Multipolar Continuum Mechanics

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Abstract

A general theory of multipolar displacement and velocity fields with corresponding multipolar body and surface forces and multipolar stresses is developed using an energy principle, an entropy production inequality and invariance conditions under superposed rigid body motions. Constitutive equations for the multipolar stresses are discussed and explicit results are given for an elastic medium. Work in a previous paper by the present authors (1964) is shown to be a special case of that given here.

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1. Introduction

In a previous paper GREEN & RIVLIN (1964) have developed a general theory of simple force and stress multipoles which were defined with the help of velocity components and their spatial derivatives. In that paper we indicated directions in which the theory could be generalized. Here we lay the foundations of a theory of considerable generality which includes the work of the previous paper as a special case.

The starting point of the present investigation rests on some ideas of TRUESDELL & TOUPIN (1960, sections 166, 205, 232). These authors introduced generalized velocities, body and surface forces, and generalized stresses.* They

* Special types of generalized displacement and velocity fields have been used by ERICKSEN (1960a, 1960b, 1960c, 1961).

postulated equations of motion in terms of generalized stresses and body forces, and they postulated surface conditions. TRUESDELL & TOUPIN also discussed a general type of virtual work theorem and showed that it was equivalent to their equations of motion and surface conditions. In the present paper we use, essentially, the same definitions of generalized body and surface forces and stresses as those of TRUESDELL & TOUPIN, but a new condition is imposed on our definition of generalized displacement and velocity. We find that the equations of motion and surface conditions given by TRUESDELL & TOUPIN are not necessarily always satisfied. Sufficient conditions under which these equations are valid are discussed in section 16. The same conditions are then sufficient for the validity of the virtual work equation.

Kinematics of ordinary displacement and velocity fields, now called monopolar kinematics, is briefly reviewed in sections 2, 3. The theory of multipolar displacements and velocities is developed in section 4. Multipolar body forces are defined in section 5, and multipolar surface forces and stresses in section 6. Appropriate expressions for kinetic energy corresponding to multipolar velocities are given in section 7. The fundamental dynamical theory of multipolar forces and stresses is considered in section 8 using only an energy equation, an entropy production inequality, and invariance conditions under superposed rigid body motions. An alternative form for this theory is given in section 9. A general theory of elasticity for multipolar stresses and forces is developed in section 10, with an alternative form in section 11.

Questions concerning constitutive equations for materials which are not elastic are considered in sections 12, 13. In section 14 we show that the elasticity theory given previously (1964) is a special case of the theory of elasticity given in section 10. In section 15 we derive the approximate theory of infinitesimal elasticity appropriate to elastic materials acted on by monopolar and dipolar stresses.*

2. Monopolar kinematics

We refer the motion of the continuum to a fixed system of rectangular cartesian axes. The position of a typical particle of the continuum at time τ is denoted by $x_i(\tau)$ where

$$x_i(\tau) = x_i(X_1, X_2, X_3, \tau) \qquad (-\infty < \tau \leqq t), \tag{2.1}$$

and X_A is a reference position of the particle. We also use the notation

$$x_i = x_i(t). \tag{2.2}$$

If this deformation is to be possible in a real material then

$$\det\left[\frac{\partial x_i(\tau)}{\partial X_A}\right] > 0. \tag{2.3}$$

* After completing the present paper the authors saw a report by R. D. MINDLIN on "Microstructure in Linear Elasticity" in which he develops a theory which is essentially the same as that contained in § 15 of our paper. MINDLIN has applied his theory to wave propagation and this application has not been studied here. — This paper has now been published in Arch. Rational Mech. Anal. **16**, 51—78 (1964).

For some purposes it is convenient to express $x_i(\tau)$ in terms of the current position of the particle at time t so that

$$x_i(\tau) = x_i(x_1, x_2, x_3, t, \tau) \tag{2.4}$$

and

$$\det\left[\frac{\partial x_i(\tau)}{\partial x_j}\right] > 0. \tag{2.5}$$

Displacement gradients taken with respect to the position X_A are denoted by

$$x_{i,A_1 A_2 \ldots A_\beta}(\tau) = \frac{\partial^\beta x_i(\tau)}{\partial X_{A_1}\,\partial X_{A_2} \ldots \partial X_{A_\beta}} \qquad (\beta = 1, 2, \ldots), \tag{2.6}$$

and we use the notation

$$x_{i,A_1 \ldots A_\beta} = x_{i,A_1 \ldots A_\beta}(t). \tag{2.7}$$

Displacement gradients taken with respect to the current position x_i at time t are

$$x_{i,i_1 i_2 \ldots i_\beta}(\tau) = \frac{\partial^\beta x_i(\tau)}{\partial x_{i_1}\,\partial x_{i_2} \ldots \partial x_{i_\beta}} \qquad (\beta = 1, 2, \ldots). \tag{2.8}$$

We observe that

$$\begin{aligned} x_{i,i_1}(t) &= \delta_{i i_1}, \\ x_{i,i_1 \ldots i_\beta}(t) &= 0 \qquad (\beta > 1), \end{aligned} \tag{2.9}$$

and that the gradients in (2.7) and (2.8) are symmetric with respect to $A_1, A_2, \ldots, A_\beta$ and $i_1, i_2, \ldots, i_\beta$ respectively.

The components of velocity at the point $x_i(\tau)$ are denoted by $v_i^{(1)}(\tau) = v_i(\tau)$ so that

$$v_i^{(1)}(\tau) = D\,x_i(\tau)/D\tau, \qquad v_i^{(1)}(t) = v_i(t) = v_i,$$

where $D/D\tau$ denotes differentiation with respect to τ holding X_A fixed in (2.1), or $x_j(t)$ and t fixed in (2.4). More generally, n^{th} velocity components may be defined as

$$v_i^{(n)}(\tau) = D^n\,x_i(\tau)/D\tau^n, \qquad v_i^{(n)}(t) = v_i^{(n)}, \qquad v_i^{(0)}(\tau) = x_i(\tau). \tag{2.10}$$

From (2.8) and (2.10) we have

$$\frac{D^n x_{i,i_1 \ldots i_\beta}(\tau)}{D\tau^n} = \frac{\partial^\beta v_i^{(n)}(\tau)}{\partial x_{i_1}\,\partial x_{i_2} \ldots \partial x_{i_\beta}} = v_{i,i_1 \ldots i_\beta}^{(n)}(\tau), \tag{2.11}$$

and we use the notation

$$v_{i,i_1 \ldots i_\beta}^{(n)}(t) = v_{i,i_1 \ldots i_\beta}^{(n)} \tag{2.12}$$

for gradients of the n^{th} velocity components at time t with respect to coordinates at time t. Also

$$v_{i,i_1 \ldots i_\beta}^{(0)}(\tau) = x_{i,i_1 \ldots i_\beta}(\tau), \qquad v_{i,i_1 \ldots i_\beta}^{(0)} = 0 \qquad (\beta > 1). \tag{2.13}$$

In view of (2.3) we may write $x_{i,A}(\tau)$ in the polar form

$$x_{i,A}(\tau) = R_{iB}(\tau)\,M_{BA}(\tau), \tag{2.14}$$

where $M_{BA}(\tau)$ is a positive definite symmetric tensor and $R_{iB}(\tau)$ is a rotation tensor, so that

$$R_{iB}(\tau)\,R_{iA}(\tau) = \delta_{AB}, \qquad R_{iA}(\tau)\,R_{jA}(\tau) = \delta_{ij}, \qquad \det R_{iA}(\tau) = 1. \tag{2.15}$$

Also

$$R_{iB}=R_{iB}(t), \qquad M_{AB}=M_{AB}(t). \tag{2.16}$$

In general, throughout the paper, lower case Latin indices $i, i_1, \dots$ are associated with coordinates $x_i(\tau)$ or x_i and take the values 1, 2, 3; upper case Latin indices $A, A_1, \dots$ are associated with coordinates X_A and take the values 1, 2, 3. The usual cartesian summation convention is used, and commas denote partial differentiation.

3. Superposed rigid-body motions

We consider motions of the continuum which differ from those given by (2.1) only by superposed rigid-body motions, at different times. Thus

$$x_i^*(\tau^*)=c_i^*(\tau^*)+Q_{ij}(\tau)\left[x_j(\tau)-c_j(\tau)\right], \tag{3.1}$$

where $c_i(\tau)$, $c_i^*(\tau^*)$ are vector functions of τ and $\tau^*(=\tau+a)$ respectively, a is an arbitrary constant and $Q_{ij}(\tau)$ is a proper orthogonal tensor which depends on τ. In section 2 vectors and tensors are defined in terms of the motion (2.1) and we denote corresponding quantities defined from (3.1) by the same letter to which we add an asterisk. From (3.1) we have

$$v_i^*(\tau^*)=\dot{c}_i^*(\tau^*)+Q_{ij}(\tau)\left[v_j(\tau)-\dot{c}_j(\tau)\right]+\Omega_{ir}(\tau)\left[x_r^*(\tau^*)-c_r^*(\tau^*)\right], \tag{3.2}$$

where a dot denotes differentiation with respect to τ or τ^* and

$$\begin{aligned} \dot{Q}_{ij}(\tau)&=\Omega_{ir}(\tau)\,Q_{rj}(\tau), \qquad & \Omega_{ij}(\tau)&=-\Omega_{ji}(\tau),\\ Q_{ij}&=Q_{ij}(t), & \Omega_{ij}&=\Omega_{ij}(t). \end{aligned} \tag{3.3}$$

Also

$$x^*_{m,A_1\dots A_\alpha}(\tau^*)=Q_{mn}(\tau)\,x_{n,A_1\dots A_\alpha}(\tau), \tag{3.4}$$

and

$$\frac{\partial^\alpha x_m^*(\tau)}{\partial x_{i_1}^*\dots\partial x_{i_\alpha}^*}=Q_{mn}(\tau)\,Q_{i_1 j_1}\cdots Q_{i_\alpha j_\alpha}\,x_{n,j_1\dots j_\alpha}(\tau), \tag{3.5}$$

for $\alpha=1, 2, \dots$.

We summarize some results obtained in a previous paper (GREEN & RIVLIN 1964). If

$$\begin{aligned} E_{AA_1\dots A_\alpha}(\tau)&=x_{m,A}(\tau)\,x_{m,A_1\dots A_\alpha}(\tau),\\ E_{ii_1\dots i_\alpha}(\tau)&=x_{m,i}(\tau)\,x_{m,i_1\dots i_\alpha}(\tau), \end{aligned} \tag{3.6}$$

for $\alpha=1, 2, \dots$, then

$$\begin{aligned} E^*_{AA_1\dots A_\alpha}(\tau^*)&=E_{AA_1\dots A_\alpha}(\tau),\\ E^*_{ii_1\dots i_\alpha}(\tau^*)&=Q_{ij}Q_{i_1j_1}\cdots Q_{i_\alpha j_\alpha}E_{jj_1\dots j_\alpha}(\tau). \end{aligned} \tag{3.7}$$

Also, if

$$A^{(\mu)}_{ii_1\dots i_\alpha}=\sum_{\beta=1}^{\mu}\binom{\mu}{\beta}v^{(\mu-\beta)}_{m,i}\,v^{(\beta)}_{m,i_1\dots i_\alpha}, \tag{3.8}$$

for $\alpha=2, 3, \dots$; $\mu=1, 2, \dots$, then

$$A^{*(\mu)}_{ii_1\dots i_\alpha}=Q_{ij}Q_{i_1j_1}\cdots Q_{i_\alpha j_\alpha}A^{(\mu)}_{jj_1\dots j_\alpha}. \tag{3.9}$$

In addition, if

$$A_{ij}=v_{i,j}+v_{j,i}, \qquad \omega_{ij}=v_{i,j}-v_{j,i}, \tag{3.10}$$

then

$$2v_{i,j} = A_{ij} + \omega_{ij}, \tag{3.11}$$

and

$$A^*_{ij} = Q_{im} Q_{jn} A_{mn}, \qquad \omega^*_{ij} = Q_{im} Q_{jn} \omega_{mn} + 2\Omega_{ij}. \tag{3.12}$$

4. Multipolar kinematics

The displacement function $x_i(\tau)$ can be regarded either as a function of X_A, τ as in (2.1) or as a function of x_j, t, τ as in (2.4). The form (2.1) is appropriate to continua in which a reference position is required and (2.4) is convenient when there is no preferred reference state. We now define a simple 2^β-pole displacement field in two forms.* Let

$$x_{iB_1 \ldots B_\beta}(\tau) = x_{iB_1 \ldots B_\beta}(X_1, X_2, X_3, \tau) \qquad (-\infty < \tau \leqq t), \tag{4.1}$$

be a tensor function under changes of rectangular cartesian axes for $\beta = 1, 2, \ldots$. The set of tensors (4.1) is a set of kinematic variables which may be changed independently of the motion (2.1), but when the motion (2.1) is changed these tensors will, in general, be altered. When the motion is altered from (2.1) to (3.1) we denote the corresponding tensor (4.1) by $x^*_{iB_1 \ldots B_\beta}(\tau^*)$. If, in addition to the above assumptions about the tensor (4.1),

$$x^*_{mB_1 \ldots B_\beta}(\tau^*) = Q_{mn}(\tau)\, x_{nB_1 \ldots B_\beta}(\tau) \qquad (\beta \geqq 1), \tag{4.2}$$

then we may say that $x_{iB_1 \ldots B_\beta}(\tau)$ is a *simple 2^β-pole displacement field*. For example, if

$$x_{iB_1 \ldots B_\beta}(\tau) = x_{i,B_1 \ldots B_\beta}(\tau), \tag{4.3}$$

then the tensor (4.3) satisfies the postulated conditions. Returning to the general tensor (4.1) we use the notation

$$x_{iB_1 \ldots B_\beta} = x_{iB_1 \ldots B_\beta}(t) \tag{4.4}$$

and we observe that the tensor in (4.1) does not necessarily have symmetries in any of its indices.

Again let

$$x_{ij_1 \ldots j_\beta}(\tau) = x_{ij_1 \ldots j_\beta}(x_1, x_2, x_3, t, \tau) \qquad (-\infty < \tau \leqq t) \tag{4.5}$$

be a tensor function for $\beta = 1, 2, \ldots$ which is such that

$$x^*_{mj_1 \ldots j_\beta}(\tau^*) = Q_{mn}(\tau)\, Q_{j_1 i_1} \cdots Q_{j_\beta i_\beta} x_{ni_1 \ldots i_\beta}(\tau). \tag{4.6}$$

Then we say that $x_{ij_1 \ldots j_\beta}(\tau)$ is also a *simple 2^β-pole displacement field* and we use the notation

$$x_{ij_1 \ldots j_\beta} = x_{ij_1 \ldots j_\beta}(t). \tag{4.7}$$

An example of such a displacement field is

$$x_{ij_1 \ldots j_\beta}(\tau) = x_{i,j_1 \ldots j_\beta}(\tau). \tag{4.8}$$

A 2^β-pole displacement field of the type (4.5) can be obtained from the field (4.1) in many different ways, and conversely, as indicated in the appendix.

* A possible motivation for the definitions given here is indicated in the Appendix.

One simple method of relating the two fields is by the equation

$$x_{i j_1 \ldots j_\beta}(\tau) = x_{j_1, B_1} \cdots x_{j_\beta, B_\beta} x_{i B_1 \ldots B_\beta}(\tau), \tag{4.9}$$

but this may not always be the relevant relation to use. In most of this paper we assume that (4.1) and (4.5) are independent descriptions of multipolar displacement fields.

We define 2^β-*pole velocity fields* from the 2^β-pole displacements (4.1) or (4.5) by the equations

$$v_{i B_1 \ldots B_\beta}(\tau) = \dot{x}_{i B_1 \ldots B_\beta}(\tau), \tag{4.10}$$

$$v_{i j_1 \ldots j_\beta}(\tau) = \dot{x}_{i j_1 \ldots j_\beta}(\tau), \tag{4.11}$$

for $\beta = 1, 2, \ldots$, where a dot denotes material time differentiation with respect to τ holding X_A fixed in (4.10) and t and x_i fixed in (4.11). We use the notation

$$\begin{aligned} v_{i B_1 \ldots B_\beta} &= v_{i B_1 \ldots B_\beta}(t), \\ v_{i j_1 \ldots j_\beta} &= v_{i j_1 \ldots j_\beta}(t), \end{aligned} \tag{4.12}$$

where we put $\tau = t$ after differentiation. It follows from (4.2) and (4.6) that

$$v^*_{m B_1 \ldots B_\beta}(\tau^*) = Q_{mn}(\tau)\, v_{n B_1 \ldots B_\beta}(\tau) + \Omega_{mn}(\tau)\, x^*_{n B_1 \ldots B_\beta}(\tau^*), \tag{4.13}$$

and

$$v^*_{m j_1 \ldots j_\beta}(\tau^*) = Q_{mn}(\tau)\, Q_{j_1 i_1} \cdots Q_{j_\beta i_\beta}\, v_{n i_1 \ldots i_\beta}(\tau) + \Omega_{mn}(\tau)\, x^*_{n j_1 \ldots j_\beta}(\tau). \tag{4.14}$$

Similarly, 2^β-*pole* n^{th} *velocity fields* may be defined as*

$$\begin{aligned} v^{(n)}_{i B_1 \ldots B_\beta}(\tau) &= \overset{(n)}{x}_{i B_1 \ldots B_\beta}(\tau), \\ v^{(n)}_{i j_1 \ldots j_\beta}(\tau) &= \overset{(n)}{x}_{i j_1 \ldots j_\beta}(\tau), \end{aligned} \tag{4.15}$$

where (n) over a symbol denotes n^{th} material time differentiation with respect to τ, and we use the notation

$$\begin{aligned} v^{(n)}_{i B_1 \ldots B_\beta} &= v^{(n)}_{i B_1 \ldots B_\beta}(t), \\ v^{(n)}_{i j_1 \ldots j_\beta} &= v^{(n)}_{i j_1 \ldots j_\beta}(t). \end{aligned} \tag{4.16}$$

For convenience we call 2^β-pole displacement and n^{th} velocity fields $(n = 1, 2, \ldots)$ *multipolar displacement and* n^{th} *velocities*. We define gradients of multipolar displacements by the equations

$$\begin{aligned} x_{i B_1 \ldots B_\beta, A_1 \ldots A_\alpha}(\tau) &= \frac{\partial^\alpha x_{i B_1 \ldots B_\beta}(\tau)}{\partial X_{A_1} \ldots \partial X_{A_\alpha}}, \\ x_{i B_1 \ldots B_\beta, A_1 \ldots A_\alpha} &= x_{i B_1 \ldots B_\beta, A_1 \ldots A_\alpha}(t), \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} x_{i j_1 \ldots j_\beta, i_1 \ldots i_\alpha}(\tau) &= \frac{\partial^\alpha x_{i j_1 \ldots j_\beta}(\tau)}{\partial x_{i_1} \ldots \partial x_{i_\alpha}}, \\ x_{i j_1 \ldots j_\beta, i_1 \ldots i_\alpha} &= x_{i j_1 \ldots j_\beta, i_1 \ldots i_\alpha}(t), \end{aligned} \tag{4.18}$$

for $\beta = 1, 2, \ldots;\ \alpha = 1, 2, \ldots$.

* $n = 0$ corresponds to a 2^β-pole displacement and $n = 1$ to a 2^β-pole velocity; in this latter case the superscript $^{(1)}$ is often omitted.

The behaviour of the multipolar displacement gradients (4.17) and (4.18) when the motion is changed by superposed rigid-body motions can be found at once from (4.2) and (4.6). If

$$E_{B_1\dots B_\beta:AA_1\dots A_\alpha}(\tau)=x_{m,A}(\tau)\,x_{mB_1\dots B_\beta,A_1\dots A_\alpha}(\tau),\tag{4.19}$$

$$E_{j_1\dots j_\beta:ii_1\dots i_\alpha}(\tau)=x_{m,i}(\tau)\,x_{mj_1\dots j_\beta,i_1\dots i_\alpha}(\tau),\tag{4.20}$$

then

$$E^*_{B_1\dots B_\beta:AA_1\dots A_\alpha}(\tau^*)=E_{B_1\dots B_\beta:AA_1\dots A_\alpha}(\tau),\tag{4.21}$$

and

$$E^*_{j_1\dots j_\beta:ii_1\dots i_\alpha}(\tau^*)=Q_{j_1m_1}\cdots Q_{j_\beta m_\beta}Q_{ij}Q_{i_1n_1}\cdots Q_{i_\alpha n_\alpha}E_{m_1\dots m_\beta:jn_1\dots n_\alpha}(\tau),\tag{4.22}$$

where

$$E^*_{j_1\dots j_\beta:ii_1\dots i_\alpha}(\tau^*)=\frac{\partial x^*_m(\tau^*)}{\partial x^*_i}\,\frac{\partial^\alpha x^*_{mj_1\dots j_\beta}(\tau^*)}{\partial x^*_{i_1}\dots\partial x^*_{i_\alpha}}.$$

From (4.19), (4.20) and (3.6) we see that

$$\begin{aligned}E_{:AA_1\dots A_\alpha}(\tau)&=E_{AA_1\dots A_\alpha}(\tau),\\ E_{:ii_1\dots i_\alpha}(\tau)&=E_{ii_1\dots i_\alpha}(\tau).\end{aligned}\tag{4.23}$$

Multipolar n^{th} velocities were defined in (4.15) and from the second form we define multipolar n^{th} velocity gradients

$$v^{(n)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}(\tau)=\frac{\partial^\alpha v^{(n)}_{ij_1\dots j_\beta}(\tau)}{\partial x_{i_1}\dots\partial x_{i_\alpha}},\tag{4.24}$$

for $\beta=1,2,\dots;\ \alpha=1,2,\dots$ and we use the notation

$$\begin{aligned}v^{(n)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}&=v^{(n)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}(t),\\ v^{(0)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}(\tau)&=x_{ij_1\dots j_\beta,i_1\dots i_\alpha}(\tau).\end{aligned}\tag{4.25}$$

If we differentiate both sides of equation (4.22) μ-times with respect to τ and then put $\tau^*=\tau=t$ we have

$$B^{*(\mu)}_{j_1\dots j_\beta:ii_1\dots i_\alpha}=Q_{j_1m_1}\cdots Q_{j_\beta m_\beta}Q_{ij}Q_{i_1n_1}\cdots Q_{i_\alpha n_\alpha}B^{(\mu)}_{m_1\dots m_\beta:jn_1\dots n_\alpha},\tag{4.26}$$

where

$$B^{(\mu)}_{j_1\dots j_\beta:ii_1\dots i_\alpha}=\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}v^{(\mu-\lambda)}_{m,i}\,v^{(\lambda)}_{mj_1\dots j_\beta,i_1\dots i_\alpha}.\tag{4.27}$$

In particular we see from (3.8) and (4.27) that

$$B^{(\mu)}_{:ii_1\dots i_\alpha}=A^{(\mu)}_{ii_1\dots i_\alpha}.\tag{4.28}$$

From (4.27) we have

$$B^{(\mu)}_{j_1\dots j_\beta:ii_1\dots i_\alpha}=v^{(\mu)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}+\sum_{\lambda=0}^{\mu-1}\binom{\mu}{\lambda}v^{(\mu-\lambda)}_{m,i}\,v^{(\lambda)}_{mj_1\dots j_\beta,i_1\dots i_\alpha}\tag{4.29}$$

for $\mu=1,2,\dots$ and given α, β, and hence, by repeated application of this formula,

$$\begin{aligned}v^{(\mu)}_{ij_1\dots j_\beta,i_1\dots i_\alpha}&=B^{(\mu)}_{j_1\dots j_\beta:ii_1\dots i_\alpha}+\text{a polynomial}\\ &\quad\text{in}\quad v^{(\lambda)}_{m,i},\ B^{(\varrho)}_{j_1\dots j_\beta:ii_1\dots i_\alpha}\quad\text{and}\quad x_{ij_1\dots j_\beta,i_1\dots i_\alpha},\end{aligned}\tag{4.30}$$

for $\lambda=1,2,\dots,\mu;\ \varrho=1,2,\dots,\mu-1$.

Again, if we differentiate both sides of (4.21) μ-times with respect to τ we see that

$$B^{*(\mu)}_{B_1\ldots B_\beta:AA_1\ldots A_\alpha}(\tau^*)=B^{(\mu)}_{B_1\ldots B_\beta:AA_1\ldots A_\alpha}(\tau) \tag{4.31}$$

where

$$B^{(\mu)}_{B_1\ldots B_\beta:AA_1\ldots A_\alpha}(\tau)=\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}v^{(\mu-\lambda)}_{m,A}(\tau)\,v^{(\lambda)}_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}(\tau), \tag{4.32}$$

and

$$v^{(\lambda)}_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}(\tau)=\frac{\partial^\alpha v^{(\lambda)}_{mB_1\ldots B_\beta}(\tau)}{\partial X_{A_1}\ldots\partial X_{A_\alpha}}, \tag{4.33}$$

for $\lambda=0,1,\ldots;\ \beta=1,2,\ldots;\ \alpha=1,2,\ldots$. Also,

$$\begin{aligned} v^{(0)}_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}(\tau)&=x_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}(\tau),\\ v^{(\lambda)}_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}&=v^{(\lambda)}_{mB_1\ldots B_\beta,A_1\ldots A_\alpha}(t). \end{aligned} \tag{4.34}$$

5. Multipolar body forces

Multipolar body forces of the first kind associated with velocity components v_i at time t and their spatial derivatives were defined previously (GREEN & RIVLIN 1964). Here we define multipolar body forces of the $(\beta+1)^{\text{th}}$ kind associated with multipolar velocities and their spatial derivatives, evaluated at time t.

If $F_{ij_1\ldots j_\beta}$ is a tensor* and $v_{ij_1\ldots j_\beta}$ an arbitrary 2^β-pole velocity at time t, and if the scalar

$$F_{ij_1\ldots j_\beta}\,v_{ij_1\ldots j_\beta} \tag{5.1}$$

is a rate of work per unit mass, then the tensor $F_{ij_1\ldots j_\beta}$ is called a *body force 2^β-pole of the $(\beta+1)^{\text{th}}$ kind, per unit mass.* The total rate of work of a body force 2^β-pole of the $(\beta+1)^{\text{th}}$ kind, per unit mass, distributed throughout a volume V at time t, is

$$\int_V \varrho F_{ij_1\ldots j_\beta}\,v_{ij_1\ldots j_\beta}\,dV, \tag{5.2}$$

where ϱ is density. When $\beta=0$ we recover the rate of work of a classical body force vector F_i in a vector velocity field. If $F_{ij_1\ldots j_\beta:i_1\ldots i_\alpha}$ is a tensor of order $\alpha+\beta+1$ and $v_{ij_1\ldots j_\beta,i_1\ldots i_\alpha}$ is an arbitrary 2^β-pole velocity gradient, and if,

$$F_{ij_1\ldots j_\beta:i_1\ldots i_\alpha}\,v_{ij_1\ldots j_\beta,i_1\ldots i_\alpha} \tag{5.3}$$

is a rate of work per unit mass, then the tensor $F_{ij_1\ldots j_\beta:i_1\ldots i_\alpha}$ is called a *body force $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind,* per unit mass. The total rate of work of such a body force distributed throughout a volume V is

$$\int_V \varrho F_{ij_1\ldots j_\beta:i_1\ldots i_\alpha}\,v_{ij_1\ldots j_\beta,i_1\ldots i_\alpha}\,dV. \tag{5.4}$$

Without loss of generality the tensor $F_{ij_1\ldots j_\beta:i_1\ldots i_\alpha}$ may be taken to be completely symmetric in the indices $i_1,\ldots,i_\alpha$. When $\beta=0$ we recover a body force 2^α-pole of the first kind,** $F_{i:i_1\ldots i_\alpha}$.

* Owing to the greater generality of the present work we have not always been able to follow the notation which we used previously (GREEN & RIVLIN, 1964).

** This was denoted by $F_{i_1\ldots i_\alpha i}$ in the previous paper but this notation is now abandoned.

The multipolar forces have been defined with the help of $v_{ij_1\dots j_\beta, i_1\dots i_\alpha}$ which is regarded as a function of x_i and t and so the body forces may also be regarded as functions of these variables, and distributed throughout a material volume V at time t. For some purposes it is more convenient to define multipolar body forces associated with a volume V but measured as functions of X_A and t, where X_A are coordinates of points in a material volume V_0 at time t_0, which correspond to points of V. If $F_{iB_1\dots B_\beta:A_1\dots A_\alpha}$ is a tensor function of X_A, t, of order $\alpha+\beta+1$, and $v_{iB_1\dots B_\beta, A_1\dots A_\alpha}$ is an arbitrary 2^β-pole velocity gradient, also a function of X_A, t, and if

$$F_{iB_1\dots B_\beta:A_1\dots A_\alpha} v_{iB_1\dots B_\beta, A_1\dots A_\alpha} \tag{5.5}$$

is a rate of work per unit mass, then the tensor $F_{iB_1\dots B_\beta:A_1\dots A_\alpha}$ is a body force $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind, per unit mass. The total rate of work of such a body force multipole distributed throughout V is

$$\int_{V_0} \varrho_0 F_{iB_1\dots B_\beta:A_1\dots A_\alpha} v_{iB_1\dots B_\beta, A_1\dots A_\alpha} dV_0, \tag{5.6}$$

where ϱ_0 is the density of the volume V_0. The multipolar body force is completely symmetric in the indices $A_1, \dots, A_\alpha$.

Since the multipolar velocity gradients $v_{ij_1\dots j_\beta, i_1\dots i_\alpha}$ can be regarded as a special case of a $2^{\beta+\alpha}$-pole velocity it follows that a body force $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind can be regarded as a special case of a body force $2^{\alpha+\beta}$-pole of the $(\alpha+\beta+1)^{\text{th}}$ kind.

6. Multipolar surface forces and stresses

Consider a surface A whose unit normal at the point x_i at time t, in a specified direction, is n_i. If $t_{ij_1\dots j_\beta:i_1\dots i_\alpha}$ is a tensor function of x_i, t of order $\alpha+\beta+1$ and if, for all arbitrary 2^β-pole velocity gradients $v_{ij_1\dots j_\beta, i_1\dots i_\alpha}$, the scalar

$$t_{ij_1\dots j_\beta:i_1\dots i_\alpha} v_{ij_1\dots j_\beta, i_1\dots i_\alpha} \tag{6.1}$$

is a rate of work per unit area of A, then the tensor $t_{ij_1\dots j_\beta:i_1\dots i_\alpha}$ is called a *surface force* $2^{\alpha+\beta}$*-pole of the* $(\beta+1)^{\text{th}}$ *kind*, per unit area. Without loss of generality the tensor may be taken to be completely symmetric in the indices $i_1, \dots, i_\alpha$. When $\beta=0$ we have a surface force 2^α-pole of the first kind* $t_{i:i_1\dots i_\alpha}$. When $\alpha=0$, $t_{ij_1\dots j_\beta}$ is called a *surface force* 2^β*-pole of the* $(\beta+1)^{\text{th}}$ *kind*, per unit area, with $\beta=0$ corresponding to the classical surface force vector t_i. The total rate of work of the surface force $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind, per unit area, over the surface A, is

$$\int_A t_{ij_1\dots j_\beta:i_1\dots i_\alpha} v_{ij_1\dots j_\beta, i_1\dots i_\alpha} dA. \tag{6.2}$$

The tensor $t_{ij_1\dots j_\beta:i_1\dots i_\alpha}$ at x_i is associated with a surface whose unit normal at the point is n_k. When n_k is a unit normal to the x_k-plane through the point we denote the corresponding tensor by

$$\sigma_{kij_1\dots j_\beta:i_1\dots i_\alpha}. \tag{6.3}$$

* Denoted previously (Green & Rivlin, 1964) by $t_{i_1\dots i_\alpha i}$.

These are the components of a *surface stress tensor* $2^{\alpha+\beta}$*-pole of the* $(\beta+1)^{\text{th}}$ *kind* on an element of area at the point normal to the x_k-axis. The rate of work of this tensor is

$$\sigma_{k\,i\,j_1\ldots j_\beta : i_1\ldots i_\alpha}\, v_{i\,j_1\ldots j_\beta,\,i_1\ldots i_\alpha} \tag{6.4}$$

per unit area of the surface normal to the x_k-axis. The first index k is not necessarily a tensor index under change of axes, but indicates the surface on which the stress tensor acts, the surface being fixed. When $\alpha=\beta=0$ we recover the classical stress tensor $\sigma_{k\,i}$ which we shall see later is a tensor with respect to both indices.

Suppose now that the surface A, containing an arbitrary material volume V at time t, was a surface A_0 at time t_0 containing a corresponding volume V_0. The coordinates of corresponding points in V_0 and V are X_i and x_i respectively and N_K is the unit outward normal at the surface A_0. Let $p_{iB_1\ldots B_\beta : A_1\ldots A_\alpha}$ be a tensor function of X_A, t, associated with the surface A but measured per unit area of A_0. If, for all arbitrary 2^β-pole velocity gradients $v_{iB_1\ldots B_\beta, A_1\ldots A_\alpha}$, the scalar

$$p_{iB_1\ldots B_\beta : A_1\ldots A_\alpha}\, v_{iB_1\ldots B_\beta, A_1\ldots A_\alpha} \tag{6.5}$$

is a rate of work per unit area of A_0, then the tensor $p_{iB_1\ldots B_\beta : A_1\ldots A_\alpha}$ is called a surface force* $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind, per unit area of A_0. The total rate of work of this surface force over A is

$$\int_{A_0} p_{iB_1\ldots B_\beta : A_1\ldots A_\alpha}\, v_{iB_1\ldots B_\beta, A_1\ldots A_\alpha}\, dA_0 . \tag{6.6}$$

The surface force multipole $p_{iB_1\ldots B_\beta : A_1\ldots A_\alpha}$ is associated with a surface A but measured per unit area of A_0 whose unit normal is N_A. When N_K is a unit normal at X_A to the X_K-plane through this point we denote the corresponding stress multipole by

$$\pi_{K\,iB_1\ldots B_\beta : A_1\ldots A_\alpha} . \tag{6.7}$$

These are the components of a stress tensor $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind associated with an element of area at the point x_i in V, which in V_0 was perpendicular to the X_K-axis, measured per unit area of this surface in V_0. The rate of work of this stress tensor is

$$\pi_{K\,iB_1\ldots B_\beta : A_1\ldots A_\alpha}\, v_{iB_1\ldots B_\beta, A_1\ldots A_\alpha} \tag{6.8}$$

per unit area of surface in V_0 normal to the X_K-axis. The first index K is not necessarily a tensor index under change of axes, but indicates the surface on which the stress tensor acts, the surface being fixed. The classical stress tensor $\pi_{K\,i}$ corresponds to $\alpha=\beta=0$ and we shall see that this is a tensor with respect to both indices.

A surface $2^{\alpha+\beta}$-pole of the $(\beta+1)^{\text{th}}$ kind may be regarded as a special case of a surface force $2^{\alpha+\beta}$-pole of the $(\alpha+\beta+1)^{\text{th}}$ kind.

* A simple surface force 2^α-pole of the first kind is denoted by $p_{i:A_1\ldots A_\alpha}$ instead of $p_{A_1\ldots A_\alpha i}$ used previously (GREEN & RIVLIN, 1964). When $\alpha=0$, $p_{iB_1\ldots B_\beta}$ is called a surface force 2^β-pole of the $(\beta+1)^{\text{th}}$ kind, per unit area of A_0.

7. Kinetic energy

Kinetic energy per unit mass at time τ, corresponding to velocity $v_i(\tau)$ is

$$\tfrac{1}{2}\, v_i(\tau)\, v_i(\tau) \tag{7.1}$$

and its material rate of change is

$$v_i(\tau)\, v_i^{(2)}(\tau). \tag{7.2}$$

In particular, its rate of change at time t, per unit mass, is

$$v_i\, v_i^{(2)}. \tag{7.3}$$

When we have, in addition, 2^β-pole velocity fields $v_{ij_1\ldots j_\beta}(\tau)$ $(\beta=1,\ldots,\nu)$ we postulate that the corresponding kinetic energy, per unit mass, is*

$$\tfrac{1}{2}\sum_{\alpha,\beta=1}^{\nu} y_{i_1\ldots i_\alpha : j_1\ldots j_\beta}\, v_{i i_1\ldots i_\beta}(\tau)\, v_{i j_1\ldots j_\beta}(\tau), \tag{7.4}$$

where $y_{i_1\ldots i_\alpha : j_1\ldots j_\beta}$, independent of τ, is a tensor function of x_i and t, and we can put

$$y_{i_1\ldots i_\alpha : j_1\ldots j_\beta} = y_{j_1\ldots j_\beta : i_1\ldots i_\alpha} \tag{7.5}$$

without loss of generality. The rate of change of this kinetic energy at time t, per unit mass, is found by differentiating (7.4) with respect to τ and then putting $\tau=t$, to give

$$\sum_{\alpha,\beta=1}^{\nu} y_{i_1\ldots i_\alpha : j_1\ldots j_\beta}\, v^{(2)}_{i i_1\ldots i_\alpha}\, v_{i j_1\ldots j_\beta}. \tag{7.6}$$

Similarly, when the 2^β-pole velocity field is $v_{iB_1\ldots B_\beta}(\tau)$ $(\beta=1,\ldots,\nu)$, the corresponding kinetic energy, per unit mass, is

$$\tfrac{1}{2}\sum_{\alpha,\beta=1}^{\nu} Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta}\, v_{iA_1\ldots A_\alpha}(\tau)\, v_{iB_1\ldots B_\beta}(\tau), \tag{7.7}$$

where $Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta}$, independent of τ, is a tensor function of X_A, and

$$Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta} = Y_{B_1\ldots B_\beta : A_1\ldots A_\alpha}. \tag{7.8}$$

The material rate of change of (7.7) at time t is

$$\sum_{\alpha,\beta=1}^{\nu} Y_{A_1\ldots A_\alpha : B_1 \ldots B_\beta}\, v^{(2)}_{iA_1\ldots A_\alpha}\, v_{iB_1\ldots B_\beta}. \tag{7.9}$$

8. The energy equation and entropy production inequality

We consider an arbitrary material volume V of the continuum bounded by a surface A at time t. We assume** that body force 2^β-poles of the $(\beta+1)^{\text{th}}$ kind $F_{ij_1\ldots j_\beta}$ $(\beta=0,1,\ldots,\nu)$, per unit mass, act throughout V and that surface force 2^β-poles of the $(\beta+1)^{\text{th}}$ kind $t_{ij_1\ldots j_\beta}$ $(\beta=0,1,\ldots,\nu)$, per unit area, act across A. We also assume that there is an internal energy function U per unit mass, an entropy function S, per unit mass, a heat supply function r per unit mass and unit time, a local temperature T, which is assumed to be always

* See the Appendix for a motivation for this definition.

** The remarks at the ends of sections 5, 6 indicate that there is no essential loss of generality in restricting our discussion to body and surface force 2^β-poles of the $(\beta+1)^{\text{th}}$ kind.

positive, a heat flux h across A per unit area, per unit time, and a heat flux Q_i, where Q_i is the flux of heat across a plane at x_j perpendicular to the x_i-axis, per unit area, per unit time. All these functions depend on x_1, x_2, x_3, t or, alternatively, on X_1, X_2, X_3, t when a preferred position for the continuum exists.

We postulate an energy balance at time t in the form

$$\int_V \varrho\, v_i v_i^{(2)} dV + \int_V \varrho\, \dot{U}\, dV = \int_V \varrho \left[r + F_i v_i + \sum_{\beta=1}^{\nu} \bar{F}_{i j_1 \dots j_\beta} v_{i j_1 \dots j_\beta} \right] dV + \\ + \int_A \left[t_i v_i + \sum_{\beta=1}^{\nu} t_{i j_1 \dots j_\beta} v_{i j_1 \dots j_\beta} \right] dA - \int_A h\, dA, \tag{8.1}$$

where a dot denotes the material time derivative and where

$$\bar{F}_{i j_1 \dots j_\beta} = F_{i j_1 \dots j_\beta} - \sum_{\alpha=1}^{\nu} y_{i_1 \dots i_\alpha : j_1 \dots j_\beta}\, v_{i i_1 \dots i_\alpha}^{(2)}. \tag{8.2}$$

The second term in (8.2) arises from the contribution (7.6) to the energy equation from the kinetic energy. We also postulate an entropy production inequality

$$\int_V \varrho\, \dot{S}\, dV - \int_V \varrho \frac{r}{T}\, dV + \int_A \frac{h}{T}\, dA \geqq 0. \tag{8.3}$$

We suppose that the continuum has arrived at the given state at time t through some prescribed motion. We consider a second motion which differs from the given motion only by a *constant* superposed rigid body translational velocity*, the continuum occupying the same position at time t. We assume that $\dot{U}$, t_i, F_i, $t_{i j_1 \dots j_\beta}$, $\bar{F}_{i j_1 \dots j_\beta}$ $(\beta = 1, \dots, \nu)$, h and r are unaltered by such superposed rigid body velocity; and we observe from section 4 that $v_{i j_1 \dots j_\beta}$ $(\beta = 1, 2, \dots, \nu)$ and $v_{i j_1 \dots j_\beta}^{(2)}$ $(\beta = 0, 1, \dots, \nu)$ are also unaltered but that v_i is changed to $v_i + a_i$, where a_i is constant. Thus equation (8.1) is also true when v_i is replaced by $v_i + a_i$, all other terms being unaltered, so that, by subtraction

$$\left[\int_V \varrho\, F_i\, dV + \int_A t_i\, dA - \int_V \varrho\, v_i^{(2)}\, dV \right] a_i = 0 \tag{8.4}$$

for all arbitrary constant a_i. Since the quantity in the square brackets in (8.4) is independent of a_i it follows that

$$\int_V \varrho F_i\, dV + \int_A t_i\, dA = \int_V \varrho\, v_i^{(2)} dV. \tag{8.5}$$

If the components of stress across the coordinate planes are σ_{ji} it follows from (8.5) that

$$\sigma_{ji,j} + \varrho\, F_i = \varrho\, v_i^{(2)}, \tag{8.6}$$

$$t_i = n_j\, \sigma_{ji}. \tag{8.7}$$

In view of (8.7), σ_{ji} is a tensor with respect to both indices j, i under changes of rectangular cartesian axes, where the stresses in each coordinate system are associated with the three coordinate planes in that system.

* The independent thermodynamic variable, which can be taken to be S, is unaltered.

With the help of (8.6) and (8.7), equation (8.1) becomes

$$\int_V \varrho \dot{U} dV = \int_V \left[\varrho r + \sum_{\beta=1}^{\nu} \bar{F}_{i j_1 \ldots j_\beta} v_{i j_1 \ldots j_\beta} + \sigma_{ji} v_{i,j}\right] dV + \\ + \int_A \sum_{\beta=1}^{\nu} t_{i j_1 \ldots j_\beta} v_{i j_1 \ldots j_\beta} dA - \int_A h dA. \tag{8.8}$$

We apply this equation to an arbitrary tetrahedron bounded by coordinate planes through the point x_i and by a plane whose unit normal is n_k, to obtain the result

$$\sum_{\beta=1}^{\nu} (t_{i j_1 \ldots j_\beta} - n_k \sigma_{k i j_1 \ldots j_\beta}) v_{i j_1 \ldots j_\beta} - h + n_i Q_i = 0. \tag{8.9}$$

Then, using (8.9) in (8.8) and applying the resulting equation to an arbitrary volume, gives

$$\varrho r - Q_{i,i} - \varrho \dot{U} + \sigma_{ji} v_{i,j} + \sum_{\beta=1}^{\nu} (\varrho \bar{F}_{i j_1 \ldots j_\beta} + \sigma_{k i j_1 \ldots j_\beta, k}) v_{i j_1 \ldots j_\beta} + \\ + \sum_{\beta=1}^{\nu} \sigma_{k i j_1 \ldots j_\beta} v_{i j_1 \ldots j_\beta, k} = 0. \tag{8.10}$$

From (4.27) we have

$$v_{i j_1 \ldots j_\beta} = B_{j_1 \ldots j_\beta : i} - v_{m,i} x_{m j_1 \ldots j_\beta}, \\ v_{i j_1 \ldots j_\beta, k} = B_{j_1 \ldots j_\beta : ik} - v_{m,i} x_{m j_1 \ldots j_\beta, k}, \tag{8.11}$$

where

$$B_{j_1 \ldots j_\beta : i} = B^{(1)}_{j_1 \ldots j_\beta : i}, \\ B_{j_1 \ldots j_\beta : ik} = B^{(1)}_{j_1 \ldots j_\beta : ik},$$

and with the help of (3.11) equations (8.11) become

$$v_{i j_1 \ldots j_\beta} = B_{j_1 \ldots j_\beta : i} - \tfrac{1}{2} (A_{mi} + \omega_{mi}) x_{m j_1 \ldots j_\beta}, \\ v_{i j_1 \ldots j_\beta, k} = B_{j_1 \ldots j_\beta : ik} - \tfrac{1}{2} (A_{mi} + \omega_{mi}) x_{m j_1 \ldots j_\beta, k}. \tag{8.12}$$

If we substitute the first of equations (8.12) into equation (8.9), we see that

$$\sum_{\beta=1}^{\nu} \bar{t}_{i j_1 \ldots j_\beta} (B_{j_1 \ldots j_\beta : i} - \tfrac{1}{2} A_{mi} x_{m j_1 \ldots j_\beta}) - \bar{h} - \tfrac{1}{2} \omega_{mi} \sum_{\beta=1}^{\nu} \bar{t}_{i j_1 \ldots j_\beta} x_{m j_1 \ldots j_\beta} = 0, \tag{8.13}$$

where

$$\bar{h} = h - n_i Q_i, \\ \bar{t}_{i j_1 \ldots j_\beta} = t_{i j_1 \ldots j_\beta} - n_k \sigma_{k i j_1 \ldots j_\beta}. \tag{8.14}$$

Also, with the help of (8.12) and (3.11) equation (8.10) becomes

$$\varrho r - Q_{i,i} - \varrho \dot{U} + \tfrac{1}{2} A_{mi} \sigma'_{im} + \sum_{\beta=1}^{\nu} \bar{\sigma}_{i j_1 \ldots j_\beta} B_{j_1 \ldots j_\beta : i} + \\ + \sum_{\beta=1}^{\nu} \sigma_{k i j_1 \ldots j_\beta} B_{j_1 \ldots j_\beta : ik} + \tfrac{1}{2} \omega_{mi} \sigma'_{im} = 0, \tag{8.15}$$

where

$$\bar{\sigma}_{i j_1 \ldots j_\beta} = \varrho \bar{F}_{i j_1 \ldots j_\beta} + \sigma_{k i j_1 \ldots j_\beta, k}, \tag{8.16}$$

and

$$\sigma'_{im} = \sigma_{im} - \sum_{\beta=1}^{\nu} \bar{\sigma}_{i j_1 \ldots j_\beta} x_{m j_1 \ldots j_\beta} - \sum_{\beta=1}^{\nu} \sigma_{k i j_1 \ldots j_\beta} x_{m j_1 \ldots j_\beta, k}. \tag{8.17}$$

We now consider a motion of the continuum which is such that the velocities differ from those of the given motion only by a superposed uniform rigid body angular velocity, the continuum occupying the same position at time t, and we assume that $\bar{h}$, Q_i, $\bar{t}_{ij_1\ldots j_\beta}$, r, $\dot{U}$, σ_{im}, $\bar{\sigma}_{ij_1\ldots j_\beta}$ and $\sigma_{kij_1\ldots j_\beta}$ are unaltered by such motions. Equations (8.13) and (8.15) hold for all velocity and multipolar velocity fields, so the equations hold when ω_{mi} is replaced by $\omega_{mi}+2\Omega_{mi}$ with all other kinematic quantities unaltered, in view of results in section 4, where Ω_{mi} is a constant arbitrary skew symmetric tensor. Hence

$$\Omega_{mi}\sum_{\beta=1}^{\nu}\bar{t}_{ij_1\ldots j_\beta}x_{mj_1\ldots j_\beta}=0,$$
$$\Omega_{mi}\sigma'_{im}=0,$$

and therefore

$$\sum_{\beta=1}^{\nu}(\bar{t}_{ij_1\ldots j_\beta}x_{mj_1\ldots j_\beta}-\bar{t}_{mj_1\ldots j_\beta}x_{ij_1\ldots j_\beta})=0, \tag{8.18}$$
$$\sigma'_{im}=\sigma'_{mi}. \tag{8.19}$$

Equations (8.13) and (8.15) then reduce to

$$\sum_{\beta=1}^{\nu}\bar{t}_{ij_1\ldots j_\beta}(B_{j_1\ldots j_\beta:i}-\tfrac{1}{2}A_{mi}x_{mj_1\ldots j_\beta})-\bar{h}=0, \tag{8.20}$$

and

$$\begin{aligned}\varrho r-Q_{i,i}-\varrho\dot{U}+\tfrac{1}{2}A_{mi}\sigma'_{im}+\sum_{\beta=1}^{\nu}\bar{\sigma}_{ij_1\ldots j_\beta}B_{j_1\ldots j_\beta:i}+&\\+\sum_{\beta=1}^{\nu}\sigma_{kij_1\ldots j_\beta}B_{j_1\ldots j_\beta:ik}=0&\end{aligned} \tag{8.21}$$

respectively.

9. Energy and entropy production: alternative form

The work of the previous section is sufficiently general to be applied to any continuum, whether solid or fluid. When the continuum has a reference configuration X_A through which it passes at time t_0 it is convenient to have an alternative form of the theory in which multipolar forces and stresses are measured with respect to this configuration.

We consider an arbitrary volume V at time t bounded by a surface A and we suppose that V_0 is the corresponding volume at time t_0, bounded by a surface A_0. Points of V_0 have coordinates X_A. Recalling the definitions in sections 5–7, the energy equation (8.1) is replaced by

$$\begin{aligned}\int_{V_0}\varrho_0 v_i v_i^{(2)}dV_0+\int_{V_0}\varrho_0\dot{U}dV_0=\int_{V_0}\varrho_0\Big[r+F_i v_i+\sum_{\beta=1}^{\nu}\bar{F}_{iB_1\ldots B_\beta}v_{iB_1\ldots B_\beta}\Big]dV_0+&\\+\int_{A_0}\Big[p_i v_i+\sum_{\beta=1}^{\nu}p_{iB_1\ldots B_\beta}v_{iB_1\ldots B_\beta}\Big]dA_0-\int_{A_0}h_0 dA_0,&\end{aligned} \tag{9.1}$$

where h_0 is the flux of heat across A, measured per unit area of A_0, and

$$\bar{F}_{iB_1\ldots B_\beta}=F_{iB_1\ldots B_\beta}-\sum_{\alpha=1}^{\nu}Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}v^{(2)}_{iA_1\ldots A_\alpha}. \tag{9.2}$$

The entropy production inequality (8.3) becomes

$$\int_{V_0} \varrho_0 \dot{S}\, dV_0 - \int_{V_0} \varrho_0 \frac{r}{T}\, dV_0 + \int_{A_0} \frac{h_0}{T}\, dA_0 \geqq 0. \tag{9.3}$$

If we follow an argument similar to that used at the beginning of section 8, we may deduce the classical equation of motion

$$\int_{V_0} \varrho_0 F_i\, dV_0 + \int_{A_0} p_i\, dA_0 = \int_{V_0} \varrho_0 v_i^{(2)}\, dV_0. \tag{9.4}$$

Hence

$$\pi_{Ki,K} + \varrho_0 F_i = \varrho_0 v_i^{(2)}, \tag{9.5}$$

$$p_i = N_K \pi_{Ki}, \tag{9.6}$$

where N_K is the unit outward normal vector to the surface A_0. In view of (9.6), π_{Ki} is a tensor with respect to both indices, under changes of rectangular cartesian axes, where the stresses in each coordinate system are associated with the three surfaces in that system in V which correspond to coordinate planes in V_0.

Using (9.5) and (9.6), equation (9.1) can be reduced to

$$\begin{aligned}\int_{V_0} \varrho_0 \dot{U}\, dV_0 = \int_{V_0} \Big[\varrho_0 r + \varrho_0 \sum_{\beta=1}^{\nu} \overline{F}_{iB_1\ldots B_\beta} v_{iB_1\ldots B_\beta} + \pi_{Bi} v_{i,B}\Big] dV_0 + \\ + \int_{A_0} \sum_{\beta=1}^{\nu} p_{iB_1\ldots B_\beta} v_{iB_1\ldots B_\beta}\, dA_0 - \int_{A_0} h_0\, dA_0.\end{aligned} \tag{9.7}$$

We apply this equation to a volume V which in the reference state V_0 was a tetrahedron bounded by coordinate planes through the point X_A and by a plane whose unit normal is N_K, to obtain the result

$$\sum_{\beta=1}^{\nu} (p_{iB_1\ldots B_\beta} - N_K \pi_{KiB_1\ldots B_\beta}) v_{iB_1\ldots B_\beta} - h_0 + N_K q_K = 0. \tag{9.8}$$

Then, using (9.8) in (9.7) and applying the equation to an arbitrary volume, gives

$$\begin{aligned}\varrho_0 r - q_{K,K} - \varrho_0 \dot{U} + \pi_{Ki} v_{i,K} + \sum_{\beta=1}^{\nu} (\varrho_0 \overline{F}_{iB_1\ldots B_\beta} + \pi_{KiB_1\ldots B_\beta,K}) v_{iB_1\ldots B_\beta} + \\ + \sum_{\beta=1}^{\nu} \pi_{KiB_1\ldots B_\beta} v_{iB_1\ldots B_\beta,K} = 0,\end{aligned} \tag{9.9}$$

where q_K is the flux of heat across surfaces in V which were originally coordinate planes perpendicular to the X_K-axes through the point X_B, measured per unit area of these planes, per unit time.

From (4.32) and (3.11) we have

$$\begin{aligned} v_{iB_1\ldots B_\beta} &= X_{A,i} B_{B_1\ldots B_\beta:A} - \tfrac{1}{2}(A_{mi} + \omega_{mi}) x_{mB_1\ldots B_\beta}, \\ v_{iB_1\ldots B_\beta,K} &= X_{A,i} B_{B_1\ldots B_\beta:AK} - \tfrac{1}{2}(A_{mi} + \omega_{mi}) x_{mB_1\ldots B_\beta,K}. \end{aligned} \tag{9.10}$$

With the help of (9.10) equations (9.8) and (9.9) become

$$\begin{aligned}\sum_{\beta=1}^{\nu} \overline{p}_{iB_1\ldots B_\beta} (X_{A,i} B_{B_1\ldots B_\beta:A} - \tfrac{1}{2} A_{mi} x_{mB_1\ldots B_\beta}) - \overline{h}_0 - \\ - \tfrac{1}{2}\omega_{mi} \sum_{\beta=1}^{\nu} \overline{p}_{iB_1\ldots B_\beta} x_{mB_1\ldots B_\beta} = 0,\end{aligned} \tag{9.11}$$

and

$$\varrho_0 r - q_{K,K} - \varrho_0 \dot{U} + \tfrac{1}{2} A_{mi} \pi'_{Km} x_{i,K} + \tfrac{1}{2} \omega_{mi} \pi'_{Km} x_{i,K} + \\ + X_{A,i} \sum_{\beta=1}^{\nu} \bar{\pi}_{iB_1 \ldots B_\beta} B_{B_1 \ldots B_\beta : A} + X_{A,i} \sum_{\beta=1}^{\nu} \pi_{KiB_1 \ldots B_\beta} B_{B_1 \ldots B_\beta : AK} = 0, \tag{9.12}$$

where

$$\pi'_{Am} = \pi_{Am} - X_{A,i} \sum_{\beta=1}^{\nu} (\bar{\pi}_{iB_1 \ldots B_\beta} x_{mB_1 \ldots B_\beta} + \pi_{KiB_1 \ldots B_\beta} x_{mB_1 \ldots B_\beta, K}), \tag{9.13}$$

$$\bar{\pi}_{iB_1 \ldots B_\beta} = \varrho_0 \bar{F}_{iB_1 \ldots B_\beta} + \pi_{KiB_1 \ldots B_\beta, K}, \tag{9.14}$$

and

$$\bar{h}_0 = h_0 - N_K q_K, \\ \bar{p}_{iB_1 \ldots B_\beta} = p_{iB_1 \ldots B_\beta} - N_K \pi_{KiB_1 \ldots B_\beta}. \tag{9.15}$$

We consider a motion of the continuum which is such that the velocities differ from those of the given motion only by a superposed rigid body angular velocity, the continuum occupying the same position at time t, and we assume that $\bar{h}_0$, q_K, $\bar{p}_{iB_1 \ldots B_\beta}$, r, $\dot{U}$, π_{Km}, $\bar{\pi}_{iB_1 \ldots B_\beta}$ and $\pi_{KiB_1 \ldots B_\beta}$ are unaltered by such motions. Equations (9.11) and (9.12) hold for all velocity and multipolar velocity fields, so the equations hold when ω_{mi} is replaced by $\omega_{mi} + 2\Omega_{mi}$ with all other kinematic quantities unaltered in view of results in section 4, where Ω_{mi} is a constant arbitrary skew-symmetric tensor. Hence

$$\Omega_{mi} \sum_{\beta=1}^{\nu} \bar{p}_{iB_1 \ldots B_\beta} x_{mB_1 \ldots B_\beta} = 0, \\ \Omega_{mi} \pi'_{Km} x_{i,K} = 0,$$

and therefore

$$\sum_{\beta=1}^{\nu} (\bar{p}_{iB_1 \ldots B_\beta} x_{mB_1 \ldots B_\beta} - \bar{p}_{mB_1 \ldots B_\beta} x_{iB_1 \ldots B_\beta}) = 0, \tag{9.16}$$

$$\pi'_{Km} x_{i,K} = \pi'_{Ki} x_{m,K}. \tag{9.17}$$

Equations (9.11) and (9.12) then reduce to

$$\sum_{\beta=1}^{\nu} \bar{p}_{iB_1 \ldots B_\beta} (X_{A,i} B_{B_1 \ldots B_\beta : A} - \tfrac{1}{2} A_{mi} x_{mB_1 \ldots B_\beta}) - \bar{h}_0 = 0, \tag{9.18}$$

and

$$\varrho_0 r - q_{K,K} - \varrho_0 \dot{U} + \tfrac{1}{2} A_{mi} \pi'_{Km} x_{i,K} + \\ + X_{A,i} \sum_{\beta=1}^{\nu} (\bar{\pi}_{iB_1 \ldots B_\beta} B_{B_1 \ldots B_\beta : A} + \pi_{KiB_1 \ldots B_\beta} B_{B_1 \ldots B_\beta : AK}) = 0. \tag{9.19}$$

10. Elasticity

We use the work of section 9 and suppose that S, x_i and $x_{iB_1 \ldots B_\beta}$ ($\beta = 1, 2, \ldots, \nu$) are functions of X_A, t. Inspection of equations (9.5), (9.6), (9.18) and (9.19) suggests that constitutive equations are required for T, $\bar{h}_0$, q_K, U, π'_{Ki}, $\pi_{KiB_1 \ldots B_\beta}$, $\bar{\pi}_{iB_1 \ldots B_\beta}$ and $\bar{p}_{iB_1 \ldots B_\beta}$ ($\beta = 1, 2, \ldots, \nu$). We define an elastic body as one for which the following constitutive equations* hold at each material point X_A and for all time t:

$$U = U(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}), \tag{10.1}$$

$$\pi'_{Ki} = \pi'_{Ki}(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}), \tag{10.2}$$

$$\pi_{KiB_1\ldots B_\beta} = \pi_{KiB_1\ldots B_\beta}(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}), \tag{10.3}$$

$$\bar{\pi}_{iB_1\ldots B_\beta} = \bar{\pi}_{iB_1\ldots B_\beta}(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}), \tag{10.4}$$

$$T = T(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}), \tag{10.5}$$

$$\bar{p}_{iB_1\ldots B_\beta} = \bar{p}_{iB_1\ldots B_\beta}(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}, N_K), \tag{10.6}$$

$$\bar{h}_0 = \bar{h}_0(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}, N_K), \tag{10.7}$$

$$q_K = q_K(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}, T_{,A}, T_{,AA_1}, \ldots, T_{,AA_1\ldots A_\mu}), \tag{10.8}$$

for $\beta = 1, 2, \ldots, \nu$; $\gamma = 1, 2, \ldots, \mu$; $\mu \geqq \nu + 1$, and all functions are assumed to be single-valued and sufficiently smooth.

For a given deformation and entropy the 2^β-pole velocities $v_{iB_1\ldots B_\beta}$ may be chosen arbitrarily and independently of each other so that, from (9.8) or (9.18),

$$\bar{h}_0 = 0, \qquad \bar{p}_{iB_1\ldots B_\beta} = 0,$$

or

$$\begin{aligned} h_0 &= N_K\, q_K, \\ p_{iB_1\ldots B_\beta} &= N_K\, \pi_{KiB_1\ldots B_\beta} \qquad (\beta = 1, 2, \ldots, \nu). \end{aligned} \tag{10.9}$$

The second equation in (10.9) shows that $\pi_{KiB_1\ldots B_\beta}$ transforms as a tensor with respect to all indices, including K, under changes of rectangular cartesian axes, where the multipolar stresses in each coordinate system are associated with the three surfaces in that system which were coordinate planes $X_K =$ constant before deformation. The first equation in (10.9) shows that q_K transforms as a vector. Equations (9.13) and (9.14) then show that π'_{Km}, $\bar{\pi}_{iB_1\ldots B_\beta}$ are tensors with respect to all indices.

If we use $(10.9)_1$ in (9.3) and apply the equation to an arbitrary volume we have

$$\varrho_0 \dot{S}\, T - \varrho_0 r + q_{K,K} - \frac{q_K T_{,K}}{T} \geqq 0, \tag{10.10}$$

with the usual smoothness assumptions, recalling also that $T > 0$. If we then substitute for r from (9.19) into (10.10) we obtain the inequality

$$\begin{aligned} &\varrho_0 (T \dot{S} - \dot{U}) - \frac{q_K T_{,K}}{T} + \frac{1}{2} \pi'_{Km} x_{i,K} A_{mi} + \\ &\quad + X_{A,i} \sum_{\beta=1}^{\nu} (\bar{\pi}_{iB_1\ldots B_\beta} B_{B_1\ldots B_\beta : A} + \pi_{KiB_1\ldots B_\beta} B_{B_1\ldots B_\beta : AK}) \geqq 0. \end{aligned} \tag{10.11}$$

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* The independent variables are all unchanged by superposed rigid body translations at all times. The form of equation (9.9) suggests that multipolar displacements and their gradients, as well as displacement gradients, should appear as independent variables. By a method similar to that used in this section and in a previous paper (GREEN & RIVLIN, 1964) it can be shown that gradients of multipolar displacements of an order higher than the first cannot occur in the constitutive equations (10.1)—(10.6).

We assume that the internal energy function U is unaltered when the continuum undergoes a deformation which differs from the given deformation only by superposed rigid body motions at all times. This assumption includes those already made about $\dot{U}$. With the help of section 4 it follows that

$$U(S, x_{i,A}, x_{iB_1\ldots B_\gamma}, x_{iB_1\ldots B_\gamma,A}) = U(S, Q_{ij}x_{j,A}, Q_{ij}x_{jB_1\ldots B_\gamma}, Q_{ij}x_{jB_1\ldots B_\gamma,A})$$

for all proper orthogonal values of Q_{ij}. It follows* that U must be expressible in the different functional form

$$U = U(S, E_{:AB}, E_{B_1\ldots B_\gamma:A}, E_{B_1\ldots B_\gamma:AK}) \tag{10.12}$$

where

$$E_{:AB} = E_{AB} = E_{AB}(t),$$

and

$$E_{B_1\ldots B_\gamma:A_1\ldots A_\alpha} = E_{B_1\ldots B_\gamma:A_1\ldots A_\alpha}(t), \tag{10.13}$$

are defined in (4.19). Recalling the results (4.31) and (4.32), it follows that

$$\begin{aligned} \dot{U} &= \frac{\partial U}{\partial S}\dot{S} + \frac{\partial U}{\partial E_{AB}} x_{i,A}x_{j,B}A_{ij} + \\ &+ \sum_{\gamma=1}^{\mu}\left(\frac{\partial U}{\partial E_{B_1\ldots B_\gamma:A}} B_{B_1\ldots B_\gamma:A} + \frac{\partial U}{\partial E_{B_1\ldots B_\gamma:AK}} B_{B_1\ldots B_\gamma:AK}\right) \end{aligned} \tag{10.14}$$

where U is written as a symmetric function of E_{AB} in the indices A, B and E_{AB} is understood to mean $\frac{1}{2}(E_{AB}+E_{BA})$ in $\partial U/\partial E_{AB}$. The inequality (10.11) can now be written in the form

$$\begin{aligned} &\varrho_0\left(T - \frac{\partial U}{\partial S}\right)\dot{S} - \frac{q_K T_{,K}}{T} + \frac{1}{2}x_{i,K}\left(\pi'_{Km} - 2\varrho_0 x_{m,A}\frac{\partial U}{\partial E_{AK}}\right)A_{mi} + \\ &\quad + X_{A,i}\sum_{\beta=1}^{\nu}\left(\bar{\pi}_{iB_1\ldots B_\beta} - \varrho_0 x_{i,B}\frac{\partial U}{\partial E_{B_1\ldots B_\beta:B}}\right)B_{B_1\ldots B_\beta:A} + \\ &\quad + X_{A,i}\sum_{\beta=1}^{\nu}\left(\pi_{KiB_1\ldots B_\beta} - \varrho_0 x_{i,B}\frac{\partial U}{\partial E_{B_1\ldots B_\beta:BK}}\right)B_{B_1\ldots B_\beta:AK} - \\ &\quad - \varrho_0\sum_{\beta=\nu+1}^{\mu}\left(\frac{\partial U}{\partial E_{B_1\ldots B_\beta:A}} B_{B_1\ldots B_\beta:A} + \frac{\partial U}{\partial E_{B_1\ldots B_\beta:AK}} B_{B_1\ldots B_\beta:AK}\right) \geqq 0. \end{aligned} \tag{10.15}$$

For a given state of deformation and entropy this inequality is to be valid for all arbitrary values of $\dot{S}$, A_{mi}, $B_{B_1\ldots B_\beta:A}$, $B_{B_1\ldots B_\beta:AK}$ $(\beta = 1, 2, \ldots, \mu)$ which can be chosen independently of each other. It follows that

$$\frac{\partial U}{\partial E_{B_1\ldots B_\beta:A}} = 0, \qquad \frac{\partial U}{\partial E_{B_1\ldots B_\beta:AK}} = 0,$$

for $\beta = \nu+1, \nu+2, \ldots, \mu$ so that U in (10.12) reduces to

$$U = U(S, E_{AB}, E_{B_1\ldots B_\beta:A}, E_{B_1\ldots B_\beta:AK}) \tag{10.16}$$

* This is analogous to a result obtained by GREEN & RIVLIN (1964) and may be obtained by the methods of that paper.

with $\beta = 1, 2, \ldots, \nu$. In addition,

$$T = \frac{\partial U}{\partial S}, \tag{10.17}$$

$$\pi'_{Km} = 2\varrho_0 \, x_{m,A} \frac{\partial U}{\partial E_{AK}}, \tag{10.18}$$

$$\bar{\pi}_{iB_1\ldots B_\beta} = \varrho_0 \, x_{i,B} \frac{\partial U}{\partial E_{B_1\ldots B_\beta : B}}, \tag{10.19}$$

$$\pi_{KiB_1\ldots B_\beta} = \varrho_0 \, x_{i,B} \frac{\partial U}{\partial E_{B_1\ldots B_\beta : BK}}, \tag{10.20}$$

the last two results holding for $\beta = 1, 2, \ldots, \nu$. Also

$$-q_K T_{,K} \geqq 0 \tag{10.21}$$

and with the help of (10.17)–(10.20) equation (9.19) reduces to

$$\varrho_0 r - q_{K,K} - \varrho_0 T \dot{S} = 0. \tag{10.22}$$

Because of $(10.9)_2$ and (10.18) equations (9.16) and (9.17) are satisfied identically.

If we introduce the Helmholtz free energy function

$$A = U - TS \tag{10.23}$$

and express A in the form

$$A = A(T, E_{AB}, E_{B_1\ldots B_\beta : A}, E_{B_1\ldots B_\beta : AK}), \tag{10.24}$$

then

$$S = -\frac{\partial A}{\partial T}, \tag{10.25}$$

$$\pi'_{Km} = 2\varrho_0 \, x_{m,A} \frac{\partial A}{\partial E_{AK}}, \tag{10.26}$$

$$\bar{\pi}_{iB_1\ldots B_\beta} = \varrho_0 \, x_{i,B} \frac{\partial A}{\partial E_{B_1\ldots B_\beta : B}}, \tag{10.27}$$

$$\pi_{KiB_1\ldots B_\beta} = \varrho_0 \, x_{i,B} \frac{\partial A}{\partial E_{B_1\ldots B_\beta : BK}}. \tag{10.28}$$

Equations (9.14) and (10.27), together with (9.5), form a basic set of equations of motion for the stresses π_{Ki} and multipolar stresses $\pi_{KiB_1\ldots B_\beta}$, the constitutive equations for these stresses being given by (10.26), (10.28) where, from (9.13)

$$\pi_{Am} = \pi'_{Am} + X_{A,i} \sum_{\beta=1}^{\nu} (\bar{\pi}_{iB_1\ldots B_\beta} \, x_{mB_1\ldots B_\beta} + \pi_{KiB_1\ldots B_\beta} \, x_{mB_1\ldots B_\beta,K}). \tag{10.29}$$

11. Elasticity: alternative form

Before considering constitutive equations of a more general type, based on the work of section 8, we obtain results for elasticity in the notation of section 8. We suppose that the continuum is in a reference state X_B at time t_0 and we assume that the internal energy U at some time τ ($t_0 \leqq \tau \leqq t$) has the form*

$$U(\tau) = U[S, x_{i,A}(\tau), x_{ij_1\ldots j_\beta}(\tau), x_{ij_1\ldots j_\beta,k}(\tau), x_{i,A}, x_{ij_1\ldots j_\beta}(t_0), x_{ij_1\ldots j_\beta,k}(t_0)] \tag{11.1}$$

* Although $U(\tau)$ is expressed in terms of the variables in (11.1) for convenience in this section it must essentially be such that it is a function of kinematic variables at times τ and t_0.

for $\beta = 1, 2, \ldots, \mu$. We consider a motion (3.1) which differs from the given motion by superposed rigid body translations and rotation and we assume that U is unaltered by such rigid body motions. Then

$$\begin{aligned} U[S, x_{i,A}(\tau), x_{mj_1\ldots j_\beta}(\tau), x_{mj_1\ldots j_\beta,k}(\tau), x_{i,A}, x_{mj_1\ldots j_\beta}(t_0), x_{mj_1\ldots j_\beta,k}(t_0)] \\ = U[S, Q_{ij}(\tau)\, x_{j,A}(\tau), Q_{mn}(\tau)\, Q_{j_1 i_1}\cdots Q_{j_\beta i_\beta}\, x_{n i_1\ldots i_\beta}(\tau), \\ Q_{mn}(\tau)\, Q_{j_1 i_1}\cdots Q_{j_\beta i_\beta}\, Q_{ks}\, x_{n i_1\ldots i_\beta,s}(\tau), Q_{ij}\, x_{j,A}, \\ Q_{j_1 i_1}\cdots Q_{j_\beta i_\beta}\, x_{m i_1\ldots i_\beta}(t_0), Q_{j_1 i_1}\cdots Q_{j_\beta i_\beta}\, Q_{ks}\, x_{m i_1\ldots i_\beta,s}(t_0)], \end{aligned}$$

for all proper orthogonal values of $Q_{ij}(\tau)$.

It follows that

$$\begin{aligned} U(\tau) = U[S, E_{AB}(\tau), \bar{E}_{B_1\ldots B_\beta:A}(\tau), \bar{E}_{B_1\ldots B_\beta:AK}(\tau), E_{AB}, \\ \bar{E}_{B_1\ldots B_\beta:A}(t_0), \bar{E}_{B_1\ldots B_\beta:AK}(t_0)], \end{aligned} \tag{11.2}$$

where

$$\bar{E}_{B_1\ldots B_\beta:A}(\tau) = \bar{E}_{j_1\ldots j_\beta:i}(\tau)\, x_{i,A}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}, \tag{11.3}$$

$$\bar{E}_{B_1\ldots B_\beta:AK}(\tau) = \bar{E}_{j_1\ldots j_\beta:ik}(\tau)\, x_{i,A}\, x_{k,K}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}. \tag{11.4}$$

Using a dot to denote material time differentiation with respect to τ, and recalling (4.27), we have

$$[\dot{\bar{E}}_{B_1\ldots B_\beta:A}(\tau)]_{\tau=t} = B_{j_1\ldots j_\beta:i}\, x_{i,A}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}, \tag{11.5}$$

$$[\dot{\bar{E}}_{B_1\ldots B_\beta:AK}(\tau)]_{\tau=t} = B_{j_1\ldots j_\beta:ik}\, x_{i,A}\, x_{k,K}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}, \tag{11.6}$$

and

$$\begin{aligned} [\dot{U}(\tau)]_{\tau=t} = \frac{\partial U}{\partial S}\dot{S} + \left[\frac{\partial U}{\partial E_{AB}(\tau)}\right]_{\tau=t} A_{ij}\, x_{i,A}\, x_{j,B} + \\ + \sum_{\beta=1}^{\mu}\left[\frac{\partial U}{\partial \bar{E}_{B_1\ldots B_\beta:A}(\tau)}\right]_{\tau=t} B_{j_1\ldots j_\beta:i}\, x_{i,A}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta} + \\ + \sum_{\beta=1}^{\mu}\left[\frac{\partial U}{\partial \bar{E}_{B_1\ldots B_\beta:AK}(\tau)}\right]_{\tau=t} B_{j_1\ldots j_\beta:ik}\, x_{i,A}\, x_{k,K}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}. \end{aligned} \tag{11.7}$$

The development of elasticity equations from (8.20) and (8.21) is similar to that given in section 10, so we omit the details and we quote the final results. Thus

$$h = n_i\, Q_i, \tag{11.8}$$

$$t_{ij_1\ldots j_\beta} = n_k\, \sigma_{kij_1\ldots j_\beta}, \tag{11.9}$$

$$T = \frac{\partial U}{\partial S}, \tag{11.10}$$

$$\sigma_{im} = 2\varrho\, x_{i,A}\, x_{m,B}\left[\frac{\partial U}{\partial E_{AB}(\tau)}\right]_{\tau=t}, \tag{11.11}$$

$$\bar{\sigma}_{ij_1\ldots j_\beta} = \varrho\, x_{i,A}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}\left[\frac{\partial U}{\partial \bar{E}_{B_1\ldots B_\beta:A}(\tau)}\right]_{\tau=t}, \tag{11.12}$$

$$\sigma_{kij_1\ldots j_\beta} = \varrho\, x_{i,A}\, x_{k,K}\, x_{j_1,B_1}\cdots x_{j_\beta,B_\beta}\left[\frac{\partial U}{\partial \bar{E}_{B_1\ldots B_\beta:AK}(\tau)}\right]_{\tau=t}, \tag{11.13}$$

where U is given by (11.2) and β in (11.2), (11.12), (11.13) takes the values $1, 2, \dots, \nu$. Also

$$-Q_i T_{,i} \geqq 0 \tag{11.14}$$

and

$$\varrho r - Q_{i,i} - \varrho T \dot{S} = 0. \tag{11.15}$$

The expression for U is symmetrized with respect to the indices A, B in $E_{AB}(\tau)$ and $E_{AB}(\tau)$ is understood to mean $\frac{1}{2}[E_{AB}(\tau) + E_{BA}(\tau)]$ before (11.11) is used, and then the symmetry condition (8.19) is satisfied. In view of (11.9) the condition (8.18) is satisfied identically.

12. Constitutive equations*

For convenience we collect here all the fundamental equations of section 8, namely (8.6), (8.16), (8.17), (8.19), and (8.21), together with (8.7), (8.14), (8.18), and (8.20), and the entropy production inequality (8.3). Thus

$$\sigma_{ji,j} + \varrho F_i = \varrho v_i^{(2)}, \tag{12.1}$$

$$\bar{\sigma}_{ij_1\dots j_\beta} = \varrho \bar{F}_{ij_1\dots j_\beta} + \sigma_{kij_1\dots j_\beta,k}, \tag{12.2}$$

$$\sigma'_{im} = \sigma'_{mi} = \sigma_{im} - \sum_{\beta=1}^{\nu} \bar{\sigma}_{ij_1\dots j_\beta} x_{mj_1\dots j_\beta} - \sum_{\beta=1}^{\nu} \sigma_{kij_1\dots j_\beta} x_{mj_1\dots j_\beta,k}, \tag{12.3}$$

$$\begin{aligned} \varrho r - Q_{i,i} - \varrho \dot{U} + \tfrac{1}{2} A_{mi} \sigma'_{im} + \sum_{\beta=1}^{\nu} \bar{\sigma}_{ij_1\dots j_\beta} B_{j_1\dots j_\beta : i} + \\ + \sum_{\beta=1}^{\nu} \sigma_{kij_1\dots j_\beta} B_{j_1\dots j_\beta : ik} = 0, \end{aligned} \tag{12.4}$$

$$t_i = n_j \sigma_{ji}, \tag{12.5}$$

$$\begin{aligned} \bar{h} &= h - n_i Q_i, \\ \bar{t}_{ij_1\dots j_\beta} &= t_{ij_1\dots j_\beta} - n_k \sigma_{kij_1\dots j_\beta}, \end{aligned} \tag{12.6}$$

$$\sum_{\beta=1}^{\nu} (\bar{t}_{ij_1\dots j_\beta} x_{mj_1\dots j_\beta} - \bar{t}_{mj_1\dots j_\beta} x_{ij_1\dots j_\beta}) = 0, \tag{12.7}$$

$$\sum_{\beta=1}^{\nu} \bar{t}_{ij_1\dots j_\beta} (B_{j_1\dots j_\beta : i} - \tfrac{1}{2} A_{mi} x_{mj_1\dots j_\beta}) - \bar{h} = 0, \tag{12.8}$$

and

$$\int_V \varrho \dot{S}\, dV - \int_V \varrho \frac{r}{T}\, dV + \int_A \frac{h}{T}\, dA \geqq 0. \tag{12.9}$$

In equation (12.8) $B_{j_1\dots j_\beta : i} - \frac{1}{2} A_{mi} x_{mj_1\dots j_\beta}$ transforms as a tensor of order $\beta + 1$ under changes of rectangular cartesian axes. We assume that $\bar{t}_{ij_1\dots j_\beta}$ also transforms as a tensor of order $\beta + 1$ and that $\bar{h}$ is a scalar, under change of axes, so that the left hand side of equation (12.8) is then a scalar. Since, for a given surface, $t_{ij_1\dots j_\beta}$ is a tensor and h a scalar, it follows from (12.6) that Q_i transforms as a vector and $\sigma_{kij_1\dots j_\beta}$ as a tensor under change of rectangular axes, where the appropriate quantities in each system of axes refer to the

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* See also section 16.

coordinate surfaces in that system. Thus, if Q_i is the flux of heat across x_i-planes at x_j, and Q_i^* is the flux across x_i^*-planes at the same point, then

$$Q_i^* = \frac{\partial x_i^*}{\partial x_j} Q_j, \tag{12.10}$$

the transformation from x_i^* to x_j being orthogonal. A similar result holds for the multipolar stress tensor. It follows from (12.2) and (12.3) that $\bar{\sigma}_{ij_1\ldots j_\beta}$ and σ'_{im} transform as tensors under changes of rectangular cartesian axes and that the left-hand side of (12.4) is a scalar under such transformations.

We now suppose that σ_{ji}, $\sigma_{kij_1\ldots j_\beta}$, $\bar{\sigma}_{ij_1\ldots j_\beta}$, $\bar{t}_{ij_1\ldots j_\beta}$, $\bar{h}$, Q_i correspond to a deformation of the continuum given by (2.1), and that σ^*_{ji}, $\sigma^*_{kij_1\ldots j_\beta}$, $\bar{\sigma}^*_{ij_1\ldots j_\beta}$, $\bar{t}^*_{ij_1\ldots j_\beta}$, $\bar{h}^*$, Q_i^* correspond to the motion (3.1), the entropy S being unaltered. If the superposed rigid body motions for all time do not change the values of $\sigma_{ji}, \ldots, Q_i$, except for orientation at time t, then

$$\sigma^*_{ji} = Q_{jm} Q_{in} \sigma_{mn}, \tag{12.11}$$

$$\sigma^*_{kij_1\ldots j_\beta} = Q_{km} Q_{in} Q_{j_1 i_1} \cdots Q_{j_\beta i_\beta} \sigma_{mni_1\ldots i_\beta}, \tag{12.12}$$

$$\bar{\sigma}^*_{ij_1\ldots j_\beta} = Q_{ij} Q_{j_1 i_1} \cdots Q_{j_\beta i_\beta} \bar{\sigma}_{ji_1\ldots i_\beta}, \tag{12.13}$$

$$\bar{t}^*_{ij_1\ldots j_\beta} = Q_{ij} Q_{j_1 i_1} \cdots Q_{j_\beta i_\beta} \bar{t}_{ji_1\ldots i_\beta}, \tag{12.14}$$

$$\bar{h}^* = \bar{h}, \tag{12.15}$$

$$Q_i^* = Q_{ij} Q_j. \tag{12.16}$$

It follows from (12.3) and (4.6) that σ'_{im} satisfies an equation of the form (12.11). Also, recalling (4.6) and (4.26) we see that the left-hand sides of equations (12.4) and (12.8) are then unaltered by superposed rigid body motions, if r and U are unchanged by such motions.

In order to make any further progress, constitutive equations must be obtained for U, σ_{ji}, $\sigma_{kij_1\ldots j_\beta}$, $\bar{\sigma}_{ij_1\ldots j_\beta}$, $\bar{t}_{ij_1\ldots j_\beta}$, $\bar{h}$, and Q_i which will represent different material properties of the continuum, and these equations can then be reduced to canonical forms with the help of the invariance conditions (12.11)—(12.16). Results for an elastic material have already been obtained in section 11, and, in a different notation, in section 10. Other aspects of this problem are discussed in section 16.

13. Constitutive equations: alternative form

We first collect together the fundamental formulae of section 9, and introduce some further notation. Thus

$$\pi_{Ki,K} + \varrho_0 F_i = \varrho_0 v_i^{(2)}, \tag{13.1}$$

$$\bar{\pi}_{iB_1\ldots B_\beta} = \varrho_0 \bar{F}_{iB_1\ldots B_\beta} + \pi_{KiB_1\ldots B_\beta,K}, \tag{13.2}$$

$$\pi_{Ai} = x_{i,K}\, s_{AK}, \qquad \pi'_{Ai} = x_{i,K}\, s'_{AK}, \tag{13.3}$$

$$\pi_{KiB_1\ldots B_\beta} = x_{i,A}\, s_{KAB_1\ldots B_\beta}, \qquad \bar{\pi}_{iB_1\ldots B_\beta} = x_{i,A}\, \bar{s}_{AB_1\ldots B_\beta}, \tag{13.4}$$

$$s'_{BA} = s'_{AB} = s_{AB} - X_{B,m} \sum_{\beta=1}^{\nu} (\bar{s}_{AB_1\ldots B_\beta}\, x_{mB_1\ldots B_\beta} + s_{KAB_1\ldots B_\beta}\, x_{mB_1\ldots B_\beta,K}), \tag{13.5}$$

$$\varrho_0 r - q_{K,K} - \varrho_0 \dot{U} + \tfrac{1}{2} s'_{KA}\, x_{m,A}\, x_{i,K} A_{mi} + \\ + \sum_{\beta=1}^{\nu} (\bar{s}_{AB_1\ldots B_\beta} B_{B_1\ldots B_\beta:A} + s_{KAB_1\ldots B_\beta} B_{B_1\ldots B_\beta:AK}) = 0. \tag{13.6}$$

Also, if

$$p_i = x_{i,A} r_A, \quad p_{iB_1\ldots B_\beta} = x_{i,A} r_{AB_1\ldots B_\beta}, \quad \bar{p}_{iB_1\ldots B_\beta} = x_{i,A} \bar{r}_{AB_1\ldots B_\beta}, \tag{13.7}$$

then

$$r_A = N_K s_{KA}, \\ \bar{h}_0 = h_0 - N_K q_K, \\ \bar{r}_{AB_1\ldots B_\beta} = r_{AB_1\ldots B_\beta} - N_K s_{KAB_1\ldots B_\beta}, \tag{13.8}$$

$$\sum_{\beta=1}^{\nu} \bar{r}_{AB_1\ldots B_\beta} (x_{i,A}\, x_{mB_1\ldots B_\beta} - x_{m,A}\, x_{iB_1\ldots B_\beta}) = 0, \tag{13.9}$$

$$\sum_{\beta=1}^{\nu} \bar{r}_{AB_1\ldots B_\beta} (B_{B_1\ldots B_\beta:A} - \tfrac{1}{2} A_{mi} x_{i,A}\, x_{mB_1\ldots B_\beta}) - \bar{h}_0 = 0, \tag{13.10}$$

and

$$\int_{V_0} \varrho_0 \dot{S}\, dV_0 - \int_{V_0} \varrho_0 \frac{r}{T}\, dV_0 + \int_{A_0} \frac{h_0}{T}\, dA_0 \geqq 0. \tag{13.11}$$

In (13.10), $B_{B_1\ldots B_\beta:A} - \frac{1}{2} A_{mi} x_{i,A}\, x_{mB_1\ldots B_\beta}$ transforms as a tensor of order $\beta+1$ under changes of rectangular cartesian axes and is also unaltered by superposed rigid-body motions at all times. We assume that $\bar{r}_{AB_1\ldots B_\beta}$ is unaltered by these rigid body motions and that it transforms as a tensor of order $\beta+1$. We also assume that $\bar{h}_0$ transforms as a scalar and is unaltered by superposed rigid body motions. It follows that the left hand side of equation (13.10) is a scalar which is unaltered when rigid body motions are superposed on the given motion. Since, for a given surface, $r_{AB_1\ldots B_\beta}$ is a tensor and h_0 a scalar, it follows from (13.8) and (13.4) that $s_{KAB_1\ldots B_\beta}$ and $\pi_{KiB_1\ldots B_\beta}$ transform as tensors under changes of rectangular axes, and that q_K transforms as a vector, with respect to all indices including K. Also, from (13.2), (13.3) and (13.5) we see that $\bar{\pi}_{iB_1\ldots B_\beta}$, π'_{Ai}, $s_{AB_1\ldots B_\beta}$ and s'_{AK} transform as tensors under changes of rectangular axes and that the left hand side of (13.6) is a scalar. Moreover, $B_{B_1\ldots B_\beta:A}$, $B_{B_1\ldots B_\beta:AK}$ and $x_{m,A}\, x_{i,K} A_{mi}$ are unchanged when superposed rigid motions at all times are added to the given motion. We therefore assume that $s_{KAB_1\ldots B_\beta}$, $\bar{s}_{AB_1\ldots B_\beta}$, s'_{BA}, q_K, U, and r are unaltered by such rigid body motions. It follows that s_{BA} and the left hand side of equation (13.6) are also unaltered.

Constitutive equations must now be postulated for s'_{BA}, $s_{AB_1\ldots B_\beta}$, $s_{KAB_1\ldots B_\beta}$, q_K, U, $\bar{r}_{AB_1\ldots B_\beta}$ and $\bar{h}_0$ which will represent different material properties of the continuum and these equations can then be reduced to canonical form, with the help of the condition that they are all unaltered when rigid body motions are superposed on the given motion.

Results for elasticity have already been obtained in section 10, but we add here some other results derived from (10.8)—(10.20), and (13.3) and (13.4), namely

$$s'_{KA} = s'_{AK} = 2\varrho_0 \frac{\partial U}{\partial E_{AK}}, \tag{13.12}$$

$$\bar{s}_{AB_1\ldots B_\beta} = \varrho_0 \frac{\partial U}{\partial E_{B_1\ldots B_\beta:A}}, \tag{13.13}$$

$$s_{KAB_1\ldots B_\beta} = \varrho_0 \frac{\partial U}{\partial E_{B_1\ldots B_\beta:AK}}. \tag{13.14}$$

14. Elasticity: relation to previous theory

In a previous paper (GREEN & RIVLIN, 1964) which was concerned with the theory of simple multipolar forces and stresses of the first kind, associated with monopolar displacements and velocities, explicit formulae were obtained for elasticity. We now show that these elastic equations can be obtained as a special case of the present theory, and for this purpose we use the form of the theory given in section 10.

The tensors $E_{B_1\dots B_\beta:B}$ and $E_{B_1\dots B_{\beta-1}:BB_\beta}$ may be expressed in the form

$$\begin{aligned} E_{B_1\dots B_\beta:B} &= E_{(B_1\dots B_\beta):B} + E^*_{B_1\dots B_\beta:B}, \\ E_{B_1\dots B_{\beta-1}:BB_\beta} &= E_{(B_1\dots B_{\beta-1}):B(B_\beta)} + E^*_{B_1\dots B_{\beta-1}:BB_\beta}, \end{aligned} \tag{14.1}$$

for $\beta=2, 3, \dots, \nu+1$, where $E_{(B_1\dots B_\beta):B}$ is the part of $E_{B_1\dots B_\beta:B}$ which is completely summetric with respect to $B_1, \dots, B_\beta$ and $E_{(B_1\dots B_{\beta-1}):B(B_\beta)}$ is the part of $E_{B_1\dots B_{\beta-1}:BB_\beta}$ which is completely symmetric with respect to the same indices. The tensors $E^*\dots$ are then defined by (14.1). With a similar notation we also have

$$\begin{aligned} \bar{\pi}_{iB_1\dots B_\beta} &= \bar{\pi}_{i(B_1\dots B_\beta)} + \bar{\pi}^*_{iB_1\dots B_\beta}, \\ \pi_{B_\beta iB_1\dots B_{\beta-1}} &= \pi_{(B_\beta)i(B_1\dots B_{\beta-1})} + \pi^*_{B_\beta iB_1\dots B_{\beta-1}}, \end{aligned} \tag{14.2}$$

for $\beta=2, 3, \dots, \nu+1$. Equations (10.19) and (10.20) may now be written in the alternative forms

$$\bar{\pi}_{i(B_1\dots B_\beta)} = \varrho_0 x_{i,B} \frac{\partial U}{\partial E_{(B_1\dots B_\beta):B}}, \tag{14.3}$$

$$\pi_{(B_\beta)i(B_1\dots B_{\beta-1})} = \varrho_0 x_{i,B} \frac{\partial U}{\partial E_{(B_1\dots B_{\beta-1}):B(B_\beta)}}, \tag{14.4}$$

$$\bar{\pi}^*_{iB_1\dots B_\beta} = \varrho_0 x_{i,B} \frac{\partial U}{\partial E^*_{B_1\dots B_\beta:B}}, \tag{14.5}$$

$$\pi^*_{B_\beta iB_1\dots B_{\beta-1}} = \varrho_0 x_{i,B} \frac{\partial U}{\partial E^*_{B_1\dots B_{\beta-1}:BB_\beta}}, \tag{14.6}$$

for $\beta=2, \dots, \nu+1$, and

$$\begin{aligned} \bar{\pi}_{iB_1} &= \varrho_0 x_{i,B} \frac{\partial U}{\partial E_{B_1:B}}, \\ \pi'_{Ki} &= 2\varrho_0 x_{i,A} \frac{\partial U}{\partial E_{AK}}. \end{aligned} \tag{14.7}$$

In (14.3) and (14.5), $\partial U/\partial E_{(B_1\dots B_\beta):B}$ denotes the part of $\partial U/\partial E_{B_1\dots B_\beta:B}$ which is completely symmetric with respect to $B_1 \dots B_\beta$ and $\partial U/\partial E^*_{B_1\dots B_\beta:B}$ denotes the remaining part. Similar notations are used in (14.4) and (14.6).

Next we take special values

$$x_{iB_1\dots B_\beta} = x_{i,B_1\dots B_\beta} \qquad (\beta=1, \dots, \nu) \tag{14.8}$$

for the multipolar displacements. It follows from (4.19) and (14.1) that

$$\begin{aligned} E_{(B_1\dots B_\beta):B} &= E_{BB_1\dots B_\beta}, \\ E_{(B_1\dots B_{\beta-1}):B(B_\beta)} &= E_{BB_1\dots B_\beta}, \end{aligned} \tag{14.9}$$

for $\beta=2, \dots, \nu$ in $(14.9)_1$, $\beta=2, \dots, \nu+1$ in $(14.9)_2$, and

$$E_{B_1:B} = E_{BB_1} = E_{B_1B}, \tag{14.10}$$

where $E_{BB_1\ldots B_\beta}$ is defined in (3.6) and is completely symmetric with respect to $B_1, \ldots, B_\beta$. Also

$$E^*_{B_1\ldots B_\beta:B} = 0, \qquad E^*_{B_1\ldots B_{\beta-1}:BB_\beta} = 0. \tag{14.11}$$

The function U in (10.16) reduces to

$$U(S, E_{AB}, E_{B_1\ldots B_\beta:A}, E_{B_1\ldots B_\beta:AK}) = \overline{U}(S, E_{AB}, E_{AA_1\ldots A_\alpha}) \quad \text{(say)} \tag{14.12}$$

where $\beta = 1, \ldots, \nu$; $\alpha = 2, \ldots, \nu+1$, and $\overline{U}$ is expressed as a symmetric function of E_{AB} and of $E_{AA_1\ldots A_\alpha}$ as far as the indices $A_1, \ldots, A_\alpha$ are concerned. From (14.9) and (14.12) we see that

$$\frac{\partial \overline{U}}{\partial E_{BB_1\ldots B_{\nu+1}}} = \frac{\partial U}{\partial E_{(B_1\ldots B_\nu):B(B_{\nu+1})}}, \tag{14.13}$$

$$\frac{\partial \overline{U}}{\partial E_{BB_1\ldots B_\beta}} = \frac{\partial U}{\partial E_{(B_1\ldots B_{\beta-1}):B(B_\beta)}} + \frac{\partial U}{\partial E_{(B_1\ldots B_\beta):B}} \tag{14.14}$$

for $\beta = 2, \ldots, \nu$ and

$$\frac{\partial \overline{U}}{\partial E_{BB_1}} = \frac{\partial U}{\partial E_{BB_1}} + \frac{\partial U}{\partial E_{B_1:B}}. \tag{14.15}$$

From (14.4) and (14.13) we have

$$\pi_{(B_{\nu+1})i(B_1\ldots B_\nu)} = \varrho_0 x_{i,B} \frac{\partial \overline{U}}{\partial E_{BB_1\ldots B_{\nu+1}}}. \tag{14.16}$$

Again, from (9.14), (14.3), (14.4), and (14.14), we obtain the formula

$$\pi_{(B_\beta)i(B_1\ldots B_{\beta-1})} + \pi_{Ki(B_1\ldots B_\beta),K} + \varrho_0 \overline{F}_{i(B_1\ldots B_\beta)} = \varrho_0 x_{i,B} \frac{\partial \overline{U}}{\partial E_{BB_1\ldots B_\beta}} \tag{14.17}$$

for $\beta = 2, \ldots, \nu$. Next, from (9.13), (9.14), and the formulae of this section, we see that

$$\begin{aligned} &\pi_{Ai} + \pi_{KiA,K} + \varrho_0 \overline{F}_{iA} \\ &\quad = 2\varrho_0 x_{i,B} \frac{\partial \overline{U}}{\partial E_{AB}} + \varrho_0 \sum_{\beta=2}^{\nu+1} \frac{\partial \overline{U}}{\partial E_{AB_1\ldots B_\beta}} x_{i,B_1\ldots B_\beta} \qquad (\nu \geqq 1). \end{aligned} \tag{14.18}$$

In deriving (14.16)—(14.18) we have assumed that U takes a definite value $\overline{U}$ when conditions (14.9), (14.10) and (14.11) apply, and that the derivatives of $\overline{U}$ in (14.16)—(14.18) can be evaluated. Formulae (14.5) and (14.6), however, contain derivatives of U with respect to the tensors $E^*_{B_1\ldots B_\beta:B}$ and $E^*_{B_1\ldots B_{\beta-1}:BB_\beta}$ at the zero values of these tensors. If U depends on elastic coefficients which tend to infinity when $E^*_{B_1\ldots B_\beta:B}$ and $E^*_{B_1\ldots B_{\beta-1}:BB_\beta}$ tend to zero, in such a way that U tends to the value (14.12) but the right hand sides of (14.5) and (14.6) tend to arbitrary functions, then the values of $\pi^*_{B_\beta iB_1\ldots B_{\beta-1}}$ and $\overline{\pi}^*_{iB_1\ldots B_\beta}$ are undetermined. This situation is analogous to that which arises when equations for incompressible elasticity are derived from those for compressible elasticity by a limiting process. Equations (14.16)—(14.18) agree with those obtained previously (1964) except for a change in notation.*

* The inertia terms were not included explicitly in the previous paper.

15. Infinitesimal elasticity

Elasticity theory appropriate to a continuum in which the displacements and multipolar displacements are infinitesimal can be obtained at once from section 10. For simplicity we restrict our attention here to the theory in which only displacements and dipolar displacements, and their corresponding stresses, are present. Then, using the Helmholtz function A,

$$A = A(T, E_{AB}, E_{B:A}, E_{B:AK}), \tag{15.1}$$

$$\pi'_{Km} = 2\varrho_0 x_{m,A} \frac{\partial A}{\partial E_{AK}}, \tag{15.2}$$

$$\bar{\pi}_{iB} = \varrho_0 x_{i,A} \frac{\partial A}{\partial E_{B:A}}, \tag{15.3}$$

$$\pi_{KiB} = \varrho_0 x_{i,A} \frac{\partial A}{\partial E_{B:AK}}, \tag{15.4}$$

$$\pi'_{Km} x_{i,K} = \pi'_{Ki} x_{m,K}, \tag{15.5}$$

$$\pi'_{Am} = \pi_{Am} - X_{A,i}(\bar{\pi}_{iB} x_{mB} + \pi_{KiB} x_{mB,K}), \tag{15.6}$$

$$\bar{\pi}_{iB} = \varrho_0 \bar{F}_{iB} + \pi_{KiB,K}, \tag{15.7}$$

$$\pi_{Ki,K} + \varrho_0 F_i = \varrho_0 v_i^{(2)}, \tag{15.8}$$

$$S = -\frac{\partial A}{\partial T}, \qquad -q_K T_{,K} \geqq 0, \tag{15.9}$$

$$\varrho_0 r - q_{K,K} - \varrho_0 T \dot{S} = 0, \tag{15.10}$$

and

$$\begin{aligned} p_i &= N_K \pi_{Ki}, \qquad h_0 = N_K q_K, \\ p_{iB} &= N_K \pi_{KiB}. \end{aligned} \tag{15.11}$$

In (15.1),

$$\begin{aligned} E_{AB} &= x_{i,A} x_{i,B}, \\ E_{B:A} &= x_{i,A} x_{iB}, \\ E_{B:AK} &= x_{i,A} x_{iB,K}. \end{aligned} \tag{15.12}$$

Let X_{iA} denote the value of x_{iA} in the reference state X_A and let

$$\begin{aligned} \widetilde{E}_{AB} &= E_{AB} - \delta_{AB}, \\ \widetilde{E}_{B:A} &= E_{B:A} - X_{AB}, \\ \widetilde{E}_{B:AK} &= E_{B:AK} - X_{AB,K}. \end{aligned} \tag{15.13}$$

We shall consider that A is a polynomial in $\widetilde{E}_{AB}$, $\widetilde{E}_{B:A}$ and $\widetilde{E}_{B:AK}$ and if these latter quantities are small enough we may approximate A by*

$$\begin{aligned} \varrho_0 A = C &+ \alpha_{AB} \widetilde{E}_{AB} + \beta_{BA} \widetilde{E}_{B:A} + \gamma_{BAK} \widetilde{E}_{B:AK} + \\ &+ \lambda_{ABCD} \widetilde{E}_{AB} \widetilde{E}_{CD} + \mu_{ABCD} \widetilde{E}_{AB} \widetilde{E}_{C:D} + \nu_{ABCDK} \widetilde{E}_{AB} \widetilde{E}_{C:DK} + \\ &+ \xi_{ABCD} \widetilde{E}_{A:B} \widetilde{E}_{C:D} + \eta_{ABCDK} \widetilde{E}_{A:B} \widetilde{E}_{C:DK} + \zeta_{ABCDEF} \widetilde{E}_{A:BC} \widetilde{E}_{D:EF}, \end{aligned} \tag{15.14}$$

* We assume here that the temperature T is constant. Alternatively, if we replace A by the internal energy U then the entropy S is constant.

where C and the coefficients $\alpha_{AB}, \ldots, \zeta_{ABCDEF}$ are constants if the body is initially homogeneous. We may omit the constant C without loss of generality. If, when the body is in its reference state, π'_{Km}, $\bar{\pi}_{iB}$ and π_{KiB} vanish and the body is in equilibrium under the action of no body or surface forces, and no multipolar body or surface forces, then α_{AB}, β_{AB}, γ_{BAK} in (15.14) are zero and A reduces to

$$\begin{aligned}\varrho_0 A = {}& \lambda_{ABCD}\tilde{E}_{AB}\tilde{E}_{CD}+\mu_{ABCD}\tilde{E}_{AB}\tilde{E}_{C:D}+\nu_{ABCDK}\tilde{E}_{AB}\tilde{E}_{C:DK}+\\ &+\xi_{ABCD}\tilde{E}_{A:B}\tilde{E}_{C:D}+\eta_{ABCDK}\tilde{E}_{A:B}\tilde{E}_{C:DK}+\zeta_{ABCDEF}\tilde{E}_{A:BC}\tilde{E}_{D:EF},\end{aligned} \tag{15.15}$$

where without loss of generality

$$\begin{aligned}&\lambda_{ABCD}=\lambda_{BACD}=\lambda_{ABDC}=\lambda_{CDBA},\\ &\mu_{ABCD}=\mu_{BACD}, \qquad \nu_{ABCDK}=\nu_{BACDK},\\ &\xi_{ABCD}=\xi_{CDAB}, \qquad \zeta_{ABCDEF}=\zeta_{DEFABC}.\end{aligned} \tag{15.16}$$

We now write

$$\begin{aligned}x_i&=X_i+\varepsilon\, u_i,\\ x_{iA}&=X_{iA}+\varepsilon\, u_{iA},\end{aligned} \tag{15.17}$$

in the expressions (15.13), and neglect terms of higher degree than the first in ε. We then obtain

$$\begin{aligned}\tilde{E}_{AB}&=\varepsilon(u_{A,B}+u_{B,A})=e_{AB},\\ \tilde{E}_{B:A}&=\varepsilon(u_{AB}+u_{i,A}X_{iB})=f_{BA},\\ \tilde{E}_{B:AK}&=\varepsilon(u_{AB,K}+u_{i,A}X_{iB,K})=f_{BAK}.\end{aligned} \tag{15.18}$$

If we introduce (15.15) into (15.2)—(15.4) and use (15.13), (15.18) and retain only terms of order ε, we have*

$$\pi'_{Km}=2(2\lambda_{KmCD}e_{CD}+\mu_{KmCD}f_{CD}+\nu_{KmCDL}f_{CDL}), \tag{15.19}$$

$$\bar{\pi}_{iB}=\mu_{CDBi}e_{CD}+2\xi_{BiCD}f_{CD}+\eta_{BiCDK}f_{CDK}, \tag{15.20}$$

$$\pi_{KiB}=\nu_{CDBiK}e_{CD}+\eta_{CDBiK}f_{CD}+2\zeta_{BiKCDF}f_{CDF}. \tag{15.21}$$

Also, from (15.5) and (15.6), we have, to order ε,

$$\pi'_{im}=\pi'_{mi}, \tag{15.22}$$

$$\pi_{Am}=\pi'_{Am}+\bar{\pi}_{AB}X_{mB}+\pi_{KAB}X_{mB,K}. \tag{15.23}$$

If the continuum in its undeformed state is isotropic with a center of symmetry (holohedral) then the coefficients in (15.15) take the special forms

$$4\lambda_{ABCD}=\lambda\delta_{AB}\delta_{CD}+\mu(\delta_{AC}\delta_{BD}+\delta_{AD}\delta_{BC}), \tag{15.24}$$

$$\mu_{ABCD}=\lambda_1\delta_{AB}\delta_{CD}+\mu_1(\delta_{AC}\delta_{BD}+\delta_{AD}\delta_{BC}), \tag{15.25}$$

$$2\xi_{ABCD}=\xi_1\delta_{AB}\delta_{CD}+\xi_2\delta_{AC}\delta_{BD}+\xi_3\delta_{AD}\delta_{BC}, \tag{15.26}$$

* We can also put $\varepsilon=1$ now without loss of generality. A must be written as a symmetric function of E_{AB} before equation (15.2) is used.

$$
\begin{aligned}
2\zeta_{ABCDEF} = \zeta_1 (\delta_{AB}\delta_{CD}\delta_{EF} + \delta_{DE}\delta_{AF}\delta_{BC}) + \\
+ \zeta_2 (\delta_{AB}\delta_{CE}\delta_{DF} + \delta_{DE}\delta_{BF}\delta_{AC}) + \\
+ \zeta_3 (\delta_{AC}\delta_{BD}\delta_{EF} + \delta_{DF}\delta_{AE}\delta_{BC}) + \\
+ \zeta_4 (\delta_{AE}\delta_{BF}\delta_{CD} + \delta_{BD}\delta_{CE}\delta_{AF}) + \\
+ \zeta_5 \delta_{AB}\delta_{CF}\delta_{DE} + \zeta_6 \delta_{AC}\delta_{BE}\delta_{DF} + \\
+ \zeta_7 \delta_{AD}\delta_{BC}\delta_{EF} + \zeta_8 \delta_{AD}\delta_{BE}\delta_{CF} + \\
+ \zeta_9 \delta_{AD}\delta_{BF}\delta_{CE} + \zeta_{10} \delta_{AE}\delta_{BD}\delta_{CF} + \\
+ \zeta_{11} \delta_{AF}\delta_{BE}\delta_{CD},
\end{aligned}
\tag{15.27}
$$

$$
\nu_{ABCDK} = \eta_{ABCDK} = 0, \tag{15.28}
$$

where the coefficients $\lambda, \mu, \ldots, \zeta_{11}$ are constants when T is constant. The expressions (15.19)—(15.21) then become

$$
\pi'_{Km} = \lambda \delta_{Km} e_{CC} + 2\mu e_{Km} + 2\lambda_1 \delta_{Km} f_{CC} + 2\mu_1 (f_{Km} + f_{mK}), \tag{15.29}
$$

$$
\bar{\pi}_{iB} = \lambda_1 \delta_{iB} e_{CC} + 2\mu_1 e_{iB} + \xi_1 \delta_{iB} f_{CC} + \xi_2 f_{Bi} + \xi_3 f_{iB}, \tag{15.30}
$$

$$
\begin{aligned}
\pi_{KiB} = \zeta_1 (\delta_{iB} f_{KDD} + \delta_{iK} f_{DDB}) + \zeta_2 (\delta_{iB} f_{DKD} + \delta_{BK} f_{DDi}) + \\
+ \zeta_3 (\delta_{iK} f_{DBD} + \delta_{BK} f_{iDD}) + \zeta_4 (f_{KBi} + f_{iKB}) + \\
+ \zeta_5 \delta_{iB} f_{DDK} + \zeta_6 \delta_{BK} f_{DiD} + \zeta_7 \delta_{iK} f_{BDD} + \\
+ \zeta_8 f_{BiK} + \zeta_9 f_{BKi} + \zeta_{10} f_{iBK} + \zeta_{11} f_{KiB}.
\end{aligned}
\tag{15.31}
$$

We consider one simple application of these results. Suppose X_{iB} is constant and that we have a homogeneous deformation

$$
u_i = b_{iA} X_A, \qquad b_{Ai} = b_{iA}, \tag{15.32}
$$

with constant values of the dipolar displacement u_{AB}. We suppose that the body is in equilibrium under the action of zero body and dipolar body forces. Then

$$
f_{BAK} = 0, \qquad \pi_{KiB} = 0, \qquad e_{AB} = 2b_{AB}, \tag{15.33}
$$

and, from (15.7), we must then have

$$
\bar{\pi}_{iB} = 0. \tag{15.34}
$$

Equation (15.34) can then be satisfied if

$$
(\xi_2 + \xi_3) f_{iB} = (\xi_2 + \xi_3) f_{Bi} = -2\mu_1 e_{iB} - \frac{\lambda_1(\xi_2+\xi_3) - 2\xi_1\mu_1}{3\xi_1+\xi_2+\xi_3} \delta_{iB} e_{DD} \tag{15.35}
$$

provided

$$
3\xi_1 + \xi_2 + \xi_3 \neq 0, \qquad \xi_2 \pm \xi_3 \neq 0. \tag{15.36}
$$

From (15.32), (15.35), and (15.29) we see that π'_{Km} are constants and, from (15.23)

$$
\pi_{Am} = \pi'_{Am} \tag{15.37}
$$

so that the equations of equilibrium (15.8) are satisfied.

Equation (15.35) gives the dipolar displacements in terms of the homogeneous deformation coefficients b_{iA}. Such a deformation can be maintained by the application of surface forces alone and no dipolar surface forces at the boundary of the body.

16. Equations of motion and variational equations

TRUESDELL & TOUPIN (1960, sections 166, 205, 232) introduced the idea of a generalized velocity, which in our terminology is called a multipolar velocity and corresponds to (4.11), but they did not include the restriction (4.14) which describes the behavior of such a velocity when rigid-body motions are superposed on the continuum. They defined generalized body and surface forces and stresses, called here body and surface 2^β-pole forces of the $(\beta+1)^{\text{th}}$ kind and surface 2^β-pole stresses of the $(\beta+1)^{\text{th}}$ kind, and they postulated equations of motion and an equivalent variational equation. In this section we examine the relation of the ideas of TRUESDELL & TOUPIN to those presented here and for this purpose we use the basic equations of sections 8, 12.

The condition (4.6) on multipolar displacements, or the equivalent condition (4.14) on multipolar velocities, under superposed rigid body motions implies, in particular, that multipolar displacements and velocities are unaltered by superposed rigid body translations at any speed, and at any time. This condition, together with the other assumptions made in section 8, enabled us to obtain the classical equations of motion (8.5) from the energy equation (8.1). Without this condition the classical equations (8.5) would not have the same form.

Because multipolar displacements and velocities are unaltered when the continuum receives superposed rigid body translations at any speeds, it is possible for the quantities U, σ'_{im}, $\sigma_{kij_1\ldots j_\beta}$, $\bar{\sigma}_{ij_1\ldots j_\beta}$, and Q_i to depend explicitly on these displacements and velocities but not, of course, on the ordinary monopolar displacement and velocity which are altered by rigid body translations at various speeds.

We now consider the special situation in which U does not depend explicitly on multipolar displacements or velocities and σ'_{im}, $\sigma_{kij_1\ldots j_\beta}$, $\bar{\sigma}_{ij_1\ldots j_\beta}$, Q_i (and r) do not depend explicitly on multipolar velocities. We consider a second motion of the continuum which is such that its position and the multipolar displacements at time t are unaltered, but it now has multipolar velocities $v_{ij_1\ldots j_\beta}+v'_{ij_1\ldots j_\beta}$ $(\beta=1,\ldots,\nu)$, where $v'_{ij_1\ldots j_\beta}$ are constants (in space and time). The corresponding energy equation will differ from (12.4) only by arbitrary constant values $v'_{ij_1\ldots j_\beta}$ added to $B_{j_1\ldots j_\beta:i}$ so that, by subtraction,

$$\sum_{\beta=1}^{\nu}\bar{\sigma}_{ij_1\ldots j_\beta}\,v'_{ij_1\ldots j_\beta}=0.$$

Since $\bar{\sigma}_{ij_1\ldots j_\beta}$ is independent of $v'_{ij_1\ldots j_\beta}$ which can be chosen arbitrarily, it follows that

$$\bar{\sigma}_{ij_1\ldots j_\beta}=0$$

or

$$\sigma_{kij_1\ldots j_\beta,k}+\bar{F}_{ij_1\ldots j_\beta}=0. \tag{16.1}$$

If we recall (8.2), we see that (16.1) are the equations of motion postulated by TRUESDELL & TOUPIN (1964, section 205). It should be emphasized that these equations are not always satisfied, and, in particular, are not necessarily satisfied for an elastic medium, as is seen by (11.12) when U depends on multipolar displacements.

Again, if $\bar{h}$ and $\bar{t}_{ij_1\ldots j_\beta}$ do not depend explicitly on multipolar velocities, we can show, from equation (12.8), that

$$\bar{t}_{ij_1\ldots j_\beta}=0$$

or

$$t_{ij_1\ldots j_\beta}=n_k\,\sigma_{kij_1\ldots j_\beta}. \tag{16.2}$$

Equations (16.2) were postulated by TRUESDELL & TOUPIN (1964, section 205). From (12.8) it then follows that $\bar{h}=0$ or

$$h=n_i Q_i. \tag{16.3}$$

We have established sufficient conditions under which the equations of motion (16.1) and the surface conditions (16.2), postulated by TRUESDELL & TOUPIN are valid. Since these equations are completely equivalent to the variational equations studied by TRUESDELL & TOUPIN (1964, section 232), we have also established sufficient conditions under which the variational equations hold. In general, however, the variational equations are incomplete unless they include variations of the internal energy and the heat conduction vector.

Throughout this section we have assumed that the multipolar displacements and velocities of all orders are independent and we have not considered degenerate or special cases. For example, if multipolar velocities have the special gradient form

$$v_{ij_1\ldots j_\beta}=v_{i,j_1\ldots j_\beta} \tag{16.4}$$

then equation (16.2) would still follow if we assume that $\bar{h}$ and $\bar{t}_{ij_1\ldots j_\beta}$ do not depend explicitly on velocity gradients of all orders, as shown previously by GREEN & RIVLIN (1964). Even if we assume that U does not depend explicitly on displacement or velocity gradients and that σ'_{im}, $\sigma_{kij_1\ldots j_\beta}$, $\bar{\sigma}_{ij_1\ldots j_\beta}$ and Q_i do not depend explicitly on velocity gradients we do not obtain equations (16.1).

17. Appendix

We suppose that N particles with masses $m^{(P)}$ $(P=1,2,\ldots,N)$ are situated at the points $X_i^{(P)}$ at time t_0. At a subsequent time τ $(t_0\leqq\tau\leqq t)$ we assume that the masses are at points $x_i^{(P)}(\tau)$ $(P=1,\ldots,N)$ and we use the notation

$$x_i^{(P)}=x_i^{(P)}(t), \qquad X_i^{(P)}=x_i^{(P)}(t_0). \tag{17.1}$$

The center of mass G of the N particles at time τ is denoted by $x_i(\tau)$ where

$$M x_i(\tau)=\sum_{P=1}^{N} m^{(P)} x_i^{(P)}(\tau), \qquad M=\sum_{P=1}^{N} m^{(P)} \tag{17.2}$$

and we write

$$x_i=x_i(t), \qquad X_i=x_i(t_0). \tag{17.3}$$

If

$$\begin{aligned} y_i^{(P)}(\tau)&=x_i^{(P)}(\tau)-x_i(\tau),\\ y_i^{(P)}&=y_i^{(P)}(t)=x_i^{(P)}-x_i,\\ Y_i^{(P)}&=y_i^{(P)}(t_0)=X_i^{(P)}-X_i, \end{aligned} \tag{17.4}$$

then

$$\sum_{P=1}^{N} m^{(P)} y_i^{(P)}(\tau)=0, \qquad \sum_{P=1}^{N} m^{(P)} y_i^{(P)}=0, \qquad \sum_{P=1}^{N} m^{(P)} Y_i^{(P)}=0. \tag{17.5}$$

The motion of each particle and the motion of G is given by

$$x_i^{(P)}(\tau) = x_i^{(P)}(\tau, X_r^{(P)}),\qquad x_i(\tau) = x_i(\tau, X_r), \tag{17.6}$$

or by

$$x_i^{(P)}(\tau) = x_i^{(P)}(\tau, t, x_r^{(P)}),\qquad x_i(\tau) = x_i(\tau, t, x_r), \tag{17.7}$$

since

$$x_i^{(P)} = x_i^{(P)}(t, X_r^{(P)}),\qquad x_i = x_i(t, X_r). \tag{17.8}$$

The velocity of the mass $m^{(P)}$ at time τ is defined as

$$v_i^{(P)}(\tau) = \dot{x}_i^{(P)}(\tau) = v_i(\tau) + \dot{y}_i^{(P)}(\tau), \tag{17.9}$$

where a dot denotes derivative with respect to τ holding $X_r^{(P)}$ fixed in (17.6) or t and $x_r^{(P)}$ fixed in (17.7), and $v_i(\tau)$ is the velocity of G. We use the notation

$$v_i^{(P)} = v_i^{(P)}(t), \qquad v_i = v_i(t). \tag{17.10}$$

It follows from (17.2) that

$$M v_i(\tau) = \sum_{P=1}^{N} m^{(P)} v_i^{(P)}(\tau), \tag{17.11}$$

and, from (17.5)

$$\sum_{P=1}^{N} m^{(P)} \dot{y}_i^{(P)}(\tau) = 0, \qquad \sum_{P=1}^{N} m^{(P)} \dot{y}_i^{(P)} = 0. \tag{17.12}$$

Suppose each mass is acted on by a force $F_i^{(P)}(\tau)$ per unit mass, where

$$F_i^{(P)}(\tau) = F_i^{(P)}[\tau, x_r^{(P)}(\tau)].$$

In view of (17.4), (17.6) and (17.7) this can be expressed in the alternative forms

$$F_i^{(P)}(\tau) = F_i^{(P)}(\tau, X_r + Y_r^{(P)}) \tag{17.13}$$

or

$$F_i^{(P)}(\tau) = F_i^{(P)}(\tau, t, x_r + y_r^{(P)}). \tag{17.14}$$

The rate of work of these forces is

$$W = \sum_{P=1}^{N} m^{(P)} F_i^{(P)}(\tau)\, v_i^{(P)}(\tau). \tag{17.15}$$

Adopting the form (17.13), we define a continuous function of τ and $X_r + Y_r$, $F^*(\tau, X_r + Y_r)$ say, with continuous derivatives up to order $\mu + 1$, such that

$$F_i^*(\tau) = F_i^*(\tau, X_r), \qquad F_i^{(P)}(\tau) = F_i^*(\tau, X_r + Y_r^{(P)})$$

for each value of P. Then,

$$F_i^{(P)}(\tau) = F_i^*(\tau, X_r) + \sum_{\beta=1}^{\mu} \frac{1}{\beta!} F^*_{i,B_1\ldots B_\beta}(\tau, X_r)\, Y_{B_1}^{(P)} \ldots Y_{B_\beta}^{(P)} + R_i, \tag{17.16}$$

where R_i is a remainder term and

$$F^*_{i,B_1\ldots B_\beta}(\tau, X_r) = \frac{\partial^\beta F_i^*(\tau)}{\partial X_{B_1}\ldots\partial X_{B_\beta}}. \tag{17.17}$$

From (17.15) and (17.9) we have

$$W = M F_i(\tau)\, v_i(\tau) + \sum_{P=1}^{N} m^{(P)} F_i^{(P)}(\tau)\, \dot{y}_i^{(P)}(\tau), \tag{17.18}$$

where

$$M F_i(\tau) = \sum_{P=1}^{N} m^{(P)} F_i^{(P)}(\tau). \tag{17.19}$$

If we substitute (17.16) into (17.18) and use (17.12) we see that

$$W = M F_i(\tau)\, v_i(\tau) + M \sum_{\beta=1}^{\mu} F^*_{i,B_1\ldots B_\beta}(\tau, X_r)\, v_{iB_1\ldots B_\beta}(\tau), \tag{17.20}$$

if the remainder term can be neglected, where

$$M v_{iB_1\ldots B_\beta}(\tau) = \frac{1}{\beta!} \sum_{P=1}^{N} m^{(P)} \dot{y}_i^{(P)}(\tau)\, Y_{B_1}^{(P)} \ldots Y_{B_\beta}^{(P)}. \tag{17.21}$$

If we define $x_{iB_1\ldots B_\beta}(\tau)$ by

$$M x_{iB_1\ldots B_\beta}(\tau) = \frac{1}{\beta!} \sum_{P=1}^{N} m^{(P)} y_i^{(P)}(\tau)\, Y_{B_1}^{(P)} \ldots Y_{B_\beta}^{(P)}, \tag{17.22}$$

then

$$v_{iB_1\ldots B_\beta}(\tau) = \dot{x}_{iB_1\ldots B_\beta}(\tau). \tag{17.23}$$

We observe that $x_{iB_1\ldots B_\beta}(\tau)$ satisfies an equation of the form (4.2), when all the particles receive an additional rigid body motion, for all times τ. In particular it is unaltered when all the particles receive the same additional translation or translational velocity. Regarded as a function of τ and X_i the expression $x_{iB_1\ldots B_\beta}(\tau)$ in (17.22) is a special case of the multipolar displacement defined in (4.1) and (4.2). The form (17.22) is completely symmetric in $B_1, \ldots, B_\beta$.

We now define a continuous function of τ and $X_r + Y_r$, $x_i^*(\tau, X_r + Y_r)$ say, with continuous derivatives up to order $\mu + 1$, such that

$$x_i^*(\tau) = x_i^*(\tau, X_r) \quad \text{and} \quad x_i^{(P)}(\tau) = x_i^*(\tau, X_r + Y_r^{(P)})$$

for all values of P. Then,

$$\begin{aligned} \dot{y}_i^{(P)}(\tau) &= \sum_{\alpha=1}^{\mu} \frac{1}{\alpha!} v^*_{i,A_1\ldots A_\alpha}(\tau, X_r)\, Y_{A_1}^{(P)} \ldots Y_{A_\alpha}^{(P)} + \overline{R}_i, \\ v_i(\tau) &= \dot{x}_i^*(\tau). \end{aligned} \tag{17.24}$$

With the help of (17.24) the rate of work (17.18) becomes

$$W = M F_i(\tau)\, v_i(\tau) + \sum_{\alpha=1}^{\mu} F_{i:A_1\ldots A_\alpha}(\tau)\, v^*_{i,A_1\ldots A_\alpha}(\tau, X_r) \tag{17.25}$$

if we neglect the remainder term, where

$$F_{i:A_1\ldots A_\alpha}(\tau) = \frac{1}{\alpha!} \sum_{P=1}^{N} m^{(P)} F_i^{(P)}(\tau)\, Y_{A_1}^{(P)} \ldots Y_{A_\alpha}^{(P)}. \tag{17.26}$$

The tensors $F_{i:A_1\ldots A_\alpha}(\tau)$ are 2^α-pole body force tensors of the first kind. Equations (17.20) and (17.25) show that the rate of work of 2^α-pole body forces ($\alpha=1,\ldots,\mu$) of the first kind is equal to the rate of work of monopolar force gradients in multipolar velocity fields.

Equations (17.21) for $\beta=1,2,\ldots,\mu$, together with the first set of equations in (17.12), may, for a given value of i, be considered as $\frac{1}{6}(\mu+1)(\mu+2)(\mu+3)$ equations for N velocities $\dot{y}_i^{(P)}(\tau)$ $(P=1,2,\ldots,N)$. If

$$\frac{1}{6}(\mu+1)(\mu+2)(\mu+3)\geqq N \tag{17.27}$$

we can, in general, express $\dot{y}_i^{(P)}(\tau)$ as a linear combination of multipolar velocities $v_{iB_1\ldots B_\beta}(\tau)$ $(\beta=1,2,\ldots,\mu)$. When $\frac{1}{6}(\mu+1)(\mu+2)(\mu+3)>N$ there will be relations between these multipolar velocities. Thus

$$\dot{y}_i^{(P)}(\tau)=\sum_{\alpha=1}^{\mu} Y^{(P)}_{A_1\ldots A_\alpha} v_{iA_1\ldots A_\alpha}(\tau), \tag{17.28}$$

where $Y^{(P)}_{A_1\ldots A_\alpha}$ is completely symmetric in $A_1,\ldots,A_\alpha$ and depends on $Y_B^{(P)}$ and $m^{(P)}$. Some of these coefficients may be taken to be zero when $\frac{1}{6}(\mu+1)(\mu+2)(\mu+3)>N$.

The kinetic energy T of the N masses $m^{(P)}$ is given by

$$\begin{aligned} 2T&=\sum_{P=1}^{N} m^{(P)} v_i^{(P)}(\tau)\, v_i^{(P)}(\tau)\\ &=M\,v_i(\tau)\,v_i(\tau)+\sum_{P=1}^{N} m^{(P)}\dot{y}_i^{(P)}(\tau)\,\dot{y}_i^{(P)}(\tau)\end{aligned} \tag{17.29}$$

and, using (17.28), this becomes

$$2T=M\,v_i(\tau)\,v_i(\tau)+\sum_{\alpha,\beta=1}^{\mu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\, v_{iA_1\ldots A_\alpha}(\tau)\, v_{iB_1\ldots B_\beta}(\tau) \tag{17.30}$$

where

$$\begin{aligned} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}&=\sum_{P=1}^{N} m^{(P)}\, Y^{(P)}_{A_1\ldots A_\alpha}\, Y^{(P)}_{B_1\ldots B_\beta}\\ &=Y_{B_1\ldots B_\beta:A_1\ldots A_\alpha}.\end{aligned}$$

The coefficients in (17.30) are also completely symmetric with respect to the indices $A_1,\ldots,A_\alpha$ and with respect to $B_1,\ldots,B_\beta$. The expression (17.30) for the kinetic energy is a special case of the kinetic energy given by (7.1) and (7.7).

Starting with the expressions (17.15) for the rate of work we may develop similar results using (17.14) and (17.7). For given t, F^* may now be regarded as a function of τ and x_r+y_r. Thus

$$W=M\,F_i(\tau)\,v_i(\tau)+M\sum_{\beta=1}^{\mu} F^*_{i,j_1\ldots j_\beta}(\tau,x_r)\, v_{ij_1\ldots j_\beta}(\tau), \tag{17.31}$$

where

$$M\,v_{ij_1\ldots j_\beta}(\tau)=\frac{1}{\beta!}\sum_{P=1}^{N} m^{(P)}\dot{y}_i^{(P)}(\tau)\, y_{j_1}^{(P)}\ldots y_{j_\beta}^{(P)}, \tag{17.32}$$

and

$$F^*_{i,j_1\ldots j_\beta}(\tau,x_r)=\frac{\partial^\beta F_i^*(\tau,x_r)}{\partial x_{j_1}\ldots\partial x_{j_\beta}}. \tag{17.33}$$

Also, if

$$M x_{i j_1 \dots j_\beta}(\tau) = \frac{1}{\beta!} \sum_{P=1}^{N} m^{(P)} y_i^{(P)}(\tau)\, y_{j_1}^{(P)} \dots y_{j_\beta}^{(P)}, \tag{17.34}$$

then

$$v_{i j_1 \dots j_\beta}(\tau) = \dot{x}_{i j_1 \dots j_\beta}(\tau). \tag{17.35}$$

We see from (17.34) that $x_{i j_1 \dots j_\beta}(\tau)$ satisfies equation (4.6) when all the particles receive an additional rigid body motion for all times τ. In particular, it is unaltered when the particles receive the same additional translation or translational velocity. Regarded as a function of τ, t and x_i, the multipolar displacement $x_{i j_1 \dots j_\beta}(\tau)$ in (17.34) is a special case of a multipolar displacement defined by (4.5) and (4.6).

We now regard the function $x_i^*(\tau)$ as a function of $x_r + y_r$. Then,

$$W = M F_i(\tau)\, v_i(\tau) + \sum_{\alpha=1}^{\mu} F_{i: i_1 \dots i_\alpha} v^*_{i, i_1 \dots i_\alpha}(\tau, x_r) \tag{17.36}$$

apart from a remainder term, where

$$F_{i: i_1 \dots i_\alpha} = \frac{1}{\alpha!} \sum_{P=1}^{N} m^{(P)} F_i^{(P)}(\tau)\, y_{i_1}^{(P)} \dots y_{i_\alpha}^{(P)}. \tag{17.37}$$

Equations (17.31) and (17.36) show that the rate of work of 2^α-body forces $(\alpha = 1, \dots, \mu)$ of the first kind is equal to the rate of work of monopolar force gradients in multipolar velocity fields.

Equations (17.32) for $\beta = 1, 2, \dots, \mu$, together with the first set of equations in (17.12) may, for a given value of i, be considered as $\frac{1}{6}(\mu+1)(\mu+2)(\mu+3)$ equations for N velocities $\dot{y}_i^{(P)}(\tau)$ $(P = 1, \dots, N)$. If condition (17.27) is satisfied we can, in general, solve for $\dot{y}_i^{(P)}(\tau)$ in the form

$$\dot{y}_i^{(P)}(\tau) = \sum_{\alpha=1}^{\mu} y_{i_1 \dots i_\alpha}^{(P)} v_{i i_1 \dots i_\alpha}(\tau), \tag{17.38}$$

where $y_{i_1 \dots i_\alpha}^{(P)}$ is completely symmetric in $i_1, \dots, i_\alpha$ and depends on $y_i^{(P)}$, and $m^{(P)}$, and not on τ. Some of these coefficients may be taken to be zero when $\frac{1}{6}(\mu+1)(\mu+2)(\mu+3) > N$.

The kinetic energy of the N masses can now be expressed as

$$2T = M v_i(\tau)\, v_i(\tau) + \sum_{\alpha, \beta=1}^{\mu} y_{i_1 \dots i_\alpha : j_1 \dots j_\beta} v_{i i_1 \dots i_\alpha}(\tau)\, v_{i j_1 \dots j_\beta}(\tau), \tag{17.39}$$

where

$$y_{i_1 \dots i_\alpha : j_1 \dots j_\beta} = y_{j_1 \dots j_\beta : i_1 \dots i_\alpha} = \sum_{P=1}^{N} m^{(P)} y_{i_1 \dots i_\alpha}^{(P)} y_{j_1 \dots j_\beta}^{(P)} \tag{17.40}$$

and the coefficients in (17.39) are completely symmetric with respect to $i_1, \dots, i_\alpha$ and with respect to $j_1, \dots, j_\beta$. The expression (17.39) for the kinetic energy is a special case of the kinetic energy given in (7.1) and (7.4).

The multipolar displacements defined in (17.22) and (17.34) can be related to each other when we know the relation between the vector $Y_B^{(P)}$ and the vector $y_j^{(P)}$, for each $P=1, 2, \ldots, N$, and such a relation will be independent of the time τ. For example, if

$$y_j^{(P)} = a_{iB} Y_B^{(P)}, \qquad (17.41)$$

then

$$x_{i j_1 \ldots j_\beta}(\tau) = a_{j_1 B_1} \ldots a_{j_\beta B_\beta} x_{i B_1 \ldots B_\beta}(\tau), \qquad (17.42)$$

where a_{iB} depend only on the initial and final positions (at time t) of the particle P and the center of gravity G.

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