Spectral and Scattering Theory for Maxwell's Equations in an Exterior Domain

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Introduction

The purpose of this work is to treat the spectral theory and the scattering theory of electromagnetic fields in the exterior of bounded obstacles within the abstract framework developed by LAX and PHILLIPS in their book [4]. There the theory is applied to the wave equation and to a certain class of linear systems of first order equations. As was pointed out in an appendix to this book (SCHMIDT [9]), MAXWELL's equations are not covered by the class of linear systems considered there, but require a separate, though in many respects analogous, treatment. This we now provide.

We begin by sketching the main features of the abstract theory. Let $U(t)$ be a group of unitary operators on a Hilbert space H. Suppose that there exist two orthogonal subspaces D_+ and D_- such that

(a) $U(t)D_+ \subset D_+, \ t \geqq 0$;

(b) $\bigcap_{t \geqq 0} U(t)D_+ = \{0\}$;

(c) $\bigcup_{t \leqq 0} U(t)D_+$ is dense in H.

(a′) $U(t)D_- \subset D_-, \ t \leqq 0$;

(b′) $\bigcap_{t \leqq 0} U(t)D_- = \{0\}$;

(c′) $\bigcup_{t \geqq 0} U(t)D_-$ is dense in H.

These subspaces are respectively said to be *outgoing* and *incoming* with respect to the group $U(t)$. As a consequence of the existence of these subspaces one can prove that there are two corresponding unitary translation representations of H on $L_2(-\infty, \infty; N)$[1] (where N is an auxiliary Hilbert space), such that $U(t)$ corresponds to the group $T(t)$ of translation operators. In the case of one representation, the so-called *outgoing translation representation*, D_+ corresponds to $L_2(0, \infty; N)$, while in the second case, the *incoming translation representation* D_- corresponds to $L_2(-\infty, 0; N)$. To any element f in H there correspond two representers in $L_2(-\infty, \infty; N)$, an incoming representer k_- and an outgoing representer k_+. We introduce a unitary map S of $L_2(-\infty, \infty; N)$ onto itself which maps k_- to k_+; S is called the *scattering operator*.

The physical significance of the operator S is not immediately evident. Scattering theory usually involves two groups, an unperturbed group $U_0(t)$ and a perturbed group $U(t)$. The wave operators $W_\pm$ are defined as the strong limits of

[1] We shall use the notation $S(D; R)$ to denote a space of functions with domain D and taking their values in R and with norm appropriate to S. In the case that the functions are complex valued we shall simply write $S(D)$.

$U(-t)\,U_0(t)$ as t goes to $\pm\infty$ respectively, in the case that these limits exist. The scattering operator is then defined by $S = W_+^{-1}\,W_-$. In the theory described above the unperturbed group is not explicitly mentioned; however, implicitly, it plays a role, for in actual practice D_+ and D_- are subspaces on which the perturbed and the unperturbed groups act in the same way for positive and negative t respectively. In fact LAX and PHILLIPS show that the operator they define is essentially the same as the usual scattering operator.

It is convenient to introduce *outgoing and incoming spectral representations* of H on $L_2(-\infty, \infty; N)$ by composing the previous representation maps with a Fourier transformation F of $L_2(-\infty, \infty; N)$. In this way H is mapped onto $L_2(-\infty, \infty; N)$ in such a way that $U(t)$ corresponds to multiplication by $e^{i\sigma t}$, and D_+ and D_- are mapped onto $A_+ = FL_2(0, \infty; N)$ and $A_- = FL_2(-\infty, 0; N)$ in the outgoing and incoming cases respectively. The scattering operator in this representation is denoted by $\mathscr{S}$. One can show that $\mathscr{S}$ can be realized as a multiplicative operator valued function $\mathscr{S}(\sigma)$ such that

(a) For each σ, $\mathscr{S}(\sigma)$ maps N into N;

(b) $\mathscr{S}(\sigma)$ is unitary for almost all σ;

(c) $\mathscr{S}(\sigma)$ is the boundary value of an operator-valued function $\mathscr{S}(z)$ analytic for $\operatorname{Im} z < 0$, which converges strongly along the lines $\operatorname{Re} z = \sigma$ to $\mathscr{S}(\sigma)$ for almost all σ;

(d) $|\mathscr{S}(z)| \leqq 1$ for all z with $\operatorname{Im} z < 0$.

Further analysis is needed in order to obtain greater insight into the analytic behavior of $\mathscr{S}(z)$. LAX and PHILLIPS introduce a semi-group $Z(t) = P_+\,U(t)\,P_-$ which acts on the subspace $(D_+ + D_-)^\perp$, where P_+ and P_- are projection operators onto $D_+^\perp$ and $D_-^\perp$. It turns out that there is an intimate connection between the behavior of the resolvent of the generator B of $Z(t)$ and $\mathscr{S}(z)$. More explicitly, the following is true:

(a) If $\operatorname{Re}\mu < 0$, then μ belongs to the resolvent set of B if and only if $\mathscr{S}(i\,\bar{\mu})$ is regular.

(b) A purely imaginary μ_0 belongs to the resolvent set of B if and only if $\mathscr{S}(z)$ can be continued analytically across the real axis at $\sigma_0 = i\,\bar{\mu}_0$.

As a final result given us by the abstract theory we state the following: If for some positive values of T and κ, the operator $Z(T)\,(\kappa\,I - B)^{-1}$ is compact, then the scattering matrix $\mathscr{S}(z)$ is holomorphic on the real axis and meromorphic in the whole plane, having a pole at each point z for which $i\,z$ belongs to the spectrum of B. Compactness of the operator $Z(T)$ for some T would allow a stronger conclusion, but since this has not been proved in any of the applications we do not write down the details.

In applying the abstract theory to concrete situations it is necessary to prove the existence of incoming and outgoing subspaces, to obtain as much information as possible about $Z(t)$, and to identify the spectrum of B. Not surprisingly, the unperturbed problem, in our case MAXWELL's equations in free space, plays an important role, even though, as was noted before, it remains submerged in the general theory. In the proof of the applicability of the abstract theory there is a delicate interplay between the many facts known about the unperturbed problem (such as domain of dependence properties and HUYGHEN'S principle) and a few fundamental results concerning the perturbed problem (such as the absence of point spectrum from the generator of the perturbed group, and a local compactness property).

One would expect the spectral and scattering properties of MAXWELL'S equations in an exterior domain to be very close to those of the wave equation, for the electro-magnetic fields satisfy the wave equation in the absence of free charges and currents. For a number of reasons the theory of Maxwell fields cannot be entirely subsumed under the corresponding theory for the wave equation:

(a) Both in free space and in an exterior domain MAXWELL'S equations have stationary solutions with finite energy (i.e., 0 is an eigenvalue of the generator of the unitary group).

(b) The energy density function corresponding to electro-magnetic fields is different from that usually associated with the wave equation.

(c) In the exterior problem the electro-magnetic fields are required to satisfy quite different boundary conditions from those which arise naturally for the wave equation.

We therefore have to follow a path different from, but parallel to, that taken by LAX and PHILLIPS in their treatment of the wave equation. Accordingly our work falls into three parts:

1) A detailed discussion of MAXWELL'S equations in free space.

2) The proof of several fundamental results about the exterior problem.

3) The proof of the applicability of the abstract theory and a listing of the conclusions allowed by that theory.

In free space it turns out that we can exploit the connection between MAXWELL'S equations and the wave equation, once this has been formulated. Section 1.1 is devoted to a study of the various function and distribution spaces to which the initial data for MAXWELL'S equations and the wave equation belong. Decomposition of these spaces corresponding to the later decomposition into stationary and non-stationary fields are considered, and a theorem (1.1.9) is proved which allows the introduction of an electromagnetic potential in the Hilbert Space setting. In Section 1.2 we recall, and suitably formulate, various properties of the wave equation. We then introduce MAXWELL'S equations (which have the form $\partial_t m = A_0 m$) in Section 1.3. The space of initial data is decomposed into stationary and non-stationary components; for the latter we introduce electro-magnetic potentials (Theorem 1.3.2), thus establishing the connection with the wave equation. This allows us to obtain fairly directly several properties of the equations in free space (Theorems 1.3.4 to 1.3.7). In Section 1.4 the existence of outgoing and incoming subspaces is proved, and the translation representation for MAXWELL'S equations together with some of its properties is obtained from the corresponding representation of the wave equation. It is necessary to consider distribution-valued initial data; this is done in 1.5. The translation representation is extended to a certain class of such data. An existence theorem, (1.5.3), is obtained for the equation $(A_0-\mu)\,m=g$, where g is divergence free and has compact support and m is required to be eventually outgoing (i.e. $U_0(t)\,m(x)$ vanishes for $|x|<t-R$ $(t>R)$). That theorem is used in Section 1.6 to obtain the fundamental outgoing solution for MAXWELL's equations. In Theorem 1.6.5 several equivalent statements of the outgoing condition for electro-magnetic fields m for which $(A_0-\mu)\,m$ has compact support are given; the first of these clarifies the dynamic significance of the Sommerfeld radiation condition. These results allow μ to be any complex number.

The exterior problem requires essentially different methods from those used for the wave equation. The first section of Part 2 is devoted to the relevant initial boundary value problem. The skew adjointness of the generator A corresponding to MAXWELL's equations and certain general classes (Theorem 2.1.3) of boundary conditions is proved using the results of LAX and PHILLIPS in [6]. The absence of non-zero point spectrum is established with the aid of Theorem 1.3.6. The local compactness theorem (2.2.9) which is crucial to further developments is proved for a particular boundary condition (corresponding essentially to an obstacle which is a perfect conductor) and in the complement of the null space of A, using an inequality of FRIEDRICHS [3] and RELLICH's compactness theorem.

The application of the abstract theory in Part 3 offers few difficulties since the proofs of the various assertions can virtually be read off word for word (with a few minor modifications, some of which were noted in Appendix 4 of [10]) from the corresponding proofs for the wave equation in [4]. The scope of the techniques due to LAX and PHILLIPS and applied here can best be judged by a perusal of the results stated in the last section of this paper.

A somewhat amplified form of this paper has appeared as a technical report [10], and will frequently be referred to for more complete proofs.

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1. The Free Space Problem

§ 1.1. Some Mathematical Preliminaries

In this section we introduce the spaces from which the initial data for the Cauchy problems corresponding to the wave equation and to MAXWELL's equations will be taken, and study their relevant properties.

We begin by defining the Beppo-Levi space[2] $BL(R^3)$; this is, roughly speaking, the closure of $C_0^\infty(R^3)$ in $\|f\|_{BL}=\|\nabla f\|_{L_2}$, where ∇ is the vector operator $(\partial_1, \partial_2, \partial_3)$ and ∂_i denotes $\partial/\partial x_i$. The manner in which the space $C_0^\infty(R^3)$ is to be completed is clarified by Lemma 1.1.1, according to which smooth functions with compact support satisfy the inequality $(*)$. Hence a sequence of smooth functions with compact support, Cauchy in $\|\cdot\|_{BL}$, is also Cauchy in $L_2^{\text{loc}}(R^3)$ and hence determines uniquely a limiting function. In this way $C_0^\infty(R^3)$ is completed to give $BL(R^3)$, and the inequality $(*)$ continues to hold for all functions in the completed space.

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[2] For a detailed study of more general spaces of this type see DENY and LIONS [2].

Lemma 1.1.1. *Suppose $f\in BL(R^3)$. Then*

$$\int_{|\omega|=1} |f(r\omega)|^2\, d\omega \leqq \frac{1}{r}\, \|f\|_{BL}^2. \qquad (*)$$

This is well known and is proved, for instance, on p. 95 of [4] or in [10].

Clearly, if f lies in $BL(R^3)$ then ∇f lies in $L_2(R^3;\mathscr{C}^3)$. It is evident from Lemma 1.1.1 that the converse does not hold; the constants, for instance, have square integrable gradients but do not belong to $BL(R^3)$. This situation will be clarified by Lemma 1.1.5. We precede this by three lemmas which state simple facts which we shall use repeatedly. Differentiation is always to be understood in the distribution sense.

Lemma 1.1.2.

a) If $f\in L_2(R^3)$ and $\Delta f=0$, then $f=0$.

b) If $f\in L_2(R^3;\mathscr{C}^3)$, $\nabla\cdot f=0$ and $\nabla\times f=0$, then $f=0$.

c) If $f\in L_2(R^3;\mathscr{C}^3)$, $\nabla\cdot f\in L_2(R^3)$ and $\nabla\times f\in L_2(R^3;\mathscr{C}^3)$, then $\nabla f_i\in L_2(R^3;\mathscr{C}^3)$ $(i=1,2,3)$; and furthermore if g satisfies the same conditions,

$$\sum_{j=1}^{3}\int \nabla f_j\cdot\overline{\nabla g_j}\,dx=\int \nabla\times f\cdot\overline{\nabla\times g}\,dx+\int(\nabla\cdot f)\overline{(\nabla\cdot g)}\,dx.$$

Lemma 1.1.3.

a) If $f\in BL(R^3)$ and $\Delta f=0$, then $f=0$.

b) If $f\in BL(R^3,\mathscr{C}^3)$, $\nabla\cdot f=0$ and $\nabla\times f=0$, then $f=0$.

c) If f and g lie in $BL(R^3,\mathscr{C}^3)$, then

$$\sum_{j=1}^{3}\int \nabla f_j\cdot\overline{\nabla g_j}\,dx=\int \nabla\times f\cdot\overline{\nabla\times g}\,dx+\int(\nabla\cdot f)\overline{(\nabla\cdot g)}\,dx.$$

Lemma 1.1.4. *The following is an orthogonal decomposition of $L_2(R^3;\mathscr{C}^3)$ into closed subspaces:*

$$L_2(R^3;\mathscr{C}^3)=L_2^0(R^3;\mathscr{C}^3)\oplus L_2^1(R^3;\mathscr{C}^3),$$

where

$$L_2^0(R^3;\mathscr{C}^3)=\{f\in L_2(R^3;\mathscr{C}^3);\ \nabla\times f=0\}$$
$$=\text{closure in } L_2 \text{ of } S^0=\{\nabla\varphi;\ \varphi\in C_0^\infty(R^3)\},$$

and

$$L_2^1(R^3;\mathscr{C}^3)=\{f\in L_2(R^3;\mathscr{C}^3);\ \nabla\cdot f=0\}$$
$$=\text{closure in } L_2 \text{ of } S^1=\{\nabla\times\psi;\ \psi\in C_0^\infty(R^3;\mathscr{C}^3)\}.$$

These lemmas are easily verified, as is done in [10], by using Fourier transforms and elementary vector identities. We can now prove

Lemma 1.1.5. *Let g be a distribution such that ∇g can be represented by a function in $L_2(R^3;\mathscr{C}^3)$. Then $g=f+c$, where f belongs to $BL(R^3)$ and c is a constant.*

Proof. Since ∇g is in $L_2(R^3;\mathscr{C}^3)$ and is curl free it must lie in $L_2^0(R^3;\mathscr{C}^3)$. Consequently there is a sequence $\{\varphi_n\}$ of functions in $C_0^\infty(R^3)$ such that the sequence $\{\nabla\varphi_n\}$ converges to ∇g in L_2. The sequence $\{\varphi_n\}$ converges in the Beppo-

Levi norm and hence determines a function f in $BL(R^3)$ in the manner described at the beginning of this section. Clearly $\nabla f = \nabla g$, and therefore $g = f + c$.

From this we easily obtain

Lemma 1.1.6. *If* $f \in L_2(R^3)$ *and* $\nabla f \in L_2(R^3; \mathscr{C}^3)$, *then* $f \in BL(R^3)$.

Proof. By the previous lemma, $f = f_1 + c$ where $f_1 \in BL(R^3)$. Because of the behavior of the functions f and f_1 at infinity we must have $c = 0$, and thus $f \in BL(R^3)$.

Recalling part *c)* of Lemma 1.1.2 we find the following corollary:

Corollary 1.1.7. *If* $f \in L_2(R^3, \mathscr{C}^3)$, $\nabla \cdot f \in L_2(R^3)$, *and* $\nabla \times f \in L_2(R^3; \mathscr{C}^3)$, *then* $f \in BL(R^3; \mathscr{C}^3)$.

We now prove a decomposition lemma for $BL(R^3; \mathscr{C}^3)$ analogous to Lemma 1.1.4.

Lemma 1.1.8. *The following is an orthogonal decomposition of* $BL(R^3; \mathscr{C}^3)$ *into closed subspaces:*

$$BL(R^3; \mathscr{C}^3) = BL^0(R^3; \mathscr{C}^3) \oplus BL^1(R^3; \mathscr{C}^3)$$

where

$$BL^0(R^3; \mathscr{C}^3) = \{f \in BL(R^3; \mathscr{C}^3); \nabla \times f = 0\} = \text{closure in } BL \text{ of } S^0$$

and

$$BL^1(R^3; \mathscr{C}^3) = \{f \in BL(R^3; \mathscr{C}^3); \nabla \cdot f = 0\} = \text{closure in } BL \text{ of } S^1.$$

Proof. We first assert that if f lies in $BL(R^3; \mathscr{C}^3)$ and is orthogonal to S^0 in that space, then $\nabla \cdot f = 0$; similarly if f is orthogonal to S^1 it must satisfy $\nabla \times f = 0$. We prove the second assertion; the proof of the first one is analogous.

If f is orthogonal to S^1 in $BL(R^3; \mathscr{C}^3)$ we have

$$0 = \sum_j \int \nabla f_j \cdot \overline{\nabla (\nabla \times \psi)_j} \, dx, \qquad \text{for } \psi \in C_0^\infty(R^3; \mathscr{C}^3).$$

We use the identity of Lemma 1.1.3, part *c)*, to obtain

$$0 = \int \nabla \times f \cdot \overline{\nabla \times (\nabla \times \psi)} \, dx + \int (\nabla \cdot f) \overline{(\nabla \cdot \nabla \times \psi)} \, dx.$$

Because $\nabla \times \nabla \times \psi = -\Delta \psi + \nabla(\nabla \cdot \psi)$ and $\nabla \cdot \nabla \times \psi = 0$, we find $0 = \int \nabla \times f \cdot \overline{\Delta \psi} \, dx$. Thus $\nabla \times f$ is harmonic. Since $\nabla \times f$ lies in L_2 (f lying in BL), Lemma 1.1.3, part *a)*, tells us that $\nabla \times f = 0$.

It is now clear that S^0 is dense in BL^0, for if f is in BL^0 but is orthogonal to S^0 we have that $\nabla \times f = 0$ and that $\nabla \cdot f = 0$, so that Lemma 1.1.3, part *b)*, applies to give $f = 0$. Similarly S^1 is dense in BL^1. The spaces BL^0 and BL^1 are orthogonal because of the identity

$$\sum_j \int \nabla (\nabla \varphi)_j \cdot \overline{\nabla (\nabla \times \psi)_j} \, dx = \int \nabla \times \nabla \varphi \cdot \overline{\nabla \times \nabla \times \psi} \, dx + \int (\nabla \cdot \nabla \varphi) \overline{(\nabla \cdot \nabla \times \psi)} \, dx;$$

the right side vanishes since $\nabla \times \nabla \varphi = 0$ and $\nabla \cdot \nabla \times \psi = 0$. That BL^0 and BL^1 span all of BL is clear since if f is orthogonal both to S^0 and to S^1, then $\nabla \cdot f = 0$ and $\nabla \times f = 0$, which again imply that $f = 0$.

The following is the most important result of this section and will allow us to introduce a potential for the electromagnetic fields and thus to establish the connection between MAXWELL's equation and the wave equation within the Hilbert space setting.

Theorem 1.1.9. *There exists a unique one-to-one norm preserving map $\bar{J}$ of $L_2^1(R^3;\mathscr{C}^3)$ onto $BL^1(R^3;\mathscr{C}^3)$, such that if f is in $L_2^1(R^3;\mathscr{C}^3)$ then $f=\nabla\times\bar{J}f$.*

Proof. The work lies in defining $\bar{J}$ suitably. Given f we have to define $\bar{J}f$ so that $f=\nabla\times\bar{J}f$, $\bar{J}f$ lies in $BL(R^3;\mathscr{C}^3)$, and $\nabla\cdot\bar{J}f=0$. Let

$$g(\xi)=|\xi|^{-2}(i\,\xi\times\hat{f}(\xi)),$$

where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$. Since $\nabla\cdot f=0$ we have $\xi\cdot\hat{f}(\xi)=0$, and thus

$$i\,\xi\times g(\xi)=|\xi|^{-2}[i\,\xi\times(i\,\xi\times\hat{f}(\xi))]=|\xi|^{-2}(|\xi|^2\hat{f}(\xi)-\xi(\xi\cdot\hat{f}(\xi)))=\hat{f}(\xi).$$

We now show that $g(\xi)$ is a tempered distribution in $\mathscr{S}'(R^3;\mathscr{C}^3)$. Letting $g(\xi)=g_1(\xi)+g_2(\xi)$, where $g_1(\xi)$ and $g_2(\xi)$ are the restrictions of $g(\xi)$ to $|\xi|\leqq 1$ and $|\xi|>1$ respectively, it is enough to show that $g_1(\xi)$ and $g_2(\xi)$ lie in $L_1(R^3;\mathscr{C}^3)$ and $L_2(R^3;\mathscr{C}^3)$, respectively, and hence are tempered distributions. Since $\xi\cdot\hat{f}(\xi)=0$, $|g(\xi)|=|\xi|^{-1}|\hat{f}(\xi)|$. Thus

$$\int_{R^3}|g_1(\xi)|\,d\xi=\int_{|\xi|\leqq 1}|g(\xi)|\,d\xi=\int_{|\xi|\leqq 1}|\xi|^{-1}|\hat{f}(\xi)|\,d\xi\leqq\left[\int_{|\xi|\leqq 1}|\hat{f}(\xi)|^2\,d\xi\int_{|\xi|\leqq 1}\frac{1}{|\xi|^2}\,d\xi\right]^{\frac{1}{2}};$$

the last expression is finite since $f(x)$ and $\hat{f}(\xi)$ are square integrable and since $d\xi=|\xi|^2\,d|\xi|\,d\omega$ in R^3. Also

$$\int_{R^3}|g_2(\xi)|^2\,d\xi=\int_{|\xi|>1}|\xi|^{-2}|\hat{f}(\xi)|^2\,d\xi\leqq\int_{|\xi|>1}|\hat{f}(\xi)|^2\,d\xi,$$

which is again finite since $\hat{f}(\xi)$ is square integrable.

Since $g(\xi)$ is a tempered distribution it is the Fourier transform of a tempered distribution t; i.e. $\hat{t}(\xi)=|\xi|^{-2}(i\xi\times\hat{f}(\xi))$. Since $i\xi\cdot\hat{t}(\xi)=0$ and $i\xi\times\hat{t}(\xi)=\hat{f}(\xi)$, we have $\nabla\cdot t=0$ and $\nabla\times t=f$. We cannot define $\bar{J}f=t$, since we cannot be sure that t lies in $BL(R^3;\mathscr{C}^3)$. We shall see, however, that by subtracting a suitable constant from t we obtain a function satisfying all the requirements. Clearly

$$\sum\xi\,\hat{t}_j(\xi)\cdot\overline{\xi\,\hat{t}_j(\xi)}=\xi\times\hat{t}(\xi)\cdot\overline{\xi\times\hat{t}(\xi)}=|\hat{f}(\xi)|^2. \qquad (*)$$

Hence the $\xi_i\,\hat{t}_j$'s and thus also the $\partial_i\,t_j$'s are square integrable for $i,j=1,2,3$. According to Lemma 1.1.5 applied componentwise, $t=\bar{t}+c$, where $\bar{t}$ is in $BL(R^3;\mathscr{C}^3)$ and c is a constant vector. Obviously $\nabla\cdot\bar{t}=\nabla\cdot t=0$, and $\nabla\times\bar{t}=\nabla\times t=f$, so that we can define $\bar{J}f=\bar{t}$.

That $\bar{J}$ is an isometry is obvious from the identity $(*)$ above, since $\nabla\bar{t}=\nabla t$. $\bar{J}$ is onto, for if h lies in $BL^1(R^3;\mathscr{C}^3)$ then $\nabla\times h$ is in $L_2(R^3;\mathscr{C}^3)$ and $\bar{J}\nabla\times h=h$. The map $\bar{J}$ is unique, for if $\bar{J}'$ is a second such map we have for any h in $L_2^1(R^3;\mathscr{C}^3)$ that $\nabla\cdot(\bar{J}-\bar{J}')\,h=0$ and $\nabla\times(\bar{J}-\bar{J}')\,h=0$; since $(\bar{J}-\bar{J}')\,h$ is in $BL(R^3;\mathscr{C}^3)$ it must vanish identically. This completes the proof.

We shall need the following approximation lemma in Section 1.4.

Lemma 1.1.10. *Suppose that f and $\partial_j f$ both lie in $L_2(R^3)$ (or $BL(R^3)$). Then there exists a sequence $\{\varphi_n\}\subset C_0^\infty(R^3)$ such that $\{\varphi_n\}$ and $\{\partial_j\,\varphi_n\}$ converge to f and $\partial_j f$ respectively in $L^2(R^3)$ (or $BL(R^3)$).*

Proof. The lemma is well known and easily proved for functions in $L_2(R^3)$ by a "cut-off and mollification" procedure, or by a method similar to that which we now present for functions in $BL(R^3)$.

We consider the Hilbert Space

$$\mathscr{H} = \{f \in BL(R^3); \partial_j f \in BL(R^3)\}$$

with inner product $(f, g)_{\mathscr{H}} = \int \nabla f \cdot \overline{\nabla g}\, dx + \int \nabla \partial_j f \cdot \overline{\nabla \partial_j g}\, dx$. For the case of functions in $BL(R^3)$ the assertion is equivalent to the statement that $C_0^\infty(R^3)$ is dense in $\mathscr{H}$. To prove this we show that any g lying in $\mathscr{H}$ but orthogonal to $C_0^\infty(R^3)$ vanishes identically. The explicit orthogonality condition and integration by parts show us that g is a weak solution of $\Delta(1+\partial_j^2)g=0$. Hence the $\partial_i g$ $(i=1,2,3)$ are likewise weak solutions of that equation, and since they are square integrable we may take Fourier transforms and find $|\xi|^2(1+\xi_j^2)\,\widehat{\partial_i g}(\xi)=0$, so that $\partial_i g=0$ as an element of $L_2(R^3)$. Thus $\partial_i g=0$, so that g is a constant; since g lies in $BL(R^3)$ this means that $g\equiv 0$, as was to be proved.

Note. We shall in fact need the following assertion which is proved in the same way: Suppose that f and $\nabla\times f$ both lie in $L_2(R^3;\mathscr{C}^3)$ (or $BL(R^3;\mathscr{C}^3)$). Then there exists a sequence $\{\psi_n\}\subset C_0^\infty(R^3;\mathscr{C}^3)$ such that $\{\psi_n\}$ and $\{\nabla\times\psi_n\}$ converge to f and $\nabla\times f$ respectively in $L_2(R^3;\mathscr{C}^3)$ (or $BL(R^3;\mathscr{C}^3)$).

Our initial data will on occasion be taken to be distributions. Sometimes it is necessary to consider arbitrary vector-valued distributions in $\mathscr{D}'(R^3;\mathscr{C}^3)$.[3] For some purposes, however, it is convenient to restrict our attention to a class of distributions smaller than $\mathscr{D}'$. This is true in particular when we wish to decompose vector-valued distributions into curl free and divergence free parts.

Let $\mathscr{D}_{L_2}=\{\psi;\, D^\alpha\psi\in L^2(R^3;\mathscr{C}^3)$, for any $\alpha\}$ topologised by the semi-norms

$$\|\psi\|_m = \sum_{|\alpha|\leqq m} \|D^\alpha\psi\|_{L_2};$$

here we have employed the familiar multi-index notation. By SOBOLEV's inequalities we know that each ψ in $\mathscr{D}_{L_2}$ is infinitely differentiable. We shall be concerned with distributions in $\mathscr{D}'_{L_2}$. It is well known that $\mathscr{E}'\subset\mathscr{D}'_{L_2}\subset\mathscr{S}'\subset\mathscr{D}'$. The following lemma, and more particularly its dual, indicates the usefulness of $\mathscr{D}'_{L_2}$ for our purposes.

Lemma 1.1.11. *Let $\psi\in\mathscr{D}_{L_2}$. Then there is a unique continuous decomposition $\psi=\psi^0+\psi^1$, such that ψ^0 and ψ^1 belong to $\mathscr{D}_{L_2}$, and are respectively curl free and divergence free. We write $\mathscr{D}_{L_2}=\mathscr{D}^0_{L_2}+\mathscr{D}^1_{L_2}$.*

Proof. From the orthogonal decomposition of $L_2(R^3;\mathscr{C}^3)$ (Lemma 1.1.4) we know that $\psi=\psi^0+\psi^1$, where ψ^0 and ψ^1 are both square integrable and respectively curl and divergence free. We prove that in the case that ψ is in $\mathscr{D}_{L_2}$ the other assertions also hold.

[3] $\mathscr{D}(R^3;\mathscr{C}^3)$ is simply $C_0^\infty(R^3;\mathscr{C}^3)$ supplied with the usual distribution topology, the euclidean norm replacing the usual absolute value in the definition of the relevant semi-norms. $\mathscr{D}'(R^3;\mathscr{C}^3)$ is the set of continuous linear functionals on $\mathscr{D}(R^3;\mathscr{C}^3)$ and can be identified with $\mathscr{D}'(R^3)\times\mathscr{D}'(R^3)\times\mathscr{D}'(R^3)$.

We proceed inductively. Suppose that $D^\alpha\psi^0$ is square integrable for $|\alpha|\leqq n$. In the sense of distributions $D^\alpha\psi=D^\alpha\psi^0+D^\alpha\psi^1$, so that $\nabla\times D^\alpha\psi^0=0$ and $\nabla\cdot D^\alpha\psi^0=\nabla\cdot D^\alpha\psi$ which is square integrable. Thus by Corollary 1.1.7 $D^\alpha\psi^0$ lies in $BL(R^3;\mathscr{C}^3)$ for any α with $|\alpha|\leqq n$. But then $D^\beta\psi^0$ is square integrable for $|\beta|=n+1$. In this way we see that $D^\alpha\psi^0$, and hence also $D^\alpha\psi^1$, are square integrable for all α, so that both ψ^0 and ψ^1 lie in $\mathscr{D}_{L_2}$. That the decomposition is continuous is obvious since $\|D^\alpha\psi\|_{L_2}^2=\|D^\alpha\psi^0\|_{L_2}^2+\|D^\alpha\psi^1\|_{L_2}^2$. The uniqueness of the decomposition is a consequence of the corresponding uniqueness of the decomposition for $L_2(R^3;\mathscr{C}^3)$.

We can now unambiguously define a dual decomposition for $\mathscr{D}'_{L_2}$.

Lemma 1.1.12. *There is a continuous decomposition of $\mathscr{D}'_{L_2}(R^3;\mathscr{C}^3)$ into curl free and divergence free components,*

$$\mathscr{D}'_{L_2}(R^3;\mathscr{C}^3)=[\mathscr{D}'_{L_2}(R^3;\mathscr{C}^3)]^0+[\mathscr{D}'_{L_2}(R^3;\mathscr{C}^3)]^1.$$

More explicitly, if T is in $\mathscr{D}'_{L_2}$ then $T=T^0+T^1$, where $T^0(\varphi)=T(\varphi^0)$ and $T^1(\varphi)=T(\varphi^1)$. Restricting T^0 and T^1 to be distributions in $\mathscr{D}'(R^3;\mathscr{C}^3)$ we have $T^0=-(1/r)*\nabla(\nabla\cdot T)$ and $T^1=(1/r)*\nabla\times(\nabla\times T)$.

Proof. The first statements follow trivially from the definitions and the properties of the decomposition of $\mathscr{D}_{L_2}$. The last assertion is seen as follows. It is well known that

$$\delta=-\frac{1}{4\pi}\Delta\frac{1}{r}.$$

Consequently we have the following decomposition for φ in $\mathscr{D}(R^3;\mathscr{C}^3)$:

$$\varphi=-\frac{1}{4\pi}\frac{1}{r}*\Delta\varphi=\frac{1}{4\pi}\frac{1}{r}*\nabla\times\nabla\times\varphi-\frac{1}{4\pi}\frac{1}{r}*\nabla(\nabla\cdot\varphi).$$

We show that

$$\varphi^0=-\frac{1}{4\pi}\frac{1}{r}\nabla(\nabla\cdot\varphi)$$

by proving that the latter curl free expression is square integrable. By the divergence theorem we have

$$\begin{aligned}\int\frac{1}{4\pi}\frac{1}{|x-y|}\nabla(\nabla\cdot\varphi)(y)\,dy&=\lim_{\substack{\varepsilon\to0\\R\to\infty}}\frac{1}{4\pi}\int_{\varepsilon<|x-y|<R}\frac{1}{|x-y|}\nabla(\nabla\cdot\varphi)(y)\,dy\\&=\lim_{\substack{\varepsilon\to0\\R\to\infty}}\Bigg[\int_{\varepsilon<|x-y|<R}\nabla\frac{1}{|x-y|}(\nabla\cdot\varphi)(y)\,dy\\&\quad+\int_{|x-y|=\varepsilon}n\frac{\nabla\cdot\varphi(y)}{\varepsilon}\varepsilon^2\,d\omega+\int_{|x-y|=R}n\frac{\nabla\cdot\varphi(y)}{|x-y|}\,dS_y\Bigg]\\&=\int\frac{1}{|x-y|^2}\frac{x-y}{|x-y|}(\nabla\cdot\varphi)(y)\,dy,\end{aligned}$$

which is square integrable by a Sobolev inequality ([1], p. 220). We therefore also have $\varphi^1=(1/r)*\nabla\times(\nabla\times\varphi)$. Thus for φ in $\mathscr{D}$

$$T^0(\varphi)=T(\varphi^0)=-T\left(\frac{1}{r}*\nabla(\nabla\cdot\varphi)\right)=-\frac{1}{r}*\nabla(\nabla\cdot T)(\varphi)$$

and similarly

$$T^1(\varphi)=\frac{1}{r}*\nabla\times(\nabla\times T)(\varphi).$$

Note. Let e_i be the i-th vector of an orthonormal basis in $\mathscr{C}^3$ and define the distribution $\delta_i=\delta e_i$ in $\mathscr{D}'_{L_2}(R^3;\mathscr{C}^3)$ by $\delta_i(\varphi)=\varphi_i(0)$; we then have

$$\delta_i^0(\varphi)=-\frac{1}{r}*\nabla(\nabla\cdot\delta_i)(\varphi)=-\delta_i\left[\frac{1}{r}*\nabla(\nabla\cdot\varphi)\right]=-\int\frac{1}{|x|}[\nabla(\nabla\cdot\varphi)]_i(x)\,dx,$$

and hence δ_i^0 is the distribution $-\nabla(\nabla\cdot(e_i/r))$ in $\mathscr{D}'(R^3;\mathscr{C}^3)$. Similarly δ_i^1 is given by $\nabla\times(\nabla\times[e_i/r])$.

§ 1.2. The Wave Equation in Free Space

We shall now review some properties of the wave equation, all of which are proved by LAX and PHILLIPS in [4]. We shall formulate the relevant results in terms of the vector-valued wave equation; they follow immediately from the corresponding results for the scalar equation.

We introduce the following space of complex valued functions defined in all of 3-space: $W_0=BL(R^3;\mathscr{C}^3)\times L_2(R^3;\mathscr{C}^3)$. The norm in W_0 is evidently the usual energy norm

$$\|w\|_{W_0}=\left[\tfrac{1}{2}\int_{R^3}(|\nabla w_1|^2+|w_2|^2)\,dx\right]^{\frac{1}{2}}.$$

Let $w=(w_1,w_2)$ belong to W_0 and consider the Cauchy problem

$$\begin{cases} w_{tt}(x,t)=\Delta w(x,t) & (x\in R^3,\ t\in R^1),\\ w(x,0)=w_1(x), & w_t(x,0)=w_2(x).\end{cases}$$

The wave equation can be expressed as

$$\frac{\partial}{\partial t}\begin{pmatrix} w\\ w_t\end{pmatrix}=\begin{pmatrix}0 & I\\ \Delta & 0\end{pmatrix}\begin{pmatrix} w\\ w_t\end{pmatrix};$$

this motivates the definition of an operator B_0 which acts like

$$\begin{pmatrix}0 & I\\ \Delta & 0\end{pmatrix}\quad\text{on}\quad D(B_0)=\left\{w\in W_0;\begin{pmatrix}0 & I\\ \Delta & 0\end{pmatrix}w\in W_0\right\},$$

where the Laplacian is understood in the distribution sense, and where we make the convention that a vector $w=(w_1,w_2)$ is always to be interpreted as a column vector when it is acted on by a matrix. B_0 turns out to be skew-adjoint and therefore generates a group $V_0(t)=\exp(B_0 t)$ of unitary operators on W_0. For given initial data w in W_0, the function $w(x,t)=[V_0(t)w]_1(x)$ is a solution of the initial value problem, and $w_t(x,t)=[V_0(t)w]_2(x)$.

The following theorems are crucial.

Theorem 1.2.1. *If $w(x)$ has its support in $|x-x_0|\geqq r$, then for $|t|<r$ the function $(V_0(t)\,w)\,(x)$ has its support in $|x-x_0|\geqq r-|t|$.*

Theorem 1.2.2. *If $w(x)$ has its support in $|x-x_0|\leqq r$, then for $|t|>r$ the function $(V_0(t)\,w)\,(x)$ has its support in $|x-x_0|\geqq|t|-r$.*

Theorem 1.2.3. *If $(V_0(t)\,w)\,(x)=0$ for $|x|<t$ (for all $t>0$), and $(V_0(t)\,\tilde{w})\,(x)=0$ for $|x|<-t$ (for all $t<0$), then w and $\tilde{w}$ are orthogonal in W_0.*

Theorem 1.2.4. *Let w be an element of W_0. If for some real, but non-zero, value of μ the function $(B_0-i\mu)\,w$ (which is to be understood in the distribution sense) vanishes for $|x|>r$, then $w(x)$ also vanishes for $|x|>r$.*

These theorems are all proved by LAX and PHILLIPS within the framework of a detailed analysis of the outgoing and incoming spectral and translation representations for the wave equation in free space. They can also be proved more directly. Theorem 1.2.1 describes domains of dependence for the wave equation. Theorem 1.2.2 essentially expresses HUYGHENS' principle, which is generally established from an explicit representation of the solution in terms of the initial values. Theorem 1.2.3 is proved directly in a paper by LAX, MORAWETZ and PHILLIPS [7], while Theorem 1.2.4 follows immediately from a theorem of RELLICH concerning solutions of the reduced wave equation $(\Delta+\mu^2)\,u=0$ (which is presented, for example, by MIRANKER in [8][4]).

LAX and PHILLIPS extend the domain of the group $V_0(t)$ to include all distribution valued initial data. We summarize their procedure since it will prove to be useful.

Given a pair $f=(f_1,f_2)$ of distributions, we define $V_0(t)f$ component-wise by

$$[V_0(t)\,f]_1\,(\varphi)=f_1([V_0(-t)\,(0,\varphi)]_2)-f_2([V_0(-t)\,(0,\varphi)]_1),$$

and

$$[V_0(t)\,f]_2\,(\varphi)=-f_1([V_0(-t)\,(\varphi,0)]_2)+f_2([V_0(-t)\,(\varphi,0)]_1).$$

This definition depends strongly on the fact guaranteed by the above theorems that $V_0(t)$ takes compactly supported data to compactly supported data, and also on the classical result that smooth data leads to smooth solutions. It is easy to verify that $w=[V_0(t)f]_1$ is a distribution solution of the wave equation and that $[V_0(t)f]_2=w_t$. The inital conditions are clearly satisfied. That there is only one solution corresponding to given initial data is also readily established.

Theorems 1.2.1 and 1.2.2 continue to hold for the extended group $V_0(t)$ and distribution data. We prove the extension of Theorem 1.2.1. Suppose that the data f has its support in $|x-x_0|\geqq r$. Let φ be a smooth function with support in $|x-x_0|\leqq r-|t|$. Then according to Theorem 1.2.1[5], $V_0(-t)\,(0,\varphi)$ has its support in $|x-x_0|\leqq r$. Looking at the explicit expressions for $[V_0(t)f]_1$ and $[V_0(t)f]_2$ and recalling that the support of f lies in $|x-x_0|\geqq r$, we see that $V_0(t)f$

[4] The main advantage of the method of LAX and PHILLIPS is that it leads to a generalization of the theorem stated here.

[5] It is an easy consequence of Theorem 1.2.1 that if the data w has its support in $|x-x_0|\leqq r$, then $V_0(t)\,w$ has its support in $|x-x_0|\leqq r+|t|$. This is what we use here.

has its support in $|x-x_0| \geqq r-|t|$. Theorem 1.2.2 can be extended in similar fashion. We formulate two further lemmas.

Lemma 1.2.5. *Let w be data such that*[6] $\nabla \cdot w (\nabla \times w)$ *has its support in* $|x-x_0| \geqq r$, *then for* $|t|<r$, $\nabla \cdot V_0(t)\, w (\nabla \times V_0(t)\, w)$ *has its support in* $|x-x_0| \geqq r-|t|$.

Lemma 1.2.6. *Let w be data such that* $\nabla \cdot w (\nabla \times w)$ *has its support in* $|x-x_0| \leqq r$, *then for* $|t|>r$, $\nabla \cdot V_0(t)\, w (\nabla \times V_0(t)\, w)$ *has its support in* $|x-x_0| \geqq |t|-r$.

These lemmas follow from Theorems 1.2.1 and 1.2.2 once we note that $\nabla \cdot V_0(t)\, w (\nabla \times V_0(t)\, w)$ are solutions of the initial value problems with initial data $\nabla \cdot w (\nabla \times w)$ vanishing in the specified regions.

To complete this section on the wave equation let us define

$$W_0^1 = \{w \in W_0;\ \nabla \cdot w = (0,0)\}.$$

As an immediate consequence of Lemma 1.2.5 we then have

Theorem 1.2.7. *The subspace* W_0^1 *of* W_0 *reduces the group* $V_0(t)$ *of unitary operators on* W_0; *i.e., if we define* $V_0^1(t)$ *and* B_0^1 *to be the restrictions of* $V_0(t)$ *and* B_0 *to* W_0^1, *then* $V_0^1(t)$ *is a group of unitary operators on* W_0^1 *and* B_0^1 *is the skew-adjoint generator of that group.*

§ *1.3.* MAXWELL's *Equations in Free Space*

MAXWELL's equations in the presence of a charge distribution $\rho(x,t)$ and a corresponding current distribution $\mu(x,t)$ (satisfying an equation of continuity) but in the absence of obstacles are

$$\begin{cases} \partial_t m_1(x,t) = \nabla \times m_2(x,t) + \mu(x,t), \text{ [7]} \\ \partial_t m_2(x,t) = -\nabla \times m_1(x,t), \end{cases}$$

subject to the initial conditions

$$m_1(x,0) = m_1(x) \quad \text{and} \quad m_2(x,0) = m_2(x),$$

where the pair $(m_1(x), m_2(x))$ belongs to the space of initial data

$$M_0 = L_2(R^3; \mathscr{C}^3) \times L_2(R^3; \mathscr{C}^3), \quad \text{with} \quad \|m\| = [\tfrac{1}{2} \textstyle\int (|m_1|^2 + |m_2|^2)\, dx]^{\frac{1}{2}}.$$

These equations describe the dynamic behavior of the electric field m_1 and the magnetic field m_2. The vector fields $m_1(x,t)$ and $m_2(x,t)$ are generally required to satisfy the additional conditions

$$\nabla \cdot m_1(x,t) = \rho(x,t), \qquad \nabla \cdot m_2(x,t) = 0.$$

We shall assume throughout that no currents are present, i.e., that $\mu(x,t)=0$. In this case the charge distribution is time independent and the divergence conditions on $m_1(x,t)$ and $m_2(x,t)$ turn out to be conditions on the initial data

[6] Here we use the notation $\nabla \cdot w = (\nabla \cdot w_1, \nabla \cdot w_2)$. Similarly $\nabla \times w = (\nabla \times w_1, \nabla \times w_2)$.

[7] In more familiar notation $m_1(x,t)$ is the electric field E, and $m_2(x,t)$ is the magnetic field H.

alone; the following heuristic argument indicates that this is so:

$$\partial_t \nabla \cdot m_1(x,t) = \nabla \cdot \partial_t m_1(x,t) = \nabla \cdot \nabla \times m_2(x,t) = 0,$$

so that $\nabla \cdot m_1(x, t) = \nabla \cdot m_1(x)$.

Since we are here concerned with MAXWELL's equations in free space we could immediately assume that $\rho(x, t)$ vanishes, and hence restrict ourselves to divergence free initial data. Instead we shall use a procedure which points the way to our subsequent treatment of the perturbed problem.

MAXWELL's equations can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} m_1(x,t) \\ m_2(x,t) \end{pmatrix} = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} \begin{pmatrix} m_1(x,t) \\ m_2(x,t) \end{pmatrix},$$

and this suggests the definition of an operator A_0 which acts like

$$\begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}$$

on

$$D(A_0) = \{m = (m_1, m_2) \in M_0 ; (\nabla\times m_2, -\nabla\times m_1) \in M_0\}.$$

We now introduce the following orthogonal decomposition of the space of initial data M_0 along the lines of Lemma 1.1.4: $M_0 = M_0^0 \oplus M_0^1$, where

$$M_0^0 = L_2^0(R^3; \mathscr{C}^3) \oplus L_2^0(R^3; \mathscr{C}^3) \quad \text{and} \quad M_0^1 = L_2^1(R^3; \mathscr{C}^3) \oplus L_2^1(R^3; \mathscr{C}^3).$$

One then has that $M_0^0 \subset D(A_0)$ and that A_0 annihilates M_0^0. It is trivial to check that the following holds:

Lemma 1.3.1. *The decomposition* $M_0 = M_0^0 \oplus M_0^1$ *reduces the operator* A_0.

We shall now establish a correspondence between the part A_0^1 of A_0 acting on M_0^1, and B_0^1, defined in Theorem 1.2.7 and acting on W_0^1. This clarifies the relationship between MAXWELL's equations and the wave equation.

Theorem 1.3.2. *There exists a one-to-one, norm-preserving map J of M_0^1 onto W_0^1 such that*

a) if $Jm = w$, then $m_1 = \nabla\times w_1$ and $m_2 = w_2$;

b) $JD(A_0^1) = D(B_0^1)$, and $A_0^1 = J^{-1} B_0^1 J$.

Proof. We use the map $\bar{J}$ introduced in Theorem 1.1.9 to define $J = \bar{J} \times I$. That J is a one-to-one, norm preserving map of M_0^1 onto W_0^1 satisfying *a)* is immediately evident. It remains to prove the relationship between A_0^1 and B_0^1. Suppose that m lies in $D(A_0^1)$ and that $w = Jm$. Then $m_1 = \nabla\times w_1$, $m_2 = w_2$, $\nabla\times m_1 = \nabla\times\nabla\times w_1$, and $\nabla\times m_2 = \nabla\times w_2$ are all square integrable. We shall show that w is in $D(B_0^1)$ or, explicitly, that $w_1 \in BL(R^3; \mathscr{C}^3)$, $w_2 \in L_2(R^3; \mathscr{C}^3)$, $w_2 \in BL(R^3; \mathscr{C}^3)$, $\Delta w_1 \in L_2(R^3; \mathscr{C}^3)$. The first two conditions are fulfilled because J maps M_0^1 into W_0^1. That w_2 lies in $BL(R^3; \mathscr{C}^3)$ is a consequence of Corollary 1.1.7, since $\nabla \cdot w_2$ (which vanishes), $\nabla\times w_2$ and w_2 are square integrable. Finally Δw_1 is in $L_2(R^3; \mathscr{C}^3)$ since $\Delta w = -\nabla\times\nabla\times w_1$ (w_1 being divergence free). Thus $JD(A_0^1) \subset D(B)_0^1$. That

J is onto follows since the above steps are reversible. Finally we have

$$J A_0^1 J^{-1} w = J A_0^1 J^{-1}(w_1, w_2) = J A_0^1(\nabla \times w_1, w_2)$$
$$= J(\nabla \times w_2, -\nabla \times \nabla \times w_1) = (w_2, \Delta w_1) = B_0^1 w.$$

This completes the proof of the theorem.

It is interesting to note that although the mapping J of M_0^1 onto W_0^1 is an isometry, locally the energy density of the "potential field" w is not the same as the energy density of the electro-magnetic field m. In order to illustrate this, two examples were presented in Appendix 1 of [10] which demonstrate that m may vanish inside or outside a sphere without Jm doing likewise.

Combining the last theorem with Theorem 1.2.7 we have

Theorem 1.3.3. *A_0^1 is a skew-adjoint operator on M_0^1, and $A_0 = 0 \oplus A_0^1$ is a skew-adjoint operator on $M_0 = M_0^0 \oplus M_0^1$.*

The operator A_0 generates a group $U_0(t)$ of unitary operators on M_0. For given initial data m in M_0, $U_0(t)\, m$ is the solution of the corresponding initial value problem for MAXWELL's equations. By the preceding decomposition of M_0 and A_0 we have $U_0(t) = I \oplus U_0^1(t)$, $U_0^1(t)$ being the group of unitary operators on M_0^1 generated by A_0^1. Thus fields corresponding to curl free and divergence free initial data are respectively stationary and time-dependent, while arbitrary fields can be decomposed uniquely into stationary and time-dependent parts (here fields have finite energy).

As a consequence of the relationship $A_0^1 = J^{-1} B_0^1 J$, we also have $U_0^1(t) = J^{-1} V_0^1(t) J$. We shall now exploit this fact to obtain for MAXWELL's equations corresponding to Theorems 1.2.1 up to 1.2.4 for the wave equation. It is clear from the preceding discussion that we could not expect these theorems to hold for $U_0(t)$ acting in all of M_0.

Theorem 1.3.4. *If m lies in M_0^1 and $m(x)$ has its support in $|x - x_0| \geqq r$, then for $|t| < r$, $[U_0^1(t)\, m](x)$ has its support in $|x - x_0| \geqq r - |t|$.*

Theorem 1.3.5. *If m lies in M_0^1 and $m(x)$ has its support in $|x - x_0| \leqq r$, then for $|t| > r$, $[U_0^1(t)\, m](x)$ has its support in $|x - x_0| \geqq |t| - r$.*

Theorem 1.3.6. *If m and $\bar{m}$ lie in M_0^1 and $[U_0^1(t)\, m](x)$ vanishes for $|x| < t$ (for all $t > 0$), while $[U_0^1(t)\, \bar{m}](x)$ vanishes for $|x| < -t$ (for all $t < 0$), then m and $\bar{m}$ are orthogonal.*

Theorem 1.3.7. *Let m be an element of $D(A_0^1)$. If for some real, but non-zero, value of μ $(A_0^1 - i\mu)\, m$ vanishes for $|x| > r$, then m itself vanishes for $|x| > r$.*

As before, these theorems can be generalized to apply to distribution valued data; in particular Theorem 1.3.7 can be greatly strengthened. However, the theorems as stated will be sufficient for our purpose. We proceed to prove them successively. The proofs via Theorem 1.3.2 are not quite as simple as one might anticipate; the difficulty lies in the fact, noted before, that the correspondence between m and $w = Jm$ is global rather than local. The proofs of all four theorems are given in [10]. Here we prove only Theorem 1.3.5 and 1.3.6; this involves an interesting intermediate result which clarifies the behaviour of the map J.

Lemma 1.3.8. *Suppose that w lies in W_0 and that $\nabla\times V_0(t)\,w$ and $\nabla\cdot V_0(t)\,w$ have their supports outside the cone $|x|<t$ $(t>0)$ [or $|x|<-t$ $(t<0)$]; then $V_0(t)\,w$ likewise has its support outside that cone.*

Proof. Since $\nabla\cdot V_0(t)\,w$ and $\nabla\times V_0(t)\,w$ vanish in the cone $|x|<t$, $[V_0(t)\,w]_1$ and $[V_0(t)\,w]_2$ are harmonic in their dependence on x in that cone. As a consequence of $\partial_t V_0(t)\,w=B_0\,V_0(t)\,w$ we then have

$$\frac{\partial}{\partial t}[V_0(t)\,w]_2=\Delta[V_0(t)\,w]_1=0, \qquad \text{for } |x|<t.$$

Thus $[V_0(t)\,w]_2\,(x)$ is constant in t for $t>|x|$. Let $\varphi(x)=[V_0(t)\,w]_2\,(x)$ with $t>|x|$. The function $\varphi(x)$ is then harmonic in R^3. For any $t>0$

$$\int_{|x|<t}|\varphi(x)|^2\,dx=\int_{|x|<t}|[V_0(t)\,w]_2(x)|^2\leqq\|w\|_{W_0}^2.$$

Thus $\varphi(x)$ is square integrable in R^3; since it is harmonic in R^3 it must vanish identically. Thus $[V_0(t)\,w]_2(x)=0$ for $|x|<t$. It remains to show that $[V_0(t)\,w]_1\,(x)$ vanishes in the same cone. Since

$$\frac{\partial}{\partial t}[V_0(t)\,w]_1=[V_0(t)\,w]_2(x)=0, \qquad \text{for } |x|<t,$$

we have that $[V_0(t)\,w]_1\,(x)$ is constant in t for $t>|x|$. As before we can define $\psi(x)=[V_0(t)\,w]_1(x)$ for $t>|x|$. Since

$$\sum_{j=1}^{3}\int_{|x|<t}|\nabla\psi_j|^2\,dx=\sum_{j=1}^{3}\int_{|x|<t}|\nabla[V_0(t)\,w]_{1j}|^2\leqq\|w\|_{W_0}^2,$$

we know that $\psi(x)$ has a square integrable gradient. Furthermore $\psi(x)$ is harmonic hence it is a constant, i.e., $\psi(x)=c$. We show that $c=0$. For any $r>0$ and $t>r$, Lemma 1.1.1 gives

$$4\pi c=\int_{|\omega|=1}|\psi(R\,\omega)|^2\,d\omega=\int_{|\omega|=1}|[V_0(t)\,w]_1(R\,\omega)|^2\,d\omega\leqq\frac{\|w\|_{W_0}^2}{R}.$$

Letting R become infinite we see that $c=0$ and that therefore $[V_0(t)\,w]_1\,(x)$ vanishes in $|x|<t$. The backwards cone $|x|<-t$ $(t<0)$ is treated similarly.

Corollary 1.3.9. *Let $w=Jm$. Then $U_0^1(t)\,m$ vanishes in $|x|<t$ $(t>0)$ [or $|x|<-t$ $(t<0)$] if and only if $V_0^1(t)\,w$ vanishes in the same cone.*

Proof. Suppose that $V_0^1(t)\,w$ vanishes for $|x|<t$. Then

$$U_0^1(t)\,m=J^{-1}\,V_0^1(t)\,J\,m=J^{-1}\,V_0^1(t)\,w=(\nabla\times[V_0^1(t)\,w]_1,[V_0^1(t)\,w]_2)$$

also vanishes there. Conversely, if $U_0^1(t)\,m$ vanishes for $|x|<t$, the identity

$$V_0^1(t)\,w=J\,U_0^1(t)\,J^{-1}\,w=J\,U_0^1(t)\,m$$

implies that $\nabla\times[V_0^1(t)\,w]_1$ and $[V_0^1(t)\,w]_2$ vanish for $|x|<t$. Noting that $\nabla\cdot V_0^1(t)w=0$ and applying the previous lemma we see that in fact $V_0^1(t)w$ vanishes there.

Theorems 1.3.5 and 1.3.6 are now easily proved.

Proof of Theorem 1.3.5. Let m be in M_0^1 and have support in $|x-x_0|\leqq r$. We have to show that $U_0^1(t)\,m$ vanishes in the cone $|x-x_0|<|t|-r$. Let $w=Jm$; then both $\nabla\cdot w$ and $\nabla\times w$ have their supports in $|x-x_0|\leqq r$. By Lemma 1.2.7 $\nabla\cdot V_0^1(t)\,w$ and $\nabla\times V_0^1(t)\,w$ vanish in $|x-x_0|<|t|-r$. Thus by the previous lemma $V_0^1(t)\,w$ itself vanishes in that cone. Hence by the above corollary $U_0^1(t)m$ vanishes for $|x-x_0|<|t|-r$.

Proof of Theorem 1.3.6. Let $w=Jm$ and $\bar{w}=J\bar{m}$. By Corollary 1.3.9 $V_0^1(t)w$ and $V_0^1(t)\bar{w}$ vanish for $|x|<t$ $(t>0)$ and $|x|<-t$ $(t<0)$ respectively. Hence by Theorem 1.2.3 w and $\bar{w}$ are orthogonal in W_0^1. Consequently, because of the properties of J, m and $\bar{m}$ are orthogonal in M_0^1.

§ *1.4. Incoming and Outgoing Subspaces for* MAXWELL'S *Equations, and the Translation Representation*

We define two subspaces of M_0 by

$$D_+^\rho=\{m\in M_0;\ U_0(t)\,m(x)=0 \text{ for } |x|<\rho+t,\ t>0\},$$

$$D_-^\rho=\{m\in M_0;\ U_0(t)\,m(x)=0 \text{ for } |x|<\rho-t,\ t<0\}.$$

In particular we denote D_+^0 and D_-^0 by D_+ and D_-, respectively. We now prove

Theorem 1.4.1. *D_+^ρ and D_-^ρ are subspaces of M_0^1. They are orthogonal, and are respectively outgoing and incoming with respect to the group $U_0^1(t)$.*

Proof. We prove the various assertions concerning D_+^ρ; the situation for D_-^ρ is analogous.

Suppose that m is in D_+^ρ and that n is in M_0^0. Then $U_0(t)n=n$ for all t and thus it is easy to verify using the unitary property of $U_0(t)$ that (see [10])

$$|(m,n)_{M_0}|\leqq\|m\|_{M_0}\Big[\int\limits_{|x|>t+\rho}(|n_1|^2+|n_2|^2)\,dx\Big]^{\frac{1}{2}};$$

the latter integral tends to zero as t tends to infinity since n_1 and n_2 are square integrable. Thus $(m,n)_{M_0}=0$ for any m in D_+^ρ and all n in M_0^0. Hence $D_+^\rho\subset M_0^1$.

That $U(t)\,D_+^\rho\subset D_+^\rho$ for $t\geqq 0$ is clear since if $U_0^1(t)\,m(x)$ vanishes for $|x|<t+\rho$ then $U_0^1(s)\,(U_0^1(t)\,m)\,(x)=U_0^1(t+s)\,m(x)$ vanishes for $|x|<t+s+\rho$, and *a fortiori* for $|x|<s+\rho$.

It is equally trivial that $\bigcap U_0^1(t)\,D_+^\rho=\{0\}$, since if m is in

$$\bigcap_{t\geqq 0} U_0^1(t)\,D_+^\rho$$

we have for all $t\geqq 0$ that $U_0^1(-t)m$ is in D_+^ρ, i.e., $m=U_0^1(t)\,(U_0^1(-t)m)$ vanishes for $|x|<t+\rho$. Thus m vanishes identically.

That $\bigcup U_0^1(t)D_+^\rho$ is dense in M_0^1 follows from the following assertions:

1) data with compact support are dense in M_0^1 (this is a consequence of Lemma 1.1.4); and

2) data with compact support lie in

$$\bigcup_{t\leq 0} U_0^1(t)D_+^\rho.$$

This second fact is a consequence of Theorem 1.3.5, for if the support of m lies in $|x|\leq r$, then $U_0^1(t)U_0^1(r+\rho)m=U_0^1(t+r+\rho)m$ vanishes in $|x|<(t+r+\rho)-r=t+\rho$, so that $U_0^1(r+\rho)m$ is in D_+^ρ and m is in $U_0^1(-r-\rho)D_+^\rho$.

Finally, the orthogonality of D_+^ρ and D_-^ρ involves simply a rephrasing of Theorem 1.3.6. This completes the proof of Theorem 1.4.1. We note that we could simply have used Corollary 1.3.9 to obtain the properties of D_+^ρ and D_-^ρ from the corresponding properties of the incoming and outgoing subspaces for the wave equation.

The properties of D_+^ρ and D_-^ρ established in the last theorem ensure that the theory of LAX and PHILLIPS is applicable. In particular we now know that there exist outgoing and incoming translation representations of $U_0^1(t)$ in M_0^1. In fact these two representations are the same (which is to be expected since no scattering is taking place), and can be derived from the translation representation of $V_0(t)$ acting in W_0. This latter transformation is obtained by the ingenious use of Fourier transformations and is described in the following theorem of LAX and PHILLIPS.

Theorem 1.4.2. *For $V_0(t)$ acting on W_0 there is a simultaneously outgoing and incoming translation representation on $L_2(-\infty,\infty;N)$, where*

$$N=\{f(\omega)\in L_2(S^2;\mathscr{C}^3)\},\quad (S^2=\{\omega\in R^3;|\omega|=1\}).$$

If w is of class[8] $\mathscr{S}$ then its translation representer $k(s,\omega)$ is given by

$$k(s,\omega)=-\partial_s^2\left[\frac{1}{4\pi}\int_{x\cdot\omega=s} w_1(x)\,dS\right]+\partial_s\left[\frac{1}{4\pi}\int_{x\cdot\omega=s} w_2(x)\,dS\right],$$

where dS is an element of surface area on the plane $x\cdot\omega=s$. Conversely if $k(s,\cdot)$ is infinitely differentiable it is the representer of data w given by

$$w_1(x)=\frac{1}{2\pi}\int_{|\omega|=1} k(x\cdot\omega,\omega)\,d\omega \quad\text{and}\quad w_2(x)=-\frac{1}{2\pi}\int_{|\omega|=1} k'(x\cdot\omega,\omega)\,d\omega$$

where $k'(s,\omega)=\partial_s k(s,\omega)$.

We denote the representation map from W_0 to $L_2(-\infty,\infty;N)$ by $\tilde{\mathscr{R}}$. We then define the map $\mathscr{R}=\tilde{\mathscr{R}}\circ J$, which maps M_0^1 onto the subspace $\tilde{\mathscr{R}}JM_0^1=\tilde{\mathscr{R}}W_0^1$ of $L_2(-\infty,\infty;N)$. We then have

[8] $\mathscr{S}$ is the class of infinitely differentiable functions such that all derivatives tend to zero at infinity faster than any polynomial of $|x|^{-1}$.

Theorem 1.4.3. *$\mathcal{R}$ defines a simultaneously outgoing and incoming translation representation of M_0^1 on $L_2(-\infty, \infty; N^1)$, where*

$$N^1 = \{f(\omega) \in N;\ \omega \cdot f(\omega) \equiv 0\}.$$

If Jm is of class $\mathcal{S}$ the translation representer of m is given by

$$\mathcal{R}\, m = k(s,\omega) = -\partial_s^2 \left[\frac{1}{4\pi} \int_{x\cdot\omega=s} [Jm]_1(x)\, dS\right] + \partial_s \left[\frac{1}{4\pi} \int_{x\cdot\omega=s} [Jm]_2(x)\, dS\right].$$

Conversely if $k(s,\cdot)$ is infinitely differentiable it is the representer of data m given by

$$m_1(x) = \frac{1}{2\pi} \int_{|\omega|=1} \omega \times k'(x\cdot\omega, \omega)\, d\omega \quad \text{and} \quad m_2(x) = -\frac{1}{2\pi} \int_{|\omega|=1} k'(x\cdot\omega, \omega)\, d\omega.$$

Proof. We have to show

a) $\mathcal{R} M_0^1 = L_2(-\infty, \infty; N^1)$;

b) $\mathcal{R}$ takes $U_0^1(t)$ in M_0^1 to the translation group in $L_2(-\infty, \infty; N^1)$;

c) $\mathcal{R} D_- = L_2(-\infty, 0; N^1)$ and $\mathcal{R} D_+ = L_2(0, \infty; N^1)$.

a) We know that $JM_0^1 = W_0^1$, so we simply need to show that $\tilde{\mathcal{R}}\, W_0^1 = L_2(-\infty, \infty; N^1)$. From Theorem 1.1.8 and Lemma 1.1.4 we know that smooth data with compact support are dense in W_0^1. For such data w we have the representation of Theorem 1.4.2, and thus

$$\omega \cdot \mathcal{R}\, w(\omega) = \omega \cdot k(s,\omega) = -\partial_s^2 \left[\frac{1}{4\pi} \int_{x\cdot\omega=s} \omega \cdot w_1(x)\, dS\right] + \partial_s \left[\frac{1}{4\pi} \int_{x\cdot\omega=s} \omega \cdot w_2(x)\, dS\right].$$

We now show that this vanishes. We have

$$\partial_s\Big[\int_{x\cdot\omega=s} \omega \cdot w_j(x)\, dS\Big] = \lim_{h\to 0} \frac{1}{h} \Big[\int_{x\cdot\omega=s+h} \omega \cdot w_j(x)\, dS - \int_{x\cdot\omega=s} \omega \cdot w_j(x)\, dS\Big].$$

Noting that ω is a unit normal to the planes $x\cdot\omega = s$ and $x\cdot\omega = s+h$, and that w_j has compact support, we can apply the divergence theorem to obtain

$$\partial_s\Big[\int_{x\cdot\omega=s} \omega \cdot w_j(x)\, dS\Big] = \lim_{h\to 0} \frac{1}{h} \Big[\int_{\partial D_h} \nabla \cdot w_j(x)\, dS\Big],$$

where D_h is the region between the planes $x\cdot\omega = s$, and $x\cdot\omega = s+h$. This last expression vanishes since w_j is divergence free. Hence $\tilde{\mathcal{R}}\, w(s)$ lies in $L_2(-\infty, \infty; N^1)$, and since the smooth data with compact support are dense in W_0^1, we have $\tilde{\mathcal{R}}\, W_0^1 \subset L_2(-\infty, \infty; N^1)$. The opposite inclusion is easily proved since infinitely differentiable functions $k(s,\omega)$ are dense in $L_2(-\infty, \infty; N^1)$; for the corresponding data in W_0 we have

$$\nabla \cdot w_1(x) = \frac{1}{2\pi} \int_{|\omega|=1} \nabla \cdot k(x\cdot\omega, \omega)\, d\omega = \frac{1}{2\pi} \int_{|\omega|=1} \omega \cdot k'(x\cdot\omega, \omega)\, d\omega = 0$$

and similarly $\nabla \cdot w_2(x) = 0$; and hence $w \in W_0^1$. This completes the proof of a).

The proof of assertion b) is immediate since $\mathcal{R}=\tilde{\mathcal{R}}\circ J$; J establishes a correspondence between $U_0^1(t)$ and $V_0^1(t)$, while $\tilde{\mathcal{R}}$ establishes a correspondence between $V_0^1(t)$ and the translation group in $L_2(-\infty,\infty;N^1)$.

Assertion c) follows from Corollary 1.3.9 and the fact that $\tilde{\mathcal{R}}$ defines a translation representation of $V_0(t)$ which is simultaneously incoming and outgoing. In fact we see that, more generally,

$$\mathcal{R}D_+^\rho=L_2(\rho,\infty;N^1),\qquad \mathcal{R}D_-^\rho=L_2(-\infty,-\rho;N^1).$$

Finally the explicit expressions for m and $\mathcal{R}m=k(s,\omega)$ in terms of each other follow directly from the definition of J and the corresponding statements in Theorem 1.4.2.

We shall now prove some further properties of the representation $\mathcal{R}=\tilde{\mathcal{R}}\circ J$ given above. In deducing the properties of $\mathcal{R}$ from the corresponding known properties of $\tilde{\mathcal{R}}$, we are as before faced by certain difficulties associated with the behavior of J. The following lemma (and particularly the remark following it) proves to be convenient.

Lemma 1.4.4. *Let w lie in W_0, and suppose that $\tilde{\mathcal{R}}w=k(s,\omega)$. Then $\partial_j w$ lies in W_0 if and only if $\omega_j k'(s,\omega)$ lies in $L_2(-\infty,\infty;N)$; if one of these equivalent conditions is fulfilled we have $\tilde{\mathcal{R}}\partial_j w=\omega_j k'(s,\omega)$.*

Proof. We suppose first that $\omega_j k'(s,\omega)$ lies in $L_2(-\infty,\infty;N)$. If $k(s,\omega)$ is infinitely differentiable we can apply the explicit equations of Theorem 1.4.2 to find that $\omega_j k'(s,\omega)$ is the representer of the data F in W_0 with components

$$F_1(x)=\frac{1}{2\pi}\int_{|\omega|=1}\omega_j k'(x\cdot\omega,\omega)\,d\omega=\frac{1}{2\pi}\int_{|\omega|=1}\partial_j k(x\cdot\omega,\omega)\,d\omega=\partial_j w_1(x),$$

$$F_2(x)=-\frac{1}{2\pi}\int_{|\omega|=1}\omega_j k''(x\cdot\omega,\omega)\,d\omega=-\frac{1}{2\pi}\int_{|\omega|=1}\partial_j k'(x\cdot\omega,\omega)\,d\omega=\partial_j w_2(x).$$

Hence $\partial_j w$ lies in W_0 and $\tilde{\mathcal{R}}\partial_j w=\omega_j k'$. If $k(s,\omega)$ is not infinitely differentiable in the s variable we mollify in the usual way to obtain a sequence $k_n(s,\omega)$ of infinitely differentiable functions such that $k_n(s,\omega)$ and $\omega_j k_n'(s,\omega)$ converge to $k(s,\omega)$ and $\omega_j k'(s,\omega)$ respectively in $L_2(-\infty,\infty;N)$. Let w_n be the element represented by k_n. Then by the preceding discussion $\partial_j w_n$ is represented by $\omega_j k_n'$. Since k_n and $\omega_j k_n'$ converge to k and $\omega_j k'$ respectively, w_n converges to w, and $\partial_j w_n$ converges to that element F of W_0 which is represented by $\omega_j k'$. Hence in the sense of distributions $\partial_j w_n$ converges both to F and to $\partial_j w$. Thus $F=\partial_j w$ and therefore $\partial_j w$ lies in W_0 and $\mathcal{R}\partial_j w=\omega_j k'(s,\omega)$.

Conversely, suppose that $\partial_j w$ lies in W_0. The proof again proceeds through two stages. If w is of class $\mathcal{S}$ we have

$$k(s,\omega)=-\partial_s^2\left[\frac{1}{4\pi}\int_{x\cdot\omega=s}w_1(x)\,dS\right]+\partial_s\left[\frac{1}{4\pi}\int_{x\cdot\omega=s}w_2(x)\,dS\right],$$

and thus

$$\omega_j k'(s,\omega)=-\partial_s^2\left[\frac{1}{4\pi}\partial_s\int_{x\cdot\omega=s}\omega_j w_1(x)\,dS\right]+\partial_s\left[\frac{1}{4\pi}\partial_s\int_{x\cdot\omega=s}\omega_j w_2(x)\,dS\right].$$

Using the same technique as in the proof of Theorem 1.4.3 (i.e. a difference quotient, the divergence theorem, and the integral mean value theorem) we find

$$\partial_s\Big[\int_{x\cdot\omega=s} \omega_j w_i(x)\,dS\Big]=\int_{x\cdot\omega=s} \partial_j w_i(x)\,dS, \qquad i=1,2,$$

so that

$$\omega_j k'(s,\omega)=-\partial_s^2\left[\frac{1}{4\pi}\int_{x\cdot\omega=s}\partial_j w_1(x)\,dS\right]+\partial_s\left[\frac{1}{4\pi}\int_{x\cdot\omega=s}\partial_j w_2(x)\,dS\right].$$

Thus we indeed have $\tilde{\mathscr{R}}\partial_j w=\omega_j k'(s,\omega)$. If w is not of class $\mathscr{S}$, we use Lemma 1.1.10 to find an approximating sequence $\{w_n\}$ of smooth data with compact support such that w_n and $\partial_j w_n$ converge to w and $\partial_j w$ respectively in W_0. An argument similar to that used in the first part of the proof shows that in the limit we have $\partial_j w$ in W_0 and $\tilde{\mathscr{R}}\partial_j w=\omega_j k'(s,\omega)$.

Note. In similar fashion we can prove the following: Let w lie in W_0, and suppose that $\tilde{\mathscr{R}}\, w=k(s,\omega)$. Then $\nabla\times w$ lies in W_0^1 [9] if and only if $\omega\times k'(s,\omega)$ lies in $L_2(-\infty,\infty;N^1)$; if one of these equivalent conditions is fulfilled we have $\tilde{\mathscr{R}}\,\nabla\times w=\omega\times k'(s,\omega)$.

Theorem 1.4.5. *Suppose that m lies in M_0^1, and that $k(s,\omega)=\mathscr{R}\, m$. Then*

a) if $k(s,\omega)=0$ for $|s|<r$, then $m(x)=0$ for $|x|<r$;

b) if $m=0$ for $|x|\geqq r$, then $k(s,\omega)=0$ for $|s|\geqq r$.

Proof. Part a) follows directly from Theorem 1.4.3 in the case that $k(s,\omega)$ is infinitely differentiable, for the explicit formulae show that the value of $m(x)$ is given entirely in terms of values of $k(s,\omega)$ in the sphere $|s|\leqq x$. If $k(s,\omega)$ is not infinitely differentiable it can be approximated through a mollification procedure by functions $k_n(s,\omega)$ with supports in $|s|\geqq r-1/n$. The corresponding elements m_n of M_0^1 have their supports in $|x|\geqq r-1/n$, and converge to m which therefore has its support in $|x|\geqq r$.

Part b) is a little more delicate. We suppose first that m is infinitely differentiable with support in $|x|\leqq r$. Then $w=Jm$ lies in W_0^1, and $\nabla\times w=(\nabla\times w_1, \nabla\times w_2)=(m_1, \nabla\times m_2)$ likewise lies in W_0^1; furthermore $\nabla\times w$ vanishes for $|x|\geqq r$. Let $\mathscr{R}\, m=\tilde{\mathscr{R}}\, w=k$; then by Lemma 1.4.4 $\tilde{\mathscr{R}}\,\nabla\times w=\omega\times k'(s,\omega)$. Since $\nabla\times w$ vanishes for $|x|\geqq r$, $\tilde{\mathscr{R}}\,\nabla\times w$ vanishes for $|s|\geqq r$ (which is directly evident from Theorem 1.4.2 and is in any case a known property of the translation representation for the wave equation). Hence $\omega\times k'(s,\omega)=0$ for $|s|\geqq r$. Since m is in M_0^1 we also know that $\omega\cdot k'(s,\omega)=0$. Consequently $k'(s,\omega)$ vanishes for $|s|\geqq r$, and thus $k(s,\omega)=n_\pm(\omega)$ for $s>r$ and $s<-r$ respectively. Since $k(s,\omega)$ is square integrable $n_\pm(\omega)$ vanish, so that part b) has been proved for infinitely differentiable $m(x)$. To obtain the proof for arbitrary elements m of M_0^1 we use an approximating sequence analogous to that used previously.

It would presumably be possible to obtain partial converses to the assertions of the last theorem analogous to those obtained by LAX and PHILLIPS for the translation representation of the wave equation. We shall not need these. It is however necessary to discuss infinite energy solutions of MAXWELL's equations,

[9] The superscripts 1 can be introduced immediately since $\nabla\cdot\nabla\times w\equiv 0$ and $\omega\cdot\omega\times k'(s,\omega)\equiv 0$.

and to extend the translation representation to a certain class of functions which do not necessarily lie in M_0^1. This will be done in the next section.

§ 1.5. MAXWELL's *Equations in the Space of Distributions*

Instead of restricting the initial data to the space M_0^1 we shall now consider distribution valued initial data. Let m lie in $\mathscr{D}'(R^3, \mathscr{C}^3) \times \mathscr{D}'(R^3; \mathscr{C}^3)$. We then define $U_0(t)\, m$ by

$$[U_0(t)\, m]_1(\varphi) = m_1([U_0(-t)(\varphi, 0)]_1) + m_2([U_0(-t)(\varphi, 0)]_2),$$

$$[U_0(t)\, m]_2(\varphi) = m_1([U_0(-t)(0, \varphi)]_1) + m_2([U_0(-t)(0, \varphi)]_2).$$

It is easy to verify that:

a) $U_0(t)\, m$ is a solution of MAXWELL's equations with initial value m.

b) This solution is unique.

c) If m is curl free, $U_0(t)\, m = m$.

d) If m is divergence free $U_0(t)\, m$ continues to be divergence free.

We denote $U_0(t)$ acting on divergence free data by $U_0^1(t)$.

The distribution data m is said to be *eventually outgoing (initially incoming)* if there exists a positive r such that $U_0(t)\, m$ vanishes in the cone $|x| < t - r$ for $t > r$ ($|x| < -t - r$ for $t < -r$). We shall now show that if m is eventually outgoing (or initially incoming) it is necessarily divergence free. We have

$$\nabla \cdot [U_0(t)\, m]_1(\varphi) = -[U_0(t)\, m]_1(\nabla\varphi)$$

$$= -m_1([U_0(-t)(\nabla\varphi, 0)]_1) - m_2([U_0(t)(\nabla\varphi, 0)]_2).$$

Since $(\nabla\varphi, 0)$ is curl free

$$[U_0(-t)(\nabla\varphi, 0)]_1 = [(\nabla\varphi, 0)]_1 = \nabla\varphi, \qquad [U_0(-t)(\nabla\varphi, 0)]_2 = 0,$$

and hence

$$\nabla \cdot [U_0(t)\, m]_1(\varphi) = -m_1(\nabla\varphi) = \nabla \cdot m_1(\varphi).$$

Consequently, since m is eventually outgoing, if the support of φ lies in $|x| \leq a$, $\nabla \cdot [U_0(t)\, m]_1(\varphi)$ and hence also $\nabla \cdot m_1(\varphi)$ vanishes for $t > r + a$. Thus $\nabla \cdot m_1$ and similarly $\nabla \cdot m_2$ both vanish as was asserted.

The question arises whether it is possible to extend the translation representation of $U_0^1(t)$ acting on M_0^1 to $U_0^1(t)$ acting on distribution valued data.

It is convenient to introduce the inverse map $\mathscr{J}$ $(= J^{-1} \circ \tilde{\mathscr{J}})$ of $\mathscr{R}$ which takes $L_2(-\infty, \infty; N^1)$ onto M_0^1. We first consider the extension of this map. The formulae

$$[\mathscr{J} k]_1(x) = \frac{1}{2\pi} \int_{|\omega|=1} \omega \times k'(x \cdot \omega, \omega)\, d\omega,$$

$$[\mathscr{J} k]_2(x) = -\frac{1}{2\pi} \int_{|\omega|=1} k'(x \cdot \omega, \omega)\, d\omega,$$

of Theorem 1.4.3 can be used to define $\mathscr{J} k(s)$ for any $k(s)$ in $C^\infty(-\infty, \infty; N^1)$ since for each x the integrals involve the values of $k(s)$ only in the compact set $|s| \leqq |x|$. A further extension of $\mathscr{J}$ by continuity to all distributions in $\mathscr{D}'(-\infty, \infty; N^1)$ is also readily justified. Then an explicit calculation with the above formulae shows that

$$A_0 \mathscr{J} = -\mathscr{J} \partial_s \quad \text{and} \quad U_0(t) \mathscr{J} = \mathscr{J} T(t).$$

Furthermore if k vanishes for $|s| \leqq r$, $\mathscr{J} k$ vanishes for $|x| \leqq r$.

There is no equally natural extension of the mapping $\mathscr{R}$ to all divergence free distribution data. However, by considering the dual of the lemma which we shall now prove, we obtain an extension of $\mathscr{R}$ to a class of data useful for our purposes.

Lemma 1.5.1. *$\mathscr{R}$ is a one-to-one map of $\mathscr{D}^1_{L_2}(R^3; \mathscr{C}^3) \times \mathscr{D}^1_{L_2}(R^3; \mathscr{C}^3)$ onto $\mathscr{D}_{L_2}(-\infty, \infty; N^1)$ and $\mathscr{J}$ is its inverse. Furthermore if $k = \mathscr{R} m$, then $\|A_0^n m\|_{M_0} = \|\partial_s^n k\|_{L_2}$.*

Proof. This follows from the basic properties of $\mathscr{R}$ and $\mathscr{J}$, from Lemma 1.4.4, and from the fact that J is an isometry between W_0^1 and M_0^1.

For m in $[\mathscr{D}'_{L_2}(R^3; \mathscr{C}^3)]^1 \times [\mathscr{D}'_{L_2}(R^3; \mathscr{C}^3)]^1$ we now define

$$\mathscr{R} m(\varphi) = \tfrac{1}{2} m(\mathscr{J} \varphi) \qquad \text{for any } \varphi \text{ in } \mathscr{D}_{L_2}(-\infty, \infty; N^1).$$

We prove immediately

Lemma 1.5.2. *$\mathscr{R}$ is a one-to-one mapping of $[\mathscr{D}'_{L_2}(R^3; \mathscr{C}^3)]^1 \times [\mathscr{D}'_{L_2}(R^3; \mathscr{C}^3)]^1$ onto $\mathscr{D}'_{L_2}(-\infty, \infty; N^1)$ and $\mathscr{J}$ is its inverse. If m^1 is the divergence free part of distribution valued data with support in $|x| \leqq r$,*[10] *then $\mathscr{R} m^1$ has its support in $|s| \leqq r$.*

Proof. The first statements follow immediately from the preceding lemma. The assertion about the supports is obtained as follows: Since for φ in $\mathscr{D}(-\infty, \infty; N^1)$, $(\mathscr{J} \varphi)^1 = \mathscr{J} \varphi$, we have

$$\mathscr{R} m^1(\varphi) = \tfrac{1}{2} m^1(\mathscr{J} \varphi) = \tfrac{1}{2} m(\mathscr{J} \varphi^1) = \tfrac{1}{2} m(\mathscr{J} \varphi).$$

For any φ which vanishes in $|s| \leqq r$, $\mathscr{J} \varphi$ vanishes in $|x| \leqq r$ so that $m(\mathscr{J} \varphi)$ and hence $\mathscr{R} m^1(\varphi)$ vanishes, which is what we need to prove.

It is possible to extend $\mathscr{R}$ further. We say that divergence free data m in $\mathscr{D}'(R^3; \mathscr{C}^3) \times \mathscr{D}'(R^3; \mathscr{C}^3)$ has a translation representer k in $\mathscr{D}'(-\infty, \infty; N^1)$ if $\mathscr{J} k = m$. The following theorem is to be understood in this way.

Theorem 1.5.3. *Let g^1 be the divergence free part of distribution valued data with compact support in $|x| \leqq r$. Then the equation $(A_0 - \mu) m = g^1$ has a unique eventually outgoing solution which has a translation representer given by the following distribution in $\mathscr{D}'(-\infty, \infty; N^1)$:*

$$k(\varphi) = \mathscr{R} g^1 \left(e^{\mu s} \int_{-\infty}^{s} \left\{ e^{-\mu \sigma} \varphi(\sigma, \omega) - \left[\int_{-\infty}^{\infty} e^{-\mu \tau} \varphi(\tau, \omega) \, d\tau \right] \theta(\sigma, \omega) \right\} d\sigma \right)$$

[10] m^1 need not have compact support!

where $\theta(\sigma, \omega)$ is a smooth non-negative function with compact support in $(-\infty, -r)$ such that

$$\int_{-\infty}^{0} \theta(\sigma, \omega)\, d\sigma = 1.$$

Proof. Proceeding formally one would expect a translation representer k of m to vanish at $-\infty$ (if m is to be eventually outgoing) and to satisfy the equation $-(\partial_s+\mu)\, k = \mathscr{R}\, g^1$. It is easily verified by explicit evaluation of $-[(\partial_s+\mu)\, k]\,(\varphi) = k([\partial_s-\mu]\,\varphi)$, that k as defined above does in fact satisfy the latter equation in the distribution sense. Furthermore, since $\mathscr{R}\, g^1$ has support in $|s| \leqq r$ it is easy to show that k vanishes for $s < -r$.

If we define $m = \mathscr{J} k$ we have

$$(A_0-\mu)\, m = (A_0-\mu)\, \mathscr{J} k = -\mathscr{J}(\partial_s+\mu)\, k = \mathscr{J}\mathscr{R}\, g^1 = g^1,$$

and m is eventually outgoing (for if $T(t+r)k = 0$ for $|s| \leqq t$ we know that $U(t+r)m = \mathscr{J} T(t+r)\, k = 0$ for $|x| \leqq t$). The solution m is unique, since otherwise there would exist a non-trivial eventually outgoing solution to $(A_0-\mu)\, d = 0$. This is impossible since $U_0(t)\, d = e^{\mu t} d$.

Corollary 1.5.4. *If m is eventually outgoing and $(A_0-\mu)\, m = 0$ for $|x| \geqq r$, m has a translation representer of the form*

$$k(s, \omega) = \begin{cases} 0 & s < -r \\ e^{-\mu s}\, n(\omega) & s > r. \end{cases}$$

Proof. That k vanishes for $s < -r$ has already been indicated. That $k(s, \omega)$ has the stated form for $s > r$ follows since $\mathscr{R}(A_0-\mu)\, m$ vanishes for $s > r$, so that for each test function with support in $s > r$

$$0 = [(\partial_s+\mu)\, k]\,(e^{\mu s}\varphi) = -k([\partial_s-\mu]\, e^{\mu s}\varphi) = -k(e^{\mu s}\partial_s\varphi) = [\partial_s(e^{\mu s} k)]\,(\varphi).$$

This implies that $e^{\mu s} k$ is independent of s for $s > r$, i.e., that $e^{\mu s} k(s, \omega)$ is in fact a function of ω alone.

Notes. a) Naturally $\omega \cdot n(\omega) = 0$.

b) We shall show later that $n(\omega)$ lies in N^1. When m is infinitely differentiable this is obvious.

c) There are analogous results for initially incoming solutions.

§ 1.6. The Fundamental Solution and Various Forms of the Outgoing and Incoming Radiation Conditions

A fundamental solution for MAXWELL's equations is a 6×6 matrix valued solution of

$$(A_0-\mu)\, G(x; \mu) = \delta(x)\, I,$$

which generally is required to satisfy some outgoing or incoming condition at infinity. The latter requirement does not have any dynamical significance for the stationary component of the equation, so in accordance with our general procedure we separate the equation into

$$(A_0-\mu)\,G^0=(\delta I)^0 \quad\text{and}\quad (A_0-\mu)\,G^1=(\delta I)^1.$$

Here δI has as its rows $(\delta_i, 0)$ and $(0, \delta_i)$ with $i=1, 2, 3$, and $(\delta I)^0$ and $(\delta I)^1$ are obtained by replacing δ_i by δ_i^0 and δ_i^1 respectively. It is easy to check, using the explicit form of δ_i^0 and δ_i^1, that $(\delta_i^0)_j=(\delta_j^0)_i$ and $(\delta_i^1)_j=(\delta_j^1)_i$, and hence that $(\delta I)^0$ and $(\delta I)^1$ are symmetric, and respectively curl and divergence free in rows and columns.

The equation for G^0 has an immediate solution; since $A_0 G^0$ must vanish we have $G^0=-(1/\mu)\,(\delta I)^0$. The equation for G^1 can be handled by means of Theorem 1.5.3. The lengthy but conceptually simple calculations necessary for the proof of the next theorem are given in [10], and will not be reproduced here.

Theorem 1.6.1. *The equation* $(A_0-\mu)\,G(x;\mu)=(\delta I)^1(x)$ *has a unique solution*

$$G(x;\mu)=\frac{1}{4\pi}\begin{pmatrix}\frac{1}{\mu}\,\nabla\times\nabla\times\left[\frac{(e^{-\mu r}-1)}{r}\,I\right] & -\nabla\times\left[\frac{e^{-\mu r}}{r}\,I\right]\\ \nabla\times\left[\frac{e^{-\mu r}}{r}\,I\right] & \frac{1}{\mu}\,\nabla\times\nabla\times\left[\frac{(e^{-\mu r}-1)}{r}\,I\right]\end{pmatrix},$$

each column of which is eventually outgoing. (The operator $\nabla\times$ *is taken to act columnwise on the square bracketed matrices.)*

Corollary 1.6.2. *Let g be divergence free distribution-valued data with compact support. Then the equation $(A_0-\mu)f=g$ has a unique eventually outgoing solution given explicitly by $f=G^1*g$.*

Before writing down further corollaries we make some observations and a definition which will make the whole procedure less cumbersome.

The equation

$$(A_0-\mu)m=g \tag{1}$$

is equivalent to the equations

$$\nabla\times\nabla\times m_1+\mu^2 m_1=-\mu\,g_1-\nabla\times g_2 \tag{2'}$$

and

$$m_2=-\frac{1}{\mu}\,\nabla\times m_1-\frac{1}{\mu}\,g_2\,. \tag{2''}$$

Definition. A solution of the equation $\nabla\times\nabla\times f+\mu^2 f=g$ (where g is a divergence free 3-vector-valued distribution with compact support) is called μ-outgoing (μ-incoming) if $(f,\ -(1/\mu)\,\nabla\times f)$ is eventually outgoing (initially incoming).

In order to find an eventually outgoing solution of equation (1), where now we assume g to be divergence free data with compact support, we need to find a μ-outgoing solution m_1 of (2′) and to define m_2 by (2″). Then $(m_1, m_2)=(m_1,\ -(1/\mu)\,\nabla\times m_1)+(0,\ -g_2/\mu)$ is eventually outgoing since m_1 is μ-outgoing and g has compact support, and is a solution of (1).

Corollary 1.6.3. *The equation $(\nabla\times\nabla\times+\mu^2)f=g$, where g is divergence free and has compact support, has a unique μ-outgoing solution $f=\gamma_\mu^1*g$, where γ_μ^1 is* *the unique* 3×3 *matrix valued distribution solution of* $(\nabla\times\nabla\times+\mu^2)\gamma_\mu^1=(\delta I)^1$ *with* μ*-outgoing columns.* γ_μ^1 *is given explicitly by*

$$\gamma_\mu^1=-\frac{1}{4\pi}\frac{1}{\mu^2}\nabla\times\nabla\times\left[\frac{(e^{-\mu r}-1)}{r}I\right].$$

In the case where we do not restrict ourselves to divergence free data we obtain the following result, which is proved in [10].

Corollary 1.6.4. *The equation* $(\nabla\times\nabla\times+\mu^2)f=g$ *(g a distribution with compact support) has a unique solution f which can be decomposed into a curl free part and a* μ*-outgoing divergence free part. That solution is given by* $f=\gamma_\mu * g$, *where*

$$\gamma_\mu=\frac{1}{4\pi\mu^2}\left[\mu^2\left(\frac{e^{-\mu r}}{r}I\right)-\nabla\nabla\cdot\left(\frac{e^{-\mu r}}{r}I\right)\right]$$

is such a solution to the equation with $g=\delta I$.

We are now in a position to discuss the various forms of the "outgoing wave" conditions for the reduced MAXWELL's equations. We assert

Theorem 1.6.5. *Let f be a solution of* $(\nabla\times\nabla\times+\mu^2)f=g$, *where g has compact support and where f is locally square integrable together with its first and second derivatives. The following statements are equivalent:*

(1) f can be written as the sum of a curl free (and hence stationary) part and a divergence free part which is μ*-eventually outgoing.*

(2) If G is a bounded region containing the support of g then in the exterior of G the solution f can be represented in terms of the boundary values of f on ∂G *by*

$$f(x)=\int_{\partial G}[n\times(\nabla\times\gamma_\mu(x-y))f(y)-\gamma_\mu(x-y)\,n\times(\nabla\times f)(y)]\,dS_y.$$

In the case that $\operatorname{Re}\mu\geqq 0$, (1) *and* (2) *are equivalent to*

(3) f satisfies the asymptotic conditions

$$\int_{S_r}\left|\frac{x}{r}\times(\nabla\times f(x))-\mu f(x)\right|dS_x=o(r),\qquad \int_{S_r}|f(x)|\,dS_x=o(r^2)$$

where $r=|x|$ *and* S_r *is the sphere of radius r.*

The proof involves the same techniques as are used by LAX and PHILLIPS for the wave equation; it is given explicitly in [10].

Notes. *a)* The assumption that f is locally square integrable together with ist first and second derivatives was needed to justify the use of GREEN's identity in proving the equivalence of (1) and (2). In fact it is easy to see the equivalence of (2) and (3), and also the assertion that (2) implies (1), without using any regularity of f in G: Outside the support of g we have $f=(1/\mu^2)\,\nabla\times\nabla\times f$; hence f is divergence free and satisfies $(-\Delta+\mu^2)f=0$, and thus is infinitely differentiable.

b) It is easy to verify that the stationary part of f, viz.

$$f^0=\gamma_\mu^0 * g=\nabla_x\left(-\frac{1}{4\pi}\int\frac{\nabla\cdot g(y)}{|x-y|}\,dy\right)$$

behaves asymptotically like $O(e^{-\mu r}/r^2)$. Consequently we have for the time-dependent part

$$f^1(x) = -\frac{1}{4\pi}\frac{e^{-\mu r}}{r}\,\omega\times\omega\times\int g(y)\,e^{\mu\,\omega\cdot y}\,dy + O\left(\frac{e^{-\mu r}}{r^2}\right).$$

We proved previously (Corollary 1.5.4) that if m is an eventually outgoing solution of $(A_0-\mu)\,m=g$, where g has support in $|x|<r$, then m has a translation representer $k(s,\omega)$ which vanishes for $s<-r$ and is of the form $e^{-\mu s}\,n(\omega)$ for $s>r$. At this point one can prove that $n(\omega)$ is in $L_2(S^2; N^1)$. In fact we can be more explicit (see [10]):

Theorem 1.6.6. *Let $n(\omega)$ be the function appearing in Corollary 1.5.4. Then*

$$n(\omega) = \frac{\mu}{4\pi}\,\omega\times a(\omega),$$

where

$$a(\omega) = \omega\times\omega\times\int\left(\mu\,g_1(y) + \nabla\times g_2(y)\right)e^{\mu\,\omega\cdot y}\,dy,$$

and

$$m_1(x) = \frac{1}{4\pi}\frac{e^{-\mu r}}{r}\,a(\omega) + O\left(\frac{e^{-\mu r}}{r^2}\right).$$

2. The Perturbed Problem

§ 2.1. MAXWELL's *Equations in the Exterior of an Obstacle*

Let G be a domain exterior to an obstacle which we suppose to have a twice continuously differentiable boundary. Let $M = L_2(G;\mathscr{C}^3)\times L_2(G;\mathscr{C}^3)$ and consider the initial value problem

$$\begin{cases} \partial_t\, m_1(x,t) = \nabla\times m_2(x,t) & ((x,t) \text{ in } G\times R),\\ \partial_t\, m_2(x,t) = -\nabla\times m_1(x,t), & \\ m_1(x,0) = m_1(x), \quad m_2(x,0) = m_2(x) & \end{cases}$$

where the pair of initial data $m=(m_1, m_2)$ lies in M. We wish to define the operator

$$A = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}$$

in such a way that it becomes skew adjoint. We shall specify the domain of A in terms of suitable boundary values.

It is convenient to introduce the following class of data:

$$\mathscr{D} = \{\varphi = (\varphi_1, \varphi_2);\ \varphi_1, \varphi_2 \in C^2(\bar{G};\mathscr{C}^3) \text{ and } \varphi,\ A\varphi\in M\}$$

where $\bar{G}$ denotes the closure of the domain G. The following is not difficult to prove with the aid of the divergence theorem, although the unboundedness of the domain requires some attention (see [10]).

Lemma 2.1.1. *For φ and ψ in $\mathscr{D}$ the following identity holds*

$$(\varphi, A\psi)_M + (A\varphi, \psi)_M = \int_{\partial G} (\varphi \cdot A_n \bar{\psi})\, dS,$$

where

$$A_n = \begin{pmatrix} 0 & n\times \\ -n\times & 0 \end{pmatrix},$$

n being the outward normal to the boundary surface ∂G of G, and dS denoting an element of surface area.

At this point we present a brief account of the relevant definitions and the main theorem contained in the paper [6] of LAX and PHILLIPS. They consider a first order matrix partial differential operator which acts on real n-vector valued functions defined on a bounded domain G in R^m:

$$L = \sum_{j=1}^{m} A^j(x)\, \partial_j + B(x),$$

where the $A^j(x)$ and $B(x)$ are real $n \times n$ matrix valued functions defined in $\bar{G}$ and satisfying the conditions:

(a) The $A^j(x)$ are symmetric, continuous and piecewise continuously differentiable.

(b) $B(x)$ is piecewise continuous.

(c) Letting $K = B - \frac{1}{2}\sum_{j=1}^{m} \partial_j A^j$, $K + K^* \leqq 0$.

Condition (a) characterizes symmetric operators, while (c) characterizes "formally dissipative operators". The boundary value problem is formulated in the following way. Let ∂G be the boundary of G, G being assumed to be bounded and ∂G to be twice continuously differentiable. For x in ∂G let

$$A_n(x) = \sum_{j=1}^{m} n_j(x)\, A^j(x),$$

where $n = (n_j)$ is the surface normal. It is assumed that:

(d) $A_n(x)$ can be continued into a neighborhood of ∂G in such a way that it has constant rank there.

With each point of ∂G we associate a subspace $N(x)$ of R^n; this boundary space is supposed to vary smoothly with x. The boundary condition on a function $u(x)$ is then "$u(x)$ lies in $N(x)$ for each x in ∂G". The boundary condition is said to be non-positive if $u(x) \cdot A_n(x)\, u(x) \leqq 0$, for all $u(x)$ in $N(x)$. It is said to be maximal non-positive if the subspace $N(x)$ cannot be extended without violating non-positivity.

For given f in $L_2(G; R^n)$ we seek a solution u of $(I - L)u = f$ with $u(x)$ in $N(x)$ for all x in ∂G. We shall say that a function u is a strong solution of that equation satisfying the boundary condition in a strong sense if there exists a sequence $\{\varphi_n\}$ of functions in

$$\mathscr{D}_N^r = \{\varphi \in C^2(\bar{G}; R^n);\ \varphi, L\varphi \in L^2(G; R^n),\ \varphi(x) \in N(x) \ \text{ for } x \in \partial G\}$$

(the superscript r reminding us that we are for the moment restricting ourselves to real valued functions) such that

$$\varphi_n \to u, \quad (I-L)\varphi_n \to f \qquad \text{in } L_2(G; R^n).$$

The central theorem is

Theorem 2.1.2. *Let L be a formally dissipative symmetric operator, G a bounded domain whose boundary is of class C^2, and $N(x)$ smoothly varying boundary spaces which are maximal non-positive. Then for every given square integrable function f, the equation $(I-L)u=f$ has a unique strong solution satisfying the given boundary conditions in the strong sense.*

Actually LAX and PHILLIPS prove the existence of a solution u approximable by functions continuous in $\bar{G}$ and piecewise continuously differentiable there. However the same arguments with only slight modifications yield a strong solution as defined above.

The theory is clearly applicable to the operator

$$L=A=\begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}.$$

In fact, since $K=0$ in this case, $-A$ is also formally dissipative (and symmetric). Hence subject to suitable boundary conditions the theorem allows us to solve both equations $(I\pm A)u=f$. We proceed to examine various possible maximal non-positive boundary conditions. Maximal non-positive subspaces $N(x)$ for which we have not only $u\cdot A_n u\leqq 0$, but in fact $u\cdot A_n u=0$, are of particular interest. For (a) in this case $N(x)$ is maximal non-positive both for A and for $-A$; and (b) the identity of Lemma 2.1.1 ensures that A is skew-symmetric when acting on functions satisfying such conditions.

We define two classes of subspaces of $R^6 \cong R^3\times R^3$. Let x be a point in ∂G, and $n(x)$ the surface normal. Let λ be any real number (or ∞) and π a plane in R^3 containing $n(x)$. Then we define

$$N_\lambda(x)=\{u=(u_1,u_2);\ n(x)\times u_1=\lambda(n(x)\times u_2)\},$$

$$(N_\infty(x)=\{u=(u_1,u_2);\ n(x)\times u_2=0\}),$$

$$N_\pi(x)=\{u=(u_1,u_2);\ u_1\in\pi,\ u_2\in\pi\}.$$

Theorem 2.1.3. *$N_\lambda(x)$ and $N_\pi(x)$ are maximal non-positive subspaces, and $u\cdot A_n u=0$ for u in $N_\lambda(x)$ or u in $N_\pi(x)$. Furthermore, if $N(x)$ is any maximal non-positive boundary space for which $u\cdot A_n u=0$, then $N(x)$ is either a subspace of the type $N_\lambda(x)$ or of the type $N_\pi(x)$.*[11]

Proof. For any $u=(u_1,u_2)$ in $R^3\times R^3$ we have

$$u\cdot A_n u=u_1\cdot n\times u_2-u_2\cdot n\times u_1=2u_1\cdot n\times u_2.$$

[11] This theorem gives a characterization of real boundary spaces $N(x)$ satisfying the requirements; it does not characterize all suitable complex boundary spaces. It is easy to see that $N^c_{\lambda(x)}=N_{\lambda(x)}+iN_{\lambda(x)}$ and $N^c_{\pi(x)}=N_{\pi(x)}+iN_{\pi(x)}$ are such complex boundary spaces; conceivably there are others.

If u is in N_λ or in N_π we know that u_1, u_2 and n are coplanar, and hence $u_1 \cdot n \times u_2 = 0$. Thus in either case we have $u \cdot A_n u = 0$.

We now show that N_λ and N_π are maximal. We could do this by a dimensionality argument looking at the eigenvalues of A_n. However we shall argue more directly.

We begin by considering N_λ. Suppose that v is a vector such that $\{v\} \cup N_\lambda$ generates a non-positive boundary space; we have to show that v lies in N_λ. For any u in N_λ and for arbitrary real k we have $(v+ku) \cdot A_n(v+ku) \leqq 0$. Since $u \cdot A_n u = 0$, and since A_n is symmetric, $v \cdot A_n v + 2kv \cdot A_n u \leqq 0$. Since this holds for each real k we must have $v \cdot A_n u = 0$ for each u in N_λ. Thus $v_1 \cdot n \times u_2 - v_2 \cdot n \times u_1 = 0$, or, since u is in N_λ, $v_1 \cdot n \times u_2 - \lambda v_2 \cdot n \times u_2 = 0$. Hence we finally have $u_2 \cdot [n \times v_1 - \lambda n \times v_2] = 0$. The vector u_2 is arbitrary since for any $u_2 (\lambda u_2, u_2)$ lies in N_λ. Hence $(n \times v_2) = \lambda(n \times v_1)$, so that v lies in N_λ.

In order to show that N_π is maximal we proceed in similar fashion to conclude that if $\{v\} \cup N_\pi$ generates a non-positive subspace then $v \cdot A_n u = 0$ for each u in N_π. Choosing $u = (u_1, 0)$ or $u = (0, u_2)$ with u_1 and u_2 in π we conclude that $v_1 \cdot n \times u_2 = 0$ and $v_2 \cdot n \times u_1 = 0$, so that v_1 and v_2 lie in π and v indeed lies in N_π.

We now consider an arbitrary maximal non-positive boundary space $N(x)$ for which $u \cdot A_n u = 0$. If $u \cdot A_n u = 0$ then $u_1 \cdot n \times u_2 = 0$, and hence u_1, u_2 and n are coplanar. We choose v in $N(x)$ so that v_1, v_2 and n are not collinear; there must exist such a v since otherwise $N(x)$ is properly contained in N_π for any π and is thus not maximal. Since v_1, v_2 and n define a plane, $n \times v_1$ and $n \times v_2$ do not both vanish. Thus $(n \times v_1) = \lambda(v)(n \times v_2)$, $\lambda(v)$ being uniquely determined (possibly infinite in the case that $n \times v_2 = 0$). For any u in $N(x)$ we define $\lambda(u)$ so that $(n \times u_1) = \lambda(u)(n \times u_2)$ in the case where u_1, u_2 and n are not collinear, and put $\lambda(u) = \lambda(v)$ when u_1, u_2 and n are collinear. If $\lambda(u) = \lambda(v)$ for all u in $N(x)$ we have $N(x) \subset N_{\lambda(v)}(x)$; since $N(x)$ is maximal this implies that $N(x) = N_{\lambda(v)}(x)$. In the alternative case there exists $\bar{v}$ in $N(x)$ such that $\lambda(\bar{v}) \neq \lambda(v)$; by the way in which v was chosen and $\lambda(\bar{v})$ was defined we know that v_1, v_2, n and $\bar{v}_1, \bar{v}_2, n$ specify uniquely two planes π and $\bar{\pi}$. We shall show that $\bar{\pi} = \pi$, and that $N(x) = N_\pi(x)$. Given any u in $N(x)$, $\lambda(u)$ is different from one or other of $\lambda(v)$ and $\lambda(\bar{v})$. Suppose first that $\lambda(u) \neq \lambda(v)$. If $\lambda(u)$ and $\lambda(v)$ are both finite then $n \times v_2$ and $n \times u_2$ do not vanish, and

$$
\begin{aligned}
0 &= \tfrac{1}{2}(u+v) \cdot A_n(u+v) = u \cdot A_n v = u_1 \cdot n \times v_2 - u_2 \cdot n \times v_1 \\
&= -v_2 \cdot n \times u_1 - u_2 \cdot n \times v_1 = \big(\lambda(u) - \lambda(v)\big)\, u_2 \cdot n \times v_2 .
\end{aligned}
$$

Hence $u_2 \cdot n \times v_2 = 0$, and thus u_2 lies in π. Since $(n \times u_1) = \lambda(u)(n \times u_2)$, u_1 likewise lies in π. The case where one or the other of $\lambda(u)$ and $\lambda(v)$ is infinite can be handled similarly. In all cases we find that $\lambda(u) \neq \lambda(v)$ implies that u lies in $N_\pi(x)$. In particular $\bar{v}$ lies in N_π and thus $\bar{\pi} = \pi$. Suppose on the other hand that $\lambda(u) = \lambda(v)$. Then $\lambda(u) \neq \lambda(\bar{v})$, and as before we can conclude that u_1 and u_2 lie in $\bar{\pi} = \pi$. Thus we again have that u is in $N_\pi(x)$. Hence $N(x) \subset N_\pi(x)$. Since $N(x)$ is maximal, $N(x) = N_\pi(x)$. This completes the proof of the theorem.

We are now in a position to define the operator A in such a way that it becomes skew adjoint.

Let $\lambda(x)$ be a smooth real valued function defined on ∂G. Let $\pi(x)$ be a smooth function which to each point x on ∂G assigns a plane $\pi(x)$ in R^3 which contains $n(x)$. Then $N_{\lambda(x)}$ and $N_{\pi(x)}$ are smoothly varying maximal non-positive boundary spaces for the operator A. Returning to our spaces of complex vector-valued functions we let

$$\mathscr{D}_{\lambda(x)}=\{\varphi\in\mathscr{D};\ \mathrm{Re}\,\varphi(x)\in N_{\lambda(x)},\ \mathrm{Im}\,\varphi(x)\in N_{\lambda(x)} \text{ for } x\in\partial G\},$$

and similarly we define $\mathscr{D}_{\pi(x)}$. The graph norm of A in the space M is $\|\varphi\|_M^2+\|A\varphi\|_M^2$. We define the operators $A_{\lambda(x)}$ and $A_{\pi(x)}$ to be

$$\begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}$$

acting in the distribution sense on

$$D(A_{\lambda(x)})=\text{closure in the graph norm of } \mathscr{D}_{\lambda(x)};$$

and

$$D(A_{\pi(x)})=\text{closure in the graph norm of } \mathscr{D}_{\pi(x)}.$$

We then have

Theorem 2.1.4. *The operators $A_{\lambda(x)}$ and $A_{\pi(x)}$ are skew adjoint.*

Proof. We prove the assertion for $A_{\lambda(x)}$. The proof for $A_{\pi(x)}$ is exactly analogous. It is sufficient to show that $A_{\lambda(x)}$ has a dense domain, is skew symmetric and closed, and that its deficiency indices are zero. That $D(A_{\lambda(x)})$ is dense in M is clear since it contains $C_0^\infty(G;\mathscr{C}^3)\times C_0^\infty(G;\mathscr{C}^3)$. The skew symmetry follows from the identity of Lemma 2.1.1 if we note that

$$\varphi\cdot A_n\bar{\psi}=[\mathrm{Re}\,\varphi+i\,\mathrm{Im}\,\varphi]\cdot A_n[\mathrm{Re}\,\psi-i\,\mathrm{Im}\,\psi],$$

which vanishes since we not only have $u\cdot A_n u=0$ for u in $N_{\lambda(x)}$, but also $u\cdot A_n v=0$ if v is likewise in $N_{\lambda(x)}$ (which is a consequence of $0=(u+v)\cdot A_n(u+v)=2u\cdot A_n v$). The identity $(\varphi, A\psi)_M+(A\varphi,\psi)_M=0$ which holds for φ and ψ in $\mathscr{D}_{\lambda(x)}$ continues to hold when we take the closure of $\mathscr{D}_{\lambda(x)}$ in the graph norm. That $A_{\lambda(x)}$ is closed is obvious since $D(A_{\lambda(x)})$ is closed in the graph norm. In order to show that the deficiency indices are zero we have to show that $I\pm A_{\lambda(x)}$ map $D(A_{\lambda(x)})$ onto M; that this is the case follows by an easy extension of Theorem 2.1.2 to unbounded domains, which is valid in the situation we are considering (as was proved in Appendix 2 of [10]); we have, of course, to apply that theorem separately to the real and imaginary parts.

Notes. *(a)* We shall usually simply write A instead of $A_{\lambda(x)}$ or $A_{\pi(x)}$, and where necessary specify which boundary conditions we are considering by saying that A corresponds to boundary values of the first or second kind, depending on whether A is of the form $A_{\lambda(x)}$ or $A_{\pi(x)}$.

(b) The boundary condition corresponding to $\lambda(x)=0$ can be expressed as $n\times m_1=0$ on ∂G; this is the boundary condition usually imposed on the electric field when the object is a perfect conductor, and expresses the continuity across

the boundary of the tangential components[12]. It does not seem clear to what physical situations the other boundary conditions correspond.

As a consequence of the last theorem, A generates a group $U(t)$ of unitary operators. For given data m in M, $m(x,t)=[U(t)\,m](x)$ is a solution of MAXWELL's equations corresponding to the initial value $m(x,0)=m(x)$. We now study the generator A and the group $U(t)$ in greater detail.

§ 2.2. Properties of the Generator

The crucial step in our development of the theory is once again the decomposition of the space M of initial data into two orthogonal subspaces, corresponding to stationary and time-dependent fields respectively.

Let $M^0=\{m;\, m\in D(A),\ Am=0\}$, and define M^1 to be the orthogonal complement of M^0 in M. The operator A and the group $U(t)$ are then reduced and we have

$$M=M^0\oplus M^1,\qquad A=0\oplus A^1,\qquad U(t)=I\oplus U^1(t).$$

We shall later have to investigate this decomposition in some detail; for the moment it suffices to note that $M^0\supset\{(\nabla\varphi_1,\nabla\varphi_2);\ \varphi_1,\varphi_2\in C_0^\infty(G)\}$, and consequently that the data in M^1 are divergence free.

At this point we can easily prove

Theorem 2.2.1. *The operator A^1 acting on M^1 has no point spectrum*[13].

Proof. Since A^1 is skew-symmetric its spectrum lies on the imaginary axis. Suppose that $(A^1-i\mu)\,m=0$, with μ real and m in $D(A^1)=D(A)\cap M^1$. We can suppose that $\mu\neq 0$, since otherwise m would lie in M^0. According to the comments preceding the theorem m is divergence free. Hence

$$(\Delta+\mu^2)\,m=(A^1+i\,\mu)(A^1-i\,\mu)\,m=0,$$

where we have used the identity $\nabla\times(\nabla\times m_i)=-\Delta m_i+\nabla(\nabla\cdot m_i)=-\Delta m_i$. Since m satisfies the above elliptic equation m must be an analytic function of x in G. We shall show that m vanishes identically outside some neighborhood of the obstacle; since m is analytic it must then vanish everywhere in G. Let $\varphi(x)$ be an infinitely differentiable function in G which vanishes in a neighborhood of the obstacle and is identically one for $|x|\geqq r$. We "cut off" $m(x)$ to obtain divergence free data vanishing near the obstacle by defining $\overline{m}(x)=-i\mu^{-1}A^1(\varphi m)$. Since $-i\mu^{-1}A^1m=m$ it is clear that $\overline{m}(x)=m(x)$ for $|x|>r$, that $\overline{m}(x)$ lies in $D(A^1)$ and that $(A^1-i\mu)\,\overline{m}(x)$ vanishes for $|x|>r$. We can therefore apply Theorem 1.3.7 and conclude that $\overline{m}(x)$ vanishes for $|x|>r$. Thus $m(x)$ vanishes there, and hence vanishes identically. Thus A^1 has no point spectrum.

We now try to clarify the sense in which the data in $D(A)$ satisfy the boundary conditions. This will at the same time be the first step towards the local compactness theorem which we shall prove subsequently. At this point it becomes necessary to restrict ourselves to boundary conditions of the first kind; furthermore

[12] The boundary condition $n\cdot m_2=0$ usually imposed on the magnetic field at the surface of a perfect conductor will arise in a natural fashion later.

[13] Later we shall see that the spectrum of A^1 is in fact absolutely continuous.

we shall suppose that $\lambda(x)$ can be extended into G in such a way that $\lambda(x)$ is once continuously differentiable in $\bar{G}$ and that $\lambda(x)$ as well as its first derivatives are uniformly bounded there.

Our analysis relies heavily on a paper by FRIEDRICHS [3]. We now specialize his results to our situation, retaining his notation. We suppose first that G is a bounded domain in R^3 which has a C^2 boundary. We define

$$\mathscr{B}_1=\{f\in L_2(G;R^3);\ \partial_j f_i\in L_2(G;R_3) \text{ in the distribution sense for } i,j=1,2,3\};$$

$$\mathscr{B}_d=\{f\in L_2(G;R^3);\ \nabla\times f\in L_2(G;R^3) \text{ in the distribution sense}\};$$

$$\mathscr{B}_\delta=\{f\in L_2(G;R^3);\ \nabla\cdot f\in L_2(G) \text{ in the distribution sense}\}.$$

We shall introduce subspaces of these spaces corresponding to functions satisfying various boundary conditions. Let

$${}^1S=\{u(x);\ u \text{ is Lipschitz continuous in } \bar{G} \text{ and } n\times u=0 \text{ on } \partial G\};$$

$${}^2S=\{u(x);\ u \text{ is Lipschitz continuous in } \bar{G} \text{ and } n\cdot u=0 \text{ on } \partial G\}.$$

We then define

$${}^k\mathscr{B}_1=\text{closure in } \|f\|^2+\sum_i\|\nabla f_i\|^2 \text{ of } {}^kS, \qquad k=1,2;$$

$${}^1\mathscr{B}_d=\text{closure in } \|f\|^2+\|\nabla\times f\|^2 \text{ of } {}^1S;$$

$${}^2\mathscr{B}_\delta=\text{closure in } \|f\|^2+\|\nabla\cdot f\|^2 \text{ of } {}^2S.$$

FRIEDRICHS proves

Lemma 2.2.2. *${}^1\mathscr{B}_1={}^1\mathscr{B}_d\cap\mathscr{B}_\delta$ and for f in ${}^1\mathscr{B}_1$*

$$\|f\|^2+\sum_{i=1}^{3}\|\nabla f_i\|^2\leqq C[\|f\|^2+\|\nabla\times f\|^2+\|\nabla\cdot f\|^2].$$

Lemma 2.2.3. *${}^2\mathscr{B}_1=\mathscr{B}_d\cap{}^2\mathscr{B}_\delta$ and for f in ${}^2\mathscr{B}_1$*

$$\|f\|^2+\sum_{i=1}^{3}\|\nabla f_i\|^2\leqq C[\|f\|^2+\|\nabla\times f\|^2+\|\nabla\cdot f\|^2].$$

We now can prove

Theorem 2.2.4. *Suppose that $A=A_{\lambda(x)}$, that m lies in $D(A)$ and that $\nabla\cdot m_1$ and $\nabla\cdot m_2$ are square integrable. Then, defining $F_1(m)=m_1(x)-\lambda(x)\,m_2(x)$, where $\lambda(x)$ is the extended function, we have*

$$\|F_1\|^2+\sum_{i=1}^{3}\|\nabla F_{1i}\|^2\leqq C[\|F_1\|^2+\|\nabla\times F_1\|^2+\|\nabla\cdot F_1\|^2].$$

Proof. Since the domain is not bounded we have to be a little careful. Let us choose two non-negative smooth functions ψ^1 and ψ^2 respectively vanishing near the obstacle and away from the obstacle and such that $\psi^1+\psi^2=1$. We then consider $\psi^1F_1(m)$ and $\psi^2F_1(m)$ separately. $\psi^1F_1(m)$ is handled by means of Lemma 1.1.2 while Lemma 2.2.2 can be applied to $\psi^2F_1(m)$ (once we note that since m is in $D(A)$ it can be approximated in the graph norm by a sequence $\{\varphi^n\}$ of data in $\mathscr{D}_{\lambda(x)}$, so that $\psi^2 F_1(m)$ can be approximated in $\|f\|^2+\|\nabla\times f\|^2$ by $\psi^2 F_1(\varphi_n)$ and hence lies in $\mathscr{B}_d^1$).

This theorem shows that the function $F_1(m)$ can be assigned a value or "trace" on the boundary of G in the sense usual to elliptic theory. We made the additional assumption that $\nabla\cdot m$ is square integrable, and deduced that not only the tangential components of $F_1(m)$ but also the normal components are properly defined on the boundary. It is not clear what happens in the absence of an assumption on $\nabla\cdot m$.

We now still require a local compactness property of the generator A^1. At this point we must unfortunately restrict ourselves to boundary conditions of the first kind for which the function $\lambda(x)$ is identically a real constant. It is evident that the method of proof used below is not directly applicable to more general boundary conditions. In fact J. RALSTON has recently shown by means of an example that the local compactness property does not hold for arbitrary $\lambda(x)$.

The following theorem provides the key to the proof of the local compactness.

Theorem 2.2.5. *If the decomposition* $M=M^0\oplus M^1$ *is associated with the operator* A_λ, *and if* m *lies in* $M^1\cap C^1(\bar{G};R^3)$, *we have that* m *is divergence free and that* $n\cdot(\lambda m_1+m_2)=0$ *on* ∂G.

Proof. We already know from a previous remark that m is divergence free. Since λ is a constant, $M^0\supset\{(\lambda\nabla\varphi,\nabla\varphi);\ \varphi\in C_0^\infty(\bar{G})\}$. Thus for m in M^1

$$\begin{aligned}0&=\int_G[\lambda\nabla\varphi\cdot m_1+\nabla\varphi\cdot m_2]\,dx=\int_G\nabla\varphi\cdot(\lambda m_1+m_2)\,dx\\&=-\int_G\varphi\nabla\cdot(\lambda m_1+m_2)\,dx+\int_{\partial G}n\cdot(\lambda m_1+m_2)\,\varphi\,dS,\end{aligned}$$

and hence, since m is divergence free and λ is a constant,

$$\int_{\partial G}n\cdot(\lambda m_1+m_2)\,\varphi\,dS=0\qquad\text{for any }\varphi\text{ in }C_0^\infty(\bar{G}).$$

Thus $n\cdot(\lambda m_1+m_2)$ vanishes on ∂G.

In the case of a perfect conductor we have $\lambda=0$, and hence $n\cdot m_2=0$, this being precisely the boundary condition which is usually imposed on the magnetic field at the boundary of a perfect conductor (expressing the continuity of its normal components across the boundary surface).

We define $F_2(m)=\lambda m_1+m_2$; the boundary condition derived above will allow us to obtain for $F_2(m)$ with m in $D(A^1)$ the same inequality obtained previously for $F_1(m)$. To do this rigorously we again rely on results proved in the paper of FRIEDRICHS. The method essentially amounts to a refinement of the previous theorem.

We introduce the following notation. Let

$$\mathfrak{U}=L_2(G);\quad \mathfrak{U}_d=\{f\in L_2(G);\ \nabla f\in L_2(G;R^3)\};\qquad d\mathfrak{U}=\{\nabla f;\ f\in\mathfrak{U}_d\}.$$

FRIEDRICHS proves

Lemma 2.2.6. *If* f *belongs to* $L_2(G;R^3)$ *and is orthogonal to* $d\mathfrak{U}$ *then* f *lies in* ${}^2\mathscr{B}_\delta$.

This allows us to prove

Lemma 2.2.7. *If m lies in* M^1, *the real and imaginary parts of* $F_2(m)$ *lie in* ${}^2\mathscr{B}_\delta$.

Proof. We show first of all that

$$M^0 \supset S = \{(\lambda g, g);\, g \in d\mathfrak{U}\} = \{(\lambda \nabla f, \nabla f);\, f \in \mathfrak{U}_d\}.$$

We apply the approximation theorem proved in the Appendix to conclude that there is a sequence $\{f_n\}$ contained in $C^\infty(\bar{G})$ such that $(\lambda \nabla f_n, \nabla f_n)$ converges to $(\lambda \nabla f, \nabla f)$ in M. Since $\nabla \times \nabla f = 0$ we conclude that $(\lambda \nabla f_n, \nabla f_n)$ lies in M^0, and that $(\lambda \nabla f_n, \nabla f_n)$ converges in the graph norm of A to $(\lambda \nabla f, \nabla f)$. Consequently we indeed have $S \subset M^0$. Hence

$$0 = \int_G [\lambda \nabla f \cdot m_1 + \nabla f \cdot m_2]\, dx = \int_G \nabla f \cdot (\lambda m_1 + m_2)\, dx.$$

Thus $F_2(m)$ is orthogonal to $d\mathfrak{U}$; by the previous lemma the real and imaginary parts of $F_2(m)$ lie in ${}^2\mathscr{B}_\delta$. This enables us to prove

Theorem 2.2.8. *Suppose that* $A = A_\lambda$, *and that m is in* $D(A) \cap M^1$. *Then*

$$\|F_2(m)\|^2 + \sum_{i=1}^{3} \|\nabla F_{2i}(m)\|^2 \leqq C[\|F_2(m)\|^2 + \|\nabla \times F_2(m)\|^2].$$

Proof. Since m is in $D(A)$, the real and imaginary parts of $F_2(m)$ are in $\mathscr{B}_d$. By the previous lemma they also lie in ${}^2\mathscr{B}_\delta$. We can thus apply Lemma 2.1.3, taking the same precautions as in the proof of Theorem 2.2.4 to deal with the unboundedness of the domain.

Besides clarifying the meaning of the boundary condition "$n \cdot F_2$ vanishes on ∂G", this last theorem finally enables us to prove the local compactness theorem.

Theorem 2.2.9. *Let S be the set of data m in* $D(A_\lambda^1)$ *(λ a constant) such that* $\|m\|^2 + \|A_\lambda^1 m\|^2 \leqq 1$. *Then S is precompact in* $\|\cdot\|_{G'}$ *for any bounded domain G′ contained in G, where* $\|\cdot\|_{G'}$ *denotes the local energy norm*

$$\|m\|_{G'}^2 = \int_{G'} (m_1 \cdot \bar{m}_1 + m_2 \cdot \bar{m}_2)\, dx.$$

Proof. It is easy to check, as is done explicitly in [10], that as a consequence of Theorems 2.2.4 and 2.2.8 each component of data m in $D(A^1)$ satisfies

$$\|m_j\|^2 + \sum_{i=1}^{3} \|\nabla m_{ji}\|^2 \leqq C \sum_{k=1}^{2} (\|m_k\|^2 + \|\nabla \times m_k\|^2). \qquad (*)$$

As a consequence we have for all m in S

$$\|m_j\|^2 + \sum_{i=1}^{3} \|\nabla m_{ji}\|^2 \leqq C, \qquad j = 1, 2.$$

The precompactness of S in the local energy norm is then an immediate consequence of RELLICH's theorem.

As a by-product of the last proof we have

Corollary 2.2.10. *If m lies in* $D(A^1_\lambda)$

$$\|[U(t)m]_j\|^2+\sum_{i=1}^{3}\|\nabla[U(t)m]_{ji}\|^2<M, \qquad j=1,2.$$

Proof. The proof is obvious, since if m lies in $D(A^1_\lambda)$, $A^1_\lambda m$ lies in M^1, and for all t,

$$\|U(t)m\|^2+\|A^1U(t)m\|^2=\|m\|^2+\|Am\|^2.$$

The assertion then follows if we apply (*) to $U(t)m$.

We suppose now that m is in $D(A^1)$. Then because of Corollary 2.2.10, $U(t)m$ and $\partial_j[U(t)m]$ are square integrable as functions of x and t in any domain $G\times[-T,T]$. Furthermore since $\partial_t[U(t)m]=U(t)Am$, and since Am lies in M^1, $\partial_t[U(t)m]$ is also square integrable in that domain. Thus the "trace" of $U(t)m$ on "nice" hypersurfaces in $G\times[-T,T]$ is properly defined, and the divergence theorem may then be applied in the usual way (see [10]) to obtain the following theorem first of all for m in $D(A^1)$, but then also for all of M^1.

Theorem 2.2.11. *Let*

$$E(m,R)=\int_{G\cap S_R}(|m_1|^2+|m_2|^2)\,dx,$$

where as before S_R denotes a sphere of radius R. Then for any m in M^1

$$E(U(T)m,R)\leqq E(m,R+T).$$

3. Conclusions from the Abstract Theory

We now have all the ingredients necessary to prove that the general scattering theory of LAX and PHILLIPS is applicable to MAXWELL's equations in the exterior of an obstacle, provided that we restrict ourselves to boundary conditions of the first kind corresponding to boundary functions $\lambda(x)$ which are identically constant. We shall state the various theorems without proof; the proofs differ only in the most trivial details from the proofs of the corresponding statements about the wave equation presented in Chapter 5 of [4]; some of these modifications were described in Appendix 4 of [10].

Let ρ be chosen so that the obstacle lies inside the region $|x|<\rho$. We consider the subspaces D^ρ_+ and D^ρ_- defined in Section 1.5 which are contained not only in M^1_0 but also in M^1. $U^1(t)$ and $U^1_0(t)$ act in the same way on D^ρ_+ for positive t and on D^ρ_- for negative t.

Theorem 3.3.1. *The subspaces D^ρ_+ and D^ρ_- of M^1 are orthogonal and are respectively outgoing and incoming with respect to $U^1(t)$.*

The assertions $\overline{\bigcup U^1(t)D^\rho_+}=M^1$ and $\overline{\bigcup U^1(t)D^\rho_-}=M^1$ which are hidden in the statement of the previous theorem are equivalent to local energy decay.

Theorem 3.3.2. *For any m in M^1, and for G' a bounded subdomain of G* $\lim_{t\to\pm\infty}E(U^1(t)m,G')=0.$

Also, since as a consequence of Theorem 3.3.1 $U^1(t)$ is unitarily equivalent to the group of translation operators on $L_2(-\infty, \infty; N^1)$, its generator has an absolutely continuous spectrum.

Theorem 3.3.3. *The spectrum of A^1 is absolutely continuous.*

An investigation of the associated semi-group $Z(t)=P_+ U^1(t) P_-$ shows that

Theorem 3.3.4. $Z(2\rho)(\kappa I - B)^{-1}$ *is a compact operator for* $\kappa > 0$.

Consequently we obtain

Theorem 3.3.5. *The generator B of $Z(t)$ has a pure point spectrum, and the resolvent B is meromorphic in the plane and holomorphic on the imaginary axis and in the right half plane.*

One says that data m belongs locally to the domain of A if φm belongs to the domain of A for any $\varphi(x)$ in $C_0^\infty(\bar{G})$. We can now also show

Theorem 3.3.6. *The generator B has μ as an eigenvalue if and only if the equation $Am=\mu m$ has a non-trivial, eventually outgoing solution which lies locally in $D(A)$.*

Putting the last two theorems together we find

Theorem 3.3.7. *Aside from a discrete set of μ's with* Re $\mu<0$, *there are no non-trivial, eventually outgoing local solutions of the equation $Am=\mu m$ which satisfy the boundary condition.*

The last result together with an extension of the proof of Theorem 3.3.6 gives

Theorem 3.3.8. *If μ belongs to the resolvent set of B and if g lies in $(D_+^c + D_-^c)^\perp$ with $c>\rho$ (in particular if g has compact support), there exists a unique eventually outgoing local solution of $(A-\mu)m=g$. The solution depends analytically on μ in the strong sense of the local energy norm.*

Theorem 3.3.9. *Aside from the discrete set of μ's* (*with* Re $\mu<0$) *for which $(A-\mu)m=0$ has a non-trivial outgoing solution, there always exists a unique eventually outgoing solution of the equation $(A-\mu)m=g$* (i.e. *a μ-outgoing solution of the reduced equation*) *satisfying the boundary condition. This solution depends analytically on μ in the local energy norm.*

Note. Corresponding results are valid when "outgoing" is replaced by "incoming".

We now exploit the connection between the poles of the scattering matrix and the spectrum of B. We recall from the introduction that the scattering operator can be represented as an operator valued function $\mathscr{S}(z)$ acting "multiplicatively" on the spectral representation of M^1.

Theorem 3.3.10. *The scattering matrix $\mathscr{S}(z)$ is well defined. It is holomorphic on the real axis and in the lower half plane, and is meromorphic in the whole plane, having a pole at exactly those points z for which there is a non-trivial, eventually outgoing local solution of the reduced* MAXWELL'S *equations $Am=izm$ satisfying the boundary conditions.*

Finally, with the aid of generalized eigenfunctions of A we shall explicitly describe the incoming and outgoing spectral representations of $U^1(t)$ acting on

M^1, and use this to represent the scattering operator as the identity plus an integral operator acting on spherical coordinates. In Appendix 4 of [10] we prove the following lemma from the corresponding statement about the wave equation:

Theorem 3.3.11. *The free space spectral representation in* $L_2(-\infty, \infty; N^1)$ *of* $U_0^1(t)$ *acting on* M_0^1 *is given by the map* $m(x) \to \overline{(\Phi_0(\cdot, \sigma, \omega), \bar{m}(\cdot))}$, *where*

$$\Phi_0(x, \sigma, \omega) = (2\pi)^{-\frac{3}{2}} \left(\nabla \times [I e^{-i\sigma x \cdot \omega}], -\frac{1}{i\sigma} \nabla \times \nabla \times [I e^{-i\sigma x \cdot \omega}] \right).$$

(The conjugated form of the inner product is used to let the matrix valued data $\Phi_0(\cdot, \sigma, \omega)$ act on the column vector $m(\cdot)$ in the by now familiar way.) For the exterior problem we then obtain

Theorem 3.3.12. *Let* σ *be real. There exist solutions* $\Psi_+(x, \sigma, \omega)$ *and* $\Psi_-(x, \sigma, \omega)$ *to the equation* $(A - i\sigma)\Psi_\pm(x, \sigma, \omega) = 0$, *which are respectively outgoing and incoming and such that* $\Psi_\pm(x, \sigma, \omega) = -\Phi_0(x, \sigma, \omega)$ *for* x *on* ∂G. *Let*

$$\Phi_\pm(x, \sigma, \omega) = \Phi_0(x, \sigma, \omega) + \Psi_\pm(x, \sigma, \omega).$$

Then the outgoing and incoming spectral representations are given by

$$m(x) \to \hat{k}_\pm(\sigma, \omega) = \overline{(\Phi_\mp(\cdot, \sigma, \omega), \bar{m}(\cdot))}.$$

According to the proof of Theorem 1.6.5 and note (1) *following it, we have the asymptotic estimate*

$$[\Psi_-(r\theta, \omega, \sigma)]_1 = \frac{e^{i\sigma r}}{r} [(2\pi)^{-\frac{3}{2}} \theta \times S(\theta, \omega; \sigma)] + O\left(\frac{1}{r^2}\right),$$

where $S(\theta, \omega; \sigma)$ *is a matrix valued function such that* $\theta \cdot S(\theta, \omega; \sigma) = 0$, *and where the bracketed phase function has been factored in a convenient way.*

Theorem 3.3.13. *The scattering operator is given by*

$$\hat{k}_+(\sigma, \omega) = [\mathscr{S}(\sigma)\hat{k}_-(\sigma, \cdot)](\omega) = \hat{k}_-(\sigma, \omega) + \frac{i\sigma}{2\pi} \int_{|\theta|=1} \overline{S(-\theta, \omega, \sigma)} \hat{k}_-(\sigma, \theta)\, d\theta.$$

As a consequence of the unitarity of the scattering operator, and the behaviour of MAXWELL's equations under time reversal and complex conjugation we have

Theorem 3.3.14. *The kernel* $S(\theta, \omega; \sigma)$ *satisfies the following identities:*

a) $\overline{S(-\theta, \omega, \sigma)} - S(-\omega, \theta, \sigma) = \frac{i\sigma}{2\pi} \int S(-\omega, \theta'; \sigma) \overline{S(-\theta, \theta'; \sigma)}\, d\theta'$

$$= \frac{i\sigma}{2\pi} \int \overline{S(-\theta', \omega, \sigma)} S(-\theta', \theta, \sigma)\, d\theta';$$

b) $S(\theta, \omega; -\sigma) = \overline{S(\omega, \theta; \sigma)}$;

c) $S(\theta, \omega; -\sigma) = \overline{S(\theta, \omega; \sigma)}$;

d) $S(\theta, \omega; \sigma) = S(\omega, \theta; \sigma)$.

Appendix

We prove an approximation theorem which was used at one point in Section 2.2.

Theorem. *Given f such that f and* ∇f_i $(i=1, 2, 3)$ *lie in* $L_2(G; R^3)$. *Then there exists a sequence* $\{f^n\}$ *of functions such that*

(a) f^n *lies in* $C^\infty(\bar{G}; R^3)$;

(b) f^n *converges to f, and* ∇f_i^n *converges to* ∇f_i $(i=1, 2, 3)$ in $L_2(G; R^3)$.

Proof. We cover ∂G with a finite number of open sets O_α $(\alpha=1, \ldots, n)$ in such a way that for each O_α we can find an open cone C_α, such that

(a) for x in $\partial G \cap O_\alpha$, $x+C_\alpha$ lies outside G and $x-C_\alpha$ lies inside G;

(b) for x in $O_\alpha \cap G$, $x-C_\alpha$ lies inside G.

Then we choose an additional open set O_0 with $\bar{O}_0 \subset G$, and such that $\{O_0\} \cup \{O_\alpha\}$ covers $\bar{G}$. We construct a finite partion of unity $\{\psi_\alpha\}$ for $\bar{G}$ subordinate to this covering. Then in $\bar{G}$

$$f=\sum_\alpha f_\alpha, \qquad \text{where } f_\alpha=\psi_\alpha f.$$

For each f_α we have f_α, $\nabla f_{\alpha i}$ in $L_2(G; R^3)$. We prove the assertions of the theorem for the f_α. Once we have constructed suitable sequences $\{f_\alpha^n\}$ corresponding to the f_α we define

$$f^n=\sum_\alpha f_\alpha^n;$$

since this is a finite sum the sequence $\{f^n\}$ has the desired properties.

We now consider a particular f_α. If the support of f_α is contained in O_0 we simply mollify it in the usual fashion to obtain a suitable approximating sequence. If the support is contained in some boundary patch O_α we have to proceed more carefully. We choose a mollifier $j_\varepsilon(x)$ whose support lies in the cone C_α. Then $f_\alpha^\varepsilon=j_\varepsilon * f_\alpha$ satisfies (a) in the usual way and f_α^ε converges to f_α in $L_2(G; R^3)$. We consider now

$$\nabla f_{\alpha i}^\varepsilon=\nabla \int_G f_{\alpha i}(y)\, j_\varepsilon(x-y)\, dy=\int_G f_{\alpha i}(y)\, \nabla_x j_\varepsilon(x-y)\, dy$$
$$=-\int_G f_{\alpha i}(y)\, \nabla_y j_\varepsilon(x-y)\, dy.$$

Let U be an open set such that $\operatorname{supp} f \subset U$ and $\bar{U} \subset O_\alpha$. Let φ be a smooth function such that φ is identically one on the support of f_α and vanishes identically outside U. Then

$$\nabla f_{\alpha i}^\varepsilon(x)=-\int_G f_{\alpha i}(y)\, \nabla_y[\varphi(y)\, j_\varepsilon(x-y)]\, dy.$$

We consider the boundary values of $\varphi(y)\, j_\varepsilon(x-y)$ for fixed x in G. If y lies outside U, $\varphi(y)=0$. If y lies in $\partial G \cap U$ we shall show that $\varphi(y)\, j_\varepsilon(x-y)$ vanishes for ε small enough, uniformly for x in G. Suppose first that x is in $G-O_\alpha$. Let $\varepsilon_0=\operatorname{dist}(U, G-O_\alpha)$. Let $\varepsilon<\varepsilon_0$. Then $|x-y| \geqq \varepsilon_0>\varepsilon$ and hence $j_\varepsilon(x-y)=0$. If on the other hand x is in $G \cap O_\alpha$, $j_\varepsilon(x-y) \neq 0$ means that $x-y$ is in C_α and hence that $y=x-(x-y)$ lies in $x-C_\alpha$, and consequently in G; this contradicts the assumption that y is on ∂G. Hence in this case too, $j_\varepsilon(x-y)=0$. Thus in all cases $\varphi(y)\, j_\varepsilon(x-y)=0$ for x in G and y on ∂G, providing that $\varepsilon<\varepsilon_0$. Similarly, derivatives of $\varphi(y)\, j_\varepsilon(x-y)$ with respect to y vanish there.

For ψ in $C_0^\infty(G)$ we have

$$-\int_G f_{\alpha i}(y)\cdot \nabla_y \psi(y)\, dy = \int_G \nabla_y f_{\alpha i}(y)\, \psi(y)\, dy$$

and this identity continues to hold for functions such as $\varphi(y)\, j_\varepsilon(x-y)$ lying in $C_0^\infty(\bar{G})$. Hence we finally have

$$\nabla f_{\alpha i}^\varepsilon = (\nabla f_{\alpha i}) * j_\varepsilon$$

and thus $\nabla f_{\alpha i}^\varepsilon(x)$ converges to $\nabla f_{\alpha i}(x)$.

Note. By a slight modification of the usual argument it can easily be shown that the approximating sequence $\{f^n\}$ constructed in the above way converges uniformly to f on compact subsets of $\bar{G}$ on which f is continuous.

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