A Markov Property for Gaussian Processes with a Multidimensional Parameter

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Introduction

In this paper we discuss a class of Gaussian processes $\{X_t\colon t\in T\}$ for which the time domain T is an n-dimensional space rather than the usual real interval. Of special interest to us are those processes which we call Markovian. These are intuitively described by requiring that for any smooth surface Γ which separates T into complementary domains, what happens inside is independent of what happens outside when conditioned by knowledge of X_t on (and near) Γ.

The discussion centers on the reproducing kernel Hilbert space associated with the covariance function of the process. Those spaces associated with Markov processes are characterized by a locality condition on their inner products that requires functions with disjoint support to be orthogonal. The discussion of these matters is very general and together with the introduction of local reproducing kernel spaces appears in Part I.

With appropriate additional assumptions the inner product in the reproducing kernel space can be identified as coming from a non-negative Dirichlet form

$$\langle u, v\rangle = \sum_{\alpha,\beta} \int_T a_{\alpha\beta} D^\alpha u \overline{D^\beta v}.$$

Moreover, if this form is elliptic the associated process $\{X_t\}$ is found to have certain generalized normal derivatives on the surface and the "boundary" σ-field is seen to be generated by these normal derivatives. We call such a process Markovian of finite order.

It is then shown that the least squares prediction problem for the process is intimately related to the Dirichlet problem for an associated elliptic operator. Formulas which solve the Dirichlet problem may be reinterpreted to give a solution of the prediction problem. These matters are discussed in Part II. Those readers familiar with the work of MCKEAN [8] and MOLCHAN [9] on Brownian motion with a multidimensional time parameter will easily recognize this as an extension of their work.

In Part III we have done some spectral theory for stationary Gaussian Markov processes on R_n. This material is not complete but does contain the characterization of the spectral densities of Markov processes of finite order as being inverses of non-negative elliptic polynomials. Further work in this direction will appear in [11].

We have attempted to make this paper accessible to an audience of probabilists that is neither used to processes with a several-dimensional time nor familiar with Dirichlet problems for higher order operators. For this reason we have not stressed formal completeness and generality, but we have included such items as the example in Section 1 of a Markov process whose time parameter runs over the circle. This example is the simplest possible and shows in the absence of all technical problems the identity of the least squares prediction problem for the process and the Dirichlet problem for the associated differential operator.

This work was begun at Rockefeller University during the year 1969–70. The ideas of H. P. McKEAN, Jr. appear throughout this paper, and with gratitude I acknowledge his influence.

1. An Example: A Stationary Markov Process on the Circle

Let $\{X_t: -\infty<t<\infty\}$ be a real continuous stationary Gaussian process that is periodic with period 2π and has zero expectation. We consider $\{X_t\}$ to be a process defined for t on the circle $C=R_1(\text{mod}\, 2\pi)$, and call $\{X_t\}$ *simply Markovian* if*:

For each interval $I=[\alpha,\beta]$ *on the circle* C, *the* "past" $=\{X_t: t\in I\}$ *and the* "future" $=\{X_t \in t\notin I\}$ *are conditionally independent given the* "present" $=\{X_\alpha, X_\beta\}$.

Since the process is Gaussian this is equivalent to the condition that for $t\notin I$,

$$E\{X_t\,|\,X_s: s\in I\}=E\{X_t\,|\,X_\alpha, X_\beta\}.$$

Our process $\{X_t\}$ is described by the periodic covariance function $R(t-s)=EX_tX_s$. Covariance functions of Markov processes are easily characterized.

Proposition 1.1. *A non-constant covariance function* $R(t-s)$ *on* C *corresponds to a Markov process if and only if* R *is the Green's function of a second order operator*

$$Lf=\alpha f-\gamma f''; \quad \alpha,\gamma>0^{\dagger} \tag{1.1}$$

on C.

Note. Our proof makes no use of the reproducing kernel spaces introduced later. The relationship with the Dirichlet problem for L and the fact that as operators $R=L^{-1}$ are, however, still to be stressed.

Proof. First suppose that L is given and R is the Green's function for L. For any proper sub-interval $I=[\alpha,\beta]$ of C, the space of functions on I satisfying $Lf=0$ has dimension two and is spanned by the functions $R(s-\alpha)$ and $R(s-\beta)$. For $t\notin I$ we have $LR(\cdot-t)=0$ on I, and thus $R(s-t)$ satisfies a relation

$$R(s-t)=A(t)R(s-\alpha)+B(t)R(s-\beta); \quad s\in I,\ t\in I \tag{1.2}$$

* This definition is a natural extension of the classical definition for processes with time domain $T=[0,\infty)$. In both cases the topological present is a point set separating the time domain into complementary domains.

† The restrictions on α and γ are consequences of the fact that R is positive definite.

where the functions A and B are continuous. But $R(s-t)=EX_tX_s$ so that (1.2) implies that $X_t-A(t)X_\alpha-B(t)X_\beta$ and X_s are uncorrelated for $s\in I$. Since the process is Gaussian, this implies independence.

Equivalently we have for $t\notin I$

$$E\{X_t|X_s\colon s\in I\}=A(t)X_\alpha+B(t)X_\beta. \tag{1.3}$$

Taking conditional expectations on both sides given X_α and X_β, it follows that

$$E\{X_t|X_s\colon s\in I\}=E\{X_t|X_\alpha, X_\beta\} \tag{1.4}$$

and thus $\{X_t\}$ is Markovian.

Conversely we note that the steps leading from (1.2) to (1.4) are reversible, and we conclude that if $\{X_t\}$ is Markovian then functions A and B must exist, which satisfy (1.2). Differentiating* $R(s-t)$ twice with respect to s, equation (1.2) shows that as functions of t each of the three functions $R(s-t)$, $R'(s-t)$ and $R''(s-t)$ is a linear combination of $A(t)$ and $B(t)$. $R(t)$ thus satisfies an equation

$$\lambda R(t)+\mu R'(t)+R''(t)=0 \quad \text{on } (0,2\pi). \tag{1.5}$$

Moreover, R is even about π since X_t has period 2π and the odd and even parts of (1.5) must vanish separately. We conclude that

$$\lambda R(t)+R''(t)=0 \quad \text{on } (0,2\pi). \tag{1.6}$$

Without loss of generality we may assume that $R(0)=EX_t^2=1$. With this normalization the unique solution of (1.6), which is even about π, is

$$R(t)=\begin{cases}\dfrac{\operatorname{Cosh}[(t-\pi)\sqrt{-\lambda/v}]}{\operatorname{Cosh}(\pi\sqrt{-\lambda/v})} & \text{if } \lambda\cdot v<0\\[2ex] \dfrac{\operatorname{Cos}[(t-\pi)\sqrt{\lambda/v}]}{\operatorname{Cos}(\pi\sqrt{\lambda/v})} & \text{if } \lambda\cdot v>0.\end{cases} \tag{1.7}$$

The first is a Green's function as required, while the second is easily eliminated as a possibility.

In fact, we must have $|R(t)|\leq 1$, and for $\lambda\cdot v>0$ we have $R(\pi)=1/\cos(\pi\sqrt{\lambda/v})$. The only possibility is that $R(\pi)=\pm 1$, but this would imply $X_{\pi+h}=X_h$. This clearly cannot happen for a non-constant Markov process and the proof is complete.

Before leaving this example we make two further observations. First, note that equation (1.2) implies $LA(t)=LB(t)=0$. It then follows from (1.3) that the function $u(t)=E\{X_t|X_s\colon s\in I\}$ satisfies

$$\begin{aligned} Lu(t)&=0 \qquad \text{for } t\notin I\\ u(\alpha)&=X_\alpha, \quad u(\beta)=X_\beta. \end{aligned} \tag{1.8}$$

That is, the prediction of X_t given $\{X_s\colon s\in I\}$ is the solution of the Dirichlet problem $Lu(t)=0$ on $C-I$ with the boundary data $u(\alpha)=X_\alpha$ and $u(\beta)=X_\beta$.

* $R(t)$ is easily shown to be smooth on the interval $(0, 2\pi)$ by considering the smooth convolutions $\int R(s-t)\phi(t)dt=R(s-\alpha)\int A(t)\phi(t)dt+R(s-\beta)\int B(t)\phi(t)dt$, where ϕ is a smooth function vanishing on I.

Lastly, we comment that an obvious generalization of our definition of a Markov process is available. We call $\{X_t: t\in C\}$ Markovian of order p if the sample paths have $p-1$ continuous derivatives and if for each interval I, the past $=\{X_t: t\in I\}$ and the future $=\{X_t: t\notin I\}$ are conditionally independent when given the present $=\{X_\alpha, X'_\alpha, \ldots, X_\alpha^{(p-1)}, X_\beta, \ldots, X_\beta^{(p-1)}\}$. Our proposition and its proof then have easy extensions to the effect that $\{X_t\}$ is Markovian of order p if and only if $R(t)$ is the Green's function of an operator of order $2p$.

This result is well-known when $T=R_1$ instead of the circle. The Fourier version states that a stationary Gaussian process is Markov of order p if and only if its spectral density is the inverse of a polynomial of degree $2p$; see LEVINSON & MCKEAN [7]. The reader may also wish to compare this example with the work of DOOB [4], HIDA [5] and DOLPH & WOODBURY [3].

I. General Theory

2. *Splitting Fields and the Markov Property*

We begin by fixing some notations and reviewing the concept of splitting fields introduced by MCKEAN [8].

A fixed probability space (Ω, Σ, P) is given. The σ-field Σ is assumed to be complete, and a field $\mathscr{F}$ is always to be understood as a sub σ-field of Σ which contains all null sets of Σ. The field generated by a collection $\{f_\alpha: \alpha\in A\}$ of random variables will be denoted by $\sigma\{f_\alpha: \alpha\in A\}$.

We will use HUNT's excellent operator formalism ([6], p. 44) and write $\mathscr{F}f$ for the conditional expectation of the random variable f given the field $\mathscr{F}$. This notation is especially appropriate when f is square integrable; $\mathscr{F}f$ is then simply the orthogonal projection of f onto the space $L^2(\Omega, \mathscr{F}, P)$.

Let $\mathscr{F}$ and $\mathscr{G}$ be two subfields of Σ and let $\mathscr{S}$ be a subfield of $\mathscr{F}$. We say that *$\mathscr{F}$ and $\mathscr{G}$ split over $\mathscr{S}$ or that $\mathscr{S}$ is a splitting field of $\mathscr{F}$ and $\mathscr{G}$ if $\mathscr{F}$ and $\mathscr{G}$ are conditionally independent given $\mathscr{S}$*, that is, if

$$\mathscr{S}(fg)=\mathscr{S}(f)\cdot\mathscr{S}(g) \tag{2.1}$$

for all bounded f and g with $\sigma\{f\}\subset\mathscr{F}$ and $\sigma\{g\}\subset\mathscr{G}$.

Splitting fields are easily characterized by the following lemma, which surprisingly seems to be new.

Lemma 2.1. *A field $\mathscr{S}\subset\mathscr{F}$ is a splitting field if and only if $\mathscr{S}$ contains the field*

$$\mathscr{S}_0=\sigma\{\mathscr{F}g: g \text{ bounded and } \mathscr{G}\text{-measurable}\}. \tag{2.2}$$

Proof. If $\mathscr{S}_0\subset\mathscr{S}\subset\mathscr{F}$ we will show that (2.1) holds. In fact, since $\mathscr{S}\subset\mathscr{F}$ we have $\mathscr{S}(f\cdot g)=\mathscr{S}\mathscr{F}(f\cdot g)=\mathscr{S}(f\cdot\mathscr{F}g)$. But $\mathscr{S}_0\subset\mathscr{S}$ implies that $\mathscr{S}g=\mathscr{F}g$ and thus $f\cdot\mathscr{F}g=f\cdot\mathscr{S}g$. Hence $\mathscr{S}(f\cdot g)=\mathscr{S}(f\cdot\mathscr{S}g)=\mathscr{S}(f)\cdot\mathscr{S}(g)$.

The converse is equivalent to showing that $\mathscr{S}g=\mathscr{F}g$ for any bounded $\mathscr{G}$-measurable function g and any splitting field $\mathscr{S}$. But $E(\mathscr{S}g)^2=E\mathscr{S}(g)\cdot\mathscr{F}(g)\leqq E(\mathscr{F}g)^2$, and the equality $\mathscr{S}g=\mathscr{F}g$ will follow from the converse of Schwarz's inequality if we prove $E(\mathscr{S}g)^2=E(\mathscr{F}g)^2$. This, however, follows from (2.1) by $E(\mathscr{S}g)^2=E\mathscr{S}(\mathscr{F}g\cdot g)=E\mathscr{F}(g)\cdot g=E(\mathscr{F}g)^2$.

The elementary properties of splitting fields proved by McKEAN [8] are now immediate consequences of Lemma 2.1. In particular, we have

Corollary 2.2. *(i)* $\mathscr{F}$ *is a splitting field*

(ii) $\mathscr{S}_0$ *is the minimal splitting field*

(iii) $\mathscr{F} \cap \mathscr{G} \subset \mathscr{S}_0$.

Simplifications occur when the fields $\mathscr{F}$ and $\mathscr{G}$ are generated by Gaussian systems. These are described in the following known lemma, which we include here for completeness.

Lemma 2.3. *Suppose* $\mathscr{F} = \sigma\{X_\phi:\ \phi \in \Phi\}$ *and* $\mathscr{G} = \sigma\{X_\gamma:\ \gamma \in \Gamma\}$ *where* $\{X_\alpha: \alpha \in \Phi \cup \Gamma\}$ *is a Gaussian family. Then*

$$\mathscr{S}_0 = \sigma\{\mathscr{F} X_\gamma: \gamma \in \Gamma\}.$$

Proof. Since polynomials in the variables $\{X_\gamma:\ \gamma \in \Gamma\}$ are dense in $L^2(\Omega, \mathscr{G}, P)$, it is enough to show for each monomial $X_{\gamma_1}^{k_1} \dots X_{\gamma_n}^{k_n}$ that $\mathscr{F}(X_{\gamma_1}^{k_1} \dots X_{\gamma_n}^{k_n})$ is measurable over $\sigma\{\mathscr{F} X_\gamma:\ \gamma \in \Gamma\}$. To this end we write $X_{\gamma_i} = Y_i + Z_i$ where $Y_i = \mathscr{F} X_{\gamma_i}$. Then the Z_i's are independent of $\mathscr{F}$, and we observe that $\mathscr{F}\left(\prod_1^n X_{\gamma_i}^{k_i}\right) = \mathscr{F}\left(\prod_1^n (Y_i + Z_i)^{k_i}\right)$ may be written as a sum of terms of the form $\mathscr{F}(\prod Y_i^{a_i} \cdot \prod Z_i^{b_i}) = \prod Y_i^{a_i} \mathscr{F}(\prod Z_i^{b_i})$. Since the Z_i's are independent of $\mathscr{F}$ we see that $\mathscr{F}(\prod Z_i^{b_i}) = E(\prod Z_i^{b_i})$ is a constant and the result follows.

We turn now to our definition of the Markov property. As formulated here the concept is perhaps too broad, and we must introduce further assumptions later to obtain deeper implications. The present generality has the advantage, however, of yielding the easy characterization in Theorem 3.3 of the next section.

Let $\{X_t:\ t \in T\}$ be a real or complex valued process whose time domain T is a smooth open set in some Euclidian space R_n. Everything that follows could easily be done when T is a smooth manifold. Because the changes necessary are easy to make we will not pursue the added generality. Let $D_- \subset T$ be an open set whose boundary in T is a smooth $n-1$ dimensional surface Γ in T. We write $\bar{D}_-$ for the closure $D_- \cup \Gamma$ of D_- and D_+ for the complement in T of $\bar{D}_-$.

With these conventions we introduce the following fields of events:

$$\begin{aligned}
\text{the "past"} &= \Sigma(D_-) \equiv \sigma\{X_t:\ t \in D_-\} \\
\text{the "future"} &= \Sigma(D_+) \equiv \sigma\{X_t:\ t \in D_+\} \\
\text{the "present"} &= \Sigma(\Gamma) \equiv \bigcap \{\Sigma(0): 0 \supset \Gamma, 0 \text{ is open}\} \\
\Sigma(\Gamma_-) &= \bigcap \{\Sigma(D_- \cap 0): 0 \supset \Gamma, 0 \text{ is open}\} \\
\Sigma(\Gamma_+) &= \bigcap \{\Sigma(D_+ \cap 0): 0 \supset \Gamma, 0 \text{ is open}\}.
\end{aligned}$$

We then define the Markov property.

Definition 2.4. $\{X_t:\ t \in T\}$ is called Markovian if for each open set $D_- \subset T$ with smooth boundary Γ we have

$$\Sigma(\Gamma_+) = \Sigma(\Gamma_-) = \Sigma(\Gamma), \tag{2.3}$$

and

(2.4) $\Sigma(\Gamma)$ is the minimal splitting field of $\Sigma(D_-)$ and $\Sigma(D_+)$.

Condition (2.3) is of a technical nature and says we see the same things in the very near future as in the very near past. Thus $\Sigma(\Gamma)\subset\Sigma(D_+)\cap\Sigma(D_-)$ and the definition requires that the splitting field of $\Sigma(D_-)$ and $\Sigma(D_+)$ is as small as possible and involves only boundary data.

The analogy with classical definitions of a Markov process is clear. In one dimension, however, our definition allows many processes that are not classically Markovian. For example, the integral of a Wiener process is Markovian by our definition but not classically so. In HIDA's terminology [5] it is a 2-ple Markov process.

For Gaussian processes Lemma 2.3 gives a useful translation of this definition into Hilbert space language. Each of the previously introduced fields is associated with a closed subspace of $L^2(\Omega, P)$. Namely, for the open set D_- we introduce:

$$H(D_-)=\text{closed linear span of } \{X_t\colon t\in\bar{D}_-\},$$

and the boundary space

$$H(\Gamma_-)=\bigcap\{H(D_-\cap 0)\colon 0\supset\Gamma, 0 \text{ is open}\}.$$

Analogously we define $H(D_+)$, $H(\Gamma_+)$ and $H(\Gamma)$. We will write H for $H(T)$.

When applied to variables in H, the conditional expectation operator $\Sigma(D_-)$ is simply the orthogonal projection onto the space $H(D_-)$, and Lemma 2.3 implies that the minimal splitting field of $\Sigma(D_-)$ and $\Sigma(D_+)$ is generated by the variables $\{\Sigma(D_-)X\colon X\in H(D_+)\}$. In this language the definition of the Markov property becomes:

Definition 2.4′. A Gaussian process $\{X_t\}$ is Markovian if for each open set $D_-\subset T$ with smooth boundary Γ,

(2.5) $$H(\Gamma_+)=H(\Gamma_-)=H(\Gamma)$$

and

(M.1) The projection of $H(D_+)$ onto $H(D_-)$ is $H(\Gamma)$.

Because of its later importance we introduce the special notation of $H_0(D_\pm)$ for the orthogonal complement of $H(\Gamma_\pm)$ in $H(D_\pm)$. Assuming condition (2.5) is satisfied, then each of the following two statements is equivalent to the Markov property for $\{X_t\}$.

(M.2) The orthogonal complement of $H(D_+)$ in H is $H_0(D_-)$.

(M.3) $$H=H_0(D_-)\oplus H(\Gamma)\oplus H_0(D_+).$$

3. Local Reproducing Kernel Spaces and the Markov Property for Gaussian Processes

In order to obtain convenient function space representations of the spaces $H(D)$ we introduce the formalism of reproducing kernel Hilbert spaces. For

Markov processes these spaces are easily characterized; when further assumptions are introduced in Part II we will see that the inner products have the form $\langle u, v\rangle = \sum_{\alpha,\beta} \int v_{\alpha\beta} D^{\alpha} u \overline{D^{\beta} v}$. This leads us to a class of Dirichlet problems (we have tried to use a suggestive notation).

Let $\{X_t\}$ be a Gaussian process with covariance function $R(t, s) = EX_t \overline{X}_s$. For an open set $D \subset T$ we introduce the function space $\mathscr{H}(D)$ which is isomorphic to $H(D)$; $\mathscr{H}(D) = \{u(s) = EX\overline{X}_s\colon s \in D \text{ and } X \in H(D)\}$ with the inner product

$$\langle u_1, u_2\rangle = EX_1 \overline{X}_2 \quad \text{where} \quad u_i(s) = EX_i \overline{X(s)}, \qquad \text{for } i = 1, 2.$$

The map $X \to EX\overline{X}_s$ is easily seen to be an isometry of $H(D)$ onto $\mathscr{H}(D)$. Instead of $\mathscr{H}(T)$ we will usually write $\mathscr{H}$, and we list without proof the following well-known elementary properties of $\mathscr{H}(D)$ (see [2]):

(3.1) $\mathscr{H}(D)$ is spanned by the functions $R(t, \cdot)$ for $t \in D$.

(3.2) For each u in $\mathscr{H}(D)$, $u(t) = \langle u, R(t, \cdot)\rangle$; that is, R is a reproducing kernel for $\mathscr{H}(D)$.

The functions $EX\overline{X}_s$ in $\mathscr{H}(D)$ were defined only for s in D, but $H(D) \subset H$ so that it makes sense for s to vary over T. Each u in $\mathscr{H}(D)$ thus extends to a function $\tilde{u}(s)$ in $\mathscr{H}$ with $\|u\|_{\mathscr{H}(D)} = \|u\|_{\mathscr{H}}$, and we may consider $\mathscr{H}(D)$ as a subspace of $\mathscr{H}$. Also note that the restriction of a function u in $\mathscr{H}$ to D defines a function in $\mathscr{H}(D)$ which may be interpreted as the projection of u onto $\mathscr{H}(D)$. We summarize these remarks.

Each function in $\mathscr{H}(D)$ has a unique norm preserving extension to a function in $\mathscr{H}$. The projection of a function u in $\mathscr{H}$ onto $\mathscr{H}(D)$ may be interpreted either as its restriction to D or as the unique element in $\mathscr{H}$ of minimal norm which agrees with u on D.

If D_- and D_+ are a complementary pair of open sets with smooth boundary Γ, we introduce spaces $\mathscr{H}(\Gamma_\pm)$ which are isomorphic to the boundary spaces $H(\Gamma_\pm)$,

$$\mathscr{H}(\Gamma_\pm) = \{u_s = EX\overline{X}_s\colon s \in D_\pm \text{ and } X \in H(\Gamma_\pm)\}.$$

If $H(\Gamma_+) = H(\Gamma_-)$ we may identify $\mathscr{H}(\Gamma_+)$ with $\mathscr{H}(\Gamma_-)$ and write $\mathscr{H}(\Gamma) = \mathscr{H}(\Gamma_+) = \mathscr{H}(\Gamma_-)$. This makes sense because $\mathscr{H}(D_+)$ and $\mathscr{H}(D_-)$ are both contained in $\mathscr{H}$. Thus $\mathscr{H}(\Gamma)$ is a subspace both of $\mathscr{H}(D_+)$ and of $\mathscr{H}(D_-)$.

The orthogonal complement of $\mathscr{H}(\Gamma)$ in $\mathscr{H}(D_+)$ (resp. $\mathscr{H}(D_-)$) is denoted as $\mathscr{H}_0(D_+)$ (resp. $\mathscr{H}_0(D_-)$). From the definition of $H(\Gamma)$ we see that $\mathscr{H}_0(D_+)$ may be described as the closure in $\mathscr{H}(D_+)$ of

(3.3) $$\{u \in \mathscr{H}(D_+)\colon u \text{ vanishes near } \Gamma\}.$$

Intuitively, $\mathscr{H}_0(D_+)$ consists of functions with zero boundary data, while $\mathscr{H}(\Gamma) \subset \mathscr{H}(D_+)$ consists of "harmonic" functions on D_+.

For reference we translate our earlier characterizations of the Markov property as follows.

Proposition 3.1. *A Gaussian process $\{X_t\colon t \in T\}$ satisfying the condition*

(3.4) $$\mathscr{H}(\Gamma_+) = \mathscr{H}(\Gamma_-) = \mathscr{H}(\Gamma)$$

is Markovian if and only if one of the three following equivalent conditions is satisfied:

(M.1)′ *The projection of* $\mathscr{H}(D_+)$ *onto* $\mathscr{H}(D_-)$ *is* $\mathscr{H}(\Gamma)$

(M.2)′ $$\mathscr{H}_0(D_-)=\mathscr{H}\ominus\mathscr{H}(D_+)$$

(M.3)′ $$\mathscr{H}=\mathscr{H}_0(D_-)\oplus\mathscr{H}(\Gamma)\oplus\mathscr{H}_0(D_+).$$

The next definition is introduced to characterize the kernel spaces associated with Markov processes. $\mathscr{L}$ is a linear function space on T with an inner product.

Definition 3.2. $\mathscr{L}$ is said to be local if for each pair of complementary open sets D_- and D_+ with smooth boundary Γ, conditions (L.1) and (L.2) are satisfied.*

(L.1) If u_+ and u_- are in $\mathscr{L}$ with the support of $u_\pm$ in $\bar{D}_\pm$, then $\langle u_+, u_-\rangle=0$.

(L.2) If $u=u_++u_-$ is in $\mathscr{L}$ with the support of u_+ in D_+ and support of u_- in $\bar{D}_-$, then u_+ and u_- are in $\mathscr{L}$.

Theorem 3.3. *A Gaussian process* $\{X_t\colon t\in T\}$ *which satisfies* $\mathscr{H}(\Gamma_-)=\mathscr{H}(\Gamma_+)=\mathscr{H}(\Gamma)$ *is Markovian if and only if the space* $\mathscr{H}=\mathscr{H}(T)$ *is local.*

Proof. Suppose $\{X_t\}$ is Markovian. We will first show that (L.1) is satisfied. Thus let $u_\pm\in\mathscr{H}$ and assume the support of $u_\pm\subset\bar{D}_\pm$. Then $\langle u_+, R(t,\cdot)\rangle=u_+(t)=0$ for $t\in D_-$. Since the functions $\{R(t,\cdot)\colon t\in\bar{D}_-\}$ span $\mathscr{H}(D_-)$ we conclude that u_+ is orthogonal to $\mathscr{H}(D_-)$ and by (M.2′) that $u_+\in\mathscr{H}_0(D_+)\subset\mathscr{H}(D_+)$. The same reasoning shows that u_- is orthogonal to $\mathscr{H}(D_+)$. Thus $\langle u_+, u_-\rangle=0$ and (L.1) is satisfied.

Turning to (L.2), let $u=u_++u_-\in\mathscr{H}$ with the support of u_+ in D_+ and the support of u_- in $\bar{D}_-$. Write u_0 for the projection of u onto $\mathscr{H}(D_+)$. Then u_0 vanishes near Γ since u does. Thus $u_0\in\mathscr{H}_0(D_+)$, and applying (M.2′) we see that $u_0\in\mathscr{H}\ominus\mathscr{H}(D_-)$; that is, u_0 vanishes on D_-. But u_0 agrees with u on D_+ so that $u_+=u_0$ is in $\mathscr{H}$ and (L.2) is satisfied.

To prove the converse we will show that if $\mathscr{H}$ is local then (M.2′) holds; that is $\mathscr{H}\ominus\mathscr{H}(D_+)=\mathscr{H}_0(D_-)$. We begin with the inclusion $\mathscr{H}\ominus\mathscr{H}(D_+)\subset\mathscr{H}(D_-)$. Let P be the projection onto $\mathscr{H}(D_-)$ and $u\in\mathscr{H}\ominus\mathscr{H}(D_+)$. We know that $u-Pu=0$ on $\bar{D}_-$ and that the support of u is in $\bar{D}_-$. Thus by (L.1), $\|(1-P)u\|^2=\langle u-Pu, u\rangle=0$ and hence $u=Pu$ and $u\in\mathscr{H}(D_-)$.

To show $u\in\mathscr{H}_0(D_-)$ it is now sufficient to show that u is orthogonal to $\mathscr{H}(\Gamma)$. This is clear since $\mathscr{H}(\Gamma)\subset\mathscr{H}(D_+)$ and by assumption u is orthogonal to $\mathscr{H}(D_+)$.

The proof of the identity $\mathscr{H}\ominus\mathscr{H}(D_+)=\mathscr{H}_0(D_-)$ will be complete if we prove $\mathscr{H}(D_+)\cap\mathscr{H}_0(D_-)=\{0\}$, and for this it suffices to prove that any u in $\mathscr{H}(D_+)\cap\mathscr{H}_0(D_-)$ vanishes on D_-. Moreover, since such a u must be orthogonal to $\mathscr{H}(\Gamma)$ we can find a sequence 0_n of open neighborhoods of Γ for which $\lim u_n=0$, where u_n is the projection of u onto $\mathscr{H}(D_+\cap 0_n)$. The problem thus reduces to showing that $v_n=u-u_n$ vanishes on D_-.

* Conditions (L.1) and (L.2) are satisfied by such examples as $L^2(R_1)$ and the Sobolev spaces H_m^2. Condition (L.2) is introduced to eliminate such examples as the space of even L^2 functions.

But v_n vanishes on $\bar{D}_+ \cap 0_n \supset \Gamma$ so that by (L.2) we may write $v_n = v_+ + v_-$ where v_- vanishes on D_+, v_+ has support in D_+ and $v_\pm \in \mathscr{H}$. By (L.1), $\langle v_+, v_- \rangle = 0$ and $\|v_n\|^2 = \|v_+\|^2 + \|v_-\|^2$. On the other hand, $\|v_n\| \leqq \|v_+\|$ since $v_n \in \mathscr{H}(D_+)$ and v_+ agrees with v_n on D_+. Hence $\|v_-\| = 0$ and $v_n = v_+$ vanishes on D_-.

II. Markov Processes of Finite Order

4. Smooth Local Spaces of Finite Order

We now show that if the function space $\mathscr{H}$ is local and rich enough then its inner product is given by a Dirichlet form. This result follows from PEETRE'S [10] characterization of differential operators. For the sake of completeness we include PEETRE'S proof here.

First we introduce some notation. For a smooth open set $T \subset R_n$, $C_0^\infty(T)$ is the set of infinitely differential functions with compact support contained in T. We will write C_0^∞ instead of $C_0^\infty(R_n)$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index where α_i's are non-negative integers, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. D^α denotes the differential operator $D^\alpha = \frac{d^{\alpha_1}}{d x_1^{\alpha_1}} \cdots \frac{d^{\alpha_n}}{d x_1^{\alpha_n}}$, and for the generic point $x = (x_1, \ldots, x_n)$ in R_n, x^α will denote the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let Δ denote the Laplacian

$$\Delta = \frac{d^2}{d x_1^2} + \cdots + \frac{d^2}{d x_n^2}.$$

For each $N \geqq 0$ we introduce an inner product $\langle \cdot, \cdot \rangle_N$ on $C_0^\infty(T)$; namely

$$\langle f, g \rangle_N = \int_T \sum_{|\alpha| \leqq N} D^\alpha f \overline{D^\alpha g}.$$

$H_0^N(T)$ denotes the completion of $C_0^\infty(T)$ with respect to the norm $\|f\|_N^2 = \langle f, f \rangle_N$. Clearly $H_0^N(T) \subset H_0^{N+1}(T)$. Functions in $H_0^N(T)$ have strong L^2 derivatives of all orders up to N; the Fourier representation easily shows that for $|\alpha| < N - n/2$ there is a constant $C(\alpha)$ such that the Sobolev like inequality

$$\sup_{t \in R_n} |D^\alpha f(t)| \leqq C(\alpha) \|f\|_N \tag{4.1}$$

holds for all f in C_0^∞, and hence by continuity this holds for all f in $H_0^N(T)$.

Convergence in $C_0^\infty(T)$ is defined as follows: $\{f_n\}$ converges to f if there is a bounded open set U with $\bar{U} \subset T$ for which $f_n \in C_0^\infty(U)$ for each n and such that f_n converges to f in $H_0^N(U)$ for all $N \geqq 0$.

The inequality (4.1) shows that $C_0^\infty(T)$ is complete.

We will call a bilinear form

$$B(f, g) = \sum_{\alpha, \beta} \int_T a_{\alpha\beta}(t) D^\alpha f(t) \overline{D^\beta g(t)}$$

a Dirichlet form if the coefficients $a_{\alpha\beta}(t)$ are locally square summable and if on each compact set all but a finite number of the $a_{\alpha\beta}$ vanish. Such a form is well-defined and continuous on $C_0^\infty(T)$. Moreover, if B is symmetric and positive then $C_0^\infty(T)$ together with the inner product $B(f, g)$ is a local space.

The following theorem states that any reasonable positive local form on $C_0^\infty(T)$ is continuous and that if a local form is continuous then it agrees with some Dirichlet form. The proof of parts (*ii*) and (*iii*) are adopted from PEETRE [10].

Theorem 4.1. (*i*) *A non-negative form* $\langle\cdot,\cdot\rangle$ *on* $C_0^\infty(T)$, *which satisfies the honesty condition: if* $u_n \to 0$ *in* $C_0^\infty(T)$ *and* $\langle u_n - u_m, u_n - u_m\rangle \to 0$ *as* $n, m \to \infty$, *then* $\langle u_n, u_n\rangle \to 0$, *is continuous.*

(*ii*) *If* $\langle\cdot,\cdot\rangle$ *is a non-negative local form on* $C_0^\infty(T)$, *there exists a discrete set* $\Delta \subset T$ *such that* $\langle\cdot,\cdot\rangle$ *is continuous on* $C_0^\infty(T-\Delta)$.

(*iii*) *If* $\langle\cdot,\cdot\rangle$ *is a continuous local form on* $C_0^\infty(T)$, *then* $\langle\cdot,\cdot\rangle$ *is a Dirichlet form.*

Proof of (i). For any relatively compact open subset Ω of T, the space $H_0^\infty(\Omega) \equiv \bigcap N_0^N(\Omega)$ is complete with respect to the quasi-norm

$$\|f\| = \sum_{n \geqq 0} 2^{-N} \cdot \|f\|_N (1 + \|f\|_N)^{-1}.$$

Since $\langle\cdot,\cdot\rangle$ is everywhere defined on $H_0^\infty(\Omega)$ we may define another quasi-norm $|f| = \|f\| + \langle f, f\rangle^{\frac{1}{2}}$ on $H_0^\infty(\Omega)$.

It now suffices to show that the two quasi-norms $\|\cdot\|$ and $|\cdot|$ are equivalent. To see this let $\{f_n\}$ be a Cauchy sequence with respect to $|\cdot|$. Then $\{f_n\}$ is Cauchy with respect to $\|\cdot\|$ and thus has a $\|\cdot\|$ limit f in $H_0^\infty(\Omega)$. This may then be used to determine a linear map S from the $|\cdot|$ completion of $H_0^\infty(\Omega)$ into $H_0^\infty(\Omega)$. S is easily seen to be continuous and onto, and therefore by the open mapping theorem S is open. Thus if S^{-1} exists it must be continuous, and this would show that $\|\cdot\|$ and $|\cdot|$ are equivalent. But S^{-1} will exist if and only if S is one to one or, what is the same thing, if and only if ker $S = \{0\}$. This, however, is precisely the content of our honesty condition and the result follows.

Proof of (ii). We will call a point $x \in T$ singular if the form $\langle\cdot,\cdot\rangle$ is unbounded on $C_0^\infty(0)$ for each neighborhood 0 of x; that is, if there are functions f_k in $C_0(0)$ with $\|f_k\|_k \leqq 1$ and $\langle f_k, f_k\rangle \uparrow \infty$. We claim that any relatively compact open set Ω with $\bar{\Omega} \subset T$ contains only a finite number of singular points.

If this were not so we could find open sets $0_k \subset \Omega$ with disjoint closures and functions f_k in $C_0^\infty(0_k)$ which satisfy $\|f_k\|_k \leqq 1/k^2$ and $\langle f_k, f_k\rangle \geqq 1$. Then $f = \sum f_k$ is in $C_0^\infty(T)$ and because $\langle\cdot,\cdot\rangle$ is local we would have for each N,

$$\langle f, f\rangle \geqq \left\langle \sum_1^N f_k, \sum_1^N f_k \right\rangle \geqq N,$$

which is impossible.

Now let Δ be the discret set of singular points and let Ω be a relatively compact open set with $\bar{\Omega} \subset T - \Delta$. An elementary partition of unity argument easily yields the existence of an integer N and a constant C for which the inequality $\langle f, f\rangle \leqq C\|f\|_N^2$ holds for all f in $C_0^\infty(\Omega)$. Thus $\langle\cdot,\cdot\rangle$ is continuous on $C_0^\infty(\Omega)$ and hence on $C_0^\infty(T-\Delta)$.

Proof of (iii). For each bounded open set U with $\bar{U} \subset T$ there is an N' for which

$$|\langle f, g\rangle| \leqq \|f\|_{N'} \|g\|_{N'}; \quad \text{for } f, g \in C_0^{N'}(U). \tag{4.2}$$

Let $N = N' + k$ where $k > n/2$. The inclusion i: $H_0^N(U) \to H_0^{N'}(U)$ is known to be Hilbert-Schmidt so that if $\{f_j\} \subset C_0^\infty(U)$ is an orthonormal basis in $H_0^N(U)$ we have $\sum \|f_j\|_{N'}^2 < \infty$.

On finite sums $\sum b_{ij} f_i(s) f_j(t)$ we define the functional

(4.3) $$F\left(\sum b_{ij} f_i(s) f_j(t)\right) = \sum b_{ij} \langle f_i, \bar{f}_j \rangle .$$

But $\langle f_{ij}, \bar{f}_j \rangle \leqq \|f_i\|_N \|f_j\|_{N'}$ so that $\sum |\langle f_i, \bar{f}_j \rangle|^2 < \infty$, and we see that F is continuous on a dense subspace of $H_0^{2N}(U \times U)$. The functional F thus determines a distribution $F \in C_0^\infty(U \times U)'$ for which

(4.4) $$F(f(s) g(t)) = \langle f, \bar{g} \rangle .$$

Because $\langle \cdot, \cdot \rangle$ is local the support of F is contained in the diagonal of $U \times U$, and using (4.4) we conclude that there are finitely many measures $\mu_{\alpha\beta}$ on U which give

$$\langle f, \bar{g} \rangle = \sum \int_U \mu_{\alpha\beta}(dt) D^\alpha f D^\beta g .$$

The measures $\mu_{\alpha\beta}$ may be written in the form $(1-\Delta)^m a'_{\alpha\beta}$ where the $a'_{\alpha\beta}$ are continuous functions. Integrating by parts and changing g to $\bar{g}$, we obtain the desired expression

$$\langle f, g \rangle = \sum \int_U a_{\alpha\beta}(t) D^\alpha f(t) \overline{D^\beta g(t)} .$$

Because U was arbitrary the proof is complete.

Note that the set Δ of singular points for a non-negative local form on $C_0^\infty(R_n)$ which commutes with translations must be closed under translations. Since Δ is discrete by part *(ii)* we see that Δ is empty.

Corollary 4.2. *A non-negative local form on $C_0^\infty(R_n)$ which commutes with translations is a Dirichlet form.*

5. Markov Processes of Finite Order

It is reasonable to expect that if an inner product $\langle \cdot, \cdot \rangle$ on $C_0^\infty(T)$ is given by a well-behaved Dirichlet form, then there should be naturally associated with $\langle \cdot, \cdot \rangle$ a Gaussian Markov process. In this section we will investigate these problems when the form $\langle \cdot, \cdot \rangle$ is elliptic. In this case we can obtain rather complete results, and the processes that we obtain are the analogues of the Markov processes of order p discussed at the end of Section 1.

A general answer to the question of when the completion of $(C_0^\infty(T), \langle \cdot, \cdot \rangle)$ is the reproducing kernel space associated with some Gaussian process is contained in the following known

Proposition 5.1. *Let $\mathscr{H}$ be the completion of $(C_0^\infty(T), \langle \cdot, \cdot \rangle)$. Then $\mathscr{H}$ is a reproducing kernel space if and only if for each $t \in T$ the functional $u \to u(t)$ is continuous with respect to the norm $\langle u, u \rangle^{1/2}$ on $C_0^\infty(T)$.*

Proof. If $\mathscr{H}$ has a reproducing kernel $R(t, s)$, then $u \to u(t) = \langle u, R(t, \cdot) \rangle$ is continuous.

Conversely if $u \to u(t)$ is continuous and if u_n is any sequence which converges in $\mathscr{H}$, we see that $u_n(t)$ converges for each t in T. Thus $\mathscr{H}(T)$ is a function space. Moreover, since $u \to u(t)$ is continuous we know that for each t there is a function $r_t(s)$ in $\mathscr{H}(T)$ such that $u(t) = \langle u, r_t \rangle$. Setting $R(t, s) = r_t(s)$, we see that R is a reproducing kernel for $\mathscr{H}$ and the proof is complete.

Note that $R(t, s)$ is non-negative definite and thus is the correlation function of some Gaussian process $\{X_t \colon t \in T\}$.

Now let

$$\langle u, v \rangle = \sum_{|\alpha|, |\beta| \leqq p} \int_T a_{\alpha\beta}(t)\, D^\alpha u(t)\, \overline{D^\beta v(t)} \tag{5.1}$$

be a positive symmetric Dirichlet form on $C_0^\infty(T)$. We will assume that the coefficients $a_{\alpha\beta}$ are bounded, uniformly continuous and infinitely differentiable. The form $\langle \cdot, \cdot \rangle$ is called *uniformly strongly elliptic* if there is a constant $C > 0$ such that for all $x \in R_n$ and $t \in T$ we have

$$\Big| \mathrm{Re} \sum_{|\alpha|, |\beta| = p} a_{\alpha\beta}(t)\, X^{\alpha+\beta} \Big| \geqq C |X|^{2p}. \tag{5.2}$$

If the form $\langle \cdot, \cdot \rangle$ is uniformly strongly elliptic then it follows from Garding's inequality (see [1], p. 78) that for each $\lambda > 0$ the two norms $\langle u, u \rangle^{1/2} + \lambda \|u\|_0$ and $\|u\|_p$ are equivalent on $C_0^\infty(T)$, where $\| \ \|_0$ is the L^2 norm and $\| \ \|_p$ is the p^{th} order Sobolev norm. Thus if the norm $\| \ \|_0$ is continuous with respect to the inner product $\langle \cdot, \cdot \rangle$ we may conclude that the space $\mathscr{H}$ coincides as a function space with the space $H_0^p(T)$ and that the two norms are equivalent.

Theorem 5.2. *Let $\langle \cdot, \cdot \rangle$ be a uniformly strongly elliptic inner product of order p on $C_0^\infty(T)$ and let $\mathscr{H}$ be the completion of $C_0^\infty(T)$ with respect to this inner product. If the norm $\|u\| = \langle u, u \rangle^{1/2}$ is equivalent to the norm $\| \ \|_p$, then $\mathscr{H}$ is local. If moreover $2p > n$, then $u \to u(t)$ is continuous, and hence $\mathscr{H}$ is a reproducing kernel space.*

Proof. We have noted that $\mathscr{H}$ contains the same functions as $H_0^p(T)$, and thus for any pair $u_\pm$ in $\mathscr{H}$ we have

$$\langle u_+, u_- \rangle = \sum_{|\alpha|, |\beta| \leqq p} \int_T a_{\alpha\beta}(t)\, D^\alpha u_+(t)\, \overline{D^\beta u_-(t)};$$

and these integrals converge absolutely.

It follows that if Γ is a smooth surface which separates T into open sets $D_\pm$, then $\langle u_+, u_- \rangle = 0$ for any two functions u_+ and u_- in $\mathscr{H}$ with the support of $u_\pm$ in $D_\pm$. (L.1) is thus satisfied.

(L.2) follows from the observation that if $u = u_+ + u_-$ is in $\mathscr{H}$ and the support of u_+ is in D_+ while the support of u_- is in $\bar{D}_-$, then $u_+ \in H_0^p(D_+)$.

The last statement follows from inequality (4.1) with $\alpha = 0$.

Now let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be as in Theorem 5.2 with $2p > n$ and let $R(t, s)$ be the associated reproducing kernel. For technical reasons we need to know that $R(t, s)$ is jointly continuous or, what is the same, that $t \to R_t$ is continuous from T to $\mathscr{H}$. This follows from the general theory of fundamental solutions for elliptic operators, but it also admits an elementary Hilbert space proof which we now give.

On $\mathscr{H}$ the inner products $\langle\cdot,\cdot\rangle$ and

$$(u,v)_p=\sum_{|\alpha|\leqq p}\int_T D^\alpha u(t)\overline{D^\beta v(t)}$$

are equivalent. Thus there is a reproducing kernel $\rho(t,s)$ in $(\mathscr{H},(\cdot,\cdot)_p)$ and a bounded operator B on $\mathscr{H}$ for which

$$(u,v)_p=\langle u,Bv\rangle \quad \text{for all } u \text{ and } v \text{ in } \mathscr{H}\,.$$

The identity

$$u(t)=(u,\rho_t)_p=\langle u,R_t\rangle \quad \text{for all } u \text{ in } \mathscr{H}$$

implies that $R_t=B\rho_t$, and since B is continuous it suffices to show $t\to\rho_t$ is continuous.

But $\rho_t=P\tilde{\rho}_t$ where P: $H_0^p(R_n)\to H_0^p(T)$ is the projection and $\tilde{\rho}(t,s)$ is the reproducing kernel in $H_0^p(R_n)$. The inequality (4.1) shows $H_0^p(R_n)$ is populated with continuous functions $u(t)=(u,\tilde{\rho}_t)_p$. This implies $t\to\tilde{\rho}_t$ is weakly continuous. It is thus strongly continuous if $t\to\|\tilde{\rho}_t\|$ is continuous. But $\|\tilde{\rho}_t\|$ is constant since $(\cdot,\cdot)_p$ is translation invariant on $H_0^p(R_n)$.

Actually we can prove considerably more with these methods. By the translation invariance of $(\cdot,\cdot)_p$ it follows that $\tilde{\rho}(t,s)$ is a function $\tilde{\rho}(t-s)$ of the difference $t-s$, and elementary manipulations show that

$$\tilde{\rho}(t)=c\int_{R_n} e^{it\cdot x}\Big(\sum_{|\alpha|\leqq p} x^{2\alpha}\Big)^{-1}$$

and, moreover, that $\tilde{\rho}(t)$ is Lipschitz continuous. It then follows from

$$\|R_t-R_s\|^2=\|BP\tilde{\rho}_t-BP\tilde{\rho}_s\|^2\leqq\|B\|^2\,\|\tilde{\rho}_t-\tilde{\rho}_s\|_p^2$$

that $R(t,s)$ is Lipschitz.

Standard arguments (see e.g. [12]) show that any centered Gaussian process $\{X_t\colon t\in T\}$ with a Lipschitz covariance function R may be modified to have continuous sample paths. We will assume this has been done and will investigate the boundary spaces $H(\Gamma)$ associated with a smooth $p-1$ dimensional surface Γ in T.

The first result says that in an appropriate weak sense $\{X_t\}$ has $p-1$ normal derivatives on Γ.

To formulate this we let $d\sigma$ be the surface measure on Γ and let $\dot{s}$ be a continuous choice of the unit normal vector to Γ at the point s on Γ. For each function f in $L^2(\Gamma)$ with compact support we introduce the function

$$F(h)=\int_\Gamma f(s)\,X_{s+h\dot{s}}\,d\sigma \tag{5.3}$$

of the real variable h.

Lemma 5.3. *As an H valued function $F(h)$ is $p-1$ times continuously differentiable.*

Proof. Let $X\in H$ and let $u(s)=EX\overline{X}_s$. Then $u\in\mathscr{H}$ and hence is in $H_0^p(T)$. It now follows from the Imbedding Theorem of Sobolev ([13], p. 69) that for $|\beta|\leqq p-1$, $D^\beta u(s)\in L^2_{\text{loc}}(\Gamma)$ and that as a function of h, with values in $L^2_{\text{loc}}(\Gamma)$,

$(D^\beta u)(s+h\dot{s})$ is continuous. In particular, for $f \in L^2(\Gamma)$ with compact support,

$$\int_\Gamma \bar{f}(s)\, u(s+h\dot{s})\, d\sigma$$

has $p-1$ continuous derivatives. But

$$\int_\Gamma \bar{f}(s)\, u(s+h\dot{s})\, d\sigma = EX \int_\Gamma \bar{f}(s)\, \bar{X}_{s+h\dot{s}}\, d\sigma,$$

and because $X \in H$ was arbitrary we conclude that $\int_\Gamma f(s) X_{s+h\dot{s}}\, d\sigma$ has $p-1$ continuous weak derivatives. But the continuity of weak derivatives implies that they are strong derivatives and the result follows.

Following McKEAN in [8] we now define a p-th order Markov process. Suppose that for all f the function $F(h)$ is for small h, $p-1$ times continuously differentiable in measure. We then introduce the "differential" σ-field

$$\Sigma_p(\Gamma) = \sigma\{F_{(0)}^{(k)}: 0 \leqq k < p, f \in L^2(\Gamma) \text{ has compact support}\}. \tag{5.4}$$

If Γ splits T into D_+ and D_- then it is clear that $\Sigma_p(\Gamma) \subset \Sigma(\Gamma_+) \cap \Sigma(\Gamma_-)$. If, in addition, for all such Γ, $\Sigma_p(\Gamma)$ is the minimal splitting field of $\Sigma(D_-)$ and $\Sigma(D_+)$ we will call $\{X_t\}$ *Markovian of order p.*

If $\{X_t\}$ is Gaussian we will also introduce the Hilbert spaces

$$H_p(\Gamma) = \operatorname{span}\{F_{(0)}^{(k)}: 0 \leqq k < p, f \in L^2(\Gamma) \text{ has compact support}\}. \tag{5.5}$$

In this case we see that a Gaussian process $\{X_t\colon t \in T\}$ is *Markovian of order p* iff for each f in $L^2(\Gamma)$ with compact support, $F(h)$ has $p-1$ continuous derivatives and the projection of $H(D_+)$ onto $H(D_-)$ is $H_p(\Gamma)$.

Theorem 5.4. *Let $\{X_t\colon t \in T\}$ be a Gaussian process and let $\mathscr{H}$ be the associated reproducing kernel space. Suppose that $\mathscr{H}$ contains $C_0^\infty(T)$ as a dense subspace, that the inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{H}$ is given by a uniformly strongly elliptic Dirichlet form of degree $p > n/2$, and that the norm on $\mathscr{H}$ is equivalent to $\|\ \|_p$. Then $\{X_t\}$ is Markovian of order p.*

Proof. By combining Theorem 5.2 and Lemma 5.3 it only remains to identify $H_p(\Gamma)$ with $H(\Gamma)$. But $H_p(\Gamma) \subset H(\Gamma)$, and we will show that $H_p(\Gamma) = H(\Gamma)$ by showing that the orthogonal complement of $H_p(\Gamma)$ in $H(\Gamma)$ is $\{0\}$. Thus we let $X \in H(\Gamma) \ominus H_p(\Gamma)$ and set $u(s) = EX\bar{X}_s$. Then $u \in \mathscr{H}(\Gamma)$ and because $\{X_t\}$ is Markovian we know that $C_0^\infty(D_-) \subset \mathscr{H}_0(D_-)$. Thus, for any $\phi \in C_0(D_-)$ we have $0 = \langle u, \phi \rangle$.

Now let $f \in L^2(\Gamma)$ have compact support and observe that for $k \leqq p-1$:

$$\int_\Gamma \overline{f(s)} \frac{\partial^k}{\partial \eta^k} u(s)\, d\sigma = EX \left[\frac{d^k}{dh^k} \int_\Gamma \overline{f(s)}\, \bar{X}_{s+h\dot{s}}\, d\sigma\right]_{h=0} \tag{5.6}$$

where $\frac{\partial^k}{\partial \eta^k} u(s)$ is the k-th normal derivative of u at $s \in \Gamma$. But by assumption the right side equals zero and we conclude that

$$\frac{\partial^k}{\partial \eta^k} u = 0 \text{ a.e. on } \Gamma \quad \text{for } k = 0, \ldots, p-1. \tag{5.7}$$

It is now a simple approximation argument to show that because of the conditions (5.1) the restriction of u to D_- is in $H_0^p(D_-)$ and thus is the limit in $\mathscr{H}(T)$ of functions ϕ_n in $C_0^\infty(D_-)$.

We have already seen, however, that $\langle u, \phi\rangle = 0$ for each $\phi \in C_0^\infty(D_-)$, and we conclude that $u(s)\equiv 0$ on D_-. By the same argument $u(s)\equiv 0$ on D_+, and we see that u is the zero function and that $X\equiv 0$.

6. *The Spaces $\mathscr{H}(D_\pm)$, $\mathscr{H}_0(D_\pm)$ and $\mathscr{H}(\Gamma)$*

The prediction problem for $\{X_t\colon t\in T\}$ when $\langle\cdot,\cdot\rangle$ is uniformly strongly elliptic of order $p>n/2$ is given in the next section. As a first step we give a new characterization of the spaces $\mathscr{H}(D_\pm)$, $\mathscr{H}_0(D_\pm)$ and $\mathscr{H}(\Gamma)$.

The key idea, which goes back to McKean [8] and Molchan [9], is to introduce the operator

$$Au = \sum_{|\alpha|,\,|\beta|\leqq p} (-1)^{|\alpha|}(D^\alpha a_{\alpha\beta} D^\beta u) \tag{6.1}$$

associated with the inner product

$$\langle u, v\rangle = \sum_{|\alpha|,\,|\beta|\leqq p} \int_T a_{\alpha\beta}(t)\, D^\alpha u(t)\, \overline{D^\beta u(t)} \tag{6.2}$$

on $\mathscr{H}$. If the form $\langle\cdot,\cdot\rangle$ is uniformly strongly elliptic and satisfies the conditions of Theorem 5.4, the operator A is also uniformly strongly elliptic and formally self-adjoint.

The various subspaces $\mathscr{H}(D_+)$, $\mathscr{H}(D_-)$ and $\mathscr{H}(\Gamma)$ of $\mathscr{H}(T)$ are easily described in terms of A.

Recall that $\mathscr{H}(D_+)$ may be identified as the orthogonal complement in $\mathscr{H}(T)$ of $\mathscr{H}_0(D_-)$ and that the space $\mathscr{H}_0(D_-)$ may be identified as the closure in $H_0^p(T)$ of the space $C_0^\infty(D_-)$. Thus if $u\in\mathscr{H}(D_+)$ and $\phi\in C_0^\infty(D_-)$ we have $0=\langle u,\phi\rangle$. Integrating by parts we see that

$$0 = \int_{D_-} u(t)\overline{A\phi(t)} \qquad \text{for all } \phi\in C_0^\infty(D_-).$$

But A is formally self-adjoint so that on D_- u is a weak solution of $Au=0$. By the regularity theorems for elliptic operators ([1], p. 131) it follows that u is actually infinitely differentiable and is a classical solution of $Au=0$ on D_-.

Conversely if $u\in\mathscr{H}$ and satisfies $Au=0$ on D_- we see that $\langle u,\phi\rangle=0$ for each $\phi\in C_0^\infty(D_-)$ and hence that $u\in\mathscr{H}(D_+)$. We thus have

Proposition 6.1. (*i*) *A function u defined on T is in the space $\mathscr{H}(D_\pm)$ if and only if $u\in H_0^p(T)$ and satisfies*

$$Au(t)=0 \qquad \textit{for } t\in D_\mp .$$

(*ii*) *A function u in $\mathscr{H}$ is in $\mathscr{H}(\Gamma)=\mathscr{H}(D_+)\cap\mathscr{H}(D_-)$ if and only if*

$$Au(t)=0 \qquad \textit{for } t\in D_+\cup D_- .$$

The question of when a function $u(t)$ defined on D_- is the restriction of a function in $\mathscr{H}$ to D_- is answered by the following

Proposition 6.2. *A function $u(t)$ defined on D_- is the restriction to D_- of some function in $\mathscr{H}$ if and only if it has an extension $\tilde{u}(t)$ in $H_0^p(T)$ which satisfies*

$$A\tilde{u}(t)=0 \quad \textit{for } t \textit{ in } D_+ .$$

The restriction of $\tilde{u}(t)$ to D_+ is the unique solution in $\mathscr{H}(D_+)$ of the exterior Dirichlet problem

$$A\tilde{u}(t)=0 \quad \textit{for } t\in D_+$$

with the same boundary data as u on Γ.

Proof. The only point which requires comment is the uniqueness of $\tilde{u}$, and for this it suffices to show that if $u\in\mathscr{H}$, u has zero boundary data on Γ, and $Au(t)\equiv 0$ on D_+ then $u(t)\equiv 0$ on D_+.

Let v be the projection of u onto $\mathscr{H}(D_+)$. Then $v(t)=EX\bar{X}_t\in\mathscr{H}(\Gamma)$ and has zero boundary data. Thus if $f\in L^2(\Gamma)$ has compact support then

$$0=\int_\Gamma \bar{f}(s)\frac{\partial^k}{\partial\eta^k}u(s)\,d\sigma=EX\left[\frac{d^k}{dh^k}\int_\Gamma \overline{f(s)}\,\bar{X}_{s+h\,\dot{s}}\,d\sigma\right]_{h=0}$$

for $k=0,\dots,p-1$, and we conclude that $EX\overline{F^k_{(0)}}=0$. Since $X\in H(\Gamma)$ and the $F^h(0)$ span $H(\Gamma)$, we conclude that $X=0$. Thus $v\equiv 0$ and since $u(t)$ agrees with $v(t)$ on D_+ we must have $u(t)\equiv 0$ on D_+.

Note. It follows from the Calderón extension theorem ([1], p. 171) and Proposition 6.2 that if D_- is bounded with $\bar{D}_-\subset T$ and if u is in the Sobolev space $H^p(D_-)$, then u extends to an element in $\mathscr{H}$.

7. Solution of the Prediction Problem

The prediction problem for $\{X_t\}$ is most easily discussed in terms of the Dirichlet problem for A. To this end we let $D_-\subset T$ be a bounded open set with smooth compact boundary Γ. Let D_+ be the complementary open set in T. For t in D_+, let $h_t(\cdot)$ be the projection of $R(t,\cdot)$ onto $\mathscr{H}(D_-)$. Then $h_t(s)$ agrees with $R(t,s)$ for s in D_- and satisfies $Ah_t(s)=0$ for s in D_+. Thus $R(t,\cdot)-h_t(\cdot)$ vanishes on D_- and hence belongs to $\mathscr{H}_0(D_+)$. Moreover, the restriction of $h_t(s)$ to each of the sets $D_-\cup\Gamma$ and $D_+\cup\Gamma$ is smooth.

Now let $u\in\mathscr{H}(D_-)$. Then

$$0=\langle u, R(t,\cdot)-h_t(\cdot)\rangle$$

and since $u(t)=\langle u, R(t,\cdot)\rangle$ we see that

$$u(t)=\langle u, h_t\rangle \quad \text{for } t\in D_+ . \tag{7.1}$$

Approximating u by a sequence $u_n\in\mathscr{H}$ with compact support we have $u(t)=\lim\langle h_t, u_n\rangle$ or

$$u(t)=\lim_{n\to\infty}\sum_{|\alpha|,|\beta|\le p}\int_T a_{\alpha\beta}(s)\,D^\alpha u_n(s)\,\overline{D^\beta h_t(s)}\,. \tag{7.2}$$

Integrating by parts to bring the derivatives onto h_t and using the fact that $Ah_t(s)=0$ for $s \notin \Gamma$, we may pass to the limit to obtain a formula

(7.3) $$u(t)=\int_\Gamma \sum_{j=0}^{p-1} b_j(t,s) \frac{\partial^i u}{\partial n_j}(s)\, d\sigma, \qquad t \in D_+$$

where the functions $b_j(t,s)$ are C^∞ functions of s on Γ.

Formula (7.3) can now be used to solve the least squares prediction problem as was illustrated by McKean [8]. That is, we can use (7.3) to calculate the projection of X_t, $t \in D_+$ onto the space $H(D_-)$.

Set

(7.4) $$\hat{X}(t)=\int_\Gamma \sum_{j=0}^{p-1} \overline{b_j}(t,s) \frac{\partial^j X}{\partial \eta_j}(s)\, d\sigma$$

where the derivatives $\dfrac{\partial^j X}{\partial \eta_j}$ are to be interpreted in the sense

(7.5) $$\int_\Gamma f(s) \frac{\partial^j X}{\partial \eta^j}(s)\, d\Gamma = \frac{d^j}{dh^j} \int f(s) X_{s+h\dot{s}}\, d\sigma .$$

We claim that $\hat{X}(t)$ is the projection of X_t onto $H(D_-)$. To see this, we only have to show that for $\tau \in D_-$,

$$EX_\tau \overline{X}_t = EX_\tau \overline{\hat{X}}_t .$$

But for $\tau \in D_-$, $EX_\tau \overline{X}_t = R(\tau, t)$ is in $\mathscr{H}(D_-)$ and hence satisfies

(7.6) $$\begin{aligned} R(\tau,t) &= \int_\Gamma \sum_{j=0}^{p-1} b_j(t,s) \frac{\partial^j}{\partial \eta^j} R(\tau,s)\, d\sigma \\ &= \overline{\sum_{j=0}^{p-1} EX_\tau \int_\Gamma \overline{b}_j(t,s) \frac{\partial^i}{\partial \eta^j} X_s\, d\sigma} = EX_\tau \overline{\hat{X}}_t . \end{aligned}$$

Since $\hat{X}_t \in H(D_-)$ it follows that $\hat{X}_t$ is the projection of X_t onto $H(D_-)$.

For completeness we formulate the

Proposition 7.1. *Assume the Gaussian process $\{X_t\colon t \in T\}$ is Markovian of order p and satisfies the conditions of Theorem 5.4. Let $\Gamma \subset T$ be a smooth compact surface which separates T into complementary open sets D_- and D_+ where D_- is bounded. Then there exist smooth functions $b_j(t,s)$, $t \in D_+$ and $s \in \Gamma$ so that for each smooth function u defined in a neighborhood of Γ,*

(7.7) $$\tilde{u}(t)=\int_\Gamma \sum_{j=0}^{p-1} b_j(t,s) \frac{\partial^j u(s)}{\partial \eta^j}\, d\sigma, \qquad t \in D_+$$

is the unique solution in $\mathscr{H}(D_+)$ of the exterior Dirichlet problem

(7.8) $$A\tilde{u}(t)=0, \qquad t \in D_+$$

with

$$\frac{\partial^j u}{\partial \eta^j}(s)=\frac{\partial^j u}{\partial \eta^j}(s), \qquad s \in \Gamma .$$

For $t\in D_+$ the least squares prediction of X_t for given $\{X_s\colon s\in D_-\}$ is then

$$X_t=\int_\Gamma \sum_{j=0}^{p-1} \overline{b_j(t,s)}\,\frac{\partial^j X(s)}{\partial \eta^j}\,d\sigma. \tag{7.9}$$

Note that just as the problem of predicting X_t for $t\in D_+$ when we are given $\{X_s\colon s\in D_-\}$ is related to exterior Dirichlet problem for the operator A, the interpolation problem of predicting X_s for s in D_- when we are given $\{X_t\colon t\in D_+\}$ is related to the standard interior Dirichlet problem for A.

III. Stationary Gaussian Markov Processes on R_n

A process $\{X_t\colon t\in R_n\}$ is called stationary if for each choice of $t_1,\dots,t_k$ in R_n the distribution of $X_{t_1+t},\dots,X_{t_k+t}$ is independent of $t\in R_n$. For stationary Gaussian processes it is possible to obtain much more detail than the general results in Part I.

The mean of a stationary process is constant and may be normalized with $EX_t\equiv 0$. The covariance function $EX_t\overline{X}_s$ is then a function $\rho(t-s)$ of the difference $t-s$. Being a covariance function, ρ is positive definite and by Bochner's theorem it is the Fourier transform of a unique positive finite measure $\Delta(dx)$ on R_n; that is,

$$\rho(t)=\int_{R_n} e^{it\cdot x}\Delta(dx).$$

In the next three sections we prove some elementary results which relate the Markovian character of $\{X_t\}$ to analytic properties of the spectral measure Δ. Section 8 is a general discussion of the Fourier representation. Section 9 is a further study of Δ under the assumption that $\{X_t\}$ has the Markov property for half-spaces. In Section 10 we prove that if $\{X_t\}$ is Markovian of degree p then Δ^{-1} is an elliptic polynomial of degree $2p$.

8. The Spectral Representation of $\mathscr{H}(T)$

The identity $EX_t\overline{X}_s=\int e^{it\cdot x}e^{-is\cdot x}\Delta(dx)$ implies that the map $X_t\to e^{it\cdot x}$ determines a linear isometry from the space H onto $L^2(R_n,\Delta)$. Thus each element X of H is associated with a unique element f of $L^2(R_n,\Delta)$. If $u(s)=EX\overline{X}_s$ is the corresponding element of $\mathscr{H}$ we have

$$u(s)=\int f(x)\,e^{-is\cdot x}\Delta(dx). \tag{8.1}$$

Hence

Proposition (8.1). *$\mathscr{H}$ consists of all functions of the form*

$$u(s)=\int f(x)\,e^{-is\cdot x}\Delta(dx)$$

where $f\in L^2(R_n,\Delta)$. The map $u(s)\to f(x)$ is an isometry of onto $L^2(R_n,\Delta)$, and thus

$$\langle u,u\rangle=\int |f(x)|^2\Delta(dx).$$

We next establish a decomposition of $\{X_t\}$ analogous to the classical decomposition of a stationary Gaussian process into deterministic and regular parts.

We introduce the spaces

$$\mathscr{H}_0 = \text{closure of } u\ \{u \in \mathscr{H}: u \text{ has compact support}\}$$

$$\mathscr{H}_\infty = \mathscr{H} \ominus \mathscr{H}_0.$$

The space $\mathscr{H}_\infty$ corresponds to the tail field $\Sigma_\infty = \bigcap \sigma\{X_t\colon |t| > N\}$, and $\mathscr{H}_0$ corresponds to the largest field independent of Σ_∞.

Proposition (8.2). *Let $\Delta(dx) = \Delta_c(x)\,dx + \Delta_s(dx)$ be the Lebesgue decomposition of Δ into absolutely continuous and singular parts. Then:*

(i) If $\mathscr{H}_0 \neq \{0\}$, then

$$\mathscr{H}_0 = \{u(s) = \int f(x)\, e^{-s\cdot x} \Delta_c(x)\, dx : f \in L^2(R_n, \Delta_c)\} \tag{8.2}$$

and

$$\mathscr{H}_\infty = \{u(s) = \int f(x)\, e^{-is\cdot x} \Delta_s(dx) : f \in L^2(R_n, \Delta_s)\}. \tag{8.3}$$

Thus the splitting $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_\infty$ corresponds to the splitting $L^2(R_n, \Delta) = L^2(R_n, \Delta_c) \oplus L^2(R_n, \Delta_s)$.

(ii) $\mathscr{H}_0 \neq \{0\}$ if and only if for some L^2 function $u(t) \not\equiv 0$ with compact support

$$\int \frac{|F(x)|^2}{\Delta_c(x)}\, dx < \infty \tag{8.4}$$

where

$$F(x) = \frac{1}{(2\pi)^{n/2}} \int e^{it\cdot x} u(t)\, dt. \tag{8.5}$$

Proof. From the Fourier representation of $\mathscr{H}$ it is clear that the translation operator $u(s) \to u_t(s) \equiv u(t+s)$ is unitary on $\mathscr{H}$. Moreover, since it leaves the space of functions with compact support invariant, the space $\mathscr{H}_0$ must be invariant under translations.

Assuming $\mathscr{H}_0 \neq \{0\}$, let $u(s) = \int f(x)\, e^{-is\cdot x} \Delta(dx)$ have compact support. We now show that a function $v(s) = \int g(x)\, e^{-is\cdot x} \Delta(dx)$ is in $\mathscr{H}_0$ if and only if $g(x)$ vanishes almost everywhere with respect to Lebesgue measure dx.

If $v \in \mathscr{H}_\infty$, then v is orthogonal to $u_t(s)$ for all t in R_n. That is,

$$0 = \langle u_t(s), v(s)\rangle = \int f(x)\, e^{-it\cdot x}\, \overline{g(x)}\, \Delta(dx),$$

and this is equivalent to saying that $g(x) = 0$ a.e. with respect to the measure $|f(x)|\, \Delta(dx)$. But $u(s) = \int e^{is\cdot x} f(x)\, \Delta(dx)$ has compact support, so by the easy half of the Paley-Wiener theorem the measure $f(x)\, \Delta(dx)$ equals $F(x)\,dx$ where

$$F(x) = \frac{1}{(2\pi)^{n/2}} \int e^{it\cdot x} u(t)\, dx$$

is an entire function of exponential type. In particular, $F(x)$ vanishes at most on a set of Lebesgue measure zero, and hence $|f(x)|\, \Delta(dx)$ is equivalent to Lebesgue measure dx. Since $u \in \mathscr{H}_0$ was an arbitrary function with compact support, the proof of (i) is complete.

To prove (*ii*) let $u(s)$ be as before and observe that

$$\infty > \langle u, u\rangle = \int |f(x)|^2 \Delta_c(x)\,dx = \int \frac{|F(x)|^2}{\Delta_c(x)}\,dx.$$

Conversely, if $u \in L^2$ has compact support and $F(x)$ defined by (8.5) satisfies $\int \frac{|F(x)|^2}{\Delta_c(x)}\,dx < \infty$, then

$$f(x) \equiv \frac{F(x)}{\Delta_c(x)} \in L^2(R_n, \Delta_c) \quad \text{and} \quad u(s) = \int f(x)\, e^{-is\cdot x} \Delta_c(x)\,dx$$

is in $\mathscr{H}_0$. This completes the proof.

Our splitting $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_\infty$ corresponds to a splitting of the space $H = H_0 \oplus H_\infty$, where

(8.6a) $$H_\infty = \bigcap_{N \geq 0} H(\{t: |t| > N\}),$$

(8.6b) $$H_0 = H \ominus H_\infty.$$

Let Y_t be the projection of X_t onto H_0 and let Z_t be the projection of X_t onto H_∞. Then $\{Y_t\}$ and $\{Z_t\}$ are independent stationary Gaussian processes and

(8.7) $$X_t = Y_t + Z_t.$$

The correlation functions of Y_t and Z_t are respectively

$$\int e^{it\cdot x} \Delta_c(x)\,dx \quad \text{and} \quad \int e^{it\cdot x} \Delta_s(dx).$$

Using a superscript notation such as H^y and H^y_∞ to denote the obvious Hilbert spaces defined for the processes $\{Y_t\}$ and $\{Z_t\}$, we have the following elementary relations:

(8.8) $$H^x_0 = H^y = H^y_0, \qquad H^y_\infty = \{0\}$$

(8.9) $$H^x_\infty = H^z = H^z_\infty, \qquad H^z_0 = \{0\}.$$

From our view point $\{Z_t\}$ is rather uninteresting, and we will assume in all that follows that $Z_t \equiv 0$ or what is the same that $H^x = H^x_0$. Thus the measure $\Delta(dx)$ is assumed absolutely continuous and we will speak of the spectral density $\Delta(x)$.

The spectral density of a Markov process is very smooth, and we will show in [11] that $\Delta^{-1}(x)$ must be an entire function of minimal exponential type. For the present, however, we will content ourselves with the following proposition. Note that $\{X_t\}$ is not assumed to be Markovian.

Proposition 8.3. *Suppose $\mathscr{H}_0 = \mathscr{H}$ and that for some bounded open set D, $\mathscr{H}(D) \cap \mathscr{H}_0 \neq \{0\}$. Then $\Delta(x)$ agrees a.e. with the ratio of two entire functions of finite exponential type.*

Proof. Let $u(s) = \int f(x)\, e^{-is\cdot x} \Delta(x)\,dx$ be in $\mathscr{H}(D)$ and have compact support. Then $f(x)\,\Delta(x)$ agrees a.e. with an entire function of exponential type. Because D is bounded we have for s in D and for large t, $u_t(s) = u(t+s) = 0$. That is, u_t

is orthogonal to $\mathscr{H}(D)$ and because $u \in \mathscr{H}(D)$ we have

$$0 = \langle u_t, u \rangle = \int f(x) e^{-it\cdot x} \bar{f}(x) \Delta(x) dx .$$

This implies that $f\bar{f}\Delta$ agrees a.e. with an entire function of exponential type so that $\Delta = (f\Delta \cdot \overline{f\Delta})/(f\bar{f}\Delta)$ agrees a.e. with the ratio of two entire functions of exponential type.

9. The Markov Property for Half-Spaces

The easiest approach for obtaining information about Δ from Markovian conditions imposed on the process $\{X_t\}$ is to restrict considerations to the special case when the domains D_- and D_+ are half-spaces. This leads to a decomposition of the spaces $\mathscr{H}(D_-)$ and $\mathscr{H}(D_+)$ which reduces considerations to the one dimensional case. We now discuss this decomposition. An application of the techniques is made in the next section.

As usual $\{X_t\colon t \in R_n\}$ is a stationary Gaussian process with spectral density Δ. We treat R_n as the direct sum $R_1 \oplus R_{n-1}$ and write $t = (\sigma, \tau)$ with $\sigma \in R_1$ and $\tau \in R_{n-1}$ for an element t of R_n. For any real number r we write

$$D_-^r = \{t \in R_n \colon \sigma < r\} \tag{9.1}$$

$$D_+^r = \{t \in R_n \colon \sigma > r\} . \tag{9.2}$$

The spectral density $\Delta(x)$ is defined on the dual of $R_1 \oplus R_{n-1}$, and we will write $x = (\xi, \eta)$ with $\xi \in R_1$ and $\eta \in R_{n-1}$. Δ is to be considered as a function $\Delta(\xi, \eta)$ of two variables. The covariance $R(t)$ takes the form

$$R(t) = R(\sigma, \tau) = \iint e^{i(\sigma\cdot\xi + \tau\cdot\eta)} \Delta(\xi, \eta) d\xi\, d\eta . \tag{9.3}$$

Corresponding to the spaces $\mathscr{H}(D_\pm^s)$ we introduce the appropriate subspaces of $L^2(R_n, \Delta)$. Namely

$$Z(D_\pm^r) = \text{closed span } \{e^{it\cdot x} \colon t \in D_\pm^r\} . \tag{9.4}$$

Then $u(s) \in \mathscr{H}(D_\pm^r)$ iff $u(s) = \int f(x) e^{-is\cdot x} \Delta(x) dx$ where $f \in Z(D_\pm^r)$.

By Fubini's theorem we may change Δ on a set of measure zero so that for all $\eta \in R_{n-1}$ the non-negative function $\Delta_\eta(\cdot) \equiv \Delta(\cdot, \eta)$ is in $L^1(R_1)$. Thus Δ_η is a permissible spectral density on R_1, and we may introduce the space $L^2(R_1, \Delta_\eta)$ and the subspaces of $L^2(R_1, \Delta_\eta)$

$$Z_\eta^{r+} \equiv \text{closed span } \{e^{i\sigma\cdot\xi} \colon \sigma > r\} \tag{9.5}$$

$$Z_\eta^{r-} \equiv \text{closed span } \{e^{i\sigma\cdot\xi} \colon \sigma < r\} . \tag{9.6}$$

The previously mentioned decomposition is based on the following proposition which shows how the spaces $Z(D_\pm^r)$ are built from the $Z_\eta^{r\pm}$. The proof we give is by no means the shortest possible but is instructive.

Proposition 9.1*. *A function f in $L^2(R_n, \Delta)$ is in $Z(D_\pm^r)$ if and only if $f(\cdot, \eta)$ is in $Z^{r\pm}$ for a.a. η.*

* In the language of direct integrals $L^2(R_n, \Delta) = \int \oplus L^2(R_1, \Delta_\eta) d\eta$; this proposition says that $Z(D_\pm^r) = \int \oplus Z_\eta^{r\pm} d\eta$.

Proof. We present the proof for $Z(D_-^r)$. Let $\mathscr{Z}^r$ denote the subspace of $L^2(R_n, \Delta)$ of functions f satisfying $f(\cdot, \eta)$ Z^{r-} for a.a. η. We must show $Z(D_-^r) = \mathscr{Z}^r$.

First we claim $\mathscr{Z}^r$ is closed. In fact, if f_k is a convergent subsequence in $\mathscr{Z}^r$ with $\lim f_k = f$ we have Fubini's theorem that

$$\int d\eta \{\int |f_k(\xi, \eta) - f(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi\} = \|f_k - f\|^2 \tag{9.7}$$

and this converges to zero as $k \to \infty$. Choosing a subsequence $k' \to \infty$ so that $\int |f_{k'}(\xi, \eta) - f(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi \to 0$ for a.a. η we see that for a.a. η, $f(\cdot, \eta)$ is the limit in $L^2(R_1, \Delta_n)$ of the functions $f_{k'}(\cdot, \eta) \in Z_\eta^{r-}$. But Z_η^{r-} is closed so $f(\cdot, \eta) \in Z_\eta^{r-}$ and $f \in \mathscr{Z}^r$, and $\mathscr{Z}^r$ is closed.

The inclusion $Z(D_-^r) \subset \mathscr{Z}^r$ is now obvious since each f in $Z^r(D_-^r)$ is a limit of finite sums of the $e^{it \cdot x}$ where $t \in D_-^r$, and each such sum is in $\mathscr{Z}^r$.

To prove the inclusion $\mathscr{Z}^r \subset Z(D_-^r)$ we observe that each function of the form $\sum_1^N e^{i\sigma_k \xi} h_k(\eta)$ where $\sigma_k \leqq r$ and $\int |h_k(\eta)|^2 \Delta(\xi, \eta)\, d\xi\, d\eta$ is in $Z(D_-^r)$. We now show that each $f \in \mathscr{Z}^r$ can be approximated by such functions and the result will follow.

For this we let $\{g_k(\xi)\}$ be an enumeration of the countable set of finite sums $\sum a_k e^{i\sigma_k \xi}$ where the $\sigma_k \leqq r$ and both the a_k and σ_k are rational. The set $\{g_k(\xi)\}$ is dense in each of the spaces Z^{r-}. For fixed $f \in \mathscr{Z}^r$ we define the approximating sequence

$$f_k(\xi, \eta) = \sum_1^k g_j(\xi)\, 1_{B_{k,j}}(\eta) \tag{9.8}$$

where $1_{B_{k,j}}$ is the indicator function of the measurable set $B_{k,j} \subset R_{n-1}$, which is defined by the condition: $\eta \in B_{k,j}$ if and only if j is the smallest integer satisfying

$$\int |f(\xi, \eta) - g_j(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi = \min_{1 \leqq i \leqq k} \{\int |f(\xi, \eta) - g_i(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi\}.$$

Then for each fixed η,

$$\int |f_k(\xi, \eta) - f(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi = \min_{1 \leqq j \leqq k} \{\int |g_j(\xi, \eta) - f(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi \tag{9.9}$$

decreases to zero as $k \to \infty$. By the monotone convergence theorem we conclude that

$$\|f_k - f\|^2 = \int d\eta \{\int |f_k(\xi, \eta) - f(\xi, \eta)|^2 \Delta(\xi, \eta)\, d\xi\} \to 0. \tag{9.10}$$

Note that if f is an arbitrary element of $L^2(R_n, \Delta)$ and the sequence f_k is defined by (9.8), then f_k converges to the projection of f on $Z(D_-^r)$ while for almost all η, $f_k(\cdot, \eta)$ converges to the projection of $f(\cdot, \eta)$ on Z_η^{r-}.

The usefulness of this proposition in studying the Markov property is that it reduces the study of the projection of $\mathscr{H}(D_+^r)$ onto $\mathscr{H}(D_-^r)$ to the study of the projections of Z_η^{r+} onto Z_η^{r-}; and for fixed η this is a one variable problem.

An example of the results obtainable by this technique is the following easily proved

Proposition 9.2. *Suppose $\{X_t\colon t \in R_n\}$ is a stationary Gaussian process with spectral density $\Delta(x)$. Suppose also that $\{X_t\}$ is Markovian. Then for a.a. η, the* *projection of Z_η^{r+} onto Z_η^{r-} is contained in*

$$\bigcap_{\varepsilon>0} \operatorname{span}\{e^{i\sigma\xi}: 0 \leqq r-\sigma<\varepsilon\}.$$

Assuming that $\mathscr{H}_\infty=\{0\}$ it then follows from results of LEVINSON & MCKEAN [7] that for a.a. η, $\Delta^{-1}(\xi, \eta)$ agrees a.e. with an entire function of minimal exponential type.

10. Stationary Markov Processes of Finite Order

We have seen in earlier sections that if $\{X_t: t \in R_n\}$ is a Gaussian Markov process, then under appropriate additional assumptions it follows that the inner product in $\mathscr{H}$ is given by a Dirichlet form. When the form is of order p, elliptic and satisfies other technical conditions, we were able to prove that $\{X_t\}$ was Markovian of order p.

In this section we assume that $\{X_t: t \in R_n\}$ is stationary and Markovian of order p. In light of the previous results a reasonable conjecture is that the inner product in $\mathscr{H}$ comes from a constant coefficient elliptic Dirichlet form. We prove the Fourier transform version of this conjecture.

Theorem 10.1. *Let $\{X_t: t \in R_n\}$ be a stationary Gaussian process with spectral density $\Delta(x)$. If $\{X_t\}$ is Markovian of order p then $\Delta^{-1}(x)$ is an elliptic polynomial* of degree $2p$.*

Proof. For $n=1$ this is well-known and was mentioned in the comment at the end of Section 1. In fact, $R(t)$ is the Green's function of the operator $\sum_0^{2p} a_\alpha \frac{d^\alpha}{dt^\alpha}$ if and only if $R(t)=\int e^{itx} p^{-1}(x)\,dx$ where

$$p(x)=\sum_0^{2p} a_\alpha(-ix)^\alpha.$$

We can also formulate the proposition for $n=1$ in terms of the space $L^2(R_1, \Delta)$. If $X_t^{(p-1)}$ exists then $x^{p-1} \in L^2(R_1, \Delta)$. Thus $\Delta^{-1}(x)$ is a polynomial if degree $2p$ if and only if x^{p-1} is in $L^2(R_1, \Delta)$ and the range of the projection of $\{e^{itx}: t \geqq 0\}$ onto $\{e^{itx}: t \leqq 0\}$ in $L^2(R_1, \Delta)$ is $\{1, x, \ldots, x^{p-1}\}$.

Turning to the case $n>1$ we know from Proposition 8.3 that $\Delta^{-1}(x)= P(x)\, Q^{-1}(x)$ where $P(x)$ and $Q(x)$ are entire functions. We can thus find some open set $U \subset R_n$ on which $Q(x)$ is non-zero. If we can show that the restriction of $P(x)\, Q^{-1}(x)$ to U is a polynomial it will then follow by analytic continuation that $P(x)\, Q^{-1}(x)$ is a polynomial.

Moreover, since $\Delta^{-1}=P Q^{-1}$ is analytic on U, we see that $\Delta^{-1}(x)$ must be a polynomial on U if we can show that in each of the separate variables $x_1, \ldots, x_n$, $\Delta^{-1}(x)=\Delta^{-1}(x_1, \ldots, x_n)$ is a polynomial. The proof of this will follow from the case $n=1$ and the reduction discussed in Section 9.

With this in mind we again write $t=(\tau, \sigma)$ for the general point t in $R_n= R_1 \oplus R_{n-1}$ and $x=(\xi, \eta)$ for the general point in the dual Euclidian space.

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* The polynomial $\sum_{|\alpha| \leqq m} a_\alpha x^\alpha$ is elliptic if there is a constant $c>0$ with $c|x|^m \leqq |\sum_{|\alpha|=m} a_\alpha x^\alpha|$.

Without loss of generality we may assume U has the form

$$U=\{(\xi,\eta)\in R_1\oplus R_{n-1}: |\xi|<\varepsilon;\ |\eta|<\varepsilon\}. \tag{10.1}$$

We must show for fixed η with $|\eta|<\varepsilon$ that as a function of ξ, $\Delta^{-1}(\xi,\eta)$ is a polynomial of degree $2p$ for $|\xi|<\varepsilon$.

Again we set

$$D_-=\{t\in R_n: \sigma<0\},\quad D_+=\{t\in R_n: \sigma>0\}$$
$$\Gamma=\{t\in R_n: \sigma=0\}. \tag{10.2}$$

If $f\in L^2(\Gamma)$ has compact support we set

$$F(\sigma)=\int_{R_{n-1}} f(\tau)X_{(\sigma,\tau)}\,d\tau.$$

Under the map $X_t\to e^{it\cdot x}$ from H onto $L^2(R_n,\Delta)$ the function $F(\sigma)$ corresponds to $e^{i\sigma\xi}\hat f(\eta)$ where

$$\hat f(\eta)=\int_\Gamma e^{i\tau\cdot\eta}f(\tau)\,d\tau,$$

and taking derivatives we have the correspondences

$$F(0)\leftrightarrow\hat f(\eta),\quad F^{(k)}_{(0)}\leftrightarrow(i\xi)^k\hat f\eta)\quad\text{for } 1\leqq k\leqq p-1. \tag{10.3}$$

Moreover, $F^{(p-1)}_{(0)}$ exists if and only if

$$\int_{R_n}|\xi^{p-1}\hat f(\eta)|^2\Delta(\xi,\eta)\,d\xi\,d\eta<\infty. \tag{10.4}$$

Thus if $\{X_t\}$ is Markovian of order p we conclude that (10.4) holds for each f in $L^2(R_{n-1})$ with compact support and that in $L^2(R_n,\Delta)$ the range of the projection of $\{e^{itx}:\ \sigma\geqq 0\}$ onto $\{e^{itx}:\ \sigma\leqq 0\}$ is contained in the closure of

$$\{\xi^k\hat f(\eta): 0\leqq k<p \text{ and } f\in L^2(\Gamma) \text{ has compact support}\}. \tag{10.5}$$

But if (10.4) holds for all permissible f we have

$$\int_{R_1}|\xi^{p-1}|^2\Delta(\xi,\eta)\,d\xi<\infty\quad\text{for a.a. }\eta. \tag{10.6}$$

Proposition 9.3. now shows that for a.a. η, the projection of $\{e^{i\sigma\xi}:\ \sigma\geqq 0\}$ onto $\{e^{i\sigma\xi}:\ \sigma\leqq 0\}$ in $L^2(R_1,\Delta_\eta)$ is contained in $\{1,\xi,\dots,\xi^{p-1}\}$. Thus for a.a. η, $\Delta^{-1}(\xi,\eta)$ is a polynomial of degree $2p$ in ξ. Since Δ^{-1} is continuous on U it follows that $\Delta^{-1}(\xi,\eta)$ is a polynomial of degree $\leqq 2p$ in ξ; and that for almost all η it is of degree $2p$. From the previous comments we see that $\Delta^{-1}(s)$ is a polynomial.

That $\Delta^{-1}(x)$ is an elliptic polynomial of degree $2p$ follows simply from the fact that any polynomial which is not elliptic of degree $2p$ may be reduced by an orthogonal change of coordinates to a polynomial which is of degree $k+2p$ in the variable x_1. This possibility contradicts the earlier discussion, and we conclude Δ^{-1} is elliptic of order $2p$. The proof is complete.

If $\Delta^{-1}(x)\geqq 0$ is an elliptic polynomial of degree $2p$ one should expect that the associated stationary Gaussian process is Markovian of order p. If $\Delta^{-1}(x)$ has no zeros this is true and upon taking Fourier transforms follows immediately from

Theorem 5.4. However, when $\Delta^{-1}(x)$ has zeros the norm in $\mathscr{H}$ is not equivalent to the norm $\| \|_p$ and Theorem 5.4 is not applicable. In this case I have been unable to prove the general result although I believe it to be true.

The basic problem consists of showing that the associated Dirichlet problem for A is well-posed in $\mathscr{H}$ for an unbounded region D_- with non-compact boundary Γ. For a general Δ^{-1} with zeros this seems very difficult. However, if we restrict considerations to the case when Γ is compact there is no real problem. For functions u in $\mathscr{H}$ with a fixed compact support the norm in $\mathscr{H}$ is equivalent to the norm $\| \|_p$, and the solution of the prediction problem given in Section 8 is valid. We thus have

Proposition 10.2. *Let $\Delta(x)$ be the spectral density of a stationary Gaussian process $\{X_t\colon\ t\in R_n\}$ and suppose that $\Delta^{-1}(x)$ is an elliptic polynomial of degree $2p$.*

(i) If $\Delta^{-1}(x)$ has no zeros then $\{X_t\}$ is Markovian of order p.

(ii) If Δ^{-1} has zeros the Markov property of order p remains valid for $\{X_t\}$ when formulated with respect to compact surfaces Γ.

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