Long-Time Behavior of Solutions to Nonlinear Evolution Equations

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Introduction

We are interested in large time behavior of solutions to nonlinear partial differential equations of evolution. More precisely we deal with the Initial-Value Problem for nonlinear equations which are perturbations of classical linear equations like: the Wave Equation, Klein-Gordon Equation, Linear Isotropic Elasticity Equations, Heat Equation, Schrödinger Equation, *etc.* The aim is to show that for sufficiently small nonlinear perturbations the corresponding solutions behave asymptotically like the solutions of the linear equations. In particular, we recover and extend our earlier results on global existence for nonlinear wave equations [1]. As in that work our method relies on a special version of the powerful Nash-Moser-Hörmander scheme which allows us to treat a very large class of perturbations. The method seems to be particularly suited for hyperbolic equations; however, we attempt here a treatement of other classical equations like the Heat Equation and the Schrödinger Equation.

For some specific examples our method is too rough and indiscriminate to give optimal results. In particular, for the special class of semilinear perturbations our results are weaker than what is available in the literature. However, we can allow here perturbations which contain, in general, derivatives of the same order as the linear part, for which the classical methods used in the semilinear case do not seem to work.

We assume the linear equations that we intend to perturb to have the form

$$u_t - \Gamma u = 0, \tag{1}$$

with the initial condition $u(x, 0) = u^{(0)}(x)$.

Here $u = u(x, t)$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, is a real (or complex) vector function. Γ is a linear differential operator with constant coefficients of order γ:

$$\Gamma = \sum_{|\alpha| \leq \gamma} \Gamma_\alpha D_x^{(\alpha)}; \quad D_x^{(\alpha)} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n};$$

Γ_α are $r \times r$ matrices with constant entries.

We assume that Γ satisfies a dissipative condition of the following type. There is a *positive definite* $r\times r$ *matrix* A^0 such that either

(D) $$\int \mathrm{Re}\,(A^0\Gamma f, f)\, dx \leqq 0$$

or

(D′) $$\int \mathrm{Re}\,(A^0\Gamma f, f)\, dx \leqq -\|\nabla f\|^2_{L^2(\mathbb{R}^n)}$$

for any $f\in C_0^\infty(\mathbb{R}^n)$.

Examples:

(LWE) *Linear Wave Equation*:

$$y_{tt}-\Delta y=0,\quad y(x,0)=f(x),\quad y_t(x,0)=g(x).$$

To write it in the form (1) we introduce $u_0=y_t$, $u_i=y_{x_i}$, $i=1,\dots,n$. We rewrite (LWE) in the form

$$u_t-\sum_{j=1}^n A_j\frac{\partial}{\partial x_j}u=0$$

where $A_j=(a_{kl})_{k,l=0,\dots,n}$ and $a_{kl}=0$ except for $a_{0j}=a_{j0}=1$. The initial data have the form

$$u^{(0)}(x)=\left(g,\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right),$$

i.e.,

(2) $$\frac{\partial}{\partial x_j}u_i^{(0)}=\frac{\partial}{\partial x_i}u_j^0.$$

The operator $\Gamma=\sum_{j=1}^n A_j\frac{\partial}{\partial x_j}$ is a first order linear operator with constant coefficients and

$$\int \mathrm{Re}\,(\Gamma f,f)\,dx=\sum_{j=1}^n \mathrm{Re}\int_{\mathbb{R}^n}\left(\frac{\partial f_j}{\partial x_j}\bar f_0+\frac{\partial f_0}{\partial x_j}\bar f_j\right)dx=0.$$

Therefore condition (D) is satisfied with $A^0=I$.

(LKGE) *Linear Klein-Gordon Equation*:

$$y_{tt}-\Delta y+m^2y=0,\quad m\neq 0,$$
$$y(x,0)=f,\quad y_t(x,0)=g.$$

As before, we introduce the unknowns $u_0=y_t$, $u_1=y_{x_1},\dots,u_n=y_{x_n}$, $u_{n+1}=y$ and rewrite (LKGE) in the form (1). The initial data have the form

$$u^0(x)=\left(g,\frac{\partial}{\partial x_1}f,\dots,\frac{\partial}{\partial x_n}f,f\right)$$

i.e.

(3) $$\frac{\partial}{\partial x_0}u_{n+1}^{(0)}=u_i^{(0)},\qquad \frac{\partial}{\partial x_j}u_i^{(0)}=\frac{\partial}{\partial x_i}u_j^{(0)}.$$

(LIEE) *Linear Isotropic Elasticity Equation*:

$$\frac{\partial^2 y_i}{\partial x^2} = \sum_k (c_1^2 - c_2^2) \frac{\partial^2 y_k}{\partial x_i\, \partial x_k} + c_2^2 \frac{\partial^2 y_i}{\partial x_k\, \partial x_k},$$

where c_1, c_2 are the speeds of propagation of waves, are given as follows in terms of the Lamé constants λ, μ, and the density ϱ:

$$c_1^2 = (\lambda + 2\mu)/\varrho, \quad c_2^2 = \mu/\varrho.$$

The initial conditions are $y_i(x, 0) = f_i(x)$, $y_{i,t}(x, 0) = g_i(x)$, $i = 1, \ldots, n$.

For convenience we rewrite the equation (L.I.E.E.) in the following form (see [6]):

$$\frac{\partial^2 y_i}{\partial t^2} = C_{ikjm} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_m} y_j$$

where $C_{ikjm} = c_{ikjm} + c_2^2(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk})$ and also

$$c_{ikjm} = (c_1^2 - 2c_2^2)\, \delta_{ik}\delta_{jm} + c^2(\delta_{ij}\delta_{km} + \delta_{kj}\delta_{im}).$$

Introducing $u_{ik} = D_k y_i$, $i = 1, 2, 3$, $k = 0, 1, 2, 3$, we obtain a system of 12 first order equations for the u_{ij}

$$D_0 u_{i0} = \sum_{k,j,m} C_{ikjm}\, D_k u_{jm},$$

$$D_0 u_{ik} = D_k u_{i0},$$

or, in vector notation,

$$D_0 u = \sum_{r=1}^{3} A^r\, D_r u,$$

where $A^r_{ikjm} = C_{irjm}\, \delta_{k0}(1 - \delta_{m0}) + \delta_{m0}\, \delta_{rk}$ is defined for $i, j = 1, 2, 3$, $k, m = 0, 1, 2, 3$. (The subscripts i, k count the rows and j, m the columns of the matrix A^r; see [6].) Setting $\Gamma = \sum_{r=1}^{3} A^r D_r u$ and taking

$$A^0_{ikjm} = (1 - \delta_{k0})\,(1 - \delta_{m0})\, C_{ikjm} + \delta_{ij}\, \delta_{k0}\, \delta_{m0},$$

one verifies (see [6]) that A^0 and $A^0 A^r$ are symmetric and A^0 is positive definite. Therefore $\mathrm{Re} \int (A^0 \Gamma f, f)\, dx = 0$ which is our condition (D).

The initial data $u^{(0)}$ have the form

$$u^{(0)}_{i0} = g_i(x), \quad u^{(0)}_{ik} = \frac{\partial}{\partial x_k} f_i(x), \quad i = 1, 2, 3, \quad k = 1, 2, 3$$

or

$$\frac{\partial}{\partial x_j} u^{(0)}_{ik} = \frac{\partial}{\partial x_k} u^{(0)}_{ij}, \quad \text{for any } i, j, k = 1, 2, 3. \tag{4}$$

In the next examples Γ will be a scalar operator of order greater than 1. Also in these examples $A^0 = I$.

(L.S.E.) *Linear Schrödinger Equation*:

$$u_t - \frac{1}{i}\Delta u = 0,$$

$$u(x, 0) = u^{(0)}(x).$$

Here $\Gamma = \frac{1}{i}\Delta$ satisfies

$$\operatorname{Re} \int (\Gamma f, f)\, dx = 0,$$

which is our condition (D), with $A^0 = I$.

(L.H.E.) *Linear Heat Equation*:

$$u_t - \Delta u = 0,$$

$$u(x, 0) = u_0(x);$$

$\Gamma = \Delta$ satisfies

$$\operatorname{Re} \int (\Gamma f, f)\, dx = -\int |\nabla f|^2\, dx,$$

which is condition (D′).

(A.E.) *The Airy Equation*:

$$u_t + u_{xxx} = 0, \qquad \Gamma = -\frac{\partial^3}{\partial x^3},$$

$$u(x, 0) = u_0(x),$$

$$\operatorname{Re} \int (\Gamma f, f)\, dx = 0.$$

In what follows we denote by $W(t)$ the fundamental solution of (1.1). Thus $W(t)\, u^{(0)}$ is a solution of (1.1) with initial data $u^{(0)}$, and it is unique. In all the examples mentioned before $W(t)$ satisfies the following basic decay property: There is a differential matrix P such that

$$(\square) \qquad |\, W(t)\, u^{(0)}|_{L^\infty} \leqq C(1 + t)^{-k_0}\, \|u^{(0)}\|_{L^1,n}$$

for all $u^{(0)} \in L^1_n(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ that satisfy

$$Pu^{(0)} = 0.$$

Indeed, one derives this estimate for all the above examples from the classical explicit form of the solutions. For (L.W.E.), (L.K.G.E.) and (L.I.E.E.) the differential matrix P is defined by the relations (2), (3), (4) and imposes the natural restrictions on the initial data $u^{(0)}$ corresponding to the initial data of the original second order equations. In all other examples $P \equiv 0$. For (L.W.E.) we have (see [1]) $k_0 = \frac{n-1}{2}$, for (L.I.E.E.) $n = 3$ and $k_0 = 1$ (see [6]) while for (L.K.G.E.), (L.S.E.) or (L.H.E.) $k_0 = \frac{n}{2}$. Finally for (A.E.) $n = 1$, $k_0 = \frac{1}{3}$ (see [7]).

We now consider nonlinear perturbations of (1)

$$\text{(N)} \qquad u_t - \Gamma_u = F(u, Du), \qquad Du = (u_t, u_{x_1}, \ldots, u_{x_n}),$$
$$u(x, 0) - u^{(0)}(x), \qquad Pu^{(0)} = 0.$$

Naturally, we assume that F is compatible with the set of constraints induced by P, *i.e.*,

$$PF(u, Du) \equiv 0, \qquad \forall u, Du.$$

We also impose the following conditions:

$$(\text{F}_1) \qquad A^0 F_{u_t}, A^0 F_{u_{x_i}}, \qquad i = 1, \ldots, n,$$

are symmetric $r \times r$ matrices and F_{u_t} is independent of u_t.

$$(\text{F}_2) \qquad |F(u, Du)| \leqq C(|u| + |Du|)^{p+1}$$

for $|u| + |Du|$ sufficiently small.

In case Γ satisfies the condition (D′) we can allow $F = F(u, D_x u, D_x^2 u)$ in our perturbation. In this case (F_1) is unnecessary.

We now state our main theorems.

Theorem 1. *Assume*

$$\text{(C)} \qquad \frac{1}{p}\left(1 + \frac{1}{p}\right) < k_0$$

where k_0, p *are the numbers defined by* (□) *and* (F_2). *Also assume* p *is an integer and that* F *is smooth. There is an integer* $N_0 > 0$ *and a small* $\eta > 0$ *such that if*

$$\|u^{(0)}\|_{L^1, N_0} \leqq \eta, \qquad \|u^{(0)}\|_{L^2, N_0} \leqq \eta,$$

then there is a unique smooth solution $u \in C^1([0, T], C^\gamma(\mathbb{R}^n))$ *of* (N). *Moreover, the solution behaves, for* t *large, like*

$$|u(x, t)| = O\left(t^{-\frac{1+\varepsilon}{p}}\right), \quad as \quad t \to \infty,$$

for some small $\varepsilon > 0$. *Also*

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = O(1), \quad as \quad t \to \infty.$$

Remarks. (1) The unpleasant requirement that p be an integer is imposed only to ensure that F is smooth near the origin. It can be avoided if the number N_0 appearing in the theorem can be made small. The value of N_0 can be made precise (see Section 4) but our results are very crude for estimating it closely in concrete examples. (2) The asymptotic behavior of the solution given in Theorem 1 can be improved if $k_0 \gg \frac{1}{p}\left(1 + \frac{1}{p}\right)$ at the expense, eventually, of requiring N_0 to be larger.

Theorem 2. *Let* $u(x, t)$ *be the solution of* (N) *introduced by Theorem* 1. *There is a unique solution* u_+ *of* (1.1) *such that*

$$\|u(t) - u_+(t)\|_{L^2} \to 0, \quad as \quad t \to \pm\infty.$$

Proof of Theorem 2. We define

(5) $$u_+(t) = u(t) + \int_t^\infty W(t-s)\, F(u(s))\, ds$$

where W is the fundamenal solution of the linear equation. The convergence of the integral in (5) follows easily from the asymptotic properties of u given in Theorem 1.

In the remaining part of this introduction we illustrate our theorems on perturbations of the linear problems presented above.

(N.W.E.) *Nonlinear Wave Equations* (*see* [1]):

$$y_{tt} - \Delta y = G(Dy, D^2y),$$
$$y(x, 0) = f(x), \quad y_t(x, 0) = g(x).$$

Rewriting the (N.W.E.) as a symmetric hyperbolic system verifies condition (F_1) *, provided that $G_{y_{tt}}$ be independent of y_{tt}. Also condition (F_2) is equivalent to

$$|G(Dy, D^2y)| \leqq C(|Dy| + |D^2y|)^{p+1}$$

for small $|Dy|$, $|D^2y|$. Since for this case $k_0 = \frac{n-1}{2}$, condition (C) becomes

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) < \frac{n-1}{2}.$$

For $p = 1$, this implies $n > 6$ which is the result obtained in [2]. For $p = 2$, $n > 2 + \frac{1}{2}$. In particular the result is valid for $n = 3$. For $p = 3, n > 1 + \frac{8}{9}$; therefore, Theorem 1 holds for $n = 2$.

(N.I.E.E.) *Nonlinear Isotropic Elasticity Equations*:

$$\frac{\partial^2 y_i}{\partial t^2} = \sum_{j,k,m=1}^{3} C_{ijkm}(\nabla y) \frac{\partial^2 y_i}{\partial x_k x_m}$$

where $y = (y_1, y_2, y_3)$ as a function of $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$ is the displacement vector (see [6]) and ∇y is the gradient $(y_{x_1}, y_{x_2}, y_{x_3})$ of y. We also have (see [6])

$$c_{ikjm} = c_{jmik} = \frac{\partial^2 E}{\partial u_{ik}\, \partial u_{jm}} \quad \text{where} \quad u_{ik} = \frac{\partial u_i}{\partial x_k},$$

$$c_{ikjm}(0) = c_{ikjm} \quad \text{defined in the (L.I.E.E.).}$$

We assume

(6) $$|c_{ikjm}(\nabla y) - c_{ikjm}(0)| \leqq C\, |\nabla y|^p \quad \forall i, k, j, m.$$

Following FRITZ JOHN [6], we can rewrite (N.I.E.E.) as a symmetric hyperbolic system:

(7) $$A^0(u)\, u_t = \sum_{r=1}^{3} A^0(u)\, A^r(u)\, D_r u$$

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* See the next example.

where $u_{ik} = D_k y_i$, $i = 1, 2, 3$, $k = 0, 1, 2, 3$. Also $A^0(0) = A^0$, $A^r(0) = A^r$, previously defined in (L.I.E.E.). Thus we can rewrite (D) in the form

$$u_t - \Gamma u = F(u, Du)$$

where, with $B^r(u) = A^0(u) \cdot A^r(u)$ and $B^r = A^0 A^r$,

$$A^0 \cdot F(u, Du) = \sum_{r=1}^{3} (B^r(u) - B^r) D_r u - (B^0(u) - B^0) D_t u.$$

The condition $PF(u, Du) = 0$ is automatically satisfied by our construction. Condition (F_1) is also satisfied since $B^r(u) - B^r$ and $B^0(u) - B^0$ are symmetric. Finally (F_2) is equivalent with condition (6). Since $n = 3$, $k_0 = 1$, condition (C) of Theorem 1 becomes:

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) < 1.$$

which holds for $p \geqq 2$.

(N.K.G.E.) *Nonlinear Klein-Gordon Equation*:

$$y_{tt} - \nabla y = G(y, Dy, D^2 y).$$

Assume that $G_{y_{tt}}$ is independent of y_{tt} and

$$|G(y, Dy, D^2 y)| \leqq C(|y| + |Dy| + |D^2 y|)^{p+1}.$$

Then Theorem 1 holds provided that

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) < \frac{n}{2}.$$

For rates of decay of solutions of (L.K.G.E.) see [8].

(N.S.E.) *Nonlinear-Schrödinger Equation*:

$$u_t - \frac{1}{i} \Delta u = F(u, Du).$$

Assume that F_{u_t} is independent on y_t and $|F(u, Du)| \leqq C(|u| + |Du|)^{p+1}$. Then Theorems 1 and 2 hold, provided that F'_{0u} is real and

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) \leqq \frac{n}{2}.$$

For $F = F(u)$ and $n = 1$ we improve the results of W. Strauss [7]. However, for this case, even better results were obtained recently in [4], [5]. (See also [10].)

Remark. Our result will also apply to coupled nonlinear Schrödinger equations $u_t - \frac{1}{i} \Delta u = F(u, Du)$, where u is an r-vector. In addition to the assumptions made before we also need

$$F'_{u_t}, F'_{u_{x_i}}, \; i = 1, \dots, n, \text{ are selfadjoint } r \times r \text{ matrices.}$$

(K.W.D.) *Korteweg-De Vries Equation*:

$$u_t + u_{xxx} = F(u, Du).$$

Here F_{u_t} is independent of u_t:

$$|F(u, Du)| \leqq C(|u| + |Du|)^{p+1},$$

$$\frac{1}{p}\left(1+\frac{1}{p}\right)<\frac{1}{3}, \quad \text{or} \quad p>\frac{3+\sqrt{21}}{2}$$

which improves the result of [7] where it was assumed $F(u, Du) = f'(u)\, u_x$ and $p > 4$.

(N.H.E.) *Coupled Nonlinear Heat Equations*:

$$u_t - \Delta u = F(u, D_x u, D_x^2 u);$$

u is an r-vector and F satisfies $|F(u, D_x u, D_x^2 u)| \leqq C(|u| + |D_x u| + |D_x^2 u|)^{p+1}$ for small $|u|$, $|D_x u|$, $|D_x^2 u|$.

Theorem (1) holds provided that

$$\frac{1}{p}\left(1+\frac{1}{p}\right)<\frac{n}{2}$$

where n is the dimension of the space of x. The linear part of the N.H.E. satisfies condition (D′). The only difference in the case where (D′) is satisfied rather than (D) is in the derivation of the energy estimate (Section 3). The modifications are rather straightforward, but a complete proof for the case (D′) will be omitted here.

The heart of the paper is Section 4, where we construct a solution to (N) using a NASH-MOSER-HÖRMANDER scheme. The scheme follows, with some modification, that presented by HÖRMANDER in [9]. The main modification consists in the presence, in addition to the usual "smoothing" operator, of a "cut off" in time operator. This is necessary in view of the "loss of decay" in the linear estimates presented in Sections 2 and 3. Further complications arise since it is necessary here to work not just with one scale of Banach spaces, as in [9], but with three double-scales of different nature corresponding to the norms

$$|\ |_{k,N} = \sup_{t \geqq 0} (1+t)^k \|u(t)\|_{L^\infty,N},$$

$$\|\ \|_{k,N} = \sup_{t \geqq 0} (1+t)^k \|u(t)\|_{L^2,N},$$

$$|||\ |||_{k,N} = \sup_{t \geqq 0} (1+t)^k \|u(t)\|_{L^1,N}.$$

However, some of these difficulties have been already encountered in [1]. Further difficulties are due, here, to decay estimates more general than those in [1].

Section 1 is devoted to a review of the calculus inequalities needed in this work. They have already been used and proved in [1]. I also present here the properties of the "smoothing" and "cut-off" operators $S^{(1)}(\theta)$, $S^2(\theta')$. Section 2 is devoted to decay estimates for the linearized problem. Due to the more general condition (F_2) on the nonlinear perturbation the estimates here are quite different from the corresponding ones derived in [1]. Finally, Section 3 is devoted to energy estimates. Condition (F_1), on the nonlinear perturbation $F(u, Du)$, assure similar type of energy estimates as for the Symmetric Hyperbolic Systems.

Notation

We use the following standard notation: $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $D_x^\alpha = D_{x_1}^{\alpha_1}, \dots, D_{x_n}^{\alpha_n}$ with $D_{x_i} = \frac{\partial}{\partial x_i} = D_i$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

For $f: \mathbb{R}^n \to \mathbb{R}^r$ we write $\|f\|_{L^p} = \max_{i=1,\dots,r} (\int |f_i(x)|^p dx)^{1/p}$, $\|f\|_{L^p,N} = \sum_{|\alpha| \le N} \|D_x^\alpha f\|_{L^p}$,

$$\|D^s f\|_{L^p} = \sum_{|\alpha| = s} \|D^\alpha f\|_{L^p} \quad \text{for} \quad s > 0.$$

For a given r-vector $u = u(x, t)$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, we set

$$Du = (u_{x_0}, u_{x_1}, \dots, u_{x_n})$$

where u_{x_0} stands for u_t. Also $\Lambda u = (u, Du)$ is the vector u together with its first derivatives. For an r-function $F(\lambda)$ which depends on the variables $\lambda = (\lambda_i, \lambda_{jk})$, $i = 1, \dots, r$, $j = 1, \dots, r$, $k = 0, \dots, r$, we shall use the following notations for the Fréchet derivatives:

$$F'_\lambda(\Lambda u)\, \Lambda v = \sum_{i=1}^r F'_{\lambda_i} v_i + \sum_{j,k} F'_{\lambda_{jk}} D_k v_j.$$

Occasionally we also use the notation F'_u, F'_{u_t}, $F'_{D_k u}$, $k = 1, \dots, n$ for the $r \times r$ matrices (F'_{i,u_j}), $(F'_{i,D_t u_j})$, $(F'_{i,D_k u_j})$, $i, j = 1, \dots, r$.

We also use the norms

$$\|u\|_{L^p;k,N} = \sup_{t \ge 0} (1 + t)^k \|u(t)\|_{L^p,N}$$

with

$$|u|_{k,N} = \|u\|_{L^\infty;k,N},$$

$$\|u\|_{k,N} = \|u\|_{L^2;k,N}, \qquad \|u\|_N = \|u\|_{0,N},$$

$$|||u|||_{k,N} = \|u\|_{L^1;k,N}.$$

§ 1. Preliminaries

Consider the nonlinear functional $\Phi(u) = u_t - \Gamma u - F(\Lambda u)$ where Λu stands for (u, Du). To solve our nonlinear equation

$$\Phi(u) = 0, \quad u(x, 0) = u^{(0)}(x), \tag{1.1}$$

by an iteration scheme, we have to be able to obtain "good" estimates for the following linearized problem:

$$L_\theta(u)\, v = g, \qquad V(x, 0) = 0 \tag{1.2}$$

where

$$L_\theta(u)\, v = v_t - \Gamma v - F'_\lambda(S_\theta \Lambda u) \cdot \Lambda v. \tag{1.3}$$

Note that $L_\theta(u)$ differs from the Fréchet derivative $\Phi'(u)$ by the mollifying operator S_θ which was defined in [1] (see Sections (6) and (7)). S_θ is the product of a "cut off" operator $S^{(1)}(\theta)$ in t with a smoothing operator $S^{(2)}(\theta')$ in x where $\theta' = \theta^{\bar\varepsilon}$, $\bar\varepsilon > 0$, and has the properties

$$\|(I - S_\theta)\, u\|_{L^q;0,0} \leqq C_{q,k,N}(\theta^{-k}\, \|u\|_{L^q;k,0} + \theta^{-\bar\varepsilon N}\, \|u\|_{L^q;0,N}), \tag{S1}$$

$$\|S_\theta u\|_{L^q;k,N} \leqq C_{q,k,N} \theta^{k-k_0} \theta^{\bar\varepsilon(N-N_0)}\, |u|_{L^q,k_0,N_0} \tag{S2}$$

for $0 \leqq k_0 \leqq k$; $0 \leqq N_0 \leqq N$. The norms here are those defined in the section on notation, above.

The following properties of the norms $|\;|_{L^2;k,N}$ will be used repeatedly:

Lemma 1.1. *Assume* $g_1, g_2 \in C^\infty([0,\infty)\times\mathbb{R}^n)$, $\dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_2}$ *and* $k = k_1 + k_2$. *Then*

$$|g_1 g_2|_{L^2;k,N} \leqq C(\|g_1\|_{L^{q_1};k_1,N} \cdot \|g_2\|_{L^{q_2};k_2,N} + \|g_1\|_{L^{q_1};k_1,N} \cdot \|g_2\|_{L^{q_2};k_2,0}).$$

Lemma 1.2. *Assume* u *is a vector function in* $(C^\infty((0,\infty)\times\mathbb{R}^N))^r$ *and* $F = F(u)$ *is a smooth function of* u. *Then*

$$\|F(u)\|_{L^q;k,N} \leqq C_{L^q;k,N}(1 + \|u\|_{L^q;k,N}),$$

provided that

$$\|u\|_{L^\infty;0,0} \leqq 1.$$

The proofs of these two propositions are based on the following two lemmas of calculus.

Lemma 1.3. *Suppose* $f_1, f_2 \in C^\infty(\mathbb{R}^n)$. *Then for* $\dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_2}$ *we have*

(i) $\|D^s f_1 f_2\|_{L^g} \leqq C_s(\|f_1\|_{L^{q_1}} \cdot \|D^s f_2\|_{L^{q_2}} + \|D^s f_1\|_{L^{q_1}}\, \|f_2\|_{L^{q_2}})$,

(ii) $\|D^s(f_1 f_2) - f_1\, D^s f_2\|_{L^q} \leqq C_s(\|Df_1\|_{L^{q_1}}\, \|D^{s-1} f_2\|_{L^{q_2}} + \|D^s f_1\|_{L^{q_1}} \cdot \|f_2\|_{L^{q_2}}$.

Lemma 1.4. *Suppose* w *is a vector function in* $(C^\infty(\mathbb{R}^n))^r$ *and* $F = F(u)$ *a smooth function of* u. *Then*

$$\|D^N f(w)\|_{L^q} \leqq C_{N,q}(\|D^N w\|_{L^q} + 1),$$

provided that $\|w\|_{L^\infty} \leqq 1$.

We also use the following interpolation lemma (see [1], Section 6).

Lemma 1.5. *Suppose* $u \in (C^\infty[0,\infty] \times \mathbb{R}^n))^r$ *be such that*

$$|u|_{0,0} \leqq C\theta^{-\beta},$$

$$|u|_{k,N} \leqq C\theta^{k+\lambda N-\beta}, \textit{ for } k+\lambda N-\beta \geqq \bar{\varepsilon} \textit{ and } 0 \leqq k \leqq \tilde{k},\ 0 \leqq \tilde{L} \leqq \tilde{L}.$$

Here $\bar{\varepsilon} > 0$, $\beta > \bar{\varepsilon}$ *and* $\tilde{k}$, $\tilde{N}$ *are fixed positive numbers such that*

$$\tilde{k} - \beta \geqq \bar{\varepsilon}, \quad \lambda\tilde{N} - \beta \geqq \bar{\varepsilon}.$$

Then

$$|u|_{k,N} \leqq (C_{\tilde{k},N} \cdot C)\, \theta^{k+\lambda N-\beta},$$

for all $0 \leqq k \leqq \tilde{k}$, $0 \leqq N \leqq \tilde{N}$.

Remark 1. The Lemmas 1.1 and 1.2 will be used specifically for the norms

(1.4)
$$|\ |_{k,N} = \|\ \|_{L^\infty;k,N},$$
$$\|\ \|_{k,N} = \|\ \|_{L^2;k,N},$$
$$|||\ |||_{k,N} = \|\ \|_{L^1;k,N}.$$

§ 2. Decay estimates for the linearized equation

As stated in the Introduction, the fundamental solution $W(t)$ of the linear equation

(2.1)
$$v_t - \Gamma v = 0$$

has the property

(□)
$$|W(t)\, v_0|_{L^\infty} \leqq C(1+t)^{-k_0} \|v_0\|_{L^1,n},$$

for all v_0 such that $Pv_0 = 0$.

We recall that, by definition, $W(t)\, v_0$ is the solution of (2.1) for initial data v_0. Also, by the dissipative condition (D) imposed on Γ we also have

(2.2)
$$\|W(t)\, v_0\|_{L^2} \leqq \|v_0\|_{L^2}.$$

In what follows we investigate the decay of solutions of the inhomogeneous equation

(2.3)
$$v_t - \Gamma v = h(t), \quad Ph(t) = 0$$

where we write the solution v of (2.3) in the form

(2.4)
$$v(t,\cdot) = \int_0^t W(t-s)\, h(s,\cdot)\, ds.$$

Splitting the integral in (2.4) into an integral between 0 and $t/2$ and one between $t/2$ and t and applying (i) on the first part and (ii) on the second, we derive

$$|v(x,t)| \leqq C \int_0^{t/2} (1+(t-s))^{-k_0} \|h(s)\|_{L^1,n}\, ds + C \int_{t/2}^{t} \|h(s)\|_{L^2,n}\, ds. \tag{2.5}$$

Applying (2.4), we can also write

$$(1+t)^{k_0} v(x,t) \leqq C \sup_{s\geqq 0} (1+s)^{k_0(p-1)} \|h(s)\|_{L^1,n} \int_0^{\theta} (1+s)^{-k_0(p-1)}\, ds$$
$$+ C \sup_{s\geqq 0} (1+s)^{k_0 p} \|h(s)\|_{L^2,n} \int_0^{\theta} (1+s)^{-k_0(p-1)}\, ds$$

and, with the notations (1.4),

$$|v|_{k_0,0} \leqq C\chi(\theta)\, |||h|||_{k_0(p-1),n} + \|h\|_{k_0 p,n}, \tag{2.6}$$

where

$$\chi(\theta) = \int_0^{\theta} (1+S)^{-k_0(p-1)}. \tag{2.7}$$

Similarly, taking space derivatives of the equations (2.3) we deduce

$$|v|_{k_0,N} \leqq C\chi(\theta)\, R_{k_0,n+N}(h), \tag{2.8}$$

where

$$R_{k_0,N}(h) = |||h|||_{k_0(p-1),N} + \|h\|_{k_0 p,N}. \tag{2.9}$$

Also from (2.3)

$$|v_t|_{k_0,N} \leqq |v|_{k_0,\gamma+N} + |h|_{k_0,N},$$

where γ is the order of the operator Γ. Thus

$$|v_t|_{k_0,N} \leqq C\chi(\theta)\, R_{k_0,n+\gamma+N}(h).$$

Hence

$$|\Lambda v|_{k_0,N} \leqq C\chi(\theta)\, R_{k_0,\bar\gamma+N}; \quad \bar\gamma = \gamma + n. \tag{2.10}$$

We are going to apply this estimate to equation (1.2) which we rewrite in the form

$$v_t - \Gamma v = F_\lambda(S_\theta \Lambda u)\cdot \Lambda v + g, \quad v(0) = 0.$$

Thus, assuming that (2.3) is satisfied for g, we deduce from (2.10) that

$$|\Lambda v|_{k_0,N} \leqq C_n\chi(\theta)\, [|||F'_\lambda(S_\theta\Lambda u)\,\Lambda v|||_{k_0(p-1),N+\bar\gamma} + \|F'_\lambda(S_\theta\Lambda u)\,\Lambda v\|_{k_0 p,N+\bar\gamma} + R_{k_0,N+\bar\gamma}(g)]. \tag{2.11}$$

By Lemma 1.1.,

$$|||F'_\lambda(S_\theta\Lambda u)\,\Lambda v|||_{k_0(p-1),N+\bar\gamma} \leqq C_N(\|F'_\lambda(S_\theta\Lambda u)\|_{k_0(p-1),0}\cdot\|\Lambda v\|_{N+\bar\gamma} + \|F'_\lambda(S_\theta\Lambda u)\|_{k_0(p-1),N+\bar\gamma}\cdot\|\Lambda v\|_0), \tag{2.12}$$

and

(2.13)

$$\| F'_\lambda(S_\theta \Lambda u)\, \Lambda u \|_{k_0, N+\bar\gamma} \leqq C_N(|F'_\lambda(S_\theta \Lambda u)|_{k_0 p, 0} \cdot \|\Lambda v\|_{N+\bar\gamma} + |F'_\lambda(S_\theta \Lambda u)|_{k_0 p, N+\bar\gamma} \|\Lambda v\|_0).$$

On the other hand, by Lemma 1.1.,

(2.14)

$$\| F'_\lambda(S_\theta \Lambda u) \|_{k_0(p-1), M} \leqq C_M(|\Lambda u|^{p-1}_{k_0, 0} \cdot \|S_\theta \Lambda u\|_M + |S_\theta \Lambda u|_{k_0, M} \cdot \|\Lambda u\|^{p-2}_{k_0, 0} \cdot \|\Lambda u\|_0),$$

$$(2.15) \qquad |F'_\lambda(S_\theta \Lambda u)|_{k_0 p, M} \leqq C_M\, |S_\theta \Lambda u|_{k_0, M} \cdot |\Lambda u|^{p-1}_{k_0, 0}$$

provided that $|\Lambda u|_{0,0} \leqq 1$.

Thus assuming that

$$|\Lambda u|_{0,0} \leqq 1, \qquad \|\Lambda u\|_0 \leqq 1,$$

we conclude from (2.11)–(2.15) that

$$(2.16) \qquad |\Lambda u|_{k_0, N} \leqq \chi(\theta)\, [R_{k_0, \bar\gamma+N}(g) + C_N \|\Lambda v\|_0 \cdot (|S_\theta \Lambda u|_{k_0, \bar\gamma+N} + \|S_\theta \Lambda u\|_{\bar\gamma+N})\,(1 + |\Lambda u|^p_{k_0, 0}) + C_N \|\Lambda v\|_{\bar\gamma+N} \cdot (1 + |\Lambda u|^p_{k_0, 0}),$$

where

$$\bar\gamma = \gamma + n$$

and

$$\chi(\theta) = \int_0^\theta (1+S)^{-k_0(p-1)}\, ds,$$

$$R_{k_0, N}(g) = |||g|||_{k_0(p-1), N} + \|g\|_{k_0 p, n+N}.$$

This estimate will be used in the proof of convergence in Section 4.

§ 3. Energy estimates for the linearized equations

Lemma 3.1. *Consider the following linear system*:

$$(3.1) \qquad Ev_t - \tilde\Gamma v = \sum_{i=1}^n A_i v_{x_i} + Bv + g,$$

where v, g are r-vector functions in $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, and $E, A_i, \dots, A_n, B$ are $r \times r$ matrices. Assume

(i) *$\tilde\Gamma$ is a linear differential operator with constant coefficients in $\mathbb{R}^n$, such that*

$$\int \operatorname{Re}(\tilde\Gamma f, f)\, dx \leqq 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^n),$$

(ii) *$E, A_1, \dots, A_n$ are self adjoint $r \times r$ matrices and*

$$\langle E(x,t)\, \xi, \xi \rangle \geqq \omega\, |\xi|^2, \qquad \omega > 0,$$

for any $\xi \in \mathbb{R}^n$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Then

(iii) $\frac{d}{dt}\|v(t)\|_E \leqq C(|D_tE(t)|_{L^\infty} + |D_xA(t)|_{L^\infty}) \cdot \|v(t)\|_E + \|H(t)\|_{L^2}$,

where $H = Bv + g$ *and*

$$\|v(t)\|_E^2 = \int_{\mathbf{R}^n} \langle E(x, t)\, v, v\rangle\, dx.$$

Also

(iv) $\frac{d}{dt}\|v(t)\|_{E,N}^2 \leqq C_N(|D_tE(t)|_{L^\infty} + |D_xA(t)|_{L^\infty} + |B(t)|_{L^\infty}) \cdot \|v(t)\|_{E,N}$

$$+ |D_xE(t)|_{L^\infty,N-1} + |D_xA(t)|_{L^\infty,N-1} + |B(t)|_{L^\infty,N}) \cdot \|v(t)\|_E + \|g(t)\|_{L^2,N},$$

where

$$\|v(t)\|_{E,N}^2 = \sum_{|\alpha|\leqq N} \|D_x^{(\alpha)}v(t)\|_E^2.$$

The proof of Lemma 3.1. is a straightforward modification of the proof for the classical energy estimates for symmetric hyperbolic systems.

By Gronwall's inequality and (ii) we derive from (iii)

$$\|v(t)\|_{L^2} \leqq C\left(\|v(0)\|_{L^2} + \int_0^t \|g(s)\|_{L^2}\, ds\right) \times \exp \int_0^t (|D_tE(s)|_{L^\infty} + |D_xA(s)|_{L^\infty} + |B(s)|_{L^\infty})\, ds.$$

Using the notation introduced above, we rewrite

(3.2)
$$\|v\|_0 \leqq C_\varepsilon(\|v(0)\|_{L^2} + \|H\|_{1+\varepsilon,0}) \cdot \exp C_3(|D_tE|_{1+\varepsilon,0} + |D_xA|_{1+\varepsilon,0} + |B|_{1+\varepsilon,0}),$$

where ε is a small positive constant. Similarly, from Lemma 3.1 (iii), and (3.2) we infer that

(3.3)
$$\|v\|_N \leqq C_{\varepsilon,N}[\|v(0)\|_{L^2,N} + \|g\|_{1+\varepsilon,N} + (|D_tE|_{1+\varepsilon,N-1} + |A|_{1+\varepsilon,N} + |B|_{1+\varepsilon,N})\, \|g\|_{1+\varepsilon,0}] \cdot \exp C_\varepsilon(|D_tE|_{1+\varepsilon,0} + |D_xA|_{1+\varepsilon,0} + |B|_{1+\varepsilon,0}).$$

We now apply (3.2) and (3.3) to the linearized form (1.2) of our nonlinear equation (1.1). In this case we have $E = A^0 - A^0F'_{u_t}(S_\theta\Lambda u)$, $A_i = A^0F'_{D_iu}(S_\theta\Lambda u)$, $i = 1, \ldots, n$, $B = A^0F'_u(S_\theta\Lambda u)$ and $\tilde{\Gamma} = A_0\Gamma$. Also, in this case, $v(0) = 0$.

To estimate $\|A\|_{1+\varepsilon,N}$, $\|B\|_{1+\varepsilon,N}$, $|DE|_{1+\varepsilon,N-1}$, $|A|_{1+\varepsilon,N}$, we apply Lemmas 1.1. and 1.2. Also, by the hypothesis (F_2) on F, we deduce

$$|D_tE|_{1+\varepsilon,N-1} + |A|_{1+\varepsilon,N} + |B|_{1+\varepsilon,N} \leqq C_N\, |S_\theta\Lambda u|_{\frac{1+\varepsilon}{p},N},$$

provided that $|\Lambda u|_{\frac{1+\varepsilon}{p},0} \leqq 1$. Since E depends only on (u, D_xu),

$$|D_tE|_{1+\varepsilon,0} + |D_xA|_{1+\varepsilon,0} \leqq C\, |\Lambda u|_{\frac{1+\varepsilon}{p},1}.$$

We thus obtain from (3.3) the following estimate:

(3.4) $$\|v\|_N \leqq C_{\varepsilon,N}\Big(\|g\|_{1+\varepsilon,N} + |S_\theta \Lambda u|_{\frac{1+\varepsilon}{p},N}\cdot \|g\|_{1+\varepsilon,0}\Big),$$

provided that $|\Lambda u|_{\frac{1+\varepsilon}{p},1} \leqq 1$.

We can estimate v_t from our linear equation (1.2):

$$v_t = (I - F'_{u_t})^{-1}\left(\Gamma v + \sum_{i=1}^{n} F'_{u_{x_i}} v_{x_i} + F'_u v\right) + g$$

and thus, for $|\Lambda u|_{0,0} \leqq 1$,

(3.5) $$\|v_t\|_N \leqq C_N(\|v\|_{N+\gamma} + |S_\theta \Lambda u|_{0,N}\cdot \|v\|_\gamma + \|g\|_N),$$

where γ is the order of our differential operator Γ.

Combining the last two inequalities we obtain

(3.6)

$$\|\Lambda v\|_N \leqq C_{\varepsilon,N}\Big(\|g\|_{1+\varepsilon,N+\gamma} + |S_\theta \Lambda u|_{\frac{1+\varepsilon}{p},N+\gamma}\cdot \|g\|_{1+\varepsilon,0} + |S_\theta \Lambda u|_{0,N}\cdot \|g\|_{1+\varepsilon,\gamma}\Big),$$

provided that $|\Lambda u|_{\frac{1+\varepsilon}{p},\gamma} \leqq 1$.

§ 4. Iteration Scheme

To solve our nonlinear problem, we construct, following HÖRMANDER [9], the iteration scheme

(4.1) $$u_{i+1} = u_i + \dot{u}_i, \qquad i = 0, 1, \ldots,$$

where $\dot{u}_i$ is the solution of the linearized equation

(4.2) $$L_i \dot{u}_i = g_i, \qquad \dot{u}_i(0) = 0$$

with $L_i = L_{\theta_i}(u_i)$ defined in § 1, *i.e.*,

(4.3) $$L_i v = v_t - \Gamma_v - F'_\lambda(S_i \Lambda u_i)\,\Lambda v.$$

Here S_i is the mollifying operator S_{θ_i} defined in § 1 where

(4.4) $$\theta_i = 2^i, \qquad i = 1, 2, \ldots.$$

The first step u_0 in our iteration is selected so that

(4.5) $$u_{0,t} - \Gamma u_0 = 0, \qquad u_0(0, x) = u^0(x).$$

Thus every u_i in our iteration scheme satisfies the given initial condition.

The g_i's in (3.2) are so defined that the u_i's converge formally to a solution of our equation. More precisely,

(4.6)
$$g_0 = -S_0\Phi(u_0),$$
$$g_i = -(S_i - S_{i-1})\,E_{i-1} - S_i e_{i-1} - (S_i - S_{i-1})\,\Phi(u_0),$$

where

(4.7)
$$E_{i-1} = \sum_{j=0}^{i-2} e_j,$$

(4.8)
$$e_j = e_j' + e_j'',$$

(4.9)
$$e_j' = (\Phi'(u_j) - L_j)\,u_j,$$

(4.10)
$$e_j'' = \Phi(u_{j+1}) - \Phi(u_j) - \Phi'(u_j)\,(u_{j+1} - u_j).$$

The g_i's depend only on $u_1, \dots, u_i$; according to (3.2) and the definition of g, they satisfy the relation

(4.11)
$$\Phi(u_{i+1}) - \Phi(u_i) = g_i + e_i.$$

Adding and applying (4.6), we find that

(4.12)
$$\Phi(u_{i+1}) = (I - S_i)\,E_i + e_i + (I - S_i)\,\Phi(u_0).$$

Since $S_i \to I$ and $e_i \to 0$ as $i \to \infty$, we have formally

$$\lim_{i\to\infty} \Phi(u_{i+1}) = 0.$$

Therefore, if $u = \lim_{i\to\infty} u_i$ exists, it will be the desired solution to our problem.

The convergence of the u_i's is an immediate consequence of the following estimates:

(E1)
$$\|\Lambda u_j\|_N \leqq \delta\theta_j^{-\beta+\bar{\varepsilon}N},$$

(E2)
$$|\Lambda \dot{u}_j|_{k,N} \leqq \delta\theta_j^{-\beta+k+\bar{\varepsilon}N},$$

valid for all $0 \leqq N \leqq N_0$, $0 \leqq k \leqq k_0$,

(E3)
$$|\Lambda u_j|_{\frac{1+\varepsilon}{p},\gamma} \leqq 1, \quad \|\Lambda u_j\|_0 \leqq 1.$$

Here β and N_0 are fixed constants which depend on k_0, p and also on γ, n, in the following way:

($\square_1$)
$$N_0 \geqq \frac{1}{\bar{\varepsilon}}\left(2\beta - \frac{1+\varepsilon}{p}\right).$$

Moreover, k_0 is the number defined by ($\square$) (see the Introduction).

To choose β we distinguish between two cases:

Case 1.

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) < k_0 \leqq \frac{1}{p-1}.$$

Then β has to satisfy the restrictions

$$(\square_2) \qquad \frac{1+\varepsilon+\bar{\varepsilon}(\bar{\gamma}+1)}{p} \leqq \beta \leqq \frac{k_0 p - \bar{\varepsilon}}{1+p}.$$

Case 2.

$$\frac{1}{p-1} < k_0.$$

Then β has to satisfy $(\square_2)$ and

$$(\square_3) \qquad \begin{aligned} k_0(p-1) + \bar{\varepsilon}\bar{\gamma} &\leqq p\beta, \\ k_0(p-2) + \bar{\varepsilon}\bar{\gamma} &\leqq (p-1)\beta. \end{aligned}$$

Here $\bar{\varepsilon}$, ε are two sufficiently small positive numbers which appeared previously in the definition of S_θ and also in the linear estimates of Sections 2, 3.

Assuming E1, E2, E3 are satisfied, we have

$$\sum_{i=0}^{\infty} \|\Lambda(u_{j+1} - u_j)\|_N \leqq C < \infty, \quad \text{if} \quad -\beta + \bar{\varepsilon}N < 0,$$

$$\sum_{j=0}^{\infty} |\Lambda(u_{j+1} - u_j)|_{k,N} \leqq C < \infty, \quad \text{if} \quad -\beta + k + \bar{\varepsilon}N < 0,$$

which implies the convergence of u_j to a solution $u \in C^1([0, T], C^\gamma(\mathbf{R}^n))$ of our problem. Moreover, since $\beta > \dfrac{1+\varepsilon}{p}$, we have

$$|u(x,t)| = O\left(t^{-\frac{1+\varepsilon}{p}}\right), \quad \text{as} \quad t \to \infty,$$

which proves Theorem 1.

Warning. In the long proof of (E1), (E2), (E3) which follows we will use the same letter C for any constant independent of θ_j. C may depend, however, on N_0, k_0, ε, $\bar{\varepsilon}$, β which are supposed to be fixed by $(\square_1)$, $(\square_2)$. Constants depending on N will be denoted by C_N.

We remind the reader of the following assumption on F (in the Introduction) which will be used repeatedly in the sequel:

$$|F(\Lambda u)| \leqq C\,|\Lambda u|^{p+1}$$

and also

$$|F'_\lambda(\Lambda u)| \leqq C\,|\Lambda u|^p,$$

$$|F''_{\lambda\lambda}(\Lambda u)| \leqq C\,|\Lambda u|^{p-1}$$

valid for

$$|\Lambda u| \leqq 1.$$

In the process of proving (E1)–(E3) we will derive the following sequence of estimates:

(E4) (a) $\|\Lambda_j u\|_N \leqq C\,\delta, \quad \text{if} \quad -\beta + \bar{\varepsilon}N \leqq -\bar{\varepsilon},$

$|\Lambda u_j|_{k,N} \leqq C\,\delta, \quad \text{if} \quad k - \beta + \bar{\varepsilon}N \leqq -\bar{\varepsilon},$

and

(b) $\|\Lambda u_j\|_N \leqq C\,\delta\theta_j^{-\beta+\bar{\varepsilon}N}, \quad \text{if} \quad -\beta + \bar{\varepsilon}N \geqq \bar{\varepsilon}; \quad N \leqq N_0,$

$|\Lambda u_j|_{k,N} \leqq C\,\delta\theta_j^{k-\beta+\bar{\varepsilon}N}, \quad \text{if} \quad k - \beta + \bar{\varepsilon}N \geqq \bar{\varepsilon}; \quad N \leqq N_0,\ k \leqq k_0,$

(E5) $|F_\lambda(S_i\Lambda u)|_{k_0 p,N} + \|F'_\lambda(S_i\Lambda u)\|_{k_0(p-1),N} \leqq C_N\,\delta^2\theta_i^{p(k_0-\beta)+\bar{\varepsilon}N}, \quad \text{for all } N \geqq 0,$

(E6) $|(I - S_j)\,\Lambda u_j|_{k,N} \leqq C\,\delta\theta_j^{k-\beta+\bar{\varepsilon}N} \quad \text{for all} \quad 0 \leqq k \leqq k_0;\ 0 \leqq N \leqq N_0,$

(E7) (a) $\|e_{j-1}\|_{1+\varepsilon,N} \leqq C\,\delta^2\theta_j^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}N},$

(b) $\|e_{j-1}\|_{k_0 p,N} \leqq C\,\delta^2\theta_j^{k_0 p-(1+p)\beta+\bar{\varepsilon}N}, \quad \text{for all} \quad 0 \leqq N \leqq N_0,$

(E8) $|||e_{j-1}|||_{(p-1)k_0,N} \leqq C\,\delta^2\theta_j^{(p-1)k_0-p\beta+\bar{\varepsilon}N},$

(E9) (a) $\|g_j\|_{1+\varepsilon,N} \leqq C\,\delta^2\theta_j^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}N},$

(b) $\|g_j\|_{pk_0,N} \leqq C\,\delta^2\theta_j^{pk_0-(1+p)\beta+\bar{\varepsilon}N},$

(c) $|||g_j|||_{k_0(p-1),N} \leqq C\,\delta^2, \quad \text{if} \quad \bar{\varepsilon}N + (p-1)\,k_0 - p\beta \leqq -\bar{\varepsilon},$

$|||g_j|||_{k_0(p-1),N} \leqq C\,\delta^2\theta_j^{(p-1)k_0-p\beta+\bar{\varepsilon}N} \quad \text{if} \quad N \leqq N_0 + \bar{\gamma},$

(E10) $\chi(\theta_j)\,R_{k,N+\bar{\gamma}}(g_j) \leqq C\,\delta^2\theta_j^{k_0-\beta+\bar{\varepsilon}N} \quad \text{for all} \quad 0 \leqq N \leqq N_0.$

The strategy for proving these estimates consists in the following steps:
Step I. We prove that (E4)–(E6) and (E9), (E10) hold for $i = 0$. These estimates depend only on the given initial data.
Step II. Assuming that (E1)–(E10) hold for all $j \leqq i - 1$, we show that (E4)–(E10) hold for $j = 1$.
Step III. We show that if (E4)–(E10) hold for $j \leqq i$, then (E1), (E2) and (E3) also hold for $j = i$.

Step I. By definition u_0 is a solution of (4.5). Thus by (□) (see the Introduction)

$$|u_0(t)|_{L^\infty,N} \leqq C(1 + t)^{-k_0}\,\|u^0\|_{L^1,N+n}$$

and immediately from this

$$|\Lambda u_0|_{k_0,N} \leqq C\,\|u^{(0)}\|_{L^1,N+\bar{\gamma}}.$$

Therefore, by the hypothesis of Theorem 1

$$|\Lambda u_0|_{k_0,N} \leqq C\,\delta, \quad \text{for all} \quad 0 \leqq N \leqq N_0.$$

Similarly,

$$\|\Lambda u_0\|_N \leqq \|u^{(0)}\|_{L^2,N+\gamma} \leqq C\,\delta \quad \text{for} \quad 0 \leqq N \leqq N_0,$$

and (E5), (E6) follow easily from (E4) (see Step II).

To prove (E9) we remark that, by (4.6) and (4.5),

$$g_0 = -S_0\Phi(u_0) = -S_0F(\Lambda u_0);$$

thus, by our hypothesis on F,

$$\|g_0\|_{1+\varepsilon,N} \leqq C\,|\Lambda u_0|_{\frac{1+\varepsilon}{p},N} \cdot \|\Lambda u_0\|_N \leqq C\,\delta,$$

$$\|g_0\|_{pk_p,N} \leqq C\,|\Lambda u_0|^p_{k_0,N} \cdot \|\Lambda u_0\|_N \leqq C\,\delta,$$

$$|||g_j|||_{k_0(p-1),N} \leqq C\,|\Lambda u_0|^{p-1}_{k_0,N} \cdot \|\Lambda u_0\|^2_N \leqq C\,\delta^2, \quad \text{for any } 0 \leqq N \leqq N_0 + \bar{\gamma}.$$

(E10) follows easily now from (E9).

Step II. From (E1) and (4.1) we derive the inequality

$$\|\Lambda u_i\|_N \leqq \|\Lambda u_0\|_N + \sum_{j=0}^{i-1} \|\Lambda \dot{u}_j\|_N \leqq C\,\delta + \delta \sum_{j=0}^{i-1} \theta_j^{-\beta+\bar{\varepsilon}N}. \tag{4.13}$$

Similarly,

$$|\Lambda u_i|_{k,N} \leqq |\Lambda u_0|_{k,N} + \delta \sum_{j=0}^{i-1} |\Lambda \dot{u}_j|_k \leqq C\,\delta + \delta \sum_{j-0}^{i-1} \theta_j^{k-\beta+\bar{\varepsilon}N}. \tag{4.14}$$

(E4) is now an immediate consequence of (4.13), (4.14).

To prove (E5) we make use of Lemmas 1.1 and 1.2. and E3 to get

(4.15)

$$\|F'_\lambda(S_i\Lambda u_i)\|_{k_0(p-1),N} \leqq C_N(\|S_i\Lambda u_i\|_N \cdot |\Lambda u_i|^{p-1}_{k_0,0} + |S_i\Lambda u_i|_{k_0,N} \cdot |\Lambda u_i|^{p-2}_{k_0,0} \cdot \|\Lambda u_i\|_0),$$

$$|F'_\lambda(S_i\Lambda u_i)|_{k_0p,N} \leqq C_N\,|S_i\Lambda u_i|_{k_0,N} \cdot |\Lambda u_i|^{p-1}_{k_0,0}. \tag{4.16}$$

Since $k_0 > \beta + \bar{\varepsilon}$ we use (E4) (b), and apply the property (S2) of the "mollifying" operator, thus obtaining for all $N \geqq 0$

$$\|F'_\lambda(S_i\Lambda u_i)\|_{k_0(p-1),N} + |F'_\lambda(S_i\Lambda u_i)|_{k_0p,N} \leqq C_N\,\delta(\theta_i^{(p-1)(k_0-\beta)+\bar{\varepsilon}N} + \theta_i^{p(k_0-\beta)+\bar{\varepsilon}N}),$$
$$\leqq C_N\,\delta\theta_i^{p(k_0-\beta)+\bar{\varepsilon}N},$$

which is precisely (E5).

To verify (E6) we use the property (S_1) of the mollifying operator,

$$|(I-S_i)\,\Lambda u_i|_{0,0} \leqq C(\theta_i^{-k_0}\,|\Lambda u_i|_{k_0,0} + \theta_i^{-\bar{\varepsilon}N_0}\,|\Lambda u_i|_{0,N_0}) \leqq C\theta_i^{-\beta+\bar{\varepsilon}N_0}. \tag{4.17}$$

Also, according to (E4),

$$(4.18) \qquad |(I - S_i)\,\Lambda u_i|_{k_0,N_0} \leqq C\,|\Lambda u_i|_{k_0,N_0} \leqq C\,\delta\theta_i^{k_0-\beta+\bar{\varepsilon}N_0},$$

valid for all $N \leqq N_0$. (E6) follows now by interpolating between (4.17) and (4.18) (see Lemma 1.5.).

We proceed now to prove (E7). By definition (see (4.8), (4.9), (4.10)),

$$e_{i-1} = e'_{i-1} + e''_{i-1},$$
$$e'_{i-1} = (F'(\Lambda u_{i-1}) - F'(S_{i-1}\Lambda u_{i-1}))\,\dot{u}_{i-1}.$$

Using Lemmas 1.1., 1.2. also (S1) and (E1)–(E4), we can write

$$\begin{aligned}(4.19) \qquad \|e'_{i-1}\|_{1+\varepsilon,0} &\leqq C\,|F'_\lambda(\Lambda u_{i-1}) - F'_\lambda(S_{i-1}\Lambda u_{i-1})|_{1+\varepsilon,0}\cdot\|\dot{u}_{i-1}\|_0\\ &\leqq C\,|(I - S_{i-1})\,\Lambda u_{i-1}|_{\frac{1+\varepsilon}{p},0}\cdot\|\dot{u}_{i-1}\|_0\\ &\leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-\beta}\cdot\theta_i^{-\beta}\\ &\leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta}.\end{aligned}$$

Also, similarly,

$$\|e'_{i-1}\|_{1+\varepsilon,N_0} \leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{2p}-2\beta+\varepsilon N_0},$$

and by interpolating between these last two inequalities (Lemma 1.5) we infer that

$$(4.20) \qquad \|e'_{i-1}\|_{1+\varepsilon,N} \leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}N_0},$$

for all $N \leqq N_0$. We also have

$$\begin{aligned}(4.21) \quad &\|e'_{i-1}\|_{pk_0,N}\\ &\leqq C(|F'_\lambda(\Lambda u_{i-1})|_{pk_0,N} + |F'_\lambda(S_{i-1}\Lambda u_{i-1})|_{pk_0,N})\cdot\|\Lambda\dot{u}_i\|_0 + C(|F'_\lambda(\Lambda u_{i-1})|_{pk_0,0}\\ &\qquad + |F'_\lambda(S_{i-1}\Lambda u_{i-1})|\,p_{k_0,0}\cdot\|\Lambda\dot{u}_i\|_N\\ &\leqq C\,\delta^2\theta_i^{pk_0-(p+1)\beta+\bar{\varepsilon}N},\end{aligned}$$

$$\begin{aligned}e''_{i-1} &= \Phi(u_i) - \Phi(u_{i-1}) - \Phi'(u_{i-1})\,\dot{u}_{i-1}\\ &= -(F(\Lambda u_i) - F(\Lambda u_{i-1}) - F'_\lambda(\Lambda u_{i-1})\cdot\Lambda\dot{u}_{i-1})\\ &= \int_0^1 (s-1)\,F''_{\lambda\lambda}(\Lambda u_{i-1} + s\Lambda u_i)\,\Lambda\dot{u}_{i-1}\cdot\Lambda\dot{u}_{i-1}\,ds.\end{aligned}$$

Therefore, using (F_3) and also Lemmas 1.1., 1.2., we see that

(4.22)

$$\begin{aligned}\|e''_{i-1}\|_{1+\varepsilon,N} \leqq C\Big[&|\Lambda\dot{u}_i|_{\frac{1+\varepsilon}{p},0}\cdot\|\Lambda\dot{u}_i\|_N\\ &+\Big(|\Lambda\dot{u}\,|_{\frac{1+\varepsilon}{p},N} + |\Lambda\dot{u}_i|_{\frac{1+\varepsilon}{p},0}\Big(|\Lambda u_{i-1}|_{\frac{1+\varepsilon}{p},N} + |\Lambda u_i|_{\frac{1+\varepsilon}{p},N}\Big)\Big)\|\Lambda\dot{u}_i\|_0\Big].\end{aligned}$$

Estimating for $N=0$ and $N=N_0$ separately, by using (E1)–(E4) and then interpolating we conclude that

$$\text{(4.23)}\qquad \|e''_{i-1}\|_{1+\varepsilon,N} \leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar\varepsilon N},$$

for all $0\leqq N\leqq N_0$. Similarly, for any $0\leqq N\leqq N_0$,

$$\text{(4.24)}\qquad \|e''_{i-1}\|_{pk_0,N} \leqq C\,\delta^2\phi_i^{pk_0-(p+1)\beta+\bar\varepsilon N}.$$

Now, (4.20) and (4.23) yield (E7) (a) while (4.21) yields (E7) (b).

We continue with (E8). Applying again Lemmas 1.1., 1.2., we can write

$$\begin{aligned}\text{(4.25)}\qquad & |||e'_{i-1}|||_{k_0(p-1),N}\\ &\leqq C_N(\|F'_\lambda(S_{i-1}Au_{i-1})\|_{k_0(p-1),0}+\|F'_\lambda(S_{i-1}Au_{i-1})\|_{k_0(p-1),0})\,\|Au_{i-1}\|_N\\ &\quad + C_N(\|F'_\lambda(S_{i-1}Au_{i-1})\|_{k_0(p-1),N}+\|F'_\lambda(S_{i-1}Au_{i-1})\|_{k_0(p-1),0})\,\|A\dot u_{i-1}\|_0\\ &\leqq C\,\delta^2\theta_i^{(p-1)k_0-p\beta+\bar\varepsilon N},\end{aligned}$$

for all $0\leqq N\leqq N_0$. Similarly,

$$\text{(4.26)}\qquad |||e''_{i-1}|||_{k_0(p-1),N}\leqq C\,\delta^2\theta_i^{(p-1)k_0-p\beta+\bar\varepsilon N},\qquad 0\leqq N\leqq N_0.$$

It remains to prove (E9) and (E10). By (4.6)

$$g_i=-(S_i-S_{i-1})\,E_{i-1}-S_ie_{i-1}-(S_i-S_{i-1})\,\Phi(u_0),$$

where $E_{i-1}=\sum_{j=0}^{i-2}e_j$. By ($S_1$) and (E6)

$$\begin{aligned}\|(S_i-S_{i-1})\,E_{i-1}\|_{1+\varepsilon,0}&\leqq C\theta_i^{1+\varepsilon-k_0p}\,\|E_{i-1}\|_{k_0p,0}+C\theta_i^{-\bar\varepsilon N_0}\,\|E_{i-1}\|_{1+\varepsilon,N_0}\\ &\leqq C\,\delta^2\theta_i^{1+\varepsilon-k_0p}\sum_{j=0}^{i-2}\theta_j^{k_0p-(1+p)\beta}+C\,\delta^2\theta_i^{-\bar\varepsilon N_0}\sum_{j=0}^{i-2}\theta_j^{\frac{1+\varepsilon}{p}-2\beta+\bar\varepsilon N_0}.\end{aligned}$$

Since by ($\square_1$), ($\square_2$),

$$k_0p-(1+p)\,\beta\geqq\bar\varepsilon,$$

$$\frac{1+\varepsilon}{p}-2\beta+\bar\varepsilon N_0\geqq\bar\varepsilon,$$

we conclude that

$$\begin{aligned}\text{(4.27)}\qquad & \|(S_i-S_{i-1})\,E_{i-1}\|_{1+\varepsilon,0}\\ &\leqq C\,\delta^2\Big(\theta_i^{1+\varepsilon-k_0p}\theta_i^{k_0p-(p+1)\beta}+\theta_i^{-\bar\varepsilon N_0}\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar\varepsilon N_0}\Big)\\ &\leqq C\,\delta^2\Big(\theta_i^{1+\varepsilon-(p+1)\beta}+\theta_i^{\frac{1+\varepsilon}{p}-2\beta}\Big)\\ &\leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta},\end{aligned}$$

by virtuĕ of $\beta > \frac{1+\varepsilon}{p}$. Also, by (E7) (a),

$$\| S_i e_{i-1} \|_{1+\varepsilon,0} \leqq C\,\delta^2 \theta_i^{\frac{1+\varepsilon}{p}-2\beta}. \tag{4.28}$$

According to the definition of u_0 (see (4.5)), $\Phi(u_0) = F(\Lambda u_0)$ and hence

$$\|(S_i - S_{i-1})\,\Phi(u_0)\|_{1+\varepsilon,0} \leqq C\,\theta^{1+\varepsilon-k_0 p}\,\|F(\Lambda u_0)\|_{k_0 p,0} + C\theta^{-\bar{\varepsilon}N_0}\,\|F(\Lambda u_0)\|_{1+\varepsilon,N_0}$$

$$\leqq C\,\delta^2(\theta_i^{1+\varepsilon-k_0 p} + \theta_i^{-\bar{\varepsilon}N_0}) \leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta}, \tag{4.29}$$

since by ($\square$), (C) and ε small

$$1 + \varepsilon - k_0 p \leqq \frac{1+\varepsilon}{p} - 2\beta,$$

$$-\bar{\varepsilon}N_0 \leqq \frac{1+\varepsilon}{p} - 2\beta.$$

The inequalities (4.27), (4.28), (4.29) yield

$$\|g_i\|_{1+\varepsilon,0} \leqq C\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta}. \tag{4.30}$$

We can get a corresponding estimate for $\|g_i\|_{1+\varepsilon,N_0+\bar{\gamma}}$ and interpolating between (4.30) we derive (E9) (a).

Similarly,

$$\|g_i\|_{k_0 p,N} \leqq C\,\delta^2\theta_i^{k_0 p-(1+p)\beta+\bar{\varepsilon},N}, \quad \text{for all} \quad 0 \leqq N \leqq N_0, \tag{4.31}$$

which is precisely (E9) (b).

Also, for all $0 \leqq N \leqq N_0$,

$$|||(S_i - S_{i-1})\,E_{i-1}|||_{k_0(p-1),N} \leqq C\,\delta^2 \sum_{j=0}^{i-2} \theta_j^{(p-1)k_0-p\beta+\bar{\varepsilon}N}.$$

Using (S_1), we have

(4.32)

$$|||(S_i - S_{i-1})\,E_{i-1}|||_{k_0(p-1),N} \leqq C\,\delta^2, \quad \text{if} \quad \bar{\varepsilon}N + (p-1)\,k_0 - p\beta \leqq -\bar{\varepsilon},$$

$$|||(S_i - S_{i-1})\,E_{i-1}|||_{k_0(p-1),N} \leqq C\,\delta^2\theta_j^{(p-1)k_0-p\beta+\bar{\varepsilon}N}, \quad \text{for} \quad N \leqq N_0 + \bar{\gamma},$$

and

$$\bar{\varepsilon}N + (p-1)\,k_0 - p\beta \geqq \bar{\varepsilon}.$$

To prove (E10) we recall that

$$R_{k_0,N}(g) = |||g_i|||_{k_0(p-1),N} + \|g_i\|_{k_0 p,N}.$$

Also

$$\chi(\theta_i) = \int_0^{\theta i} (1 + S)^{-k_0(p-1)}\,ds.$$

We distinguish two cases.

Case 1°.

$$k_0 \leqq \frac{1}{p-1};$$

then

$$\chi(\theta_i) \leqq C\theta_i^{1+\bar{\varepsilon}-k_0(p-1)}$$

(we add an extra $\bar{\varepsilon}$ for the case $k_0(p-1)=1$). Thus, by (E9) and the choice of β for this case

(4.31)

$$\chi(\theta_i)\, R_{k_0,N+\bar{\gamma}}(g_i) \leqq C\,\delta^2\theta_i^{1+\bar{\varepsilon}-k_0(p-1)}[\theta_i^{k_0p-(1+p)\beta+\bar{\varepsilon}(N+\bar{\gamma})} + \theta_i^{\bar{\varepsilon}N}] \leqq C\,\delta^2\theta_i^{k_0-\beta+\bar{\varepsilon}N},$$

provided that (see ($\square_2$))

$$1+\bar{\varepsilon}(\bar{\gamma}+1) \leqq p\beta$$

and

$$1+\bar{\varepsilon}+\beta \leqq k_0 p.$$

Case 2°.

$$k_0 > \frac{1}{p-1};$$

then

$$\chi(\theta_i) \leqq 1$$

and, according to (E9),

(4.32) $$\chi(\theta_i)\, R_{k_0,N+\bar{\gamma}}(g_i) \leqq [\theta_i^{k_0p-(1+p)\beta+\bar{\varepsilon}(N+\bar{\gamma})} + \theta_i^{(p-1)k_0-p\beta+\bar{\varepsilon}N+\bar{\varepsilon}\bar{\gamma}}]$$

for

$$\bar{\varepsilon}N+(p-1)\,k_0-p\beta+\overline{\varepsilon\gamma} \geqq \bar{\varepsilon}.$$

Also

$$\chi(\theta_i)\, R_{k_0,N+\bar{\gamma}}(g_i) \leqq C\,\delta^2[\theta_i^{k_0p-(1+p)\beta+\bar{\varepsilon}(N+\bar{\gamma})}+1],$$

for

$$\bar{\varepsilon}N+(p-1)\,k_0-p\beta+\overline{\varepsilon\gamma} \leqq -\bar{\varepsilon}.$$

Hence

$$\chi(\theta_i)\, R_{k_0,N+\bar{\gamma}}(g_i) \leqq C\,\delta^2\theta_i^{k_0-\beta+\bar{\varepsilon}N},$$

provided that (see ($\square_3$))

$$k_0(p-1)+\overline{\varepsilon\gamma} \leqq p\beta,$$

$$k_0(p-2)+\overline{\varepsilon\gamma} \leqq (p-1)\,\beta,$$

for

$$\bar{\varepsilon}N+(p-1)\,k_0-p\beta \geqq \bar{\varepsilon}.$$

Also

$$k_0(p-1)+\overline{\varepsilon\gamma} \leqq p\beta \quad \text{for} \quad \bar{\varepsilon}N+(p-1)\,k_0-p\beta \leqq -\bar{\varepsilon}.$$

Step III. First we remark that (E3) follows immediately from (E4) by just choosing δ to be small.

We proceed now with a proof of (E1). To this end we apply the estimate (3.6) to our linear equation (4.2) and show that

$$\|\Lambda\dot{u}_i\|_0 \leqq C_\varepsilon \|g\|_{1+\varepsilon,\gamma} \leqq C\,\delta^2 \theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}\gamma}. \tag{4.33}$$

Similarly,

(4.34)

$$\begin{aligned}\|\Lambda\dot{u}_i\|_{N_0} &\leqq C_{N_0}\Big(\|g_i\|_{1+\varepsilon,N_0+\gamma} + |S_i\Lambda u_i|_{\frac{1+\varepsilon}{p},N_0+\gamma}\cdot\|g_i\|_{1+\varepsilon,0} + |S_\theta\Lambda u_i|_{0,N_0}\|g_i\|_{1+\varepsilon,\gamma}\Big)\\ &\leqq C_{N_0}\,\delta^2\Big(\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}(N_0+\gamma)} + \theta_i^{\frac{1+\varepsilon}{p}-\beta+\bar{\varepsilon}(N_0+\gamma)}\,\theta_i^{\frac{1+\varepsilon}{p}-2\beta} + \theta_i^{-\beta+\bar{\varepsilon}N_0}\,\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}\gamma}\Big)\\ &\leqq C_{N_0}\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}(N_0+\gamma)}.\end{aligned}$$

By interpolating between this and (4.33) we get

$$\|\Lambda u_i\|_N \leqq C_{N_0}\,\delta^2\theta_i^{\frac{1+\varepsilon}{p}-2\beta+\bar{\varepsilon}(N_0+\gamma_0)}, \quad \text{for all} \quad 0 \leqq N \leqq N_0, \tag{4.35}$$

and (E2) now follows, by choosing δ small, from the inequality

$$\frac{1+\varepsilon}{p} - 2\beta + \bar{\varepsilon}\gamma \leqq -\beta,$$

which is satisfied since $\beta > \dfrac{1+\varepsilon+\bar{\varepsilon}\gamma}{p}$.

It remains only to prove (E2). For this reason we apply the estimate (2.16) to (4.2) and derive

$$\begin{aligned}|\Lambda\dot{u}_i|_{k_0,N} \leqq{}& \chi(\theta_i)\,R_{k_0,N+\bar{\gamma}}(g_i) + C_N\chi(\theta_i)\,[(\|F_\lambda'(S_i\Lambda u_i)\|_{k_0(p-1),0} + |F_\lambda'(S_i\Lambda_{u_i})|_{k_0p,0})\\ &\times\|\Lambda u_i\|_{N+\bar{\gamma}} + (\|F_\lambda'(S_i\Lambda u_i)\|_{k_0(p-1),N+\bar{\gamma}} + |F_\lambda'(S_i\Lambda u_i)|_{k_0p,N+\bar{\gamma}})\,\|\Lambda\dot{u}\|_{i_0}].\end{aligned}$$

We thus infer, from (E1), (E5), and (E10),

(4.36)

$$\begin{aligned}|\Lambda\dot{u}_i|_{k_0,N} &\leqq C_N\,\delta^2\theta_i^{-\beta+\bar{\varepsilon}N} + C_N\,\delta^2(1+\theta_i^{1+\bar{\varepsilon}-k_0(p-1)})\,[\theta_i^{p(k_0-\beta)}\theta_i^{-\beta+\bar{\varepsilon}(N+\bar{\gamma})}\\ &\qquad + \theta_i^{p(k_0-\beta)+\bar{\varepsilon}(N+\bar{\gamma})}\cdot\theta_i^{-\beta}]\\ &\leqq C_N\,\delta^2(\theta_i^{-\beta+\bar{\varepsilon}N} + \theta_i^{k_0+1-(p+1)\beta+\bar{\varepsilon}(N+\bar{\gamma}+1)})\\ &\leqq C_N\,\delta^2\theta_i^{-\beta+\bar{\varepsilon}N},\end{aligned}$$

provided that

$$1 + \bar{\varepsilon}(\bar{\gamma}+1) \leqq p\beta,$$

which is satisfied by our assumptions.

By SOBOLEV's Inequality we have

(4.37) $$|\Lambda u_i|_{0,0} \leqq C\,\|\Lambda \dot{u}_i\|_{\left[\frac{n}{2}\right]+1} \leqq C\,\delta^2 \theta_i^{\frac{1+\varepsilon}{p} - 2\beta + \bar{\varepsilon}\left(\gamma + \frac{n}{2} + 1\right)}$$

and since, again,

$$\beta > \frac{1 + \varepsilon + \bar{\varepsilon}\left(\gamma + \frac{n}{2} + 1\right)}{p},$$

we derive

(4.38) $$|\Lambda u_i|_{0,0} \leqq C\,\delta^2 \theta_i^{-\beta}.$$

Combining (4.38) with (4.36), and choosing δ small and dependent only on N_0, we obtain the desired estimate.

This completes the proof of (E2) and our induction. To finish the proof of Theorem 1 we remark that, (by (4.12)),

$$\|\Phi(u_i)\|_{0,0} \to 0 \quad \text{as} \quad i \to \infty.$$

Also it is easy to verify that

$$\|\Phi(u_i) - \Phi(u)\|_{0,0} \to 0 \quad \text{as} \quad i \to \infty,$$

where $u = \lim_{j\to\infty} u_j$. Therefore, u is the solution to our problem.

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