

# *Eigenfunction Expansions Associated with the Schroedinger Operators and their Applications to Scattering Theory*

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## Introduction

In the present paper we shall be concerned with eigenfunction expansions associated with the Schroedinger operator  $-\Delta + q(x)$ , where  $\Delta$  denotes the 3-dimensional Laplacian and  $q(x)$  is a real-valued potential function-defined on the whole 3-dimensional Euclidean space  $E = E^3$ , which tends to 0 at infinity. Solving the expansion problem, we go forward to apply the results obtained to clarify some properties of the spectrum of the Schroedinger operator and to show the unitary character of the S-matrix that plays an important part in describing the scattering process in quantum mechanics.

So far as ordinary differential equations are concerned, the eigenfunction expansion theory that originated from a study of WEYL [36] has been developed to a satisfactory extent by the efforts of STONE [31], TITCHMARSH [34], KODAIRA [19], YOSIDA [38], KREIN, LEVITAN and others<sup>1</sup>.

As for partial differential equations, however, it seems that no complete theory, comparable with the one for ordinary differential equations, has been presented. One of the main difficulties appears to consist in the presence of infinite multiplicities of the spectrum, which is not the case with ordinary differential equations.

In 1934 CARLEMAN [6] studied the Schroedinger operator in  $E$  under the assumption that  $q(x)$  is locally square integrable, and obtained the following result: If the unique self-adjoint extension of the operator  $-\Delta + q(x)$  exists, then there exist functions  $\vartheta(x, y; \mu)$ , called *spectral functions*, such that for any  $f(x) \in L_2(E)$  the expansion formula

$$(1) \quad f(x) = \int_{-\infty}^{\infty} d_{\mu} \int_E \vartheta(x, y; \mu) f(y) dy$$

holds, where the formal derivatives with respect to  $\mu$  of  $\vartheta(x, y; \mu)$  satisfy the Schroedinger equation

$$(2) \quad -\Delta \varphi + q(x) \varphi = \mu \varphi$$

as functions of  $x$  and also of  $y$ . The same problem was investigated also by TITCHMARSH, mainly in the 2-dimensional case<sup>2</sup>. POVZNER [28] extended CARLEMAN'S result to every self-adjoint extension of the operator  $-\Delta + q(x)$  and, moreover, using the Radon-Nikodym theorem, proved the existence of the *spectral density*  $\varrho(\mu)$ , by which  $\vartheta(x, y; \mu)$  can be represented as

$$d_{\mu} \vartheta(x, y; \mu) = \psi(x, y; \mu) d\varrho(\mu).$$

Thus (1) reduces to

$$(3) \quad f(x) = \int_{-\infty}^{\infty} d\varrho(\mu) \int_E \psi(x, y; \mu) f(y) dy,$$

where  $\psi(x, y; \mu)$  serve as eigenfunctions, *i.e.* they satisfy (2) as functions of  $x$  and also of  $y$ . But these eigenfunctions are not separated in the form of linear combinations of the products of the eigenfunctions of *one space-variable*, whereas this is the case with the ordinary differential equations. Such a separation of  $\psi(x, y; \mu)$  was effected by MAUTNER [21] and GÅRDING [9] in the form

$$(4) \quad f(x) = \int_{-\infty}^{\infty} \sum_{\nu=1}^{N_{\mu}} \psi_{\nu}(x, \mu) d\varrho(\mu) \int_E \overline{\psi_{\nu}(y, \mu)} f(y) dy,$$

where  $\psi_{\nu}(x, \mu)$  are eigenfunctions ( $N_{\mu} \leq \infty$ ).

Other approaches to eigenfunction expansions connected with more general partial differential operators were made by MAUTNER [21], GÅRDING [9], BROWDER [2, 3, 4], GEL'FAND and KOSTYUCHENKO [10], BEREZANSKI [1], FLEKSER [8] and ITO [13]. Their results can be summed up in the form (4) in the case of the Schroedinger operator in  $E$ .

<sup>1</sup> See TITCHMARSH [34, 35], NAIMARK [27] and the references cited in these books.

<sup>2</sup> See [35]. It is also abundant in references up to about 1956.

We should remark here that these authors have not specified the eigenfunctions needed for expansion. In other words, a prescription is wanting for the conditions under which we should solve (2) in order to obtain the eigenfunctions  $\varphi_\nu(x, \mu)$  in question. Another point to be noted is that in (4) a countable number of eigenfunctions associated with a given spectral point are used, while a continuous family of such functions is also known to exist in simple examples.

We shall consider the case where  $q(x)$  tends to 0 at infinity. More precisely, we assume that  $q(x)$  is Hoelder continuous except for a finite number of singularities, is square integrable and behaves like  $O(|x|^{-2-h})$  ( $h > 0$ ) as  $|x| \rightarrow \infty$ . Many an important problem of potential scattering is included in this case. Now we can find bounded solutions  $\varphi(x, k)$  of the Schroedinger equation (2) for  $\mu > 0$ , where  $k$  denotes a 3-dimensional wave vector such that  $|k|^2 = \mu$ , as unique solutions of the integral equation,

$$(5) \quad \varphi(x, k) = e^{i k \cdot x} - \frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(x) \varphi(y, k) dy,$$

$k \cdot x$  denoting the scalar product of  $k$  and  $x$ .  $\varphi(x, k)$  represents the *distorted plane wave*, i.e. the plane wave plus the outgoing scattered wave. On the other hand, in case of negative  $\mu$ , (2) is solvable only for particular values of  $\mu$ , called *eigenvalues*, which we denote by  $\mu_n$  ( $n=1, 2, \dots$ ). The associated solutions  $\varphi_n(x) \in L_2(E)$  are called *eigenvectors*, to be distinguished from the eigenfunctions  $\varphi(x, k)$ <sup>3</sup>. Here we agree to count  $\mu_n$  as many times as its multiplicity if  $\mu_n$  is degenerate. The  $\varphi_n(x)$  can be regarded as forming an orthonormal system. In terms of the eigenfunctions  $\varphi(x, k)$  and the eigenvectors  $\varphi_n(x)$  our expansion formula<sup>4</sup> reads:

$$(6) \quad f(x) = (2\pi)^{-\frac{3}{2}} \int_M \varphi(x, k) \hat{f}(k) dk + \sum_{n=1}^{\infty} \hat{f}_n \varphi_n(x),$$

where

$$\hat{f}(k) = (2\pi)^{-\frac{3}{2}} \int_E \overline{\varphi(x, k)} f(x) dx, \quad \hat{f}_n = \int_E \overline{\varphi_n(x)} f(x) dx$$

and  $M$  is the 3-dimensional space formed by vectors  $k$ , which is not essentially different from  $E$ . Formula (6) is a natural generalization of the ordinary Fourier expansion, whereas (4) is not. (6) shows that the system of functions  $\varphi(x, k)$  and  $\varphi_n(x)$  is *complete*, but it will also be shown that these functions form an orthonormal set (in a sense to be specified later).

Now we shall turn to the problem concerning the *S-matrix* that is known as a physical quantity which governs a scattering process<sup>5</sup>. The existence of the *S-matrix*, however, is not self-evident from the mathematical point of view. In order to study the *S-matrix*  $S$ , it is convenient to introduce the isometric operators  $W_{\pm}$ , called *wave operators*, by  $W_{\pm} = \lim_{t \rightarrow \pm\infty} e^{i t H} e^{-i t H_0}$ , where  $H$  and  $H_0$  are, in our case,  $-\Delta + q(x)$  and  $-\Delta$  respectively.  $S$  can be defined in terms of  $W_{\pm}$  as  $S = W_{+}^* W_{-}$  and is generally believed to be unitary.

<sup>3</sup> We agree to call  $\varphi(x, k)$  an *eigenfunction* belonging to the *eigennumber*  $|k|$ . Cf. also SHNOL' [30].

<sup>4</sup> For more precise expression see Theorem 5 (§ 8).

<sup>5</sup> See e.g. MÖLLER [26].

A mathematical formulation of the S-matrix theory has been proposed by JAUCH [14] and KURODA [20]. According to their results, the existence and the unitary character of  $S$  will be cleared up if we prove the existence of  $W_{\pm}$  and that  $W_{\pm}$  have *one and the same range*. The existence problem of the wave operators has been investigated by COOK [7], JAUCH & ZINNES [15] and HACK [11]. COOK assumed that  $q(x) \in L_2(E)$ . JAUCH & ZINNES treated a spherically symmetric potential:  $q(x) = |x|^{-c}$  ( $1 < c < \frac{3}{2}$ ). HACK considered the case where  $q(x)$  is locally square integrable and  $q(x) = O(|x|^{-c})$  ( $c > 1$ ) at infinity. No results, however, have been reported by them on the unitary property of  $S$ . Recently KURODA [20] has solved both the problems under the assumption that  $q(x) \in L_1(E) \cap L_2(E)$ . We shall give a proof that  $S$  is unitary under the conditions stated above; these conditions are weaker than KURODA's assumption in one respect. Our proof depends partly on the so-called *time-dependent theory* and partly on the eigenfunction expansion.

We shall outline here the contents of the present paper. In I (§§ 1, 2, 3, 4 and 5) we give some relations between the resolvent of  $H$  and its kernel function, whose conjugate Fourier transform is deeply connected with the eigenfunction  $\varphi(x, k)$ , and introduce the kernel equations and investigate their properties. II (§§ 6–9) deals with the eigenfunction expansion. § 6 is of a preparatory character, where the eigenfunctions  $\varphi(x, k)$ , tools for expansion, will be introduced. In § 7 we shall comment on some properties of the spectrum of the Schroedinger operator and prove that the negative part of the spectrum consists only of discrete eigenvalues. In §§ 8 and 9 we shall state the expansion theorem with a partial proof thereof and show the absolute continuity of the positive part of the spectrum. In III (§§ 10 and 11) we shall show that S-matrix is unitary and at the same time complete the proof of the expansion theorem given in § 8. IV is reserved for remarks on the 2-dimensional and higher-dimensional cases.

## I. Resolvent kernel

**§ 1. Assumptions. The resolvent kernel.** We shall consider the Schroedinger operator  $-\Delta + q(x)$  with the potential function  $q(x)$  defined on  $E = E^3$ , where  $x$  denotes a position vector in  $E$ , its length being  $|x|$ . Throughout the present paper  $q(x)$  is assumed to satisfy the following conditions:

(A)  $q(x)$  is a real-valued function which is locally Hoelder continuous except at a finite number of singularities. Furthermore,  $q(x)$  is square integrable ( $q(x) \in L_2(E)$ ) and behaves like  $O(|x|^{-2-h})$  ( $h > 0$ ) at infinity, i.e. there exist positive numbers  $h$ ,  $C_0$  and  $R_0$  such that

$$|q(x)| \leq C_0 |x|^{-2-h} \quad \text{for } |x| \geq R_0.$$

Let us first define the operator  $A$  by  $Af(x) = -\Delta f(x) + q(x)f(x)$  for  $f \in C_0^\infty(E)$ , where  $C_0^\infty(E)$  consists of all functions which are infinitely differentiable and have compact carriers.  $C_0^\infty(E)$  is contained and dense in the Hilbert space  $L_2(E)$  in the sense of the  $L_2$ -norm  $\|\cdot\|$  ( $\|\cdot\|$  will be used exclusively for the  $L_2$ -norm). Then it is known<sup>6</sup> that, under the conditions (A) imposed on  $q(x)$ ,  $A$  is lower semi-bounded and essentially self-adjoint in  $L_2(E)$ . Moreover, if we denote by

<sup>6</sup> See KATO [16], STUMMEL [32] and WIENHOLTS [37]. Our conditions (A) are more stringent than required for essential self-adjointness.

$H$  the unique self-adjoint extension of  $A$  and by  $H_0$  the corresponding operator for the case  $q(x) \equiv 0$ , then it is known<sup>7</sup> that  $D(H) = D(H_0)$ .

We shall study the resolvent  $R_\lambda = (H - \lambda)^{-1}$  of  $H$  and its kernel  $G(x, y; \lambda)$  for non-real  $\lambda$ ; later this kernel will be extended to real  $\lambda$  outside the point spectrum of  $H$ . The main results are summarized in

**Theorem 1<sup>8</sup>.** Let  $\text{Im} \lambda \neq 0$ <sup>9</sup>. i)  $R_\lambda$  is an integral operator of the Carleman type<sup>10</sup> and its kernel  $G(x, y; \lambda)$ , called the resolvent kernel, satisfies the integral equation

$$(1.1) \quad G(x, y; \lambda) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} - \frac{1}{4\pi} \int_E \frac{e^{i\sqrt{\lambda}|x-z|}}{4\pi|x-z|} q(z) G(z, y; \lambda) dz$$

as a function of  $x$  a.e.<sup>12</sup> in  $E$  for a.e. fixed  $y \in E$ . ii) If  $F(x, y; \lambda)$  is a solution of (1.1) such that  $F(\cdot, y; \lambda) \in L_2(E)$  for each fixed  $y$ , then  $F(x, y; \lambda)$  is the resolvent kernel of  $R_\lambda$ :  $F(x, y; \lambda) = G(x, y; \lambda)$  for a.e.  $x, y$  in  $E \times E$ .

We shall call (1.1) the kernel equation.

Now let  $B$  be the Banach space of all continuous functions  $u(x)$  defined on  $E$ , tending uniformly to 0 as  $|x| \rightarrow \infty$ , with the norm

$$\|u\|_B = \max_{x \in E} |u(x)|.$$

We also put for  $\text{Im} \kappa \geq 0$

$$(1.2) \quad \begin{aligned} A^{(i)}(x, y; \kappa) &= -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-z|}}{|x-z|} q(z) A^{(i-1)}(z, y; \kappa) dz, \\ A^{(0)}(x, y; \kappa) &= \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \quad (i = 1, 2, 3, 4). \end{aligned}$$

Then we have the following

**Theorem 2.**  $A^{(i)}(x, y; \kappa)$  are continuous in  $x$  for  $x \neq y$  and  $A^{(4)}(\cdot, y; \kappa) \in B$  ( $\text{Im} \kappa \geq 0$ ). Further, let  $\text{Im} \kappa > 0$  and  $\text{Im} \kappa^2 \neq 0$ . Then there exists a unique solution  $H^{(4)}(x, y; \kappa)$  in  $B$  of the modified kernel equation

$$(1.3) \quad H^{(4)}(x, y; \kappa) = A^{(4)}(x, y; \kappa) - \frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-z|}}{|x-z|} q(z) H^{(4)}(z, y; \kappa) dz$$

<sup>7</sup> See KATO [16]. We denote by  $D(T)$  the domain of the operator  $T$ .

<sup>8</sup> For the proof of Theorem 1, i) it suffices to assume only that  $q(x) \in L_2(E)$ .

<sup>9</sup>  $\text{Im} \lambda$  means the imaginary part of  $\lambda$ .

<sup>10</sup> An integral operator  $T$  as well as its kernel  $T(x, y)$  is said to be of the Carleman type if  $T(x, y)$  satisfies the conditions  $\int_E |T(x, y)|^2 dy < \infty$  for a.e.  $x$  and  $\int_E |T(x, y)|^2 dx < \infty$  for a.e.  $y$ . See also STONE [31]. POVZNER [28] showed that if  $q(x)$  is continuous on  $E$ , then the resolvent  $R_\lambda$  of every self-adjoint extension of  $A$  is of the Carleman type. This result was extended by BURNAT [5] so as to include the case where  $q(x)$  has a countable number of singularities  $x_n$ , in the neighbourhoods of which

$$q(x) = O(|x - x_n|^{-c_n}) \quad (c_n < \frac{3}{2}).$$

<sup>11</sup> Here and in the sequel by  $\sqrt{\lambda}$  is meant the branch of the square root of  $\lambda$  with  $\text{Im} \sqrt{\lambda} > 0$ .

<sup>12</sup> "a.e." means "almost every" or "almost everywhere".

<sup>13</sup> Our Banach space is somewhat different from the one introduced by POVZNER [28]. He used the Banach space  $B_y$  ( $y$  being fixed) of all functions  $u(x) = a|x-y|^{-1} + v(x)$ , where  $a$  is a constant and  $v(x)$  is continuous and tends uniformly to 0 as  $|x| \rightarrow \infty$ , with the norm  $\|u\|_{B_y} = |a| + \max_{x \in E} |v(x)|$ . Cf. also BURNAT [5].

for each fixed  $y \in E$ . If we put

$$(1.4) \quad H(x, y; \kappa) = \sum_{i=0}^3 A^{(i)}(x, y; \kappa) + H^{(4)}(x, y; \kappa),$$

then  $H(x, y; \kappa) = G(x, y; \kappa^2)$  for a.e.  $x, y$  in  $E \times E$  and, moreover,

$$(1.5) \quad \|H(\cdot, y; \kappa)\| \leq C,$$

where  $C$  is a constant dependent on  $\kappa$  but not on  $y \in E$ .

As a consequence of Theorem 2, we can regard the resolvent kernel  $G(x, y; \lambda) = H(x, y; \sqrt{\lambda})$  as a function of  $x$  defined everywhere in  $E$  except at  $x = y$  for each fixed  $y$ .

**§ 2. Proof of Theorem 1. The kernel equation.** Let us first show that  $R_\lambda$  is an integral operator of the Carleman type. Denoting by  $R_{0\lambda}$  the resolvent of  $H_0$  and by  $V$  the operator of multiplication by  $q(x)$ , we have the following operator equations (the so-called second resolvent equations<sup>14</sup>) in virtue of the relation  $D(H) = D(H_0)$ :

$$(2.1) \quad R_\lambda - R_{0\lambda} = -R_{0\lambda} V R_\lambda \quad \text{and} \quad R_\lambda - R_{0\lambda} = -R_\lambda V R_{0\lambda}.$$

$R_{0\lambda}$  is known to be an integral operator of the Carleman type with the kernel  $(4\pi|x-y|)^{-1} e^{i\sqrt{\lambda}|x-y|}$ . Let us put  $\sqrt{\lambda} = a + ib$  ( $b > 0$ ). Then we have

$$\iint_E \left| q(x) \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \right|^2 dx dy = C \int_E |q(x)|^2 dx \int_E \frac{e^{-2b|x-y|}}{|x-y|^2} dy = C \|q\|^2;$$

here and henceforth we denote by  $C$  any constant, not necessarily the same ( $C$  in the second member of the above equality is a pure constant and  $C$  in the third depends on  $b$ ). This implies that  $V R_{0\lambda}$  is a completely continuous operator of the Hilbert-Schmidt type.  $R_\lambda$  being a bounded operator,  $R_\lambda V R_{0\lambda}$  is also of the Hilbert-Schmidt type and should be an integral operator with a Hilbert-Schmidt kernel<sup>15</sup>, say  $K(x, y; \lambda)$ . Thus we have

$$R_\lambda V R_{0\lambda} f(x) = \int_E K(x, y; \lambda) f(y) dy \quad \text{a.e.},$$

where

$$\iint_{EE} |K(x, y; \lambda)|^2 dx dy < \infty.$$

Since  $R_{0\lambda}$  and  $R_\lambda V R_{0\lambda}$ , *a fortiori* by the above argument, are of the Carleman type, we see from (2.1) that  $R_\lambda$  is of the same type, too, and is representable as

$$(2.2) \quad R_\lambda f(x) = \int_E G(x, y; \lambda) f(y) dy \quad \text{a.e.}$$

by means of the *resolvent kernel*  $G(x, y; \lambda)$ , where

$$\int_E |G(x, y; \lambda)|^2 dy < \infty \quad \text{for a.e. } x, \quad \int_E |G(x, y; \lambda)|^2 dx < \infty \quad \text{for a.e. } y.$$

<sup>14</sup> See HILLE & PHILLIPS [12], p. 126.

<sup>15</sup> See *e.g.* RIESZ & NAGY [29], Chapter VI.

Let us proceed to the deduction of (1.1). For any  $f(x) \in L_2(E)$  we obtain from (2.1) and (2.2) the equation

$$(2.3) \quad \int_E G(x, y; \lambda) f(y) dy = \frac{1}{4\pi} \int_E \frac{e^{i\lambda|x-y|}}{|x-y|} f(y) dy - \\ - \frac{1}{4\pi} \int_E G(x, z; \lambda) q(z) dz \int_E \frac{e^{iV\lambda|z-y|}}{|z-y|} f(y) dy \quad \text{a.e.}$$

By repeated use of SCHWARZ'S inequality we have

$$\int_E |G(x, z; \lambda) q(z)| dz \int_E \left| \frac{e^{i\lambda|z-y|}}{|z-y|} f(y) \right| dy \leq C \|G(x, \cdot; \lambda)\| \|q\| \|f\| < \infty$$

for a.e.  $x$ . We can, therefore, interchange the order of integration in the last term of (2.3) by FUBINI'S theorem. In view of the arbitrariness of  $f \in L_2(E)$ , it follows that

$$G(x, y; \lambda) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} - \frac{1}{4\pi} \int_E G(x, z; \lambda) q(z) \frac{e^{iV\lambda|z-y|}}{|z-y|} dz$$

for a.e.  $y$  with a.e. fixed  $x$ . By use of Lemma 1.1 just below we can replace  $G(x, y; \lambda)$  by  $G(y, x; \lambda)$  and  $G(x, z; \lambda)$  by  $G(z, x; \lambda)$  and, on interchanging  $x$  and  $y$ , we obtain (1.1). Thus we have proved i) of Theorem 1.

**Lemma 2.1.**  $G(x, y; \lambda)$  is symmetric in  $x$  and  $y$ :

$$(2.4) \quad G(x, y; \lambda) = G(y, x; \lambda) \quad \text{for a.e. } x, y \text{ in } E \times E.$$

*Proof.* Let us introduce an indefinite "inner product"  $\langle \cdot, \cdot \rangle$  in  $L_2(E)$  by  $\langle f, g \rangle = \int_E f(x)g(x) = \langle f, \bar{g} \rangle$ . Then what we have to show is that  $R_\lambda$  is symmetric with respect to this inner product, i.e.  $\langle R_\lambda f, g \rangle = \langle f, R_\lambda g \rangle$ . For this purpose it suffices to show that  $\langle Hf, g \rangle = \langle f, Hg \rangle$ . But this is obvious, since we have  $\langle Hf, g \rangle = \langle H_0 f, g \rangle + \langle Vf, g \rangle = \langle H_0 f, \bar{g} \rangle + \langle f, Vg \rangle = \langle f, \overline{H_0 g} \rangle + \langle f, Vg \rangle = \langle f, H_0 g \rangle + \langle f, Vg \rangle = \langle f, Hg \rangle$ , noting that  $H_0$  is a real operator and that  $V$  is an operator of multiplying by  $q(x)$ .

Let us now prove ii) of Theorem 1. To this end consider the difference  $f(x) = F(x, y; \lambda) - G(x, y; \lambda)$  which is by assumption in  $L_2(E)$  (for a.e. fixed  $y$ ) and satisfies the equation

$$(2.5) \quad f(x) = - \frac{1}{4\pi} \int_F \frac{e^{iV\lambda|x-z|}}{|x-z|} q(z) f(z) dz \quad \text{a.e.}$$

The kernel  $(4\pi|x-z|)^{-1} e^{iV\lambda|x-z|} q(z)$  is of the Hilbert-Schmidt type (see above) and hence defines a completely continuous operator  $T$ , which is an extension of  $R_{0\lambda}V$ . Therefore, we can rewrite (2.5) in the form

$$(2.6) \quad f = -Tf \quad \text{or} \quad f = (-T)^n f \quad (n = 1, 2, \dots),$$

where  $T^n$  are also integral operators of the Hilbert-Schmidt type. Consequently we have, noting (1.2),

$$f(x) = - \int_E A^{(3)}(x, z; V\lambda) q(z) f(z) dz.$$

The function  $A^{(3)}(x, z; \sqrt{\lambda})$  is bounded in  $x$  and  $z$ , as will be shown in Lemma 2.2 that follows. This fact together with SCHWARZ'S inequality gives

$$|f(x)| \leq C \int_E |q(z) f(z)| dz \leq C \|q\| \|f\|.$$

Thus  $f(x)$  is bounded and hence  $Vf \in L_2(E)$ , which permits us to rewrite (2.6) as

$$(2.7) \quad f = -R_{0\lambda} Vf \in D(H_0).$$

Application of  $H_0 - \lambda$  to both sides of (2.7) yields

$$(2.8) \quad (H_0 - \lambda)f = -Vf \quad \text{or} \quad Hf = \lambda f.$$

Since  $H$  is self-adjoint and  $\lambda$  is non-real, we have  $f=0$ , which proves ii) of Theorem 1.

**Lemma 2.2.** *If  $\text{Im}\kappa \geq 0$ , then  $A^{(3)}(x, y; \kappa)$  and  $A^{(4)}(x, y; \kappa)$  are bounded in  $x, y$  and  $\kappa$ .*

*Proof.* Let us show first that  $A^{(1)}(x, y; \kappa) = O(|x-y|^\frac{1}{2})$ . By means of SCHWARZ'S inequality we have for  $|x-y| = \delta > 0$

$$|A^{(1)}(x, y; \kappa)|^2 \leq C \int_E \frac{dz}{|x-z|^2 |z-y|^2} \|q\|^2 = C/\delta$$

in consideration of homogeneity of the above integral in  $x-y$ . Thus we have

$$(2.9) \quad |A^{(1)}(x, y; \kappa)| \leq C |x-y|^{-\frac{1}{2}},$$

where  $C$  depends neither on  $x, y$  nor on  $\kappa$ .

Next we show that  $A^{(2)}(x, y; \kappa) = O(|\log|x-y||)$ . We have

$$|A^{(2)}(x, y; \kappa)| \leq C \left( \int_{K(R_0)} \frac{|q(z)|}{|x-z|} |A^{(1)}(z, y; \kappa)| dz + \int_{K(R_0)'} \right) = J + J'.$$

Here and in the sequel  $K(x, R)$  denotes the sphere of radius  $R$  with its centre at  $x$  ( $K(R) = K(0, R)$ ) and  $K(x, R)' = E - K(x, R)$  the exterior of  $K_R$ . (For  $R_0$  see (A) of § 1). We also denote by  $K(R) \cap K(R_1)$  the intersection of  $K(R)$  and  $K(R_1)$ . We shall now estimate  $J'$  as follows: By (2.9) and (A) we get

$$J' \leq C \left( \int_{K(R_0)' \cap K(y, 1)} \frac{dz}{|z|^{2+h} |x-z| |y-z|^\frac{1}{2}} + \int_{K(R_0)' - K(y, 1)} \right) = J'_1 + J'_2.$$

An estimation of  $J'_1$  is

$$J'_1 \leq C \int_{K(y, 1)} \frac{dz}{R_0^{2+h} |x-z| |y-z|^\frac{1}{2}} \leq C \left[ \int_{K(y, 1)} \frac{dz}{|x-z|^2} \int_{K(y, 1)} \frac{dz}{|y-z|} \right]^\frac{1}{2} \leq C.$$



For  $J'_2$  we have, introducing spherical coordinates,

$$\begin{aligned} J'_2 &\leq C \int_{K(R_0)'} \frac{dz}{|z|^{2+h} |x-z|} = C \int_{R_0}^{\infty} \int_0^{\pi} \frac{r^2 \sin \vartheta dr d\vartheta}{r^{2+h} (|x|^2 + r^2 - 2|x|r \cos \vartheta)^{\frac{1}{2}}} \\ &= C |x|^{-1} \int_{R_0}^{\infty} r^{-1-h} (|x| + r - |x| - r) dr. \end{aligned}$$

If  $|x| \leq R_0$  and  $x \neq 0$ , we obtain

$$J'_2 \leq C \int_{R_0}^{\infty} r^{-1-h} dr \leq C;$$

the case  $x=0$  can be dealt with separately, and a similar result follows. If  $|x| > R_0$

$$J'_2 \leq C |x|^{-1} \int_{R_0}^{|x|} r^{-h} dr + C |x|^{-1} \int_{|x|}^{\infty} |x| r^{-1-h} dr \leq C.$$

Thus  $J'_2$  is always bounded; hence so is  $J'$ .

Now we have to estimate  $J$ :

$$\begin{aligned} J^2 &\leq C \int_{K(R_0)} \frac{dz}{|x-z|^2 |y-z|} \int_{K(R_0)} |q(z)|^2 dz \\ &= C \left( \int_{K(R_0) \cap K(x, \delta/2)} \frac{dz}{|x-z|^2 |y-z|} + \int_{K(R_0) - K(x, \delta/2)} \right) = J_1 + J_2 \quad (\delta = |x-y|), \\ J_1 &\leq C \delta^{-1} \int_{K(x, \delta/2)} \frac{dz}{|x-z|^2} \leq C. \end{aligned}$$

If  $\delta \geq \frac{1}{2}$ , we have for  $J_2$

$$J_2 \leq C \int_{K(R_0)} \frac{dz}{|x-z|^2 |y-z|} \leq C \int_{K(x, \frac{1}{2})} \frac{dz}{|x-z|^2} + C \int_{K(y, \frac{1}{2})} \frac{dz}{|y-z|} + C \int_{K(R_0)} dz \leq C.$$

If  $\delta < \frac{1}{2}$ , we first assume that  $|x| \geq R_0 + 1$ . Then

$$J_2 \leq C \int_{K(R_0)} \frac{1}{|y-z|} dz \leq C.$$

For  $|x| < R_0 + 1$ , on translating the origin, we have

$$J_2 \leq C \int_{K(2R_0+1) - K(\delta/2)} \frac{dz}{|z|^2 |x-y-z|} = C \delta^{-1} \int_{\delta/2}^{R_0+1} r^{-1} (r + \delta - |r - \delta|) dr \leq C |\log \delta|.$$

Thus we get

$$(2.10) \quad \begin{aligned} |A^2(x, y; \kappa)| &\leq C \quad \text{if } |x-y| \geq \frac{1}{2} \\ &\leq C |\log |x-y|| \quad \text{if } |x-y| < \frac{1}{2} \end{aligned}$$

where  $C$  is independent of  $x, y$  and  $\kappa$ .

Now we shall show that  $A^{(3)}(x, y; \kappa)$  is bounded. We have from (2.10)

$$|A^{(3)}(x, y; \kappa)| \leq C \int_{K(y, \frac{1}{2})} \frac{|q(z)|}{|x-z|} |\log |z-y|| dz + C \int_{K(y, \frac{1}{2})'} \frac{|q(z)|}{|x-z|} dz = L + L'.$$

$L$  and  $L'$  can be estimated as follows:

$$\begin{aligned} L &\leq C \left[ \int_{K(y, \frac{1}{2})} \frac{|\log |z-y||^2}{|x-z|^2} dz \int_{K(y, \frac{1}{2})} |q(z)|^2 dz \right]^{\frac{1}{2}} \leq C, \\ L' &\leq C \int_E \frac{|q(z)|}{|x-z|} dz = C \left( \int_{K(R_0)} \frac{|q(z)|}{|x-z|} dz + \int_E \right) = L'_1 + L'_2, \\ L'_1 &\leq C \left[ \int_{K(R_0)} \frac{dz}{|x-z|^2} \int_{K(R_0)} |q(z)|^2 dz \right]^{\frac{1}{2}} \leq C, \\ L'_2 &\leq C \int_{K(R_0)'} \frac{dz}{|z|^{2+h} |x-z|} \leq C. \quad (\text{See } J'_2 \text{ above.}) \end{aligned}$$

These estimates show that

$$(2.11) \quad |A^{(3)}(x, y; \kappa)| \leq C,$$

where  $C$  is independent of  $x, y$  and  $\kappa$ .

From (2.11) and the estimation performed for  $L'$  we get

$$(2.12) \quad |A^{(4)}(x, y; \kappa)| \leq C,$$

$C$  being independent of  $x, y$  and  $\kappa$ . (2.11) and (2.12) prove the assertion of Lemma 2.2.

The following lemma will be needed for the proof of Theorem 2 that we shall give in § 5.

**Lemma 2.3.** *If  $\text{Im } \kappa > 0$ , then  $\|A^{(i)}(\cdot, y; \kappa)\| \leq C$  ( $i=0, 1, 2, 3, 4$ ), where  $C$  depends on  $\kappa$  but not on  $y$ .*

*Proof.* It is easily seen that  $\|A^{(0)}(\cdot, y; \kappa)\| \leq C$ , where  $C$  does not depend on  $y$ . We assume that this is the case for  $A^{(i)}(\cdot, y; \kappa)$  and prove that the assertion is true for  $A^{(i+1)}(\cdot, y; \kappa)$ .

Using the operator  $T$  defined before (see (2.8) and (2.9)), we have  $A^{(i+1)}(\cdot, y; \kappa) = T A^{(i)}(\cdot, y; \kappa)$ , whence follows

$$(2.13) \quad \|A^{(i+1)}(\cdot, y; \kappa)\| \leq C \|A^{(i)}(\cdot, y; \kappa)\| \leq C$$

in view of the boundedness of  $T$ .

**§ 3. Asymptotic behavior of the function**  $-\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} v(y) dy$ . In

order to solve the kernel and modified kernel equations, we shall study some asymptotic properties of functions of the form

$$q(x) = -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} v(y) dy \quad (\text{Im } \kappa \geq 0).$$

**Lemma 3.1.** *Let  $\text{Im } \kappa \geq 0$ , let  $v(x)$  be locally integrable and  $v(x) = O(|x|^{-2-\epsilon})$  ( $|x| \rightarrow \infty$ ) ( $\epsilon > 0$ ). Then as  $|x| \rightarrow \infty$*

$$(3.1) \quad \varphi(x) = O(|x|^{-1}) + O(|x|^{-\epsilon}).$$

*Proof.* Let  $R$  be such that  $|v(x)| \leq C|x|^{-2-\epsilon}$  for  $|x| \geq R$ . If  $|x| > R$  we have

$$\begin{aligned} \varphi(x) &= -\frac{1}{4\pi} \int_{K(R)} \frac{e^{i\kappa|x-y|}}{|x-y|} v(y) \omega_x \cdot y - \int_{K(R)'} = I + I', \\ |I'| &\leq C \int_{K(R)'} \frac{|v(y)|}{|x-y|} dy \leq C \int_R^\infty \int_0^\pi \frac{r^2 \sin \vartheta dr d\vartheta}{r^{2+\epsilon} (|x|^2 + r^2 - 2|x|r \cos \vartheta)^{\frac{1}{2}}} \\ &= C|x|^{-1} \int_R^\infty r^{-1-\epsilon} (|x| + r - ||x| - r|) dr = C|x|^{-1} + C|x|^{-\epsilon}. \end{aligned}$$

In order to give an estimate for  $I$ , it suffices to note that  $v(x)$  is integrable over  $K(R)$  and that  $|x-y| \geq |x|-R$ . Then we have  $I = O(|x|^{-1})$ . By putting  $I$  and  $I'$  together, (3.1) follows.

**Lemma 3.2**<sup>16</sup>. *Let  $v(x) = O(|x|^{-3-\epsilon})$  ( $|x| \rightarrow \infty$ ) ( $\epsilon > 0$ ),  $v(x) \in L_2(E)$  and let  $\kappa = a$  be real. Then we have*

$$(3.2) \quad \varphi(x) = -\frac{e^{ia|x|}}{4\pi|x|} \int_E e^{-ia\omega_x \cdot y} v(y) dy + O(|x|^{-1-\epsilon/2}) + O(|x|^{-2}),$$

where  $\omega_x$  denotes the unit vector with the direction of  $x$  and  $\omega_x \cdot y$  denotes the scalar product of  $\omega_x$  and  $y$ .

*Proof.* Let  $R_1$  be fixed so that  $|v(x)| \leq C|x|^{-3-\epsilon}$  for  $|x| \geq R_1$  and let  $|x|$  be so large that  $|x|^{\frac{1}{2}} = R > R_1$ . Then we have

$$(3.3) \quad \varphi(x) = -\frac{1}{4\pi} \left( \int_{K(R)} \frac{e^{ia|x-y|}}{|x-y|} v(y) dy + \int_{K(R)'} \right) = I + I'.$$

An argument similar to the one used in proving Lemma 3.1 gives

$$|I'| \leq C|x|^{-1}R^{-\epsilon} + C|x|^{-1-\epsilon} = C|x|^{-1-\epsilon/2} + C|x|^{-1-\epsilon}.$$

We proceed to estimate  $I$ . Considering  $|y| < R$  we have

$$e^{ia|x-y|} = e^{ia|x|} \left[ 1 - \left( \omega_x \cdot \frac{y}{|x|} \right) + \eta_1 \right] \quad (\eta_1 = O(|y|/|x|^2)),$$

$$|x-y|^{-1} = |x|^{-1} \left[ 1 + \left( \omega_x \cdot \frac{y}{|x|} \right) + \eta_2 \right] \quad (\eta_2 = O(|y|/|x|^2)),$$

and hence

$$\frac{e^{ia|x-y|}}{|x-y|} = \frac{1}{|x|} e^{ia|x|} \left[ 1 - ia\omega_x \cdot y + ia|y| \eta_1 + \frac{\eta_2}{|x|} e^{ia|x|} \cdot y + \frac{1}{|x|^2} \omega_x \cdot y e^{ia|x-y|} \right].$$

<sup>16</sup> Cf. POVZNER [28]. He obtained the result in which the last two terms in (3.2) are replaced by  $O(|x|^{-1-\frac{\epsilon}{2+\epsilon}})$ . Our result enables us to proceed along his line without assuming that the potential has the asymptotic form  $q(x) = O(|x|^{-3.5-\epsilon})$ , which was assumed in [28] together with the continuous differentiability of  $q(x)$ .

Thus we have

$$\begin{aligned}
I &= -\frac{e^{ia|x|}}{4\pi|x|} \int_E e^{-ia\omega_x \cdot y} v(y) dy + \frac{e^{ia|x|}}{4\pi|x|} \int_{K(R)'} e^{-ia\omega_x \cdot y} v(y) dy + \\
&\quad + \frac{e^{ia|x|}}{4\pi|x|} \int_{K(R)} e^{-ia\omega_x \cdot y} (1 - e^{ia|x|\eta_1}) v(y) dy - \\
&\quad - \frac{1}{4\pi|x|} \int_{K(R)} \eta_2 e^{ia|x-y|} v(y) dy - \frac{1}{4\pi|x|^2} \int_{K(R)} \omega_x \cdot y e^{ia|x-y|} v(y) dy \\
&= -\frac{e^{ia|x|}}{4\pi|x|} \int_E e^{-ia\omega_x \cdot y} v(y) dy + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

$I_i$  ( $i=1, 2, 3, 4$ ) are estimated as follows:

$$\begin{aligned}
|I_1| &\leq C|x|^{-1} \int_{K(R)'} |v(y)| dy \leq C|x|^{-1} R^{-\varepsilon} = C|x|^{-1-\varepsilon/2}, \\
|I_2| &\leq C|x|^{-1} \int_{K(R)} \frac{|y|^2}{|x|^2} |x| |v(y)| dy = C|x|^{-2} \left( \int_{K(R_1)} |y|^2 |v(y)| dy + \int_{K(R)-K(R_1)} \right) \\
&\leq C|x|^{-2} + C|x|^{-2} R^{2-\varepsilon} = C|x|^{-2} + C|x|^{-1-\varepsilon/2}, \\
|I_3| &\leq C|x|^{-1} \int_{K(R)} \frac{|y|^2}{|x|^2} |v(y)| dy = C|x|^{-3} \left( \int_{K(R_1)} |y|^2 |v(y)| dy + \int_{K(R)-K(R_1)} \right) \\
&\leq C|x|^{-3} + C|x|^{-3} R^{2-\varepsilon} = C|x|^{-3} + C|x|^{-2-\varepsilon/2}, \\
|I_4| &\leq C|x|^{-2} \int_{K(R)} |y| |v(y)| dy = C|x|^{-2} \left( \int_{K(R_1)} + \int_{K(R)-K(R_1)} \right) \\
&\leq C|x|^{-2} + C|x|^{-2} R^{1-\varepsilon} = C|x|^{-2} + C|x|^{-(3+\varepsilon)/2}.
\end{aligned}$$

In these estimates the constants  $C$  may or may not depend on  $a, \varepsilon$  and  $v(\cdot)$ . (3.2) now follows from (3.3) and the estimates for  $I$  and  $I'$ .

**Lemma 3.3.** *Let  $v(x)$  and  $\varphi(x)$  be as in Lemma 3.2. Then  $\varphi(x)$  satisfies the radiation condition<sup>17</sup>, i.e.*

$$(3.4) \quad \lim_{|x| \rightarrow \infty} |x| \left[ \frac{\partial \varphi}{\partial |x|} - ia\varphi \right] = 0.$$

For the proof see POVZNER [28], Chapter II, Lemma 2.

**Lemma 3.4.** *Let  $\varphi(x) \in B$  be a solution of the integral equation*

$$(3.5) \quad \varphi(x) = -\frac{1}{4\pi} \int_E \frac{e^{ia|x-y|}}{|x-y|} q(y) \varphi(y) dy,$$

where  $a$  is real. Then

$$(3.6) \quad \int_E e^{-ia\omega \cdot x} q(x) \varphi(x) dx = 0,$$

where  $\omega$  is an arbitrary unit vector.

<sup>17</sup> There are definitions of the radiation condition other than the one adopted here. See e.g. MIRANKER [22] and MUELLER [24]. Here we follow POVZNER [28].

For the proof see POVZNER [28], Chapter II, Lemma 5. But it seems necessary to add some remarks. The proof relies on Lemmas 3.2 and 3.3 and on GREEN'S formula. We maintain that if  $\varphi(x) \in B$  satisfies (3.5), then  $\varphi(x) = O(|x|^{-1})$  ( $|x| \rightarrow \infty$ ), so that Lemmas 3.2 and 3.3 may be applied. In fact  $\varphi(x) = O(|x|^{-1}) + O(|x|^{-h})$  by Lemma 3.1 because  $q(x) = O(|x|^{-2-h})$ . Hence repeated application of Lemma 3.1 to (3.5) furnishes us with the result  $\varphi(x) = O(|x|^{-1})$ .

Another remark applies to the use of GREEN'S formula. An explicit use of the Hoelder continuity of  $\dot{q}(x)$  and  $\varphi(x)$  is made to obtain from (3.5) the differential equation

$$(3.7) \quad -\Delta \varphi(x) + q(x) \varphi(x) = a^2 \varphi(x),$$

which holds for  $x$  different from the singularities of  $q(x)$  (Hoelder continuity of  $\varphi(x)$  follows from (3.5) and the proof of Lemma 4.1 given below). We apply GREEN'S formula to  $\varphi$  and  $\bar{\varphi}$ , which also satisfies (3.7), and get

$$(3.8) \quad 0 = \int_{K(R)} (\Delta \varphi \cdot \bar{\varphi} - \varphi \cdot \Delta \bar{\varphi}) dx = \int_{S(R)} \left( \bar{\varphi} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \bar{\varphi}}{\partial n} \right) ds$$

where  $S(R)$  is the surface of  $K(R)$  and  $n$  is the outer normal. (3.8) is valid if  $K(R)$  contains no singularity of  $q(x)$ , but even if singularities of  $q(x)$  exist in  $K(R)$ , it is seen to be true. For we first exclude from  $K(R)$  the singularities which are finite in number, and then by a limiting process we obtain (3.8) valid for  $K(R)$ , when we note that  $\partial \varphi / \partial n$  as well as  $\partial \bar{\varphi} / \partial n$  is  $O(|x - x_n|^{-1/2})$  ( $x_n$ : singularities of  $q(x)$ ), as is seen from differentiation of (3.5) under the integral sign, so that the surface integrals taken around the singularities are arbitrarily small.

The above remarks are enough to get to (3.6), following the proof by POVZNER [28].

**§ 4. Solution of the kernel equation.** Let us define an operator  $T_\kappa$  for functions in  $B$  by

$$(4.1) \quad T_\kappa f(x) = -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) f(y) dy \quad (\text{Im } \kappa \geq 0, f \in B).$$

**Lemma 4.1.**  $T_\kappa$  is a bounded linear operator on  $B$  to  $B$  ( $\text{Im } \kappa \geq 0$ ).

*Proof.* Let  $f \in B$ . Then  $q(x)f(x) = O(|x|^{-2-h})$  ( $|x| \rightarrow \infty$ ), so Lemma 3.1 shows that  $T_\kappa f(x) \rightarrow 0$  uniformly as  $|x| \rightarrow \infty$ .

$T_\kappa f(x)$  is bounded in  $x$ . In fact we obtain from (4.1)

$$|T_\kappa f(x)| \leq C \|f\|_B \int_E \frac{|q(y)|}{|x-y|} dy.$$

The boundedness in  $x$  of the right member follows from the estimation carried out for  $L'$  in the proof of Lemma 2.2.

Next we have to show the continuity in  $x$  of  $T_\kappa f(x)$ , and for this purpose we consider the difference

$$(4.2) \quad \begin{aligned} T_\kappa f(x) - T_\kappa f(x') &= -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|} - e^{i\kappa|x'-y|}}{|x-y|} q(y) f(y) dy - \\ &- \frac{1}{4\pi} \int_E \left[ \frac{1}{|x-y|} - \frac{1}{|x'-y|} \right] e^{i\kappa|x'-y|} q(y) f(y) dy = J_1 + J_2. \end{aligned}$$

Considering the inequalities  $|e^{i\kappa|x-y|} - e^{i\kappa|x'-y|}| \leq |\kappa||x-x'|$  and

$$\left| \frac{1}{|x-y|} - \frac{1}{|x'-y|} \right| \leq \frac{|x-x'|}{|x-y||x'-y|},$$

we can give estimates for  $J_1$  and  $J_2$ :

$$|J_1| \leq C|x-x'| \int_E \frac{|q(y)f(y)|}{|x-y|} dy \leq C\|f\|_B|x-x'|,$$

$$|J_2| \leq C|x-x'| \|f\|_B \int_E \frac{|q(y)|}{|x-y||x'-y|} dy \leq C\|f\|_B|x-x'|^{\frac{1}{2}},$$

as is seen from the estimations worked out for  $L'$  and  $A^{(1)}(x, y; \kappa)$  in the proof of Lemma 2.2. Hence it follows that

$$(4.3) \quad T_\kappa f(x) - T_\kappa f(x') = O(|x-x'|^{\frac{1}{2}}) \quad \text{for } |x-x'| \rightarrow 0.$$

(4.3) shows that  $T_\kappa f(x)$  is not merely continuous, but also Hölder continuous in  $x$ . Consequently  $T_\kappa f$  turns out to be an element of  $B$ . This completes the proof of the lemma.

*Remark.* The proof given above uses only the boundedness of  $f(x)$ . Thus  $T_\kappa f$  belongs to  $B$  and is even Hölder continuous if  $f$  is only bounded.

**Lemma 4.2.**  $T_\kappa$  is completely continuous<sup>18</sup>.

*Proof.* Let  $f_n$  ( $n=1, 2, \dots$ ) be any sequence in  $B$  such that  $\|f_n\|_B \leq 1$ . We have to show that we can take out of the sequence  $u_n = T_\kappa f_n$  a subsequence convergent in the norm of  $B$ .

Re-examining the proof of Lemma 4.1, we find out firstly that  $\|u_n\|_B$  are uniformly bounded with respect to  $n$ , because  $\|f_n\|_B \leq 1$ , and secondly that on the same account the continuity of  $u_n(x)$  is uniform with respect to  $n$ . Moreover, we see that  $u_n(x) \rightarrow 0$  uniformly in  $n$  as  $|x| \rightarrow \infty$ . Thus  $u_n(x)$  are equi-bounded and uniformly equi-continuous with respect to  $n$  in any compact domain of  $E$ . On applying the Ascoli-Arzelà theorem to  $u_n(x)$ , we can choose a subsequence  $u_{n'}(x)$  converging uniformly to a continuous function  $u(x)$  in any compact domain. Since  $u_{n'}(x) \rightarrow 0$  uniformly in  $n'$  as  $|x| \rightarrow \infty$ ,  $u(x)$  also tends to 0 at infinity, and hence we have  $\|u_{n'} - u\|_B \rightarrow 0$ , which was to be proved.

We are now in a position to make use of the Riesz-Schauder theory of completely continuous operators in a Banach space<sup>19</sup>. If  $T$  is a completely continuous operator in  $B$ , the equation  $f = g + Tf$  is solvable for any given  $g \in B$ , if and only if  $f = Tf$  implies that  $f = 0$ . Thus we have the following

**Lemma 4.3.** Let  $g \in B$ . Then the integral equation

$$(4.4) \quad f(x) = g(x) - \frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) f(y) dy \quad (\text{Im } \kappa \geq 0)$$

<sup>18</sup> Cf. also POVZNER [28], Chapter II, Lemma 3.

<sup>19</sup> See e.g. RIESZ & NAGY [29], Chapter VI or Yosida [39], § 41.

has a unique solution in  $B$  if and only if the homogeneous equation

$$(4.5) \quad f(x) = -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|^2} q(y) f(y) dy$$

has the unique solution  $f(x) \equiv 0$ .

**Lemma 4.4.** *If  $\kappa^2$  is real and positive, then (4.5) has the unique solution  $f(x) \equiv 0$ .*

*Proof.* If  $f(x)$  satisfies (4.5), (3.7) follows from the assumed Hoelder continuity of  $q(x)$  and that of  $f(x)$ , remarked in the proof of Lemma 4.1 (see also the remarks to Lemma 3.4). On the other hand, Lemmas 3.2 and 3.4 give the asymptotic order

$$(4.7) \quad f(x) = o(|x|^{-1}) \quad (|x| \rightarrow \infty).$$

According to KATO [18], however, any solution of (3.7) subject to the condition (4.7) vanishes indentially at least outside a sufficiently large sphere containing all the singularities of  $q(x)$ . Hence the unique continuation theorem for solution of an elliptic differential equation<sup>20</sup> is applied to obtain the result that  $f(x) = 0$  everywhere.

**Lemma 4.5.** *Let  $\text{Im}\kappa > 0$ . Then (4.5) has non-trivial solutions in  $B$  if and only if  $\kappa^2$  is an eigenvalue of  $H$ . In particular, if  $\text{Im}\kappa^2 \neq 0$  in addition, (4.5) has only the trivial solution  $f(x) \equiv 0$ .*

*Proof.* Let  $f(x) \in B$  satisfy (4.5). Since  $q(x) \in L_2(E)$ ,  $q(x)f(x) \in L_2(E)$ . Then from (4.5) we have  $f = -R_{0,\kappa} V f$ , which is nothing but (2.8) with  $\lambda = \kappa^2$ , and hence we get  $f = 0$  if  $\kappa^2$  is non-real, in exactly the same way as in § 2. This proves the second assertion.

Let  $f(x)$  be a non-trivial solution of (4.5) ( $\text{Im}\kappa > 0$ ). Then, as we have seen before (the proof of Theorem 1, ii) in § 2), it follows that  $f, V f \in L_2(E)$  and  $H f = \kappa^2 f$ . This proves the necessity of the first assertion.

Conversely let  $\kappa^2$  be an eigenvalue of  $H$  with an eigenvector  $f \in D(H) = D(H_0)$ . Then  $V f \in L_2(E)$  and (4.5) readily follows. Accordingly, by use of SCHWARZ'S inequality, we have

$$|f(x)| \leq C \left( \int_E \frac{e^{-2b|x-y|}}{|x-y|^2} dy \int_E |q(y) f(y)|^2 dy \right)^{\frac{1}{2}} \leq C \quad (b = \text{Im}\kappa),$$

from which and (4.5) we see that  $f \in B$ , as in the proof of Lemma 4.1. This proves the sufficiency.

**Lemma 4.6.**  *$T_\kappa(\text{Im}\kappa \geq 0)$  depends continuously on  $\kappa$ ; i.e. given any  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon)$  such that  $\|T_{\kappa_1} - T_{\kappa_2}\|_B < \varepsilon$ <sup>21</sup> if  $|\kappa_1 - \kappa_2| < \delta$ .*

For the proof see POVZNER [28], Chapter II, Lemma 8. We have adopted a norm different from the one used by POVZNER (see footnote 13), but this makes the proof of the present lemma rather simpler, for we have avoided inclusion of singular functions in  $B$ .

<sup>20</sup> See e.g. MUELLER [25].

<sup>21</sup> The norm of an operator  $T$  in  $B$  is defined by  $\|T\|_B = \sup_{\|f\|_B=1} \|Tf\|_B$ .

All the bounded operators on  $B$  form a Banach algebra  $\mathbf{B}$ , in which  $\|T_n - T\|_B \rightarrow 0$  ( $n \rightarrow \infty$ ) and the existence of  $T_n^{-1}$  and  $T^{-1} \in \mathbf{B}$  imply  $\|T_n^{-1} - T^{-1}\|_B \rightarrow 0$ <sup>22</sup>. Thus from Lemma 4.6 follows

**Lemma 4.7.** *If  $\kappa^2$  is not a non-positive eigenvalue of  $H$ ,  $(I - T_\kappa)^{-1} \in B$  exists and depends continuously on  $\kappa$  in the sense of the norm  $\|\cdot\|_B$ .*

**Lemma 4.8.** *Let  $D$  be a compact domain of the upper  $\kappa$ -plane ( $\text{Im}\kappa \geq 0$ ) such that  $D$  does not contain the square roots of the non-positive eigenvalues. Let  $f_\kappa \in B$  be a unique solution of*

$$(4.9) \quad f_\kappa = g_\kappa + T_\kappa f_\kappa,$$

where  $g_\kappa \in B$  is strongly continuous in  $\kappa \in D$ . Then  $f \in B$  is strongly continuous in  $\kappa \in D$ .

*Proof.* Since  $f_\kappa = (I - T_\kappa)^{-1}g_\kappa$  and  $(I - T_\kappa)^{-1}$  is uniformly continuous in  $\kappa \in D$  (Lemma 4.7) and also  $g_\kappa$  is by assumption strongly continuous in  $\kappa$ , the assertion follows immediately.

**§ 5. Proof of Theorem 2.** We shall show first that  $A^{(i)}(x, y; \kappa)$  are continuous in  $x$  unless  $x = y$ . By definition we have for  $i = 0, 1, 2, 3$

$$\begin{aligned} & A^{(i+1)}(x, y; \kappa) - A^{(i+1)}(x', y; \kappa) \\ &= -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-z|} - e^{i\kappa|x'-z|}}{|x-z|} q(z) A^{(i)}(z, y; \kappa) dz - \\ & \quad - \frac{1}{4\pi} \int_E \left[ \frac{1}{|x-z|} - \frac{1}{|x'-z|} \right] e^{i\kappa|x'-z|} q(z) A^{(i)}(z, y; \kappa) dz \\ &= J_1 + J_2. \end{aligned}$$

Proceeding as in the proof of Lemma 4.4, we arrive at the following inequalities for  $J_1$  and  $J_2$ :

$$\begin{aligned} |J_1| &\leq C|x - x'| \int_E \frac{|q(z) A^{(i)}(z, y; \kappa)|}{|x-z|} dz, \\ |J_2| &\leq C|x - x'| \int_E \frac{|q(z) A^{(i)}(z, y; \kappa)|}{|x-z||x'-z|} dz. \end{aligned}$$

Since  $A^{(i)}(z, y; \kappa)$  are bounded in  $z$  outside some sphere with its centre at  $y$ , as is seen in the proof of Lemma 2.2, we get

$$\begin{aligned} |J_1| &\leq C|x - x'| \int_{K(x, \delta)} \frac{|q(z)|}{|x-z|} dz + C|x - x'| \int_{K(x, \delta)'} |q(z) A^{(i)}(z, y; \kappa)| dz, \\ |J_2| &\leq C|x - x'| \int_{K(x, \delta)} \frac{|q(z)|}{|x-z||x'-z|} dz + C|x - x'| \int_{K(x', \eta)} \frac{|q(z)|}{|x-z||x'-z|} dz + \\ & \quad + C|x - x'| \int_{K(x, \delta)' \cap K(x', \eta)'} |q(z) A^{(i)}(z, y; \kappa)| dz, \end{aligned}$$

<sup>22</sup> See any book on Banach algebras, e.g. HILLE & PHILLIPS [12] and Yosida [39].



where we have chosen  $\delta$  and  $\eta$  so that  $K(x, \delta)$  and  $K(x', \eta)$  do not contain the point  $y$ . Then from the estimations in the proofs of Lemmas 2.3 and 4.1 we obtain by SCHWARZ'S inequality

$$\begin{aligned} |J_1| &\leq C|x - x'| + C|x - x'| \|q\| \|A^{(i)}(\cdot, y; \kappa)\|, \\ |J_2| &\leq C|x - x'|^{\frac{1}{2}} + C|x - x'|^{\frac{1}{2}} + C|x - x'| \|q\| \|A^{(i)}(\cdot, y; \kappa)\|, \end{aligned}$$

which prove the asserted continuity of  $A^{(i)}(x, y; \kappa)$  ( $i=1, 2, 3, 4$ ), while  $A^{(0)}(x, y; \kappa)$  is obviously continuous in  $x$  for  $x \neq y$ .

That  $A^{(4)}(\cdot, y; \kappa) \in B$  follows from (2.11) and the remark after Lemma 4.1.

The above facts together with Lemmas 4.3 and 4.5 show that a unique solution  $H^{(4)}(x, y; \kappa)$  of the modified kernel equation exists for fixed  $y$ . If we prove that  $\|H^{(4)}(\cdot, y; \kappa)\| \leq C$ , where  $C$  depends only on  $\kappa$ , we can complete the proof of the theorem in view of Theorem 1, ii).

From (1.4) we see that  $H^{(4)}(x, y; \kappa)$  is the sum of  $A^{(4)}(x, y; \kappa)$  and  $f(x) = -R_{0,x} V H^{(4)}(x, y; \kappa)$  for fixed  $y$ . It is easily seen that  $\|f\| \leq \|R_{0,x}\| \|q\| \|H^{(4)}(\cdot, y; \kappa)\|_B$  is independent of  $y$ , for we have

$$\|H^{(4)}(\cdot, y; \kappa)\|_B \leq \|(I - T_\kappa)^{-1}\|_B \|A^{(4)}(\cdot, y; \kappa)\|_B.$$

Now in virtue of (2.12) we have proved the desired result.

## II. Eigenfunction expansions

**§ 6. Existence of the eigenfunctions.** In quantum-mechanical problems of potential scattering, the *distorted plane wave* plays an important role. It is generally assumed to be a bounded solution of the Schroedinger equation

$$(6.1) \quad -\Delta \varphi(x) + q(x) \varphi(x) = |k|^2 \varphi(x)$$

with the asymptotic conditions  $v(x) = \varphi(x) - e^{i k \cdot x} = O(|x|^{-1})$  and  $|x| (\partial v(x) / \partial |x| - i |k| v(x)) \rightarrow 0$  (radiation condition) for  $|x| \rightarrow \infty$ . Here  $k$  is a 3-dimensional vector called the *wave vector* and  $|k|$  denotes its length. We shall denote by  $M$  the totality of the wave vectors  $k$ , which is, of course, isomorphic to  $E$ .  $\varphi(x)$  has the form (plane wave) + (outgoing wave). It is also possible to consider the distorted plane wave which has the form (plane wave) + (incoming wave) in quite a similar way. As is described in most of the physical literature<sup>23</sup>, the distorted plane wave  $\varphi(x)$  is obtained as a solution of the integral equation

$$(6.2) \quad \varphi(x, k) = e^{i k \cdot x} - \frac{1}{4\pi} \int_E \frac{e^{i |k| |x-y|}}{|x-y|} q(y) \varphi(y, k) dy,$$

which is more convenient for mathematically rigorous treatment than (6.1), for (6.1) is not valid at singularities of  $q(x)$ . Let us introduce the function

$$(6.3) \quad \phi(x, k) = -\frac{1}{4\pi} \int_E \frac{e^{i |k| |x-y|}}{|x-y|} q(y) e^{i k \cdot y} dy.$$

<sup>23</sup> See e.g. MOTT & MASSEY [23].

Then, if we put  $v(x, k) = \varphi(x, k) - e^{ik \cdot x}$ , (6.2) becomes

$$(6.4) \quad v(x, k) = p(x, k) - \frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(y) v(y, k) dy.$$

Since  $p(\cdot, k) \in B$  as is easily seen from the remark after Lemma 4.1, (6.4) has a unique solution  $v(\cdot, k) \in B$  for each fixed  $k$  ( $|k| > 0$ ) in virtue of Lemmas 4.3 and 4.4. Moreover, we see from Lemma 4.1 and the remark thereafter that  $v(x, k)$  as well as  $p(x, k)$  is Hoelder continuous in  $x$ . Since  $q(x)$  is Hoelder continuous,  $\varphi(x, k)$  is then seen to satisfy (6.1). Thus we arrive at the following

**Theorem 3.** *There exists a unique solution  $v(x, k) \in B$  of (6.4) for  $|k| > 0$  and  $\varphi(x, k) = e^{ik \cdot x} + v(x, k)$  is a solution of (6.1) and (6.2). Moreover,  $\varphi(x, k)$  as well as  $v(x, k)$  is bounded and uniformly continuous in  $x$  and  $k$  for  $x \in E$  and  $k \in D$ , where  $D$  is any compact domain of  $M$  not containing the origin.*

$\varphi(x, k)$  is called an *eigenfunction* associated with the *eigennumber*  $|k|^2$ , and we can speak of an  $\infty^3$ -continuous family of eigenfunctions in conformity with various magnitudes and directions of the wave vectors  $k$ . However, we should remark here that under our assumptions (A) on  $q(x)$ ,  $v(x, k)$  does not always satisfy the asymptotic conditions mentioned above <sup>24</sup>.

*Proof of Theorem 3.* We have only to prove the last statement. From the remark after Lemma 4.1 it follows that  $|p(x, k)| \leq C$ , where  $C$  is independent of  $x$  and  $k$ . Since (6.4) can be written as  $v(\cdot, k) = (I - T_{|k|})^{-1} p(\cdot, k)$ , we see from Lemma 4.7 that  $v(x, k)$  is bounded in  $x \in E$  and  $k \in D$ . Similarly it follows from Lemma 4.8 that  $v(x, k)$  is uniformly continuous in  $x$  and  $k$  in the same domain, if we show that  $p(\cdot, k)$  is strongly continuous in  $k \in D$ .

Given any  $\delta > 0$ , we have for a sufficiently large  $R = R(\delta)$  and for  $|x| \geq R$

$$(6.5) \quad |p(x, k)| \leq C \int_E \frac{|q(y)|}{|x-y|} dy < \delta/2$$

by Lemma 3.1. For  $|x| < R$  we have

$$\begin{aligned} |p(x, k_1) - p(x, k_2)| &\leq C \int_E \frac{|e^{i|k_1||x-y|} - e^{i|k_2||x-y|}}{|x-y|} |q(y)| dy + \\ &\quad + C \int_E \frac{|q(y)|}{|x-y|} |e^{ik_1 \cdot y} - e^{ik_2 \cdot y}| dy \\ &\leq C \int_{K(R_1)} ||k_1| - |k_2|| |q(y)| dy + C \int_{K(R_1)} \frac{|q(y)|}{|x-y|} dy + \\ &\quad + C \int_{K(R_1)} |k_1 - k_2| |y| \frac{|q(y)|}{|x-y|} dy + C \int_{K(R_1)} \frac{|q(y)|}{|x-y|} dy \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Here we can choose an  $R_1 = R_1(\delta)$  sufficiently large so that each of  $J_2$  and  $J_4$  may be bounded by  $\delta/4$  (see the estimation of  $J_2'$  in the proof of Lemma 2.2). On the other hand,  $J_1$  and  $J_3$  can be made arbitrarily small ( $J_1 < \delta/4$ ,  $J_3 < \delta/4$ )

<sup>24</sup> These conditions are satisfied if we assume that  $q(x) = O(|x|^{-3-h})$  ( $|x| \rightarrow \infty$ ).

by taking  $|k_1 - k_2| \geq ||k_1| - |k_2||$  small. Together with (6.5) we see that

$$|p(x, k_1) - p(x, k_2)| < \delta \quad (x \in E, k_1, k_2 \in M)$$

if  $|k_1 - k_2|$  is sufficiently small. This proves the desired continuity of  $p(\cdot, k)$  and completes the proof of Theorem 3.

**§ 7. The spectrum of  $H$ .** It is generally believed that the positive real line is occupied by what are called the *continuous eigenvalues* of  $H$  which are, in our terminology, *eigennumbers*, while in the negative half there exist possibly discrete eigenvalues but no continuous spectrum. This is certainly the case under our assumptions (A) on  $q(x)$ . But from the mathematical standpoint the non-existence of positive eigenvalues is not self-evident. This problem has been raised by POVZNER [28], though he only commented on its plausibility and proved it for the case where  $q(x)$  is a function of  $|x|$  at least for large  $|x|$ . Recently, however, KATO [18] has proved a theorem<sup>25</sup> which implies the non-existence of positive eigenvalues in our case. We shall give a proof of this fact later (Theorem 6). Concerning the features of the negative part of the spectrum of  $H$ , we have

**Theorem 4.** *On the negative real line the continuous spectrum of  $H$  is absent<sup>26</sup>. The negative eigenvalues, if they exist, are of finite multiplicity and are discrete in the sense that they form an isolated set having no limit point other than the origin 0.*

*Proof.* Let  $a$  be sufficiently large so that we may have  $H + a = H_0 + V + a > a' > 0$  (Note that  $H$  is bounded from below; see § 1.) Since the spectrum of  $H_0$  occupies the full interval  $[0, \infty)$ , that of  $(H_0 + a)^{-1}$  is the interval  $[0, 1/a]$ . Now the second resolvent equation gives  $(H + a)^{-1} = (H_0 + a)^{-1} - (H + a)^{-1}V(H_0 + a)^{-1}$ , where  $(H + a)^{-1}V(H_0 + a)^{-1}$  is a completely continuous operator, for  $-a$  lies in the resolvent sets of both  $H$  and  $H_0$  and, as we have seen in § 2,  $V(H_0 + a)^{-1}$  is of the Hilbert-Schmidt type. According to a well-known theorem of WEYL<sup>27</sup> that asserts the invariance of the set of limit points<sup>28</sup> of the spectrum of a self-adjoint operator by the addition of a completely continuous operator, the spectrum of  $(H + a)^{-1}$  consists of the whole interval  $[0, 1/a]$  and, possibly, of isolated points outside  $[0, 1/a]$  (which may have limit point  $1/a$ ). This means that, on the negative real axis,  $H$  may have only discrete eigenvalues of finite multiplicity having, possibly, limit point at 0. This completes the proof of Theorem 6.

Hereafter we shall denote by  $\mu_n$  the discrete negative eigenvalues counted according to their multiplicity and by  $\varphi_n$  the orthonormalized *eigenvectors* (see footnote 3) associated with  $\mu_n$ .

**§ 8. Expansion theorem.** As we have pointed out in Introduction, we can use the distorted plane waves  $\varphi(x, k)$  constructed in § 6 as the eigenfunctions for expanding an arbitrary function.

**Theorem 5.** *Let  $f(x)$  be an arbitrary  $L_2$ -function. i) Then the generalized Fourier transform*

$$(8.1) \quad \hat{f}(k) = (2\pi)^{-\frac{1}{2}} \text{l.i.m.} \int_E \overline{\varphi(x, k)} f(x) dx$$

<sup>25</sup> Cf. Lemma 4.4.

<sup>26</sup> Cf. also POVZNER [28], Chapter II, Theorem 6.

<sup>27</sup> See RIESZ & NAGY [29], p. 367.

<sup>28</sup> Eigenvalues of infinite multiplicity are included among the "limit points".

of  $f(x)$  exists and belongs to  $L_2(M)$ , where  $\text{l.i.m.} \int_E \dots dx$  means the limit in the mean of the function  $\int \dots dx$  as  $N \rightarrow \infty$ . Also the generalized Fourier coefficient

$$(8.2) \quad \hat{f}_n = \int_E \overline{\varphi_n(x)} f(x) dx = (f, \varphi_n)$$

of  $f(x)$  exists, and  $\sum_{n=1}^{\infty} |\hat{f}_n|^2 < \infty$ .

ii) We have the following expansion formulas:

$$(8.3) \quad f(x) = (2\pi)^{-\frac{1}{2}} \text{l.i.m.} \int_M \varphi(x, k) f(k) dk + \text{l.i.m.} \sum_{N \rightarrow \infty} \sum_{n=1}^N \hat{f}_n \varphi_n(x),$$

$$(8.4) \quad \|f\|^2 = \int_M |\hat{f}(k)|^2 dk + \sum_{n=1}^{\infty} |\hat{f}_n|^2 \quad (\text{PARSEVAL'S EQUALITY}),$$

$$(8.5) \quad (f, g) = \int_M \hat{f}(k) \overline{\hat{g}(k)} dk + \sum_{n=1}^{\infty} \hat{f}_n \overline{\hat{g}_n} \quad (\text{GENERALIZED PARSEVAL'S EQUALITY}).$$

In (8.5)  $g \in L_2(E)$ , and in (8.3)  $\text{l.i.m.} \int \dots dk$  means  $\text{l.i.m.} \int \dots dk$  as  $N \rightarrow \infty, \epsilon \rightarrow \infty$  on  $K(N) - K(\epsilon)$ .

iii) Conversely let  $F(k) \in L_2(M)$  be given. Then we can construct an  $L_2$ -function  $f(x)$  through formula (8.3) with  $\hat{f}(k) = F(k)$ ,  $\hat{f}_n = 0$  such that (8.1), (8.2), (8.4) and (8.5) hold good.

iv) Let  $E_\mu$  be the resolution of the identity associated with  $H$ , and let  $P = I - E_0$ . Then the part in  $PL_2(E)$  of  $H$  is unitarily equivalent to  $H_0$ .

v)  $f \in D(H) = D(H_0)$  if and only if  $|k|^2 \hat{f}(k) \in L_2(M)$  and  $\sum_{n=1}^{\infty} \mu_n^2 |\hat{f}_n|^2 < \infty$ . We have the following representation of  $H$ :

$$(8.6) \quad Hf(x) = (2\pi)^{-\frac{1}{2}} \text{l.i.m.} \int_M |k|^2 \varphi(x, k) \hat{f}(k) dk + \text{l.i.m.} \sum_{N \rightarrow \infty} \sum_{n=1}^N \mu_n \hat{f}_n \varphi_n(x).$$

The proof of i), ii) and v) except for the complete domain characterization of  $H$  will be given in the following section; it seems appropriate to remark here that the proof depends on some properties of the spectrum of  $H$  (Theorem 4) but not on the absolute continuity of the positive part of the spectrum (Theorem 6), which will be established in the course of the proof of ii). We shall also make use of Lemma 4.4, which is *almost* equivalent to the non-existence of the positive eigenvalues of  $H$ , but not in the strict sense, for we have not shown that every eigenvector associated with any positive eigenvalues should satisfy the homogeneous integral equation (4.7) and that it should be a  $B$ -function. Pertaining to the positive part of the spectrum of  $H$ , we have

**Theorem 6.** *There exists no positive eigenvalue of  $H$ . Moreover, the spectrum on the positive real line is absolutely continuous.*

The proof of Theorem 5, iii), iv) and the domain characterization of  $H$  will be given in connection with the proof, by the aid of the time-dependent theory, that the  $S$ -matrix is unitary.

It is implied by (8.6) that the Schrodinger operator  $H = -\Delta + V$  admits of a *diagonal representation*, or  $H$  is *diagonalizable*, in terms of the eigenvectors  $\varphi_n(x)$  and the eigenfunctions  $\varphi(x, k)$ .

§ 9. Proof of the completeness of  $\{\varphi_n(x)\}$  and  $\{\varphi(x, k)\}$ . We shall divide the discussions that follow into two parts. One concerns the projection  $P$  and the other  $E_0$ . So to speak, we shall first take up the expansion problem for the function  $Pf(x)$  and then extend our research to the function  $E_0f(x)$ .

Let us consider the conjugate Fourier transform  $g(x, k; \kappa)$  of  $H(x, y; \kappa)$  with the defining equation

$$(9.1) \quad g(x, k; \kappa) = (2\pi)^{-\frac{1}{2}} \int_E H(x, y; \kappa) e^{ik \cdot y} dy$$

for each  $x$  and  $\text{Im}\kappa > 0$ ,  $\text{Im}\kappa^2 \neq 0$ , noting that  $H(x, \cdot; \kappa) \in L_2(E)$  (see Theorem 2 and Lemma 2.4). Actually, however, "l.i.m." in the above definition is unnecessary. This will be made clear by the following

**Lemma 9.1.** *If  $\text{Im}\kappa > 0$  and  $\text{Im}\kappa^2 \neq 0$ , then  $H(x, y; \kappa)$  is absolutely integrable in  $x$  or  $y$  and  $g(x, k; \kappa)$  is a bounded function of  $x$  and  $k$  for  $x \in E$  and  $k \in M$ .*

*Proof.* According to the kernel equation (1.1) for  $H(x, y; \kappa) = G(x, y; \kappa^2)$  we have ( $\kappa = a + ib$ )

$$\int_E |H(x, y; \kappa)| dx \leq C \int_E \frac{e^{-b|x-y|}}{|x-y|} dx + C \int_E |q(z) H(z, y; \kappa)| dz \int_E \frac{e^{-b|x-z|}}{|x-z|} dx.$$

A further estimation is possible by (1.5) and SCHWARZ'S inequality, which gives

$$\int_E |H(x, y; \kappa)| dx \leq C + C \|q\| \|H(\cdot, y; \kappa)\| \leq C,$$

where  $C$  depends on  $\kappa$  but not on  $y$  and is finite if  $\text{Im}\kappa > 0$  and  $\text{Im}\kappa^2 \neq 0$  (see Theorem 2). By the symmetry stated in Lemma 2.4  $H(x, y; \kappa)$  is absolutely integrable in  $y$ , too, and hence it follows from (9.1) that  $g(x, k; \kappa)$  is bounded in  $x$  and  $k$  for  $x \in E$  and  $k \in M$ .

Set

$$(9.1)' \quad h(x, k; \kappa) = (2\pi)^{\frac{1}{2}} (|k|^2 - \kappa^2) g(x, k; \kappa) = e^{ik \cdot x} + h^{(1)}(x, k; \kappa).$$

Then we have

**Lemma 9.2.**  *$h^{(1)}(x, k; \kappa)$  is a B-function of  $x$  for  $k \in M$ ,  $\text{Im}\kappa > 0$  and  $\text{Im}\kappa^2 \neq 0$  and satisfies the integral equation*

$$(9.2) \quad h^{(1)}(x, k; \kappa) = p(x, k; \kappa) - \frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) h^{(1)}(y, k; \kappa) dy,$$

where

$$(9.3) \quad p(x, k; \kappa) = -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{|x-y|} q(y) e^{ik \cdot y} dy$$

is a B-function of  $x$  for  $k \in M$  and  $\text{Im}\kappa \geq 0$ . Moreover, we can extend the definition of  $h(x, k; \kappa)$  to the case where  $\text{Im}\kappa = 0$  and  $\text{Re}\kappa > 0$ .  $h(x, k; \kappa)$  is bounded and uniformly continuous in  $x, k$  and  $\kappa$  for  $x \in E, k \in M$  and  $\text{Im}\kappa \geq 0, 0 < \alpha \leq \text{Re}\kappa \leq \beta < \infty$ . In particular,  $h(x, k; |k|) = \varphi(x, k)$ .

*Proof.* Let us first show that  $g(x, k; \varkappa)$  satisfies the equation

$$(9.4) \quad g(x, k; \varkappa) = (2\pi)^{-\frac{3}{2}} \frac{e^{ik \cdot x}}{|k|^2 - \varkappa^2} - \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|} q(y) g(y, k; \varkappa) dy.$$

From the kernel equation (1.1) we have for any  $f \in L_2(E)$

$$\begin{aligned} \int_E H(x, y; \varkappa) f(y) dy &= \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-y|}}{|x-y|} f(y) dy - \\ &\quad - \frac{1}{4\pi} \int_E \frac{e^{i\varkappa|x-z|}}{|x-z|} q(z) dz \int_E H(z, y; \varkappa) f(y) dy, \end{aligned}$$

where we have performed interchange of the order of integrations, which is permitted since the last integral is absolutely convergent because  $\int_E \int_E |dz| |dy| \leq C \|q\| \|H(z, \cdot; \varkappa)\| \|f\| \leq C \|q\| \|f\|$  (by Theorem 2). Introducing the Fourier transform of  $f(x)$  by

$$(9.5) \quad \hat{f}_0(k) = (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_E e^{-ik \cdot x} f(x) dx$$

and making use of PARSEVAL'S equality, we get therefore

$$(9.6) \quad \begin{aligned} \int_M g(x, k; \varkappa) \hat{f}_0(k) dk &= (2\pi)^{-\frac{3}{2}} \int_M \frac{e^{ik \cdot x}}{|k|^2 - \varkappa^2} \hat{f}_0(k) dk - \\ &\quad - \frac{1}{4\pi} \int_M \left[ \int_E \frac{e^{i\varkappa|x-z|}}{|x-z|} q(z) g(z, k; \varkappa) dz \right] \hat{f}_0(k) dk \end{aligned}$$

where we have used the fact that the conjugate Fourier transform of  $(4\pi|x-y|)^{-1} \times e^{i\varkappa|x-y|}$  as function of  $y$  is  $(2\pi)^{-\frac{3}{2}} (|k|^2 - \varkappa^2)^{-1} e^{ik \cdot x}$  and have interchanged the order of integrations in the last integral, noting that  $|x-z|^{-1} e^{i\varkappa|x-z|} q(z) \in L_1$  and that  $\int_M |g(z, k; \varkappa)| |\hat{f}_0(k)| dk \leq C \|H(z, \cdot; \varkappa)\| \|f\| \leq C \|f\|$  (by Theorem 2). Since  $\hat{f}_0 \in L_2(M)$  is arbitrary, (9.6) yields (9.4). Then the equation (9.2) for  $h^{(1)}(x, k; \varkappa)$  follows at once from (9.1)' and (9.4).

From the remark after Lemma 4.1 we can see that  $p(\cdot, k; \varkappa) \in B$  and that  $p(x, k; \varkappa)$  is bounded for  $x \in E$ ,  $k \in M$  and  $\text{Im} \varkappa \geq 0$ . Similarly, the second term on the right of (9.2) is also in  $B$  if  $\text{Im} \varkappa > 0$  and  $\text{Im} \varkappa^2 \neq 0$  (see Lemma 9.1). Thus  $h^{(1)}(\cdot, k; \varkappa) \in B$  for  $\text{Im} \varkappa > 0$ ,  $\text{Im} \varkappa^2 \neq 0$ , so that we can rewrite (9.2) as  $h^{(1)}(\cdot, k; \varkappa) = (I - T_\varkappa)^{-1} p(\cdot, k; \varkappa)$ . Noting that  $p(\cdot, k; |k|) = p(\cdot, k) \in B$ , this proves the lemma in view of Lemma 4.8 and Theorem 3, if we show that  $p(\cdot, k; \varkappa) \in B$  is strongly continuous uniformly in  $k \in M$  and  $\varkappa$  for  $\text{Im} \varkappa \geq 0$ . The uniform continuity in  $k$  can be proved by an argument similar to, but simpler than, the one used for proving the uniform continuity in  $x$  and  $k$  of  $\varphi(x, k)$  in § 6. Thus it remains to prove the uniform continuity in  $\varkappa$  of  $p(\cdot, k; \varkappa)$ .

Given any  $\delta > 0$ , we can choose an  $R = R(\delta)$  so that for  $|x| \geq R$  we have

$$(9.7) \quad |p(x, k; \varkappa)| \leq C \int_E \frac{|q(y)|}{|x-y|} dy < \delta/2$$

by virtue of Lemma 3.1. For  $|x| < R$

$$\begin{aligned} |\phi(x, k; \kappa_1) - \phi(x, k; \kappa_2)| &\leq C \int_E \frac{|e^{i\kappa_1|x-y|} - e^{i\kappa_2|x-y|}|}{|x-y|} |q(y)| dy \\ &\leq C \int_{K(R_1)} \frac{|e^{i\kappa_1|x-y|} - e^{i\kappa_2|x-y|}|}{|x-y|} |q(y)| dy + C \int_{K(R_1)'} \frac{|q(y)|}{|x-y|} dy \\ &= J + J'. \end{aligned}$$

Again we can choose an  $R_1 = R_1(\delta)$  so large that  $J'$  may be bounded by  $\delta/2$  (see the estimation of  $J'_2$  in the proof of Lemma 2.2). In order to estimate  $J$ , we first note that

$$|e^{i\kappa_1|x-y|} - e^{i\kappa_2|x-y|}| = |i|x-y| \int_L e^{i\nu|x-y|} d\nu| \leq |\kappa_1 - \kappa_2| |x-y|,$$

where  $L$  denotes a straight line which starts from  $\kappa_2$  and ends in  $\kappa_1$ . Application of this inequality to  $J$  gives

$$J \leq C |\kappa_1 - \kappa_2| \int_{K(R_1)} |q(y)| dy.$$

Taking  $\kappa_1$  and  $\kappa_2$  so near to each other that  $J < \delta/2$ , we are led, noting (9.7) and the above estimates for  $J$  and  $J'$ , to the result that for any given  $\delta > 0$  we have

$$(9.8) \quad |\phi(x, k; \kappa_1) - \phi(x, k; \kappa_2)| < \delta \quad (x \in E, k \in M),$$

if  $|\kappa_1 - \kappa_2|$  is sufficiently small. (9.8) proves the required uniform continuity in  $\kappa$  of  $\phi(\cdot, k; \kappa)$ .

Now we shall enter into the expansion problem for  $(Pf)(x)$ , where we first assume that  $f(x) \in C_0^\infty(E)$ . As the first step we prove PARSEVAL'S equality.

Since  $g(x, k; \kappa)$  is the conjugate Fourier transform of  $H(x, y; \kappa)$ , it follows from PARSEVAL'S equality that for  $\text{Im} \kappa > 0$  and  $\text{Im} \kappa^2 \neq 0$

$$\begin{aligned} \int_E H(x, z; \kappa) \overline{H(z, y; \kappa)} dz &= \int_M g(x, k; \kappa) \overline{g(y, k; \kappa)} dk \\ (9.9) \quad &= \frac{1}{(2\pi)^3 (\kappa^2 - \overline{\kappa^2})} \int_M \left[ \frac{1}{|k|^2 - \kappa^2} - \frac{1}{|k|^2 - \overline{\kappa^2}} \right] \times \\ &\quad \times h(x, k; \kappa) \overline{h(y, k; \kappa)} dk. \end{aligned}$$

Let us multiply both sides of (9.9) by  $\overline{f(x)}$  and  $f(y)$  and integrate over  $E$  with respect to  $x$  and  $y$ . Then the left-hand side gives, on multiplying by  $(\kappa^2 - \overline{\kappa^2})$ ,

$$(9.10) \quad (\kappa^2 - \overline{\kappa^2}) (R_{\kappa^2} f, R_{\kappa^2} f) = (\kappa^2 - \overline{\kappa^2}) (R_{\kappa^2} R_{\overline{\kappa^2}} f, f) = ((R_{\kappa^2} - R_{\overline{\kappa^2}}) f, f),$$

where we have made use of the first resolvent equation<sup>29</sup>  $R_\lambda - R_\kappa = (\lambda - \kappa) R_\lambda R_\kappa$  and freely interchanged the order of integrations in the integral considered, noting that it is absolutely convergent:

$$\int_E \int_E \|H(\cdot, x; \kappa)\| \|H(\cdot, y; \kappa)\| |f(x)| |f(y)| dy dx < \infty.$$

<sup>29</sup> See HILLE & PHILLIPS [12], p. 126.

Here let us introduce the function

$$(9.11) \quad \Phi(k; \kappa) = (2\pi)^{-\frac{1}{2}} \int_E \overline{h(x, k; \kappa)} f(x) dx \quad (f(x) \in C_0^\infty(E)).$$

Before proceeding further we note that  $\Phi(k; \sqrt{\mu + i\varepsilon})$  is bounded for  $k \in M$ ,  $\mu \in [\alpha, \beta]$ <sup>30</sup> ( $0 < \alpha < \beta$ ) and  $0 \leq \varepsilon \leq \varepsilon_0$  ( $\varepsilon_0 > 0$ ), as is easily seen from Lemma 9.2 and that  $\Phi(k, \kappa) \rightarrow \hat{f}(k)$  as  $\kappa \rightarrow |k|$ , as is seen from (8.1) and the fact that  $f(x) \in C_0^\infty(E)$ , and hence the integral is well defined.

Let us put  $\kappa^2 = \mu + i\varepsilon$ . Since  $f(x) \in C_0^\infty(E)$  and the integrand of the right side of (9.9)

$$\frac{2i\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} \overline{h(x, k; \sqrt{\mu + i\varepsilon})} h(y, k; \sqrt{\mu + i\varepsilon})$$

is absolutely integrable in  $k$  and bounded in  $x$  and  $y$ , we can freely interchange the order of integration in forming the above-mentioned multiple integral on the right-hand side of (9.9). This enables us to obtain in view of (9.10), for  $\varepsilon > 0$ ,

$$(9.12) \quad ((R_{\mu+i\varepsilon} - R_{\mu-i\varepsilon})f, f) = \int_M \frac{2i\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} |\Phi(k; \sqrt{\mu + i\varepsilon})|^2 dk.$$

Now we can avail ourselves of the fundamental relation

$$\frac{1}{2} [(E_\beta + E_{\beta-0})f, f] - [(E_\alpha + E_{\alpha-0})f, f] = \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta ((R_{\mu+i\varepsilon} - R_{\mu-i\varepsilon})f, f) d\mu$$
<sup>31</sup>,

which, incorporated with (9.12), yields for  $\alpha$  and  $\beta$  such that  $0 < \alpha < \beta$

$$\begin{aligned} & \frac{1}{2} [(E_\beta + E_{\beta-0})f, f] - [(E_\alpha + E_{\alpha-0})f, f] \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta d\mu \int_M \frac{\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} |\Phi(k; \sqrt{\mu + i\varepsilon})|^2 dk \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_M dk \int_\alpha^\beta \frac{\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} |\Phi(k; \sqrt{\mu + i\varepsilon})|^2 d\mu, \end{aligned}$$

where we have used the above-mentioned boundedness of  $\Phi(k, \kappa)$  and FUBINI'S theorem. Let us consider the inner integral  $\int_\alpha^\beta \dots d\mu$ . In view of the equation

$$\int_\alpha^\beta \frac{\varepsilon}{(|k|^2 - \mu)^2 + \varepsilon^2} d\mu = \tan^{-1} \frac{\beta - |k|^2}{\varepsilon} - \tan^{-1} \frac{\alpha - |k|^2}{\varepsilon},$$

we have for  $\varepsilon \in [0, \varepsilon_0]$  ( $\varepsilon_0 > 0$ )

$$\begin{aligned} \left| \int_\alpha^\beta \dots d\mu \right| &\leq F(k) = C && \text{if } |k| \leq \sqrt{\beta} + 1, \\ &= \frac{C}{(|k|^2 - \beta)^2} && \text{if } |k| > \sqrt{\beta} + 1, \end{aligned}$$

<sup>30</sup>  $[\alpha, \beta]$  denotes the closed interval  $\alpha \leq \mu \leq \beta$ .

<sup>31</sup> See e.g. STONE [31], p. 183.



where the  $C$ 's do not depend on  $\varepsilon$ .  $F(k)$  is integrable, and hence by the bounded convergence theorem we can interchange  $\lim_{\varepsilon \downarrow 0}$  and the  $k$ -integration. Since  $\Phi(k; \varkappa)$  tends to  $\hat{f}(k)$  for  $\varkappa \rightarrow |k|$  as has been seen above, we obtain for  $f(x) \in C_0^\infty(E)$

$$(9.13) \quad \frac{1}{2} [(E_\beta + E_{\beta-0})f, f] - ((E_\alpha + E_{\alpha-0})f, f) = \int_{\gamma\alpha < |k| < \gamma\beta} |\hat{f}(k)|^2 dk;$$

here we have made use of the well-known relation<sup>32</sup>

$$\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta \frac{\varepsilon}{(a-\mu)^2 + \varepsilon^2} f(\mu, \varepsilon) d\mu = \begin{cases} 0 & \text{if } a < \alpha \text{ or } \beta < a \\ f(a, 0) & \text{if } \alpha < a < \beta, \end{cases}$$

in which  $f(\mu, \varepsilon)$  is a continuous function of  $\mu$  and  $\varepsilon$  for  $\mu \in [\alpha, \beta]$  and  $\varepsilon \in [0, \varepsilon_0]$  ( $\varepsilon_0 > 0$ ). Letting  $\alpha \rightarrow \beta$  in (9.13), we see that  $((E_\beta - E_{\beta-0})f, f) = 0$  for any  $f(x) \in C_0^\infty(E)$ , which is dense in  $L_2(E)$  in the  $L_2$ -norm. This shows that  $E_\beta = E_{\beta-0}$  and  $\beta$  is not an eigenvalue of  $H$ , and since  $\beta$  is positive and otherwise arbitrary, we conclude that no positive eigenvalues exist. This proves the first half of Theorem 6. Hence we have from (9.13)

$$(9.14) \quad ((E_\beta - E_\alpha)f, f) = \int_{\gamma\alpha < |k| < \gamma\beta} |\hat{f}(k)|^2 dk$$

for  $f(x) \in C_0^\infty(E)$ . Letting  $\alpha \rightarrow 0$  and  $\beta \rightarrow \infty$ , we arrive at PARSEVAL'S equality

$$(9.15) \quad \|Pf\|^2 = \int_M |\hat{f}(k)|^2 dk$$

for  $f(x) \in C_0^\infty(E)$ . We have so far assumed that  $f(x) \in C_0^\infty(E)$ . But the extension to the general case  $f(x) \in L_2(E)$  can be made in a standard manner<sup>33</sup>. Namely, we can show that  $\hat{f}(k)$  exists and lies in  $L_2(M)$  if we take the limit in the mean, as specified in Theorem 5, in forming  $\hat{f}(k)$ . Thus (9.14) and (9.15) hold true for  $f(x) \in L_2(E)$ .

Now let us step forward to the second problem, *i.e.*, the expansion of  $(E_0 f)(x)$ . According to Theorem 4 the subspace  $E_0 L_2(E)$  is spanned by the eigenvectors  $\{\varphi_n(x)\}$  belonging to the eigenvalues  $\mu_n \leq 0$ , or  $\{\varphi_n(x)\}$  form a complete orthonormal system in  $E_0 L_2(E)$ . Consequently we have

$$(9.16) \quad \|E_0 f\|^2 = \sum_{n=1}^{\infty} |\hat{f}_n|^2.$$

In view of (9.15) and (9.16) we are led to the required relation

$$(8.4) \quad \|f\|^2 = \|E_0 f\|^2 + \|Pf\|^2 = \int_M |\hat{f}(k)|^2 dk + \sum_{n=1}^{\infty} |\hat{f}_n|^2.$$

Derivation of (8.5) from (8.4) is almost obvious<sup>34</sup>.

Now let us prove (8.3). It is obvious from the completeness in  $E_0 L_2(E)$  of  $\{\varphi_n(x)\}$  that

$$(9.17) \quad (E_0 f)(x) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n=1}^N \hat{f}_n \varphi_n(x).$$

<sup>32</sup> See *e.g.* TITCHMARSH [33], p. 31.

<sup>33</sup> See *e.g.* TITCHMARSH [34], pp. 55–56.

<sup>34</sup> See any books on Fourier integrals, *e.g.* TITCHMARSH [33], Chapter III.

It remains to show that

$$(9.18) \quad (Pf)(x) = (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M \varphi(x, k) \hat{f}(k) dk.$$

We can obtain from (9.15), if  $g(x) \in C_0^\infty(E)$ ,

$$\begin{aligned} ((E_\beta - E_\alpha)f, g) &= \int_{\sqrt{\alpha} < |k| < \sqrt{\beta}} \hat{f}(k) \overline{\hat{g}(k)} dk \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\sqrt{\alpha} < |k| < \sqrt{\beta}} \hat{f}(k) dk \int_E \varphi(x, k) \overline{g(x)} dx \\ &= (2\pi)^{-\frac{3}{2}} \int_E \left[ \int_{\sqrt{\alpha} < |k| < \sqrt{\beta}} \varphi(x, k) \hat{f}(k) dk \right] \overline{g(x)} dx, \end{aligned}$$

since  $\varphi(x, k)$  is bounded and  $\hat{f}(k)$  is integrable for  $\sqrt{\alpha} < |k| < \sqrt{\beta}$  and  $x \in E$ .  $C_0^\infty(E)$  being dense in  $L_2(E)$  in the  $L_2$ -norm, we have

$$(9.19) \quad ((E_\beta - E_\alpha)f)(x) = (2\pi)^{-\frac{3}{2}} \int_{\sqrt{\alpha} < |k| < \sqrt{\beta}} \varphi(x, k) \hat{f}(k) dk.$$

If we let  $\alpha \rightarrow 0$  and  $\beta \rightarrow \infty$ , then the left side of (9.19) converges strongly to  $(Pf)(x)$ , and hence the limit in the mean of the right also exists and is equal to  $(Pf)(x)$ . This is nothing but the desired relation (9.18). We have proved i) and ii) of Theorem 5.

Now we show the diagonal representation (8.6) of  $H$  without complete characterization of the domain. For this purpose it suffices to show

$$(9.20) \quad (Hf, g) = \int_M |k|^2 \hat{f}(k) \overline{\hat{g}(k)} dk + \sum_{n=1}^{\infty} \mu_n \hat{f}_n \overline{\hat{g}_n}$$

for  $f(x) \in D(H) = D(H_0)$  and  $g(x) \in L_2(E)$ . We see from (9.14) that

$$(9.21) \quad \begin{aligned} (E_\mu f, g) &= \sum_{\mu_n \leq \mu} \hat{f}_n \overline{\hat{g}_n} & (\mu < 0) \\ &= \sum_{n=1}^{\infty} \hat{f}_n \overline{\hat{g}_n} + \int_{|k| < \sqrt{\mu}} \hat{f}(k) \overline{\hat{g}(k)} dk & (\mu \geq 0); \end{aligned}$$

hence (9.20) holds if the relation

$$(Hf, g) = \int_{-\infty}^{\infty} \mu d(E_\mu f, g)$$

is taken into account.

Finally let us prove the last half of Theorem 6. Taking  $g=f$  in (9.21), we can see that  $\varrho_f(\mu) = (E_\mu f, f)$  is absolutely continuous in  $\mu$  for  $\mu > 0$ . This implies nothing but the absolute continuity of the spectrum of  $H$  on the positive real line.

### III. S-matrices

**§ 10. Wave operators  $W_\pm$ .** As has been mentioned in the Introduction, we find it convenient to introduce the *wave operators*  $W_\pm$  not only for the mathematical treatment of the S-matrix  $S$ , but also for the proof of Theorem 5, iii).

It is known<sup>35</sup> from the time-dependent theory of the S-matrices that, under the assumption that  $q(x) \in L_2(E)$ , the wave operators  $W_{\pm}$  exist:

$$(10.1) \quad W_{\pm} = s\text{-}\lim_{t \rightarrow \pm\infty} U(t) \text{ }^{36}, \quad U(t) = e^{itH} e^{-itH_0};$$

$W_{\pm}$  are isometric operators satisfying

$$(10.2) \quad PW_{\pm} = W_{\pm},$$

which implies that the ranges of  $W_{\pm}$  are subsets of  $PL_2(E)$ . The S-matrix is defined by

$$(10.3) \quad S = W_+^* W_-,$$

and it is generally believed to be unitary, which is, however, not self-evident.  $S$  is unitary if and only if the ranges of  $W_{\pm}$  are identical. We shall prove that  $S$  is unitary by showing that the ranges of  $W_{\pm}$  are identical and coincide with  $PL_2(E)$ .

To this end we introduce the operators  $U_{\pm}$  by

$$(10.4) \quad (U_- f)(x) = (2\pi)^{-\frac{1}{2}} \text{l.i.m.} \int_M \varphi(x, k) \hat{f}_0(k) dk.$$

$$(10.5) \quad U_+ f = \overline{U_- \bar{f}},$$

where  $\hat{f}_0(k)$  is the Fourier transform of  $f(x) \in L_2(E)$  (see (9.5)), and prove

**Theorem 7.**  $U_{\pm}$ , formally defined by (10.4) and (10.5), are isometric operators with domains  $L_2(\bar{E})$  and ranges  $PL_2(E)$ , and we have  $U_{\pm} = W_{\pm}$ .  $S$  defined by (10.3) is unitary.

We shall prove the above theorem in the next section and then complete the proof of Theorem 5. Here we note that the statement iii) of Theorem 5 is equivalent to the following: Let  $Z$  be an isometric operator from  $PL_2(E)$  to  $L_2(M)$  defined by  $Zf(k) = \hat{f}(k)$  ( $f \in PL_2(E)$ ). Then  $Z$  is unitary. (The isometry of  $Z$  is already shown in (8.4).)

**§ 11. Proof of Theorem 7.** We first show that  $U_{\pm}$  are well defined operators. To this end we need only to show that this is the case with  $U_-$ , since  $U_+$  is related to  $U_-$  through (10.5).

Let us consider the integral

$$(11.1) \quad L_f(g) = \int_M \hat{f}_0(k) \overline{\hat{g}(k)} dk = (2\pi)^{-\frac{1}{2}} \int_M \hat{f}_0(k) \left[ \int_E \varphi(x, k) \overline{g(x)} dx \right] dk,$$

where  $g(x) \in C_0^{\infty}(E)$ ,  $\hat{f}_0(k) \in C_0^{\infty}(M)$  and the carrier of  $\hat{f}_0(k)$  does not contain the origin of  $M$ . The totality of such functions of  $k$  will be denoted by  $C_0^{\infty}(M)'$ . In (11.1) the  $x$ - and  $k$ -integrations are, actually, extended over compact domains and  $\varphi(x, k)$  is bounded in  $k$  in a compact domain from which the origin is excluded (see Theorem 3). This fact allows us to interchange the order of integration, yielding

$$(11.2) \quad L_f(g) = (2\pi)^{-\frac{1}{2}} \int_E \left[ \int_M \varphi(x, k) \hat{f}_0(k) dk \right] \overline{g(x)} dx.$$

<sup>35</sup> See COOK [7], KATO [17] and KURODA [20].

<sup>36</sup> "s-lim" denotes strong limit.

On the other hand it follows from (11.4) that

$$|L(g)| \leq \left( \int_M |\hat{f}_0(k)|^2 dk \int_M |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} = \|f\| \|Pg\| \leq \|f\| \|g\|.$$

Thus  $L_f(g)$  is a densely defined, bounded linear functional of  $g \in C_0^\infty(E)$ . But such a functional can be extended to the whole  $L_2(E)$ , and we shall also denote the extension by  $L_f(g)$ . Then by means of RIESZ'S theorem<sup>37</sup> there exists a unique element  $f^*$  of  $L_2(E)$  such that  $L_f(g) = (f^*, g)$ , which defines an operator  $U$  by  $Uf = f^*$  and  $\|Uf\| \leq \|f\|$ . Now in view of (11.2), again restricting  $g(x)$  to  $C_0^\infty(E)$ , we have

$$(11.3) \quad (Uf)(x) = (2\pi)^{-\frac{3}{2}} \int_M \varphi(x, k) \hat{f}_0(k) dk \quad a.e. \quad (f_0(k) \in C_0^\infty(M)).$$

A standard argument shows that (11.3) can be extended to every  $f \in L_2(E)$  by writing l.i.m.  $\int_M \dots dk$  instead of  $\int \dots dk$ . Comparison of (11.3) with (10.4) gives the result that  $U$  is nothing but the required operator  $U_-$ . Thus we have proved that  $U_-$  is an everywhere defined, bounded operator ( $D(U_-) = L_2(E)$ ) and its norm  $\|U_-\|$  does not exceed 1.

Next let us determine the adjoint  $U_-^*$  of  $U_-$ . This can be done most conveniently by using the relation  $(U_-^*f, g) = (f, U_-g)$ ; the result is easily seen to be

$$(11.4) \quad (U_-^*f)(x) = (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M e^{ik \cdot x} \hat{f}(k) dk.$$

In the derivation of (11.4) we have used reasoning quite analogous to that for getting (11.3).

$U_-$  has the property that

$$(11.5) \quad U_- U_-^* = P,$$

which follows immediately from the defining equations (10.4), (11.4) of  $U_-$ ,  $U_-^*$  and (8.3). Also we have

$$(11.6) \quad U_-^* H \subset H_0 U_-.$$

This can be seen from the following equations: For  $f(x) \in D(H) = D(H_0)$

$$\begin{aligned} (U_-^* H f)(x) &= (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M e^{ik \cdot x} (Hf)^\wedge(k) dk \\ &= (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M e^{ik \cdot x} |k|^2 \hat{f}(k) dk; \\ (H_0 U_-^* f)(x) &= (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M e^{ik \cdot x} |k|^2 (U_-^* f)^\wedge(k) dk \\ &= (2\pi)^{-\frac{3}{2}} \text{l.i.m.} \int_M e^{ik \cdot x} |k|^2 \hat{f}(k) dk, \end{aligned}$$

where we have made use of the diagonal representation of  $H$  (8.6) and the fact that  $H_0$  is a special case of  $H$  with  $V=0$  and has no eigenvalues.

Now we can complete the proof of Theorem 7. Application of  $U_-^*$  from the left to both sides of (10.4) gives

$$(11.7) \quad U_-^* W_- = \lim_{t \rightarrow -\infty} U_-^* U(t) \quad (\text{lim} = \text{s-lim}),$$

<sup>37</sup> See e.g. RIESZ & NAGY [29] and YosIDA [39].

since  $U_-^*$  is a bounded operator and  $\lim_{t \rightarrow -\infty} U(t)$  exists. On the other hand, we see, by differentiating and again integrating, that

$$(11.8) \quad \begin{aligned} (U_-^* U(t) f, g) - (U_-^* f, g) &= i \int_0^t (U_-^* e^{itH} V e^{-itH_0} f, g) dt \\ &= i \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g) dt, \end{aligned}$$

where  $f(x) \in D(H_0) = D(H)$  and  $g(x) \in L_2(E)$  and we have made use of the fact that  $U_-^* e^{itH} = e^{itH_0} U_-^*$ , which is a consequence of (11.6). In view of (11.7) and (11.8), we obtain

$$(11.9) \quad (U_-^* W_- f, g) = (U_-^* f, g) + \lim_{t \rightarrow -\infty} i \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g) dt.$$

The integrand of (11.9) is calculated as follows:

$$(11.10) \quad \begin{aligned} (e^{itH_0} U_-^* V e^{-itH_0} f, g) &= \int_M e^{it|k|^2} (U_-^* V e^{-itH_0} f)_0^\wedge(k) \overline{\hat{g}_0(k)} dk \\ &= \int_M e^{it|k|^2} (V e^{-itH_0} f)^\wedge(k) \overline{\hat{g}_0(k)} dk \\ &= \int_M (V e^{-it(H_0 - |k|^2)} f)^\wedge(k) \overline{\hat{g}_0(k)} dk \\ &= (2\pi)^{-\frac{3}{2}} \int_M \left[ \int_E \overline{\varphi(x, k)} q(x) (e^{-it(H_0 - |k|^2)} f)(x) dx \right] \overline{\hat{g}_0(k)} dk, \end{aligned}$$

where l.i.m. is not needed before  $\int$ , because  $\varphi(x, k)$  is bounded in  $x$ ,  $q(x) \in L_2(E)$  and  $(e^{-it(H_0 - |k|^2)} f)(x) \in L_2(E)$ .

Returning to (11.9), the limit for  $t \rightarrow -\infty$  of the  $t$ -integral can be replaced by the Abelian limit, *i.e.*

$$\lim_{t \rightarrow -\infty} \int_0^t (e^{itH_0} U_-^* V e^{-itH_0} f, g) dt = \lim_{\varepsilon \downarrow 0} \int_0^{-\infty} e^{\varepsilon t} (e^{itH_0} U_-^* V e^{-itH_0} f, g) dt,$$

since the existence of the ordinary limit is known. Then (11.9) and (11.10) give

$$\begin{aligned} (U_-^* W_- f, g) &= (U_-^* f, g) + \frac{i}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_0^{-\infty} e^{\varepsilon t} dt \times \\ &\quad \times \int_M \left[ \int_E \overline{\varphi(x, k)} q(x) (e^{-it(H_0 - |k|^2)} f)(x) dx \right] \overline{\hat{g}_0(k)} dk. \end{aligned}$$

If we assume here that  $\hat{g}_0(k) \in C_0^\infty(M)'$ , then we can interchange the  $t$ - and  $k$ -integrations, for  $\varphi(x, k)$  is bounded for  $x \in E$  and  $k \in D = \text{carrier of } \hat{g}_0(k)$  (see Theorem 3) while

$$\int_E |q(x)| |e^{-it(H_0 - |k|^2)} f(x)| dx \leq \|q\| \|f\|.$$

$$(U_-^* W_- f, g) = (U_-^* f, g) + \frac{i}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \overline{\hat{g}_0(k)} dk \int_0^{-\infty} (e^{-it(H_0 - (|k|^2 - i\varepsilon))} f, F(\cdot, k)) dt.$$

where we have put

$$F(x, k) = \varphi(x, k) q(x) \in L_2(E) \quad (\text{for each fixed } k \in M).$$

Hence

$$\begin{aligned}
(U_-^* W_- f, g) &= (U_-^* f, g) - \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \overline{\hat{g}_0(k)} dk \times \\
&\quad \times \lim_{t \rightarrow -\infty} ([e^{-it(H_0 - (|k|^2 - i\varepsilon))} - 1] R_{0, |k|^2 - i\varepsilon} t, F(\cdot, k)), \\
(11.11) \quad &= (U_-^* f, g) + \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M \overline{\hat{g}_0(k)} (R_{0, |k|^2 - i\varepsilon} f, F(\cdot, k)) dk \\
&= (U_-^* f, g) + \frac{1}{(2\pi)^{\frac{3}{2}}} \lim_{\varepsilon \downarrow 0} \int_M (f, R_{0, |k|^2 + i\varepsilon} \varphi(\cdot, k) q(\cdot)) \overline{\hat{g}_0(k)} dk.
\end{aligned}$$

Here we further assume that  $f(x) \in C_0^\infty(E) \subset D(H_0)$ . Now the function

$$\begin{aligned}
(R_{0, |k|^2 + i\varepsilon} \varphi(\cdot, k) q(\cdot))(x) &= \frac{1}{4\pi} \int_E \frac{e^{i|x-y|}}{|x-y|} q(y) \varphi(y, k) dy \\
&= - (T_\varepsilon \varphi(\cdot, k))(x) \quad (\varepsilon = \sqrt{|k|^2 + i\varepsilon})
\end{aligned}$$

converges to

$$\frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(y) \varphi(y, k) dy = - (T_{|k|} \varphi(\cdot, k))(x)$$

as  $\varepsilon \downarrow 0$  uniformly with respect to  $x \in E$  and  $k$  contained in the compact carrier of  $\hat{g}_0(k)$  by virtue of Theorem 3 and Lemma 4.6. Thus the convergence of the integrand of (11.11) for  $\varepsilon \downarrow 0$  is uniform, and we can interchange  $\lim$  and  $\int \dots dk$  so as to obtain

$$\begin{aligned}
\overline{(U_-^* W_- f, g)} &= \overline{(U_-^* f, g)} + \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k) dk \int_E \overline{f(x)} dx - \frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(y) \varphi(y, k) dy \\
(11.12) \quad &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k) dk \int_E \overline{f(x)} \left[ \varphi(x, k) + \frac{1}{4\pi} \int_E \frac{e^{i|k||x-y|}}{|x-y|} q(y) \varphi(y, k) dy \right] dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \hat{g}_0(k) dk \int_E e^{ik \cdot x} \overline{f(x)} dx \quad (\text{by (6.2)}) \\
&= \int_M \overline{f_0(k)} \overline{\hat{g}_0(k)} dk = \overline{(f, g)}.
\end{aligned}$$

$C_0^\infty(M)'$  is dense in  $L_2(M)$ , and hence the totality of such  $g(x)$  that  $\hat{g}_0(k) \in C_0^\infty(M)'$  is dense in  $L_2(E)$ .  $C_0^\infty(E)$  is of course dense in  $L_2(E)$ . We can, therefore, obtain from (11.12) the following relation:

$$(11.13) \quad U_-^* W_- = I.$$

If we operate with  $U_-$  on both sides of (11.13) and take account of (10.2) and (11.5), we get

$$(11.14) \quad U_- = U_- U_-^* W_- = P W_- = W_-$$

and, on substituting this relation into (11.13),

$$(11.15) \quad U_-^* U_- = W_-^* W_- = I$$

The relations (11.5) and (11.15) show that the operator  $U_-$  is an isometric operator with domain  $D(U_-) = L_2(E)$  and range  $R(U_-) = PL_2(E)$ <sup>38</sup>. Quite a similar result follows for  $U_+$  and  $W_+$ . Thus we can see that  $S = W_+^* W_- = U_+^* U_-$  and  $S$  is unitary. Now the proof of Theorem 7 is complete.

In connection with eigenfunction expansion it remains to prove that the isometric operator  $Z$  is unitary, to characterize the domain of  $H$ , and to establish the unitary equivalence between the part of  $H$  in  $PL_2(E)$  and  $H_0$ . To these ends we shall define the operators  $Y$  and  $Z'$  by

$$\begin{aligned} (Yf)(k) &= \hat{f}_0(k) && \text{(from } L_2(E) \text{ onto } L_2(M)); \\ (Z'f)(x) &= (2\pi)^{-\frac{1}{2}} \text{l.i.m.} \int_M \varphi(x, k) f(k) dk && \text{(from } L_2(M) \text{ into } L_2(E)). \end{aligned}$$

Then we have  $U_- = Z'Y$  and  $U_-^* = Y^{-1}Z$ , and we have

$$(11.16) \quad I = U_-^* U_- = Y^{-1} Z Z' Y, \quad Z Z' = Y Y^{-1} = I' \text{ }^{39}$$

On the other hand (8.3), (8.4) and (8.5) imply that

$$(11.17) \quad P = Z' Z.$$

(11.16) and (11.17) show that  $Z$  is a unitary operator from  $PL_2(E)$  onto  $L_2(M)$ , and that  $Z' = Z^*$ , the adjoint operator of  $Z$  ( $Z^*$  is a unitary operator from  $L_2(M)$  onto  $PL_2(E)$ ).

Now it follows from (8.6) that

$$(ZPHf)(k) = |k|^2 \hat{f}(k),$$

and it is known that  $H_0$  represents the operation of multiplying by  $|k|^2$  in  $L_2(M)$ . Thus we have proved iv) of Theorem 5.

Finally let us prove the first assertion of Theorem 5, v). Since the necessity is obvious from the diagonal representation (8.6) of  $H$ , it remains to show the sufficiency. We see from the unitarity of  $Z$  and (8.6) that if  $F(k) \in L_2(M)$  and  $|k|^2 F(k) \in L_2(M)$  then  $Z^* F(x) \in PD(H) \subset D(H)$ . Thus we need only to prove that for any sequence  $F_n$  such that  $\sum |F_n|^2 < \infty$  and  $\sum \mu_n^2 |F_n|^2 < \infty$ , l.i.m.  $\sum F_n \varphi_n \in E_0 D(H) \subset D(H)$ . But this follows easily from (8.6) and the fact that the correspondence between  $E_0 L_2(E)$  and  $l_2$ <sup>40</sup>, which assigns every  $f \in E_0 L_2(E)$  to  $\hat{f}_n \in l_2$ , is one-to-one and isometric, where  $l_2$  denotes the Hilbert space of all sequences  $\{F_n\}$  such that  $\sum |F_n|^2 < \infty$ .

#### IV. Concluding remarks

**§ 12. Higher-dimensional and two-dimensional cases.** Our theory, which has so far been developed for the 3-dimensional Schroedinger operator, cannot be extended directly to the case of higher dimension. One of the main difficulties is that Theorems 1 and 2, which serve as a bridge between the resolvent  $R_\lambda$  and the kernel  $G(x, y; \lambda)$ , fail to hold, because the kernel  $G(x, y; \lambda)$ , even though it exists, is not of the Carleman type in the case of more than three dimensions.

<sup>38</sup>  $R(A)$  means the range of  $A$ .

<sup>39</sup>  $I'$  denotes the identity operator in  $L_2(M)$ .

<sup>40</sup> If the dimension  $m$  of  $E_0 L_2(E)$  is finite then  $l_2$  is the totality of finite sequences of length  $m$ .

A remedy for it has been suggested by GÄRDING [9]; it is to consider the operator  $(H^t - \lambda)^{-1}$ , where  $t$  is a certain positive integer, instead of  $R_\lambda = (H - \lambda)^{-1}$ . The operator  $(H^t - \lambda)^{-1}$  is, in fact, known to have a kernel of the Carleman type. Of course, this will require many an alteration in the details of our theory.

On the contrary, in the 2-dimensional case our theory suffers no essential modification, for then the resolvent kernels can be shown to be of the Carleman type. But it should be remarked that results for the 2-dimensional case do not follow as special cases from the 3-dimensional treatment, because the potential function  $q(x)$ , which diminishes at infinity in the 2-dimensional Euclidian space  $E^2$ , does not always diminishes at infinity in  $E = E^3$ , which is an extension of  $E$ . The conditions imposed upon  $q(x)$  are to be replaced by

(A')  $q(x)$  is a real-valued function which is locally Hölder continuous except at a finite number of singularities. Furthermore,  $q(x)$  is square integrable ( $q(x) \in L_2(E^2)$ ) and behaves like  $O(|x|^{-3-h})$  ( $h > 0$ ) at infinity, i.e. there exist positive numbers  $h$ ,  $C_0$  and  $R_0$  such that

$$|q(x)| \leq C_0 |x|^{-3-h} \quad \text{for } |x| \geq R_0.$$

Not all the details of the theory need be modified if we replace the function  $(4\pi|x-y|)^{-1} e^{i\kappa|x-y|}$ , which is the resolvent kernel of  $R_{0,\kappa}$ , by  $4^{1/2} i H_0^{(1)}(\kappa|x-y|)$ , where  $H_0^{(1)}$  denotes the Hankel function of first kind of order 0.

In concluding we remark that if we had assumed in addition to (A) that  $q(x) \in L_1(E)$  or more explicitly  $q(x) = O(|x|^{-3-h})$  at infinity (in the 3-dimensional case), our arguments could have been much simplified.

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<sup>41</sup> See e.g. TITCHMARSH [35].



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