mathbb{R}^n)$, $\partial_i Pf \in H^s(\mathbb{R}^n)$ $(i=1,2,\ldots,n)$. Using the Fourier transform we see easily that $Pf \in H^{s+1}(\mathbb{R}^n)$. But then $f = Pf \mid \mathbb{R}^n_- \in H^{s+1}(\mathbb{R}^n_-)$. Q.E.D.

The remainder of this section will not be used in later ones. The following theorem is very useful in hydrodynamics. It is a particular case of a theorem of de Rham [13], §22, Theorem 17′. We note it because it is a simple consequence of the theorem above and of Theorem 2.4. In Theorem 2.4 take $\mathscr{H}_0^k = H_0^k(\Omega)$, $\mathscr{H}^k = H^k(\Omega)$, $\mathscr{K}_0^k = \vec{H}_0^k(\Omega)$, $\mathscr{K}^k = \vec{H}^k(\Omega)$, $T_0 = \text{grad}$: $H_0^\infty(\Omega) \to \vec{H}_0^\infty(\Omega)$, $S_0 = \text{div}$: $\vec{H}_0^\infty(\Omega) \to H_0^\infty(\Omega)$. The compactness of the inclusions $H^{k+1} \subset H^k$, $\vec{H}^{k+1} \subset \vec{H}^k$ follows from Theorem 16.1, Chapter 1, of [9]. The hypotheses of Theorem 2.4 are easily seen to be satisfied. In particular, if $f \in H^{-\infty}(\Omega)$ and $\operatorname{grad} f = 0$, then $f \in H^\infty(\Omega)$, so f must be locally constant on Ω. If Ω is connected, this implies that f must be a constant. We have obtained

Theorem 3.2. 1) *$u\in\vec{H}^{-\infty}(\Omega)$ is of the form* $\operatorname{grad} f$ *for some $f\in H^{-\infty}(\Omega)$ if and only if $\langle v,u\rangle=0$ for any $v\in\vec{H}_0^{\infty}(\Omega)$ with* $\operatorname{div} v=0$. *Moreover, the function f is uniquely determined modulo a locally constant function, i.e. if $f\in H^{-\infty}(\Omega)$ and* $\operatorname{grad} f=0$, *then f is constant on each connected component of Ω.* 2) *If Σ_0 is the closure in $\vec{H}(\Omega)$ of $\{v\in\vec{H}_0^{\infty}(\Omega)\,|\operatorname{div} v=0\}$ and $\Pi=\operatorname{grad} H^1(\Omega)$, then $\vec{H}(\Omega)=\Sigma_0\oplus\Pi$.*

Remark. If $m\geqq 1$ is an integer, then we have the trivial inequality $\|f\|_{m,\Omega}\leqq c\,\|\operatorname{grad} f\|_{m-1,\Omega}+c\,\|f\|_{m-1,\Omega}$. Lemma 2.1 shows that grad: $H^m(\Omega)\to\vec{H}^{m-1}(\Omega)$ has a closed image. Let div_0 be the Hilbert-space closure of the operator div: $\vec{H}_0^{\infty}(\Omega)\subset\vec{H}(\Omega)\to H(\Omega)$. Using the closed-range theorem, one can show easily that $\vec{H}(\Omega)=\operatorname{Ker}\operatorname{div}_0\oplus\Pi$. It is much more difficult to show that $\Sigma_0=\operatorname{Ker}\operatorname{div}_0$ (this is what the theorem above says).

There is another important fact about the decomposition $\vec{H}(\Omega)=\Sigma_0\oplus\Pi$, namely it enjoys the following regularity: If $u\in\vec{H}^s(\Omega)$, $s\geqq 0$, and $u=u_1+u_2$ is its orthogonal decomposition in $\vec{H}(\Omega)$, then $u_1,\ u_2\in\vec{H}^s(\Omega)$.

I have not been able to find a proof of this assertion without reducing the problem to demonstrating the regularity of a second-order elliptic operator. This is the idea of the usual proof (see [2] for example), and we recall it, since we shall use a similar reasoning in Section 4. Let $u_2=\operatorname{grad} f$, $f\in H^1(\Omega)$, be the component of u in Π. Clearly $\operatorname{div} u=\operatorname{div}\operatorname{grad} f=\Delta f\in H^{s-1}(\Omega)$. Suppose, moreover, that $s\geqq 1$, and let $g\in H^1(\Omega)$. Since $\operatorname{grad} g\in\Pi$ and $u_1\in\Sigma_0$, we shall have $(\operatorname{grad} g,u)_{0,\Omega}=(\operatorname{grad} g,\operatorname{grad} f)_{0,\Omega}$. We write νu for the function on Γ given by $\nu^i u_i|\Gamma$. It is known that $\nu u\in H^{s-\frac{1}{2}}(\Gamma)$ and a limiting procedure applied to (1) gives $(\operatorname{grad} g,u)_{0,\Omega}=-(g,\operatorname{div} u)_{0,\Omega}+(g|\Gamma,\nu u)_{0,\Gamma}$. Finally

$$(\operatorname{grad} g,\operatorname{grad} f)_{0,\Omega}=-(g,\operatorname{div} u)_{0,\Omega}+(g|\Gamma,\nu u)_{0,\Gamma}$$

for any $g\in H^1(\Omega)$. Since $f\in H^1(\Omega)$, this means that f is solution of the Neumann problem:

$$\begin{cases}\Delta f=\operatorname{div} u,\\ \dfrac{\partial f}{\partial\nu}=\nu u.\end{cases}$$

Appealing to the regularity of this elliptic boundary-value problem, we conclude that $f\in H^{s+1}(\Omega)$, i.e. $u_2=\operatorname{grad} f\in\vec{H}^s(\Omega)$. Let P_2: $\vec{H}(\Omega)\to\vec{H}(\Omega)$ be the orthogonal projection on Π. We have proved that $P_2(\vec{H}^s(\Omega))\subset\vec{H}^s(\Omega)$ if $s\geqq 1$ (or for an integer $s\geqq 1$, if you prefer). By the closed-graph theorem P_2: $\vec{H}^s(\Omega)\to\vec{H}^s(\Omega)$ is also continuous. Then we interpolate and obtain $P_2(\vec{H}^s(\Omega))\subset\vec{H}^s(\Omega)$ for any $s\geqq 0$. Q.E.D.

Another useful corollary of Theorem 3.1 is the following (see 4) of Theorem 2.4).

Corollary 3.3. *Let $s\in\mathbb{R}$, $s\neq-(k+\frac{1}{2})$ if $k\geqq 0$ is an integer. Let $\|\cdot\|_{s,\Omega}$ (respectively $\|\cdot\|_{s+1,\Omega}$) be a norm on $H^s(\Omega)$ (respectively $H^{s+1}(\Omega)$) which defines its topology, and let $|\cdot|_{s+1}$ be a continuous seminorm on $H^{s+1}(\Omega)$ such that if c is a* *constant* $\neq 0$ *then* $|c|_{s+1} \neq 0$. *We suppose* Ω *connected. Then there is a finite constant* c_s *such that, for any* $f \in H^{s+1}(\Omega)$,

$$\|f\|_{s+1,\Omega} \leqq c_s(\|\operatorname{grad} f\|_{s,\Omega} + |f|_{s+1}).$$

4. The Operator of Symmetric Differentiations

Our essential result is

Theorem 4.1. *Let* $s \in \mathbb{R}$, $s \neq -(k+\frac{1}{2})$ *for* k *an integer not less than* -1. *If* $u \in \vec{H}^{-\infty}(\Omega)$ *and* $\varepsilon(u) \in \overleftrightarrow{H}^s(\Omega)$, *then* $u \in \vec{H}^{s+1}(\Omega)$.

Proof. Suppose first that Ω is an open subset of the Euclidean space, with the induced Riemannian structure. Then the theorem is an immediate consequence of Theorem 3.1 and of the identity

$$2\,\partial_i \partial_j u_k = \partial_i\, \varepsilon(u)_{jk} + \partial_j\, \varepsilon(u)_{ik} - \partial_k\, \varepsilon(u)_{ij}. \tag{3}$$

In the general case, if $u \in \vec{H}^{-\infty}(\Omega)$, then there is $t \leqq s$ (t an integer, for example) such that $u \in \vec{H}^t(\Omega)$. In local coordinates: $\partial_i u_j + \partial_j u_i = \varepsilon(u)_{ij} +$ terms of zero order in u, so that $\partial_i u_j + \partial_j u_i \in H^t$. By use of the result just proved, it follows that $u \in \vec{H}^{t+1}(\Omega)$. We repeat the argument until we reach the required result. Q.E.D.

Theorems 4.2, 4.3 and 4.4 will now be easy corollaries of Theorems 4.1 and 2.4. They are proved exactly in the same way as Theorem 3.2 (see the remarks preceding it) using the fact that $\overleftrightarrow{H}^{k+1} \subset \overleftrightarrow{H}^k$ compactly (again use Theorem 16.1, Chapter 1, [9]). Notice that in the proof of Theorem 4.2 Korn's inequality is not used. In fact, a generalized Korn inequality follows directly from 4) of Theorem 2.4 and it is stated below as Theorem 4.4.

Theorem 4.2. $\omega \in \overleftrightarrow{H}^{-\infty}(\Omega)$ *is of the form* $\varepsilon(u)$ *for some* $u \in \vec{H}^{-\infty}(\Omega)$ *if and only if* $\langle w, \omega \rangle = 0$ *for any* $w \in \overleftrightarrow{H}_0^{\infty}(\Omega)$ *with* $\operatorname{Div} w = 0$. *In particular, if* $\mathscr{S}_0$ *is the closure in* $\overleftrightarrow{H}(\Omega)$ *of* $\{w \in \overleftrightarrow{H}_0^{\infty}(\Omega) \mid \operatorname{Div} w = 0\}$ *and* $\mathscr{P} = \varepsilon(\vec{H}^1(\Omega))$, *then* $\overleftrightarrow{H}(\Omega) = \mathscr{S}_0 \oplus \mathscr{P}$.

Theorem 4.3. *If* $u \in \vec{H}^{-\infty}(\Omega)$ *and* $\varepsilon(u) = 0$ *then* $u \in \vec{H}^{\infty}(\Omega)$. $\operatorname{Ker} \varepsilon \equiv \{u \in \vec{H}^{\infty}(\Omega) \mid \varepsilon(u) = 0\}$ *is finite-dimensional, and it coincides with the space* $\vec{Q}(\Omega)$ *of Killing vector fields on* Ω. *In particular* $\operatorname{Ker} \varepsilon$ *has dimension* $\leqq \frac{n(n+1)}{2}$ *if* Ω *is connected. Moreover, if* $\varepsilon(u) = 0$ *and* $u = 0$ *on* Γ, *then* $u = 0$.

Proof. We have only to prove the last assertion, the others being known (see Assertion 2) of Theorem 2.4). However we shall also prove the assertion concerning the dimension of $\operatorname{Ker} \varepsilon$. More precisely, we shall prove that any $\varphi \in \vec{H}^{\infty}(\Omega)$ with $\varepsilon(\varphi) = 0$ is uniquely (and linearly) determined by the values $\{\varphi(P), (\nabla \varphi)(P)\}$ in some (fixed) point $P \in \Omega$. Since $\varphi(P)$ can have n independent components and $(\nabla \varphi)(P)$, being a second-order antisymmetric tensor ($\nabla_i \varphi_j + \nabla_j \varphi_i = 0$), can have $\frac{n(n-1)}{2}$ independent components, we shall have at most $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ linearly independent φ's. In local coordinates $\varepsilon(\varphi) = 0$ is equivalent to $\partial_i \varphi_j + \partial_j \varphi_i = 2\Gamma_{ij}^k \varphi_k$. Using (3), we get

$$\partial_i \partial_j \varphi_k = \Gamma_{ik}^l \partial_j \varphi_l + \Gamma_{kj}^l \partial_i \varphi_l - \Gamma_{ij}^l \partial_k \varphi_l + (\partial_j \Gamma_{ik}^l + \partial_i \Gamma_{kj}^l - \partial_k \Gamma_{ij}^l) \varphi_l.$$

The linear system of equations

$$(4) \qquad \begin{cases} \partial_i \varphi_j = \psi_{ij} \\ \partial_i \psi_{jk} = \Gamma_{ik}^l \psi_{jl} + \Gamma_{kj}^l \psi_{il} - \Gamma_{ij}^l \psi_{kl} \\ \qquad + (\partial_j \Gamma_{ik}^l + \partial_i \Gamma_{kj}^l - \partial_k \Gamma_{ij}^l) \varphi_l \end{cases}$$

is of Frobenius type. It may have no solution, but if it has one, that solution (φ_i, ψ_{jk}) is certainly uniquely defined by its value at one point. Since $\psi_{ij} = -\psi_{ji} + 2\Gamma_{ij}^k \varphi_k$ only ψ_{ij} with $1 \leqq i < j \leqq n$ can be prescribed arbitrarily, which proves the assertion. Suppose now that $\varphi = 0$ on Γ. We shall work in the coordinates of a boundary chart, in the neighbourhood of a point $P \in \Gamma$ which has coordinates zero and is such that the equation of Γ in this neighbourhood is $x^1 = 0$; Ω is on the side where $x^1 < 0$. Since $\varphi_i(0, x'') = 0$ near $x'' = 0$, we shall also have $\partial_j \varphi_i(0, x'') = 0$ if $i \geqq 1$, $j \geqq 2$. But $\partial_1 \varphi_1(0, x'') = (\Gamma_{11}^k \varphi_k)(0, x'') = 0$ and $\partial_1 \varphi_j(0, x'') = (2\Gamma_{1j}^k \varphi_k - \partial_j \varphi_1)(0, x'') = 0$. Thus (φ_i, ψ_{jk}) is the solution of (4) having value zero at $P = 0$. The system being linear we obtain $\varphi = 0$ in the neighbourhood of P. Repeating the argument shows that $\varphi \equiv 0$. Q.E.D.

Theorem 4.4 (*Generalized Korn inequality*). *Let $s \in \mathbb{R}$, $s \neq -(k + \frac{1}{2})$ if k is an integer not less than -1. Let $\|\cdot\|_{s,\Omega}$ (respectively $\|\cdot\|_{s+1,\Omega}$) be a norm on $\overleftrightarrow{H}^s(\Omega)$ (respectively $\vec{H}^{s+1}(\Omega)$) which defines its topology, and let $|\cdot|_{s+1}$ be a continuous seminorm on $\vec{H}^{s+1}(\Omega)$ such that if $\varphi \neq 0$ is a Killing vector field on Ω then $|\varphi|_{s+1} \neq 0$. Suppose Ω connected. Then there is a finite constant c_s such that for any $u \in \vec{H}^{s+1}(\Omega)$:*

$$\|u\|_{s+1,\Omega} \leqq c_s \big(\|\varepsilon(u)\|_{s,\Omega} + |u|_{s+1} \big).$$

Example. The usual Korn inequality is obtained by taking $s = 0$ *and* $|\cdot|_{s+1} = \|\cdot\|_{0,\Omega}$. By use of Theorem 4.3 it is easily seen that for $s > -\frac{1}{2}$, $s \neq \frac{1}{2}$ we can take

$$|u|_{s+1} = \|u|\Gamma\|_{0,\Gamma} = \Big(\int_\Gamma (u|\Gamma, u|\Gamma) * 1_\Gamma \Big)^{\frac{1}{2}}.$$

Remark. We have used in the proof of Theorem 4.1 a method due to DUVAUT & LIONS [4]. They used it in their proof of Korn's inequality. Notice that the preceding generalized Korn inequality is a consequence of Theorems 2.4 (point 4) and 4.1.

Theorem 4.2 generalizes all of TING's results [14]. But we have not yet generalized BERGER & EBIN's results [1] to a manifold with boundary: we must also show that the orthogonal decomposition $\overleftrightarrow{H}(\Omega) = \mathscr{S}_0 \oplus \mathscr{P}$ enjoys regularity similar to that of $\vec{H}(\Omega) = \Sigma_0 \oplus \Pi$ (see the remarks between Theorem 3.2 and Corollary 3.3), namely that if $\omega \in \overleftrightarrow{H}^s(\Omega)$, $s \geqq 0$ and $\omega = \omega_1 + \omega_2$ is its orthogonal decomposition in $\overleftrightarrow{H}(\Omega)$, then $\omega_1, \omega_2 \in \overleftrightarrow{H}^s(\Omega)$. To do so, we shall use the same idea as in Section 3, *i.e.* we shall reduce the question to the regularity of an elliptic boundary value problem (BERGER & EBIN have used the same method without considering boundary terms, since $\Gamma = \emptyset$ in their case).

We begin by introducing some new notations. We denote by $\vec{H}^s(\Omega|\Gamma)$ (respectively $\overleftrightarrow{H}^s(\Omega|\Gamma)$) Sobolev spaces of first-order (respectively symmetric second-order) tensors on $\bar{\Omega}$ over Γ (for example, $\vec{H}^\infty(\Omega|\Gamma)$ are C^∞ sections of the fiber bundle of the first order tensors on $\bar{\Omega}$, but defined only on Γ). In particular $\vec{H}^0(\Omega|\Gamma) \equiv \vec{H}(\Omega|\Gamma)$ (respectively $\overleftrightarrow{H}^0(\Omega|\Gamma) \equiv \overleftrightarrow{H}(\Omega|\Gamma)$) is a Hilbert space with the scalar product

$$(u, v)_{0,\Gamma} = \int_\Gamma (u, v) * 1_\Gamma.$$

Since Γ has no boundary, $\vec{H}^s_0(\Omega|\Gamma) = \vec{H}^s(\Omega|\Gamma)$, $\overleftrightarrow{H}^s_0(\Omega|\Gamma) = \overleftrightarrow{H}^s(\Omega|\Gamma)$ if $s \geqq 0$. If $u \in \vec{H}^\infty(\Omega)$, let $ru \in \vec{H}^\infty(\Omega|\Gamma)$ be its restriction to Γ, *i.e.* for any $x \in \Gamma$: $(ru)(x) = u(x)$. It is known that for $s > \frac{1}{2}$ the operator r has an extension to a continuous surjection r: $\vec{H}^s(\Omega) \to \vec{H}^{s-\frac{1}{2}}(\Omega|\Gamma)$. Similarly, if $\omega \in \overleftrightarrow{H}^\infty(\Omega)$, we set $(\nu\omega)_i(x) = \nu^j(x)\,\omega_{ij}(x)$ $(x \in \Gamma)$, so that $\nu\omega \in \vec{H}^\infty(\Omega|\Gamma)$. This application also has a continuous surjective extension ν: $\overleftrightarrow{H}^s(\Omega) \to \vec{H}^{s-\frac{1}{2}}(\Omega|\Gamma)$ if $s > \frac{1}{2}$. From (2), by a limiting procedure, we obtain for $u \in \vec{H}^1(\Omega)$, $\omega \in \overleftrightarrow{H}^1(\Omega)$,

$$(5) \qquad (\varepsilon(u), \omega)_{0,\Omega} + (u, \operatorname{Div} \omega)_{0,\Omega} = (ru, \nu\omega)_{0,\Gamma}.$$

We need the following

Lemma 4.5. *If* $s > \frac{3}{2}$, *then the mapping*

$$\vec{H}^s(\Omega) \ni u \mapsto (ru, \nu\varepsilon(u)) \in \vec{H}^{s-\frac{1}{2}}(\Omega|\Gamma) \oplus \vec{H}^{s-\frac{3}{2}}(\Omega|\Gamma)$$

is a continuous surjection. If, moreover, $s \leqq \frac{5}{2}$, *then its kernel is* $\vec{H}^s_0(\Omega)$.

Proof. Let $\varphi \in \vec{H}^{s-\frac{1}{2}}(\Omega|\Gamma)$, $\psi \in \vec{H}^{s-\frac{3}{2}}(\Omega|\Gamma)$ and $\theta_1, \dots, \theta_m \in H^\infty(\Omega)$ such that $\sum_{i=1}^m \theta_i|\Gamma = 1$, each θ_i having its support in a domain of boundary chart. Then $\theta_i\varphi$, $\theta_i\psi$ have properties similar to φ, ψ. If we construct $u_i \in \vec{H}^s(\Omega)$ such that $ru_i = \theta_i\varphi$, $\nu\varepsilon(u_i) = \theta_i\psi$, then $u = \sum_{i=1}^m u_i \in \vec{H}^s(\Omega)$, $ru = \sum_{i=1}^m \theta_i\varphi = \varphi$, $\nu\varepsilon(u) = \sum_{i=1}^m \theta_i\psi = \psi$. It follows that we can suppose from the beginning $\operatorname{supp}\varphi \subset U$, $\operatorname{supp}\psi \subset U$, where U is the domain of a boundary chart. It is known that we can choose the coordinates such that the components of $\nu(x)$ be $\nu_1(x) = 1$, $\nu_i(x) = 0$ if $i \geqq 2$ $(x \in U_0 = U \cap \Gamma)$. Working in these coordinates, we see that it is sufficient to consider the case in which

$$U = \{x = (x^1, x'') \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, |x| < 1, x^1 \leqq 0\}.$$

Then φ, ψ are vector fields in $\mathbb{R}^n$ defined on $U_0 = \{x \in U \,|\, x^1 = 0\}$, and we must construct a vector field u in $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \,|\, x^1 < 0\}$, with support in U, components in H^s, and $u_j|U_0 = \varphi_j$,

$$\nu^i(\nabla_i u_j + \nabla_j u_i)|U_0 \equiv (\nabla_1 u_j + \nabla_j u_1)|U_0 \equiv (\partial_1 u_j + \partial_j u_1 + 2\Gamma^k_{1j} u_k)|U_0 = \psi_j.$$

This is equivalent to $u_j|U_0=\varphi_j$,

$$(\partial_1 u_j+\partial_j u_1)|U_0=\psi_j-2\Gamma^k_{1j}(0,\cdot)\varphi_k\equiv\xi_j.$$

In particular $\partial_1 u_1|U_0=\frac{1}{2}\xi_1$ and for $j\geqq 2$: $\partial_1 u_j|U_0=\xi_j-\partial_j\varphi_1$. Now we can use theorems 9.4 and 11.5 of [9] chapter 1. Q.E.D.

Lemma 4.7. *If for some $u\in\vec{H}^1(\Omega)$ there is a $v\in\vec{H}^0(\Omega)$ such that*

$$(6)\qquad \big(\varepsilon(\varphi),\varepsilon(u)\big)_{0,\Omega}=(\varphi,v)_{0,\Omega}$$

for any $\varphi\in\vec{H}^1(\Omega)$, then $u\in\vec{H}^2(\Omega)$.

Proof. Let $\theta\in H^\infty(\Omega)$. Then $\theta u\in\vec{H}^1(\Omega)$ and

$$\begin{aligned}\big(\varepsilon(\varphi),\varepsilon(\theta u)\big)_{0,\Omega}&=\big(\varepsilon(\varphi),\theta\varepsilon(u)\big)_{0,\Omega}+\big(\varepsilon(\varphi),(\nabla\theta\cdot u)\big)_{0,\Omega}\\&=\big(\varepsilon(\bar\theta\varphi),\varepsilon(u)\big)_{0,\Omega}-\big((\nabla\bar\theta\cdot\varphi),\varepsilon(u)\big)_{0,\Omega}+\big(\varepsilon(\varphi),(\nabla\theta\cdot u)\big)_{0,\Omega}\\&=\big(\varphi,-\varepsilon(u)\cdot\nabla\theta-\operatorname{Div}(\nabla\theta\cdot u)+\theta v\big)_{0,\Omega}+\big(r\varphi,\nu(\nabla\theta\cdot u)\big)_{0,\Gamma}\end{aligned}$$

where $(\nabla\theta\cdot u)$ has components $\partial_i\theta\cdot u_j+\partial_j\theta\cdot u_i$ and $\big(\varepsilon(u)\cdot\nabla\theta\big)_i=\varepsilon(u)_{ij}\nabla^j\theta$. Suppose θ has support in a domain of chart U and take $u_0\in\vec{H}^2(\Omega)$, $\operatorname{supp}u_0\subset U$, with $\nu\varepsilon(u_0)=\nu(\nabla\theta\cdot u)$. Then $u_1=\theta u-u_0\in\vec{H}^1(\Omega)$ and if $\varphi\in\vec{H}^1(\Omega)$

$$\begin{aligned}\big(\varepsilon(\varphi),\varepsilon(u_1)\big)_{0,\Omega}&=\big(\varphi,\theta v-\varepsilon(u)\cdot\nabla\theta-\operatorname{Div}(\nabla\theta\cdot u)+\operatorname{Div}\varepsilon(u_0)\big)_{0,\Omega}\\&\equiv(\varphi,v_1)_{0,\Omega}\end{aligned}$$

where $v_1\in\vec{H}^0(\Omega)$. We shall treat only the more complicated case when U is the domain of a boundary chart, the coordinates being chosen exactly as in the proof of Lemma 4.5. Working only in these coordinates, we have

$$\tfrac{1}{2}\int_U\varepsilon(\varphi)_{ij}\varepsilon(u_1)_{kl}g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x=(\varphi,v_1)_{0,U}$$

for any $\varphi\in\vec{H}^1(U)$ $\big(U=\{x=(x^1,x'')\in\mathbb{R}\times\mathbb{R}^{n-1}\,|\,|x|<1,\ x^1\leqq 0\}$, $U_0=\{x\in U\,|\,x^1=0\}\big)$. Since $\varepsilon(\varphi)_{ij}=\partial_i\varphi_j+\partial_j\varphi_i+2\Gamma^k_{ij}\varphi_k$, by replacing v_1 by $v_1'\in\vec{H}^0(U)$ we conclude that

$$\int_U(\partial_i\varphi_j+\partial_j\varphi_i)\varepsilon(u_1)_{kl}g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x=(\varphi,v_1')_{0,U}$$

for any $\varphi\in\vec{H}^1(U)$. But, if $\psi\in\vec{H}^1(U)$,

$$(7)\qquad\begin{aligned}&\int_U(\partial_i\varphi_j+\partial_j\varphi_i)\big(\partial_k(u_1-\psi)_l+\partial_l(u_1-\psi)_k\big)g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x\\&=\int_U(\partial_i\varphi_j+\partial_j\varphi_i)\varepsilon(u_1)_{kl}g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x\\&\quad-\int_U(\partial_i\varphi_j+\partial_j\varphi_i)(2\Gamma^m_{kl}u_{1m}+\partial_k\psi_l+\partial_l\psi_k)g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x.\end{aligned}$$

We choose $\psi\in\vec{H}^2(U)$, with support at a strictly positive distance from the curved part of ∂U and such that $(\partial_1\psi_l+\partial_l\psi_1+2\Gamma^m_{1l}u_{1m})|U_0=0$ (this is possible, according to Lemma 4.5). Then, if we integrate by parts the last integral in (7), we obtain as boundary term

$$(8)\qquad \begin{aligned}&-\int_{U_0} \varphi_j(2\Gamma_{kl}^m u_{1m}+\partial_k\psi_l+\partial_l\psi_k)g^{1k}g^{jl}\sqrt{g}\,\mathrm{d}x\\ &-\int_{U_0} \varphi_i(2\Gamma_{kl}^m u_{1m}+\partial_k\psi_l+\partial_l\psi_k)g^{ik}g^{1l}\sqrt{g}\,\mathrm{d}x.\end{aligned}$$

Since $v_1(x)=1$ and $v_i(x)=0$ if $i\geqq 2$, $x\in U_0$ in these coordinates, we have $g^{1k}(0,x'')=1$ if $k=1$ and $=0$ if $k\geqq 2$. It follows that the expression (8) vanishes, *i.e.* the boundary term disappears when we integrate by parts the last integral in (6). In conclusion, there is $b\in\vec{H}^0(U)$ such that for any $\varphi\in\vec{H}^1(U)$

$$\int_U (\partial_i\varphi_j+\partial_j\varphi_i)(\partial_k a_l+\partial_l a_k)g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x=(\varphi,b)_{0,U}$$

where $a=u_1-\psi$. We shall show by the method of "differential quotients" (see [12]) that $a\in\vec{H}^2(U)$. For $m\geqq 2$, $\lambda\in\mathbb{R}\backslash\{0\}$ and f a function defined in $\mathbb{R}^n_-=\{x\in\mathbb{R}^n|x^1\leqq 0\}$ we set $(\rho_\lambda^m f)(x)=\lambda^{-1}(f(x+\lambda e_m)-f(x))$, $(\tau_\lambda^m f)(x)=f(x+\lambda e_m)$, where e_m is the unit vector in the direction of x_m-axis. If $f\in H^0(U)$, then we define f in $\mathbb{R}^n_-\backslash U$ by 0, so that $f\in H^0(\mathbb{R}^n_-)$. If $f,g\in H^0(U)$ and one of them has support disjoint from the curved part of ∂U, then

$$\int_U \rho_\lambda^m f\cdot g\,\mathrm{d}x=-\int_U f\cdot\rho_{-\lambda}^m g\,\mathrm{d}x$$

if λ is sufficiently small. Also $\rho_\lambda^m(fg)=\rho_\lambda^m(f)\cdot\tau_\lambda^m(g)+f\rho_\lambda^m(g)$. We obtain, for small λ,

$$\begin{aligned}&\Big|\int_U (\partial_i\varphi_j+\partial_j\varphi_i)(\partial_k\rho_\lambda^m a_l+\partial_l\rho_\lambda^m a_k)g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x\Big|\\ &\quad=\Big|-\int_U(\partial_i\tau_{-\lambda}^m\varphi_j+\partial_j\tau_{-\lambda}^m\varphi_i)(\partial_k a_l+\partial_l a_k)\rho_{-\lambda}^m(g^{ik}g^{jl}\sqrt{g})\,\mathrm{d}x\\ &\qquad-\int_U(\partial_i\rho_{-\lambda}^m\varphi_j+\partial_j\rho_{-\lambda}^m\varphi_i)(\partial_k a_l+\partial_l a_k)g^{ik}g^{jl}\sqrt{g}\,\mathrm{d}x\Big|\\ &\quad\leqq c\|\varphi\|_{1,U}\|a\|_{1,U}+|(\rho_{-\lambda}^m\varphi,b)_{0,U}|\\ &\quad\leqq c\|\varphi\|_{1,U}\|a\|_{1,U}+\|\rho_{-\lambda}^m\varphi\|_{0,U}\|b\|_{0,U}\\ &\quad\leqq c\|\varphi\|_{1,U}(\|a\|_{1,U}+\|b\|_{0,U})\end{aligned}$$

for any $\varphi\in\vec{H}^1(U)$. We take $\varphi=\rho_\lambda^m a$, so that

$$\begin{aligned}\|\rho_\lambda^m a\|_{1,U}^2&\leqq c\|\rho_\lambda^m a\|_{1,U}(\|a\|_{1,U}+\|b\|_{0,U})+c\|\rho_\lambda^m a\|_{0,U}^2\\ &\leqq c\|\rho_\lambda^m a\|_{1,U}(\|a\|_{1,U}+\|b\|_{0,U})+c\|a\|_{1,U}^2\end{aligned}$$

where we have used Corollary 4.4 (case $s=0$, *i.e.* the usual Korn inequality). The last inequality implies that $\limsup_{\lambda\to 0}\|\rho_\lambda^m a\|_{1,U}<\infty$, *i.e.* $\partial_m a\in\vec{H}^1(U)$ for any $m\geqq 2$. This is equivalent to $\partial_m(\theta u)\in\vec{H}^1(U)$ if $m\geqq 2$. But $\operatorname{Div}\varepsilon(\theta u)=\theta\operatorname{Div}\varepsilon(u)+$ terms in $\vec{H}^0(\Omega)=\theta v+$ terms in $\vec{H}^0(\Omega)\in\vec{H}^0(\Omega)$. In local coordinates $\nabla^j(\nabla_i(\theta u_j)$

$+\nabla_j(\theta u_i)) \in \vec{H}^0(U)$, from which we easily obtain $\partial_1^2(\theta u) \in \vec{H}^0(U)$. In conclusion, $\theta u \in \vec{H}^2(U)$, *i.e.* ($\theta$ being any C^∞ function with small support) $u \in \vec{H}^2(\Omega)$. Q.E.D.

Theorem 4.8. *Let* $s \geqq 0$, $\alpha \in \vec{H}^{s+\frac{1}{2}}(\Omega|\Gamma)$ *and* $v \in \vec{H}^s(\Omega)$. *Then* $u \in \vec{H}^1(\Omega)$ *verifies*

(9) $$(\varepsilon(\varphi), \varepsilon(u))_{0,\Omega} = (r\varphi, \alpha)_{0,\Gamma} - (\varphi, v)_{0,\Omega}$$

for any $\varphi \in \vec{H}^1(\Omega)$ *if and only if* $u \in \vec{H}^2(\Omega)$ *and* u *is a solution of the boundary-value problem:*

(10) $$\begin{cases} \operatorname{Div} \varepsilon(u) = v, \\ \quad \nu\varepsilon(u) = \alpha. \end{cases}$$

The problem (9) *has a solution* $u \in \vec{H}^1(\Omega)$ *if and only if for any* $k \in \vec{Q}(\Omega)$ (*see* 4.3):

(11) $$(rk, \alpha)_{0,\Gamma} = (k, v)_{0,\Omega}.$$

If a solution exists in $\vec{H}^1(\Omega)$, *then it is unique modulo* $\vec{Q}(\Omega)$ *and it belongs to* $\vec{H}^{2+s}(\Omega)$.

Proof. (10) is an elliptic boundary-value problem known in elasticity theory (see [5] for example). We shall however give a simple proof of the result, in order to make the paper self-contained. Clearly $(\varepsilon(\varphi), \varepsilon(\psi))_{0,\Omega}$ is a closed, positive, sesquilinear form in $\vec{H}(\Omega)$ (with domain $\vec{H}^1(\Omega)$). Let A be the positive self-adjoint operator in $\vec{H}(\Omega)$ associated with it. Using Theorem 2.1, Chapter VI, [7] and Lemma 4.7, we obtain $D(A) = \{u \in \vec{H}^2(\Omega) | \nu\varepsilon(u) = 0\}$ and $Au = -\operatorname{Div} \varepsilon(u)$ if $u \in D(A)$. Since $\vec{H}^2(\Omega) \subset \vec{H}(\Omega)$ is a compact injection, A will have only a discrete spectrum ($(1+A)^{-1}$ is compact in $\vec{H}(\Omega)$). In particular A has closed range: $\operatorname{Im} A = \{u \in \vec{H}(\Omega) | (u, v)_{0,\Omega} = 0$ if $v \in \operatorname{Ker} A\}$. Obviously $\operatorname{Ker} A = \vec{Q}(\Omega)$. It follows that for $v \in \vec{H}(\Omega)$ there is $u \in \vec{H}^1(\Omega)$ such that $(\varepsilon(\varphi), \varepsilon(\psi))_{0,\Omega} = -(\varphi, v)_{0,\Omega}$, $\forall \varphi \in \vec{H}^1(\Omega)$ if and only if $(v, k)_{0,\Omega} = 0$ for any $k \in \vec{Q}(\Omega)$. In this case u is unique modulo $\vec{Q}(\Omega)$ and $u \in \vec{H}^2(\Omega)$, $\nu\varepsilon(u) = 0$. Then, using (5) and Lemma 4.5, we easily show that (9) and (10) are equivalent. The same argument also shows that (11) is a necessary and sufficient condition for problem (9) to have a solution. The assertion of uniqueness is obvious. Let us now prove regularity. First we apply the remark following Lemma 2.1 with $X_1 = \vec{H}^2(\Omega)$, $X_2 = \vec{H}^0(\Omega) \oplus \vec{H}^{\frac{1}{2}}(\Omega|\Gamma)$, $Tu = (\operatorname{Div} \varepsilon(u), \nu\varepsilon(u))$, $|\cdot|_1 = \|\cdot\|_{0,\Omega}$. We obtain

$$\|u\|_{2,\Omega} \leqq c\,\|\operatorname{Div} \varepsilon(u)\|_{0,\Omega} + c\,\|\nu\varepsilon(u)\|_{\frac{1}{2},\Gamma} + c\,\|u\|_{0,\Omega}$$

for any $u \in \vec{H}^2(\Omega)$, where c is some finite constant. Let $u \in \vec{H}^2(\Omega)$ such that $\operatorname{Div} \varepsilon(u) \in \vec{H}^1(\Omega)$, $\nu\varepsilon(u) \in H^{\frac{3}{2}}(\Omega|\Gamma)$. Let U be the domain of a chart on $\bar{\Omega}$ and $\theta \in H^\infty(\Omega)$, $\operatorname{supp} \theta \subset U$. Then $\theta u \in \vec{H}^2(\Omega)$, $\operatorname{Div} \varepsilon(\theta u) \in \vec{H}^1(\Omega)$, $\nu\varepsilon(\theta u) \in \vec{H}^{\frac{3}{2}}(\Omega|\Gamma)$. We suppose U to be domain of a boundary chart (this is the more complicated case) and choose the coordinates as in Lemma 4.5. In particular, we identify U with a half-ball in $\mathbb{R}^n$ and $U_0 = U \cap \Gamma$ with $U \cap \mathbb{R}^{n-1}$. Clearly, if $v \in \vec{H}^2(U)$, $\operatorname{supp} v$ being disjoint from the curved part of ∂U, then

$$\|v\|_{2,U} \leqq c\,\|\operatorname{Div} \varepsilon(v)\|_{0,U} + c\,\|\nu\varepsilon(v)\|_{\frac{1}{2},U_0} + c\,\|v\|_{0,U}.$$

Using the method of "differential quotients", we prove now that $\partial_j(\theta u)\in\vec{H}^2(U)$, $j=2,\dots,n$. With the same notations as in Lemma 4.7, for small enough $\lambda\in\mathbb{R}$ and $m\geqq 2$

$$\|\rho_\lambda^m\theta u\|_{2,U}\leqq c\,\|\mathrm{Div}\,\varepsilon(\rho_\lambda^m\theta u)\|_{0,U}+c\,\|v\,\varepsilon(\rho_\lambda^m\theta u)\|_{\frac{1}{2},U_0}+c\,\|\rho_\lambda^m\theta u\|_{0,U}.$$

By common arguments the right-hand side is dominated by

$$c\big(\|\mathrm{Div}\,\varepsilon(\theta u)\|_{1,U}+\|v\,\varepsilon(\theta u)\|_{\frac{3}{2},U_0}+\|\theta u\|_{2,U}\big)$$

for any small λ, which implies that $\partial_m(\theta u)\in\vec{H}^2(U)$ for $m\geqq 2$. Since $\mathrm{Div}\,\varepsilon(u)\in\vec{H}^1(\mathrm{U})$, we also obtain $\partial_1(\theta u)\in\vec{H}^2(U)$. In conclusion $\theta u\in\vec{H}^3(U)$, *i.e.* $u\in\vec{H}^3(\Omega)$ (use a partition of unity). Regularity has been proved for $s=1$. It is proved in a similar way for the case of any integer $s+1$ by employing the inequality:

$$\|u\|_{2+s,\Omega}\leqq c_s\,\|\mathrm{Div}\,\varepsilon(u)\|_{s,\Omega}+c_s\,\|v\,\varepsilon(u)\|_{s+\frac{1}{2},\Gamma}+c_s\,\|u\|_{0,\Omega},$$

established at step s in the same manner as for $s=0$. For general s, we interpolate (see Theorems 13.2 and 13.3, Chapter 1 [9]). Q.E.D.

Theorem 4.9. *Let* $\omega\in\overleftrightarrow{H}(\Omega)=\mathscr{S}_0\oplus\mathscr{P}$ *(see 4.2) and let* $\omega=\omega_1+\omega_2$, $\omega_1\in\mathscr{S}_0$, $\omega_2\in\mathscr{P}$, *be its orthogonal decomposition. If* $\omega\in\overleftrightarrow{H}^s(\Omega)$ *for some* $s\geqq 0$, *then* $\omega_1,\omega_2\in\overleftrightarrow{H}^s(\Omega)$. *If, moreover,* $s\neq\frac{1}{2}$, *then* $\omega_2=\varepsilon(u)$ *for some* $u\in\vec{H}^{s+1}(\Omega)$.

Proof. If $s\geqq 1$ and $\omega=\omega_1+\omega_2=\omega_1+\varepsilon(u)$, $\omega_1\in\mathscr{S}_0$, $u\in\vec{H}^1(\Omega)$, then for any $\varphi\in\vec{H}^1(\Omega)$:

$$\big(\varepsilon(\varphi),\varepsilon(u)\big)_{0,\Omega}=\big(\varepsilon(\varphi),\omega\big)_{0,\Omega}=(r\varphi,v\omega)_{0,\Gamma}-(\varphi,\mathrm{Div}\,\omega)_{0,\Omega}$$

by (5). Theorem 4.8 shows $u\in\vec{H}^{s+1}(\Omega)$. The case $0<s<1$ is proved by interpolation. In fact, the orthogonal projection of $\overleftrightarrow{H}(\Omega)$ onto $\mathscr{P}$ maps $\overleftrightarrow{H}^s(\Omega)$ into $\overleftrightarrow{H}^s(\Omega)$ for integer s; in particular (by the closed-graph theorem) it is a continuous operator $\overleftrightarrow{H}^s(\Omega)\to\overleftrightarrow{H}^s(\Omega)$. By interpolation this remains true for any real $s\geqq 0$. Q.E.D.

Remarks. Let ε_1 be the restriction to $\vec{H}^1(\Omega)$ of ε, considered as a closed, densely defined operator from $\vec{H}(\Omega)$ to $\overleftrightarrow{H}(\Omega)$. Then its Hilbert space adjoint Div_0 is the closure of the restriction of Div to $\overleftrightarrow{H}_0^\infty(\Omega)$. PEETRE's lemma and the usual Korn inequality allow us to prove that ε_1 has closed image. The closed-range theorem gives then $\overleftrightarrow{H}(\Omega)=\mathrm{Ker}\,\mathrm{Div}_0\oplus\mathrm{Im}\,\varepsilon_1$. This is a very simple proof of the orthogonal decomposition of $\overleftrightarrow{H}(\Omega)$ determined by $\mathrm{Im}\,\varepsilon_1$. But it is difficult to show that $\{\omega\in\overleftrightarrow{H}_0^\infty(\Omega)|\mathrm{Div}\,\omega=0\}$ is dense in $\mathrm{Ker}\,\mathrm{Div}_0$ (this is asserted in Theorem 4.2). On the other hand, using Theorem 4.8, one can show directly that

$$\{\omega\in\overleftrightarrow{H}^\infty(\Omega)|\,\mathrm{Div}\,\omega=0,\ v\omega=0\}$$

is dense in $\mathrm{Ker}\,\mathrm{Div}_0$. This is the result proved by TING [14]. My Lemma 4.7 is similar to Theorem 4.6 of [14]. However, I consider TING's proof a little ambiguous because his space $Z^1(M)$ is not correctly defined. In fact, by LIONS & MAGENES' method [9] we can define $v\omega$ as an element of $\vec{H}^{-\frac{1}{2}}(\Omega|\Gamma)$ for any

$\omega \in \vec{H}(\Omega)$ such that $\operatorname{Div}\omega \in \vec{H}(\Omega)$. But this is certainly not true for all ω's of the form $\varepsilon(u)$, $u \in \vec{H}^1(\Omega)$. In this context note that $\mathscr{S}_0 = \operatorname{Ker}\operatorname{Div}_0$ can also be defined by

$$\{\omega \in \overleftrightarrow{H}(\Omega) \mid \operatorname{Div}\omega = 0,\ \nu\omega = 0\}$$

(since $\operatorname{Div}\omega = 0 \in \vec{H}(\Omega)$, $\nu\omega \in \vec{H}^{-\frac{1}{2}}(\Omega|\Gamma)$ is well defined).

We close this section with three results related to the preceding ones. The first concerns the Dirichlet problem for the operator $\operatorname{Div}\varepsilon$, and we shall need it in the proof of the next theorem.

Theorem 4.10. *For any $s \geqq 1$, $s \neq \frac{3}{2}$, the mapping*

$$\vec{H}^s(\Omega) \ni u \mapsto (\operatorname{Div}\varepsilon(u), ru) \in \vec{H}^{s-2}(\Omega) \oplus H^{s-\frac{1}{2}}(\Omega|\Gamma) \tag{12}$$

is a topological isomorphism.

Proof. The mapping is continuous and injective, since $\operatorname{Div}\varepsilon(u) = 0$, $u \in \vec{H}^1(\Omega)$ and $ru = 0 \Rightarrow \varepsilon(u) = 0$ and we apply Theorem 4.3. If $u \in \vec{H}_0^1(\Omega) \equiv \{u \in \vec{H}^1(\Omega) \mid ru = 0\}$, then $\langle u, -\operatorname{Div}\varepsilon(u)\rangle = (\varepsilon(u), \varepsilon(u))_{0,\Omega} \geqq c\,\|u\|_{1,\Omega}^2$ (Theorem 4.4). The LAX-MILGRAM Lemma shows that $\operatorname{Div}\circ\varepsilon\colon \vec{H}_0^1(\Omega) \to \vec{H}^{-1}(\Omega)$ is an isomorphism. In order to show that (12) is surjective if $s = 1$, it is enough to find for any $\alpha \in \vec{H}^{\frac{1}{2}}(\Omega|\Gamma)$ some $u \in \vec{H}^1(\Omega)$ with $\operatorname{Div}\varepsilon(u) = 0$ and $ru = \alpha$. Since $\vec{H}^1(\Omega) \ni u \mapsto ru \in \vec{H}^{\frac{1}{2}}(\Omega|\Gamma)$ is a surjection with kernel $\vec{H}_0^1(\Omega)$, it is enough to prove that $\mathscr{K} = \{u \in \vec{H}^1(\Omega) \mid \operatorname{Div}\varepsilon(u) = 0\}$ is a topological supplement of $\vec{H}_0^1(\Omega)$ in $\vec{H}^1(\Omega)$. Since $\vec{H}_0^1(\Omega)$ and $\mathscr{K}$ are closed subspaces of $\vec{H}^1(\Omega)$ with zero intersection we need only show that $\vec{H}^1(\Omega) = \vec{H}_0^1(\Omega) + \mathscr{K}$. If $u \in \vec{H}^1(\Omega)$, then $\operatorname{Div}\varepsilon(u) \in \vec{H}^{-1}(\Omega)$. In particular there is a $u_0 \in \vec{H}_0^1(\Omega)$ verifying $\operatorname{Div}\varepsilon(u) = \operatorname{Div}\varepsilon(u_0)$. We get $u = u_0 + (u - u_0)$, $u_0 \in \vec{H}_0^1(\Omega)$, $u - u_0 \in \mathscr{K}$. The theorem is proved for $s = 1$. As a consequence we obtain the inequality

$$\|u\|_{1,\Omega} \leqq c\,\|\operatorname{Div}\varepsilon(u)\|_{-1,\Omega} + c\,\|ru\|_{\frac{1}{2},\Gamma}$$

for any $u \in \vec{H}^1(\Omega)$. Regularity is then proved as in Theorem 4.8. Q.E.D.

We have used in the above proof an idea of LIONS & MAGENES [10]. The next result is an orthogonal decomposition theorem similar to (but simpler than) Theorem 4.9.

Theorem 4.11. *Let $\mathscr{S} = \{\omega \in \overleftrightarrow{H}(\Omega) \mid \operatorname{Div}\omega = 0\}$ and $\mathscr{P}_0 = \varepsilon(\vec{H}_0^1(\Omega))$. Then $\overleftrightarrow{H}(\Omega) = \mathscr{S} \oplus \mathscr{P}_0$ (Hilbert direct sum). The subspace $\mathscr{S}^{(\infty)} = \{\omega \in \overleftrightarrow{H}^\infty(\Omega) \mid \operatorname{Div}\omega = 0\}$ is dense in $\mathscr{S}$. Moreover, the above orthogonal decomposition is regular, i.e. if $\omega \in \overleftrightarrow{H}^s(\Omega)$ for some $s \geqq 0$ and $\omega = \omega_1 + \omega_2$, $\omega_1 \in \mathscr{S}$, $\omega_2 \in \mathscr{P}_0$, then $\omega_1, \omega_2 \in \overleftrightarrow{H}^s(\Omega)$.*

Proof. Let $\operatorname{Div}_1\colon D(\operatorname{Div}_1) \subset \overleftrightarrow{H}(\Omega) \to \vec{H}(\Omega)$ be the operator:

$$D(\operatorname{Div}_1) = \{\omega \in \overleftrightarrow{H}(\Omega) \mid \operatorname{Div}\omega \in \vec{H}(\Omega)\}, \qquad \omega \in D(\operatorname{Div}_1) \Rightarrow \operatorname{Div}_1\omega = \operatorname{Div}\omega.$$

Then Div_1 is a closed, densely defined operator and its adjoint is equal to $\varepsilon_0\colon D(\varepsilon_0) \subset \vec{H}(\Omega) \to \overleftrightarrow{H}(\Omega)$, $D(\varepsilon_0) = \vec{H}_0^1(\Omega)$, $\varepsilon_0(u) = \varepsilon(u)$ if $u \in D(\varepsilon_0)$. Since ε_0 has closed range (see Lemma 2.1), the closed-range theorem implies that

$$\overleftrightarrow{H}(\Omega) = \operatorname{Ker}\operatorname{Div}_1 \oplus \operatorname{Im}\varepsilon_0 \equiv \mathscr{S} \oplus \mathscr{P}_0.$$

Let us now prove regularity. If $\omega=\omega_1+\omega_2=\omega_1+\varepsilon(u)\in\overleftrightarrow{H}^s(\Omega)$, $s\geqq 0$, $s\neq\frac{1}{2}$, $\omega_1\in\mathscr{S}$ and $u\in\vec{H}_0^1(\Omega)$, then $\operatorname{Div}\varepsilon(u)=\operatorname{Div}\omega\in\vec{H}^{s-1}(\Omega)$. By Theorem 4.10: $u\in\vec{H}^{s+1}(\Omega)$, which establishes regularity for any $s\geqq 0$, $s\neq\frac{1}{2}$. The case $s=\frac{1}{2}$ is treated by interpolation, as in Theorem 4.9. The density assertion is trivial now. In fact P, the orthogonal projection of $\overleftrightarrow{H}(\Omega)$ onto $\mathscr{S}$, sends a dense subspace of $\overleftrightarrow{H}(\Omega)$ onto a dense subspace of $\mathscr{S}$ and also $P(\overleftrightarrow{H}^\infty(\Omega))\subset\overleftrightarrow{H}^\infty(\Omega)$. Q.E.D.

As a corollary we get a generalization of a result by DORN & SCHILD [3] (see also [6] page 119 and [15] page 602).

Theorem 4.12. *Let $\omega\in\overleftrightarrow{H}(\Omega)$ and $\alpha\in\vec{H}^{\frac{1}{2}}(\Omega|\Gamma)$. Then there is $u\in\vec{H}^1(\Omega)$ such that $\varepsilon(u)=\omega$ and $ru=\alpha$ if and only if $(\omega,w)_{0,\Omega}=(\alpha,\nu w)_{0,\Gamma}$ for any $w\in\overleftrightarrow{H}^\infty(\Omega)$ verifying* $\operatorname{Div} w=0$. *If u exists, then it is uniquely defined, and it belongs to $\vec{H}^{s+1}(\Omega)$ if and only if $\omega\in\overleftrightarrow{H}^s(\Omega)$ $(s\geqq 0,\ s\neq\frac{1}{2})$.*

Proof. Let $u_0\in\vec{H}^1(\Omega)$ such that $ru_0=\alpha$. Then

$$(\omega-\varepsilon(u_0),w)_{0,\Omega}=(\omega,w)_{0,\Omega}-(\alpha,\nu w)_{0,\Gamma}$$

for any $w\in\mathscr{S}^{(\infty)}$. By Theorem 4.11, $\omega-\varepsilon(u_0)\in\mathscr{P}_0$ if and only if $(\omega-\varepsilon(u_0),w)_{0,\Omega}=0$ for any $w\in\mathscr{S}^{(\infty)}$, *i.e.* if and only if $(\omega,w)_{0,\Omega}=(\alpha,\nu w)_{0,\Gamma}$, $\forall\, w\in\mathscr{S}^{(\infty)}$. Q.E.D.

5. St. Venant's Compatibility Conditions and Related Topics

This section is devoted to two questions of elasticity theory. However, I begin by stating a problem which I have not been able to solve in the general case. Let us go back for a moment to Theorem 3.2. According to that theorem $u\in\vec{H}(\Omega)$ is of the form $\operatorname{grad} f$ for some $f\in H^1(\Omega)$ if and only if $(u,v)_{0,\Omega}=0$ for any $v\in\vec{H}_0^\infty(\Omega)$ such that $\operatorname{div} v=0$. This criterion is not very useful in practice since it involves an infinite number of conditions. A much more useful set of conditions on u is the following one: for some $u\in\vec{H}(\Omega)$ there is an $f\in H^1(\Omega)$ such that $u=\operatorname{grad} f$ if and only if $du=0$ and $(u,v)_{0,\Omega}=0$ for any $v\in\vec{A}_\tau(\Omega)$ (d is the exterior derivative and $\vec{A}_\tau(\Omega)$ is the finite-dimensional space of vector fields $v\in\vec{H}^\infty(\Omega)$ satisfying $dv=0$, $\operatorname{div} v=0$, $\nu v=0$). Now we pose the analogous problem for the operator ε: find a "simple and natural" differential operator $\tilde{\varepsilon}$, acting on symmetric second-order tensors, and a finite-dimensional vector space $V\subset\overleftrightarrow{H}(\Omega)$ such that for $\omega\in\overleftrightarrow{H}(\Omega)$ the condition $\omega=\varepsilon(u)$, $u\in\vec{H}^1(\Omega)$, be equivalent to $\tilde{\varepsilon}(\omega)=0$ and $(\omega,w)_{0,\Omega}=0$ for any $w\in V$. The difficulties related to this problem are explained in [15], page 351. We shall completely solve the problem in the case of Euclidean Ω, because in this case the operator $\tilde{\varepsilon}$ is known, so that the "local" question is already solved. See GURTIN [6], Section 14 and TRUESDELL & TOUPIN [15], Section 34. There is also a question "dual" to that stated above, namely, describe the "structure" of the set $\{\omega\in\overleftrightarrow{H}(\Omega)|\operatorname{Div}\omega=0\}$. We shall first give a complete solution of this last problem, also in the Euclidean case, which generalizes GURTIN's results (see [6], Section 17 especially Theorem (3); we avoid all differentiability requirements; see also [15], Sections 226 and 227), and then we solve the first problem by a duality argument.

In the rest of this section Ω will be a bounded, open subset of $\mathbb{R}^n$ with C^∞ boundary Γ, such that Ω is locally on one side of Γ. Then $\bar{\Omega}=\Omega\cup\Gamma$, provided that the Riemannian structure induced by $\mathbb{R}^n$ satisfies all the conditions imposed on $\bar{\Omega}$ in the Introduction. If u is a real vector field on $\bar{\Omega}$, we have defined $\nu u\colon \Gamma\to\mathbb{R}$ as $\nu^i u_i|\Gamma \equiv \sum_{i=1}^n \nu_i u_i|\Gamma$. We also define $\tau u\colon \Gamma\to\mathbb{R}^n$ by $(\tau u)(x)$ $=$ projection of the vector $u(x)\in\mathbb{R}^n$ on the tangent space to Γ at $x\in\Gamma$. Let $\vec{A}_\tau(\Omega)$ (respectively $\vec{A}_\nu(\Omega)$) be the finite-dimensional space of real vector fields $u\in\vec{H}^\infty(\Omega)$ satisfying $\partial_i u_j=\partial_j u_i$ $(1\leqq i,j\leqq n)$, $\operatorname{div} u=0$ and $\nu u=0$ (respectively $\tau u=0$). We shall use the following lemma, which is an easy consequence of MORREY's results [11]:

Lemma 5.1. *Let $u\in\vec{H}^s(\Omega)$, $s\geqq 0$. Then there is an antisymmetric second-order tensor $\omega=(\omega_{ij})$, $\omega_{ij}=-\omega_{ji}\in H^{s+1}(\Omega)$ such that $u_i=\sum_{j=1}^n \partial_j\omega_{ij}$, if and only if* $\operatorname{div} u=0$ *and $(v,u)_{0,\Omega}=0$ for any $v\in\vec{A}_\nu(\Omega)$.*

Theorem 5.2. *Let $s\geqq 0$ and $S=(S_{ij})$, $S_{ij}=S_{ji}$, be a symmetric second-order tensor on Ω with $S_{ij}\in H^s(\Omega)$ (any i,j). Then there is a fourth-order tensor $T=(T_{ijkl})$, with the properties $T_{ijkl}=T_{klij}=-T_{jikl}$, $T_{ijkl}\in H^{s+2}(\Omega)$ (any i,j,k,l) such that $S_{ij}=\sum_{k,l=1}^n \partial_k\partial_l T_{ikjl}$ if and only if* $\operatorname{Div} S=0$ $\left(\text{i.e. } \sum_{j=1}^n \partial_j S_{ij}=0\right)$ *and*

$$\sum_{i,j=1}^n \int_\Omega S_{ij}\,k_i a_j\,\mathrm{d}x=0 \tag{13}$$

for any Killing vector $k\in\vec{Q}(\Omega)$ and any $a\in\vec{A}_\nu(\Omega)$.

Proof. The conditions (13) are inspired from the relations defining "self-equilibrated stress fields" in GURTIN [6], Section 17. In the proof we shall follow the method of DORN & SCHILD. Remark that since Ω is Euclidean, the Killing vectors are the vector fields of the form $k_i(x)=\sum_{j=1}^n \omega_{ij}x_j+c_i$, where ω_{ij}, c_i are real constants, $\omega_{ij}=-\omega_{ji}$. In particular, from (13) we get

$$\sum_{j=1}^n \int_\Omega S_{ij}a_j\,\mathrm{d}x=0$$

for any i and $a\in\vec{A}_\nu(\Omega)$. We apply Lemma 5.1 and obtain $u_{ijk}\in H^{s+1}(\Omega)$, $u_{ijk}=-u_{ikj}$ such that $S_{ij}=\sum_{k=1}^n \partial_k u_{ijk}$. We have $\sum_{k=1}^n \partial_k(u_{ijk}-u_{jik})=0$ since $S_{ij}=S_{ji}$. Now, let $\omega_{ij}=-\omega_{ji}$ be some real constants and $k_i(x)=\sum_{j=1}^n \omega_{ij}x_j$, $k\in\vec{Q}(\Omega)$. Then

$$\begin{aligned}0&=\sum_{i,j}\int_\Omega S_{ij}k_i a_j\,\mathrm{d}x\\&=\sum_{i,j,k}\int_\Omega \partial_k(k_i u_{ijk})\cdot a_j\,\mathrm{d}x-\sum_{i,j,k}\int_\Omega u_{ijk}\partial_k k_i a_j\,\mathrm{d}x\end{aligned}$$

$$= \sum_{i,j,k} \int_{\Gamma} u_{ijk} \not{k}_i a_j v_k \, dx - \sum_{i,j,k} \int_{\Omega} u_{ijk} \omega_{ik} a_j \, dx$$
$$= \sum_{i,j,k} \tfrac{1}{2} \int_{\Gamma} u_{ijk} \not{k}_i (a_j v_k - v_j a_k) \, dx - \sum_{i<k;\, j} \int_{\Omega} \omega_{ik} (u_{ijk} - u_{kji}) a_j \, dx.$$

In the last member the first term is zero because $\tau a = 0$. Since ω_{ik} are independent constants if $i<k$, we find, using $u_{ijk} = -u_{ikj}$, that

$$\sum_{j=1}^{n} \int_{\Omega} (u_{ijk} - u_{jik}) a_k \, dx = 0 \tag{15}$$

for any $a \in \vec{A}_v(\Omega)$. Applying once again Lemma 5.1, we find $v_{ijkl} \in H^{s+2}(\Omega)$, $v_{ijkl} = -v_{ijlk}$, $v_{ijkl} = -v_{jikl}$ such that $u_{ijk} - u_{jik} = \sum_{l=1}^{n} \partial_l v_{ijkl}$. Clearly then

$$u_{ijkl} = \tfrac{1}{2} \sum_{l=1}^{n} \partial_l (v_{ijkl} + v_{kijl} - v_{jkil}).$$

Using this expression and $\sum_{k,l} \partial_k \partial_l v_{ijkl} = 0$ (antisymmetry in k, l of v_{ijkl}), we see that

$$S_{ij} = \sum_{k,l} \partial_k \partial_l \tfrac{1}{2} (v_{kijl} + v_{jkli}).$$

We set $T_{kijl} = -\frac{1}{2}(v_{kijl} + v_{jlki})$ and so obtain the required representation. Conversely, if $S_{ij} = \sum_{k,l} \partial_k \partial_l T_{ikjl}$, $T_{ijkl} = T_{klij} = -T_{jikl}$, then $\sum_j \partial_j S_{ij} = 0$, and we have (13), because of the following relations:

$$\begin{aligned}
\sum_i \not{k}_i S_{ij} = \sum_{i,k,l} [&\partial_k \partial_l (\not{k}_i T_{ikjl}) \\
&- \partial_l \not{k}_i \partial_k T_{ikjl} - \partial_k \not{k}_i \partial_l T_{ikjl}] \\
= \sum_l &\partial_l \sum_{i,k} \partial_k (\not{k}_i T_{ikjl}) \\
&- \sum_l \partial_l \sum_{i,k} \partial_k \not{k}_i T_{ikjl} - \sum_k \partial_k \sum_{i,l} \partial_l \not{k}_i T_{ikjl} \\
= \sum_k &\partial_k \sum_{i,l} (\partial_l (\not{k}_i T_{iljk}) - \partial_l \not{k}_i T_{iljk} - \partial_l \not{k}_i T_{ikjl}),
\end{aligned}$$

(note that $\partial_l \not{k}_i$ are constants). Since $\partial_l \not{k}_i = -\partial_i \not{k}_l$, the expression $\sum_{i,l}(\dots)$ is antisymmetric in j, k, so we can apply Lemma 5.1. Q.E.D.

Remark. Using the DE RHAM theorem ([13], §22, Theorem 17′), one can prove in the same way the following result: If $\vec{A}_v(\Omega) = 0$, then a symmetric second-order tensor S with components $S_{ij} \in H^{-\infty}(\Omega)$ is of the form $S_{ij} = \sum_{k,l} \partial_k \partial_l T_{ikjl}$, $T_{ijkl} = T_{klij} = -T_{jikl} \in H^{-\infty}(\Omega)$ if and only if $\sum_j \partial_j S_{ij} = 0$. (The condition $\vec{A}_v(\Omega) = 0$ essentially means "Γ consists of a single closed surface"; see GURTIN [6], Section 17 Theorem 4).

Now we shall prove the essential result of this section:

Theorem 5.3. *Let $s \geqq 0$, $s \neq \frac{1}{2}$ and $S=(S_{ij})$, $S_{ij}=S_{ji}$ be a symmetric second order tensor on Ω with components $S_{ij} \in H^s(\Omega)$. Then there is a vector field $u=(u_i)$ with components $u_i \in H^{s+1}(\Omega)$ such that $S_{ij}=\partial_i u_j+\partial_j u_i$ if and only if*

$$\partial_k \partial_l S_{ij}-\partial_k \partial_j S_{il}+\partial_i \partial_j S_{kl}-\partial_i \partial_l S_{kj}=0 \tag{16}$$

(these are St. Venant's compatibility conditions; the derivatives are in the sense of distributions) and, moreover,

$$\sum_{i,j=1}^{n} \int_\Omega S_{ij} k_i a_j \, \mathrm{d}x=0$$

for any Killing vector $k \in \vec{Q}(\Omega)$ and any $a \in \vec{A}_\tau(\Omega)$. In particular, if Ω is simply connected (which implies $\vec{A}_\tau(\Omega)=0$), then (16) are necessary and sufficient conditions.

Proof. This theorem can be proved using GURTIN's method ([6], Section 14, Theorem 2) by an argument similar to that of Theorem 5.2. However, I prefer a proof based on a duality argument. We begin by recalling some known facts. Let div_0: $D(\operatorname{div}_0) \subset \vec{H}(\Omega) \to H(\Omega)$ be the Hilbert-space closure of the restriction $\operatorname{div}|\vec{H}_0^1(\Omega)$. Part of the Hodge-Kodaira decomposition theorem says that $u=(u_i) \in \operatorname{Ker}\operatorname{div}_0$ is of the form $u_i=\sum_{j=1}^{n} \partial_j \omega_{ij}$, $\omega_{ij}=-\omega_{ji} \in H^1(\Omega)$, $\sum_{j=1}^{n} v_j \omega_{ij}|\Gamma=0$, if and only if $(u,a)_{0,\Omega}=0$ for any $a \in \vec{A}_\tau(\Omega)$ (see MORREY [11]). In fact, it can be proved that in this case we can choose $\omega_{ij} \in H_0^1(\Omega)$ for any i, j. Moreover, if $u \in \vec{H}_0^k(\Omega)$ for some integer $k \geqq 1$, then one can choose $\omega_{ij} \in H_0^{k+1}(\Omega)$ (these are particular cases of some general theorems I have proved in a paper submitted for publication). We define now $\mathscr{H}(\Omega)$ as the space of fourth-order tensors $T=(T_{ijkl})$ such that $T_{ijkl}=T_{klij}=-T_{jikl} \in H(\Omega)$, provided with the scalar product

$$(V, W)_{0,\Omega}=\sum_{i,j,k,l} \tfrac{1}{8}(V_{ijkl}, W_{ijkl})_{0,\Omega},$$

so $\mathscr{H}(\Omega)$ becomes a Hilbert space. Let $\mathscr{D}\mathrm{iv}_0$ be the closure of the operator:

$$\mathscr{H}(\Omega) \supset \mathscr{H}_0^2(\Omega) \ni (T_{ijkl}) \mapsto \Big(\sum_{k,l} \partial_k \partial_l T_{ikjl}\Big) \in \overset{\leftrightarrow}{H}(\Omega)$$

where

$$\mathscr{H}_0^2(\Omega)=\{T \in \mathscr{H}(\Omega) \mid T_{ijkl} \in H_0^2(\Omega)\}.$$

We call the operator $\mathscr{E}$: $D(\mathscr{E}) \subset \overset{\leftrightarrow}{H}(\Omega) \to \mathscr{H}(\Omega)$ its adjoint. We see easily that

$$D(\mathscr{E})=\{S \in \overset{\leftrightarrow}{H}(\Omega) \mid \partial_k \partial_l S_{ij}-\partial_k \partial_j S_{il} +\partial_i \partial_j S_{kl}-\partial_i \partial_l S_{kj} \in H(\Omega) \quad \text{for any } i, j, k, l\},$$

and for $S \in D(\mathscr{E})$,

$$\mathscr{E}(S)_{ikjl}=\partial_k \partial_l S_{ij}-\partial_k \partial_j S_{il}+\partial_i \partial_j S_{kl}-\partial_i \partial_l S_{kj}.$$

Let $\mathscr{K}_\tau(\Omega)$ be the (finite-dimensional) vector subspace of $\overset{\leftrightarrow}{H}(\Omega)$ generated by tensors of the form $k_i a_j+k_j a_i$, $k \in \vec{Q}(\Omega)$, $a \in \vec{A}_\tau(\Omega)$. Then $\operatorname{Ker}\operatorname{Div}_0=\operatorname{Im}\mathscr{D}\mathrm{iv}_0$

$\oplus\mathscr{K}_\tau(\Omega)$, a Hilbert direct sum ($\mathrm{Div}_0$ is defined in the remarks following Theorem 4.9). This result is proved exactly as is Theorem 5.2, using the remarks made at the beginning of this proof. Namely, if $S\in\mathrm{Ker}\,\mathrm{Div}_0$ and $(S,\omega)_{0,\Omega}=0$ for any $\omega\in\mathscr{K}_\tau(\Omega)$, then for any i the vector field $\vec{S}_{i\cdot}=\{S_{ij}|j=1,\dots,n\}$ clearly belongs to $\mathrm{Ker}\,\mathrm{div}_0$, and it satisfies $(S_{i\cdot},a)_{0,\Omega}=0$ for any $a\in\vec{A}_\tau(\Omega)$, so that there is $u_{ijk}\in H_0^1(\Omega)$, $u_{ijk}=-u_{ikj}$ such that $S_{ij}=\sum_k\partial_k u_{ijk}$. As in the proof of Theorem 5.2 we obtain (15) for any $a\in\vec{A}_\tau(\Omega)$, noting that the boundary term in (14) is zero because of $\sum_k v_k u_{ijk}|\Gamma=0$. Then we can choose $v_{ijkl}\in H_0^2(\Omega)$, as explained before, so the tensor T of Theorem 5.2 is now in $\mathscr{H}_0^2(\Omega)$. We have proved that $\mathrm{Ker}\,\mathrm{Div}_0=\mathscr{D}\mathrm{iv}_0(\mathscr{H}_0^2(\Omega))\oplus\mathscr{K}_\tau(\Omega)$, in particular $\mathscr{D}\mathrm{iv}_0|\mathscr{H}_0^2(\Omega)$ has a closed range. Obviously, its closure will have the same range, *i.e.* $\mathrm{Ker}\,\mathrm{Div}_0=\mathrm{Im}\,\mathscr{D}\mathrm{iv}_0\oplus\mathscr{K}_\tau(\Omega)$. But $\mathrm{Im}\,\varepsilon_1$ is $\overleftrightarrow{H}(\Omega)\ominus\mathrm{Ker}\,\mathrm{Div}_0$ (see the remarks after Theorem 4.9), *i.e.* $\mathrm{Im}\,\varepsilon_1=(\mathrm{Im}\,\mathscr{D}\mathrm{iv}_0)^\perp\cap(\mathscr{K}_\tau(\Omega))^\perp=(\mathrm{Ker}\,\mathscr{E})\cap(\mathscr{K}_\tau(\Omega))^\perp$. The proof is finished by an application of Theorem 4.1. Q.E.D.

Remark. Using GURTIN's method ([6], Section 14, Theorem 2) and the DE RHAM theorem ([13], §22, Theorem 17′), one can prove the following result: If $\vec{A}_\tau(\Omega)=0$ (in particular if Ω is simply connected), then a symmetric second order tensor S with components $S_{ij}\in H^{-\infty}(\Omega)$ is of the form $S_{ij}=\partial_i u_j+\partial_j u_i$, with $u_i\in H^{-\infty}(\Omega)$ for any i, if and only if St. Venant's compatibility conditions (16) are satisfied.

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