The Existence of the Flux Vector and the Divergence Theorem for General Cauchy Fluxes

MIROSLAV ŠILHAVÝ

To Professor W. Noll on his 60^{th} birthday

Contents

Abstract

A new proof of the existence of the flux vector is given for general Cauchy fluxes. The proof is based on an approximation theorem in the theory of functions rather than on the classical tetrahedron argument. This enables us to replace the usual assumptions of Lipschitz continuity with respect to area and volume by less restrictive assumptions so as to produce the flux vector fields with possible singularities. The classical expression for the area density of the flux is proved and the flux is shown to satisfy an appropriate version of the divergence theorem.

1. Introduction

There are many physical quantities that, at a given instant, can be associated with each surface in a body. Examples of such quantities are the contact force and the heat conducted in that body. As these quantities usually can be interpreted as fluxes through the surfaces, a mathematical object describing them is called the Cauchy flux in recent papers [1], [2], [3].

The physical interpretation requires that the flux behave additively on compatible material surfaces. Further, it is natural to assume that if two surfaces differ by a set of zero area, then the values of the flux on these two surfaces coincide. Finally, the fluxes encountered in physics usually satisfy a classical balance law which states that the flux corresponding to the boundary of any part of the body is equal to some quantity that is additive and volume-continuous with respect to parts of the body. A central result in the theory of Cauchy fluxes states that the above properties, when rendered precise and augmented with suitable technical assumptions, imply the existence of the flux vector field whose scalar product with the normal to the surface yields the surface density of the flux. The first result of this type was established by CAUCHY [4] in 1823 under an additional non-technical assumption that the density depends only on the normal. NOLL (1957) [5] showed that this assumption is essentially a consequence of the other assumptions on the flux. This stimulated new research, and the works of NOLL [6, 7], GURTIN & WILLIAMS [8], GURTIN, MIZEL & WILLIAMS [9], GURTIN & MARTINS [1], and ZIEMER [3] contain further important developments.

The most general proofs available in the literature can be carried out only under the assumption of global Lipschitz continuity with respect to the area of the surface and with respect to the volume of the part of the body. The subsequent sections contain a precise statement of these conditions. They are rather restrictive because they imply the global boundedness of the field of the flux vector and the global boundedness of its divergence. On the other hand, it seems reasonable, both from physical and mathematical points of view, to allow singularities of the flux vector at certain points. This is perhaps best motivated when the flux is visualized as the contact force: neither does the presently available theory of partial differential equations afford tools sufficient to guarantee the boundedness of the stress tensor, nor does that boundedness seem natural from the point of view of applications.

In this paper I give a new argument to prove the existence of the flux vector under less restrictive additional assumptions which, I believe, cover more situations encountered in mechanics, including those in which unbounded and singular stresses occur. The proof is based on the following observation: if the flux vector $\boldsymbol{q}$ exists and is sufficiently smooth, then by the divergence theorem,

$$\begin{aligned} \int_{\partial P} v\boldsymbol{q}\cdot\boldsymbol{n}\,dA &= \int_P \operatorname{div}(v\boldsymbol{q})\,dV \\ &= \int_P (v\cdot\operatorname{div}\boldsymbol{q} + \nabla v\cdot\boldsymbol{q})\,dV \end{aligned} \tag{1.1}$$

for every smooth (in fact, Lipschitz continuous) function v and every part P of the body. Here ∂P is the boundary of the part P and $\boldsymbol{n}$ the exterior normal to ∂P, dA is the element of the area of ∂P and dV is the element of volume of P. Now, the first expression in (1.1) can be given immediate meaning provided the Cauchy flux behaves as a measure on ∂P:

$$\int_{\partial P} v\boldsymbol{q}\cdot\boldsymbol{n}\,dA = \int_{\partial P} v\,dQ \tag{1.2}$$

and we note that the right-hand side of (1.2) does not contain the flux vector $\boldsymbol{q}$. On the other hand, the last expression in (1.1) can be used to *define* both the flux vector and its divergence. In other words, we seek to prove the existence of a vector field q and a scalar field b such that

$$\int_{\partial P} v\, dQ = \int_P (v \cdot b + \nabla v \cdot q)\, dV, \tag{1.3}$$

for every part P of the body and every Lipschitz continuous function v. Obvious candidates for the components of q are the surface densities of the flux with respect to planar material surfaces perpendicular to the coordinate axes. That with such a definition of q (and with an appropriate definition of b) the formula (1.3) holds can be proved by a surprisingly simple computation if v is affine or piecewise affine. The validity of (1.3) is then extended to general v by approximating v by a sequence of piecewise affine functions.

From (1.3) we deduce that the divergence div q of q in the sense of distributions satisfies $\operatorname{div} q = b$ and this in turn implies that

$$\int_{\partial P} v\, dQ = \int_P \operatorname{div} (vq)\, dV \tag{1.4}$$

which is the **divergence theorem** for the Cauchy flux Q. The validity of this theorem is important in considerations about energy in mechanics (q = stress tensor, v = velocity) and in manipulations with the Clausius-Duhem inequality (q = heat flux vector, $v = 1/\theta$ = the reciprocal of the absolute temperature).

Next we use the divergence theorem (1.4) to establish the usual expression for the surface densities of the flux in terms of the flux vector. A set N_0 of exceptional points of the body emerges in the proof. N_0 is small in the sense that its volume is zero, and the formula for the density of the flux can be proved only for surfaces whose intersection with N_0 has area at most zero. Whether or not N_0 is empty depends on whether or not the flux vector can be changed on a set of zero Lebesgue measure to produce a function whose Lebesgue set [10] is the whole region occupied by the body. The latter condition is certainly satisfied if the components of the flux vector can be represented by continuous functions, in which case the exceptional set is empty and the surface density is given by the usual expression for every surface. This is the position of the classical result within the present approach. Also the other results known can be recovered by using the present methods, and the present approach often permits slight generalizations of them.

The present study hence shows that each Cauchy flux satisfying the additional technical assumption gives rise to a vector field with divergence in the sense of distributions of class L^1. I do not know if the converse is also true. It can be proved that any bounded measurable vector field with bounded divergence gives rise to a Cauchy flux satisfying the conditions of Lipschitz continuity. The general vector fields with divergence of class L^1 induce certain power functionals close to the Cauchy fluxes. However, it is not clear whether the power functionals can be identified with the Cauchy fluxes.

2. Bodies, parts, and material surfaces

Throughout, we identify the N-dimensional Euclidean space with the space $\mathbb{R}^N$ of N-tuples of real numbers. We further denote by $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$;

$\mathbb{S}^{N-1} = \{n \in \mathbb{R}^N : |n| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. V denotes the Lebesgue measure in $\mathbb{R}^N$ and A the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. The values of these measures on a Borel set S are denoted by $V(S)$ and $A(S)$, respectively.

We identify the body with a bounded open region $B \subset \mathbb{R}^N$. No regularity of the boundary is assumed. To stress the generality of the present approach, I interpret the parts of the body B as sets of finite perimeter contained in B. However, the results of the present paper can be established for smaller collections of parts of B as well. An extensive treatment of sets of finite perimeter is contained in FEDERER'S book [11], and I also refer to ZIEMER [3, § 2] for a brief introduction to this topic. The treatment of sets of finite perimeter in continuum mechanics can be found in BANFI & FABRIZIO [12, 13] and ZIEMER [3].

A part P of B is any Borel subset of B of finite perimeter whose boundary ∂P in the sense of measure theory is contained in B. We denote by $\mathscr{P}$ the set of all parts of B. Each set P of finite perimeter has a well defined exterior normal n^P which is defined A-almost everywhere (A-a.e.) on ∂P. The class $\mathscr{P}$ forms a ring of subsets of B, and this ring generates the σ-ring of Borel subsets of B. Note also that actually it is natural to identify the sets $P_1, P_2 \in \mathscr{P}$ if their symmetric difference has Lebesgue measure zero and to consider them as forming the same part of the body B. That means that rather than with the ring $\mathscr{P}$ one should work with the quotient of $\mathscr{P}$ modulo the ideal of the sets of Lebesgue measure zero, as BIRKHOFF & VON NEUMANN proposed in a different context [14]. The same applies to the class of all surfaces and the ideal of sets of Hausdorff measure (area) zero. I do not follow this possibility here, although I note that in this way certain requirements of absolute continuity would be automatically satisfied.

A material surface S is any pair $S = (\hat{S}, n)$, where $\hat{S}$ is a Borel subset of B and $n : \hat{S} \to \mathbb{R}^N$ a Borel measurable function such that there is a part $P \in \mathscr{P}$ with $\hat{S} \subset \partial P$ and

$$n(x) = n^P(x)$$

for all $x \in \hat{S}$ for which the exterior normal $n^P(x)$ is defined and

$$n(x) = 0$$

otherwise. The function n is the normal to the surface S; it orients S. The opposite of the material surface $S = (\hat{S}, n)$ is the pair $-S$ given by $-S = (\hat{S}, -n)$. It may happen that $-S$ is not a material surface. However, if the material surface S is contained in some compact subset of B, then $-S$ is a material surface. We shall frequently identify the material surface S with the corresponding underlying set $\hat{S}$. For instance, if P is a part of the body, then ∂P will denote both the set ∂P and the oriented surface $(\partial P, n^P)$. If S_1, S_2 are two material surfaces, then the inclusion $S_1 \subset S_2$ will be understood to mean not only that the underlying sets satisfy the corresponding relation, but also that S_1 has the same orientation as S_2 on S_1. The set of all material surfaces is denoted $\mathscr{S}$.

A countable family $S_1, S_2, \ldots \in \mathscr{S}$ of material surfaces is said to be compatible if there is a part $P \in \mathscr{P}$ such that $S_i \subset \partial P$ $(i = 1, 2, \ldots)$. A compatible family $S_1, S_2, \dots$ is said to be disjoint if the underlying sets of all members of the family are pairwise disjoint. In this case, we denote by $\bigcup_{i=1}^{\infty} S_i$ the (oriented) union of the family, *i.e.*, the material surface whose underlying set is the union of the underlying sets of the members of the family and whose orientation is the same as that of each member of the family.

The symbol $L^1(B, V)$ denotes the usual Lebesgue space of all measurable functions f defined on B which satisfy

$$\int_B |f| \, dV < \infty.$$

Similarly, if S is a material surface, then $L^1(S, A)$ denotes the space of all Borel functions f defined on S which satisfy

$$\int_S |f| \, dA < \infty.$$

3. Cauchy fluxes

Before defining formally the concept of the Cauchy flux, we briefly discuss the definitions employed in the earlier papers [1], [2], [3]. As in these works, also here the Cauchy flux will be a function $Q : \mathscr{S} \to \mathbb{R}$ which assigns to each material surface S a number $Q(S)$. In [1], [2], [3], the function Q is subject to the following requirements: (a) additivity:

$$Q(S_1 \cup S_2) = Q(S_1) + Q(S_2) \tag{3.1}$$

whenever S_1 and S_2 are two disjoint compatible material surfaces, and, (b) Lipschitz continuity with respect to area, *i.e.*,

$$|Q(S)| \leqq C\, A(S) \tag{3.2}$$

for every $S \in \mathscr{S}$, where C is a constant independent of S. The additivity (3.1) is well justified. In contrast, I wish to discuss the role of (3.2). Examination of the proofs reveals that (3.2) serves two purposes in the theory. First, this condition is used, in conjunction with additivity, to prove [8] that Q can be extended to a countably additive measure on each material surface, defined on all Borel subsets of that surface, and absolutely continuous with respect to the area measure on the surface. Hence Q has a density on each material surface. Were this the only reason to impose (3.2), then of course, to generalize it would be a routine excercise in measure theory. (This exercise is stated as Proposition 1, below.) However, there are deeper reasons to impose (3.2), namely, certain parts of the argument proving the dependence of the density on the normal [3] cannot be carried out without (3.2); also the tetrahedron argument, even in its refined form in [1], rests on this assumption.

Now we wish to avoid (3.2) but the countable additivity and the absolute continuity drawn from (3.2) seem to be reasonable. (This in particular applies to the condition of absolute continuity, because the boundaries of disjoint parts add only to within sets of area zero.) Hence we introduce a definition of the Cauchy flux such that countable additivity and absolute continuity are satisfied but (3.2) need not hold. However, the existence of the flux vector cannot be established in this generality and certain additional assumption, much weaker than (3.2), will have to be added in Section 5.

Several equivalent versions of the present definition of the Cauchy flux can be given, and it is hard to select one of them as the basic one. The following proposition, a routine consequence of measure theory (FUGLEDE [15]) lists the equivalent versions of the definition.

Proposition 1. *For a real-valued function Q defined on $\mathcal{S}$ the following four conditions are equivalent*:

(1) *For every* $S \in \mathcal{S}$ *there is a Borel function* $q^S \in L^1(S, A)$ *such that*

$$Q(S') = \int_{S'} q^S \, dA$$

for every $S' \in \mathcal{S}$, $S' \subset S$.

(2) *The function Q is countably additive on compatible material surfaces*, i.e.,

$$Q\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} Q(S_i)$$

for every disjoint compatible family $S_1, S_2, \ldots \in \mathcal{S}$, *and moreover*,

$$Q(S) = 0$$

whenever $A(S) = 0$.

(3) *The function Q is additive on compatible material surfaces*, i.e.,

$$Q(S_1 \cup S_2) = Q(S_1) + Q(S_2)$$

for every pair S_1, S_2 *of compatible disjoint material surfaces, and moreover, for each* $S \in \mathcal{S}$ *there is a non-negative Borel function* $h^S \in L^1(S, A)$ *such that*

$$|Q(S')| \leqq \int_{S'} h^S \, dA$$

for every $S' \in \mathcal{S}$, $S' \subset S$.

(4) *The function Q is additive on compatible material surfaces and, moreover, for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$\sum_{i=1}^{l} |Q(S_i)| < \varepsilon$$

for every finite number $S_1, \ldots, S_l$ *of compatible disjoint material surfaces satisfying*

$$\sum_{i=1}^{l} A(S_i) < \delta.$$

Any function Q satisfying the above four equivalent conditions is called a **Cauchy flux**, and the function q^S as in (1) is called the **surface density** of Q. Contact with the definitions of the Cauchy flux given in the earlier papers [1], [2], [3] is established through Condition (3), as obviously every function satisfying (3.1), (3.2) satisfies also (3) with $h^S = C = \text{const}$. However, the present definition is more general, as the converse need not hold.

4. Weakly balanced Cauchy fluxes

The basic motivation for the concept of a weakly balanced Cauchy flux is the fact that the fluxes encountered in physics satisfy the integral form of the classical balance law. One possible definition of a weakly balanced Cauchy flux is precisely the condition expressing this, and it is listed below as Condition (1) in Theorem 1. However, using the countable additivity of Cauchy fluxes, we can adopt seemingly weaker but actually equivalent conditions. These are listed as the remaining two conditions in Theorem 1.

Theorem 1. *For a Cauchy flux the following three conditions are equivalent:*
(1) *There is a function* $b \in L^1(B, V)$ *such that*

$$Q(\partial P) = \int_P b\, dV \tag{4.1}$$

for every part P.
(2) *There is a function* $k \in L^1(B, V)$ *such that* $k \geqq 0$ *and*

$$|Q(\partial P)| \leqq \int_P k\, dV$$

for every part P.
(3) *For every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$\sum_{i=1}^{l} |Q(\partial P_i)| < \varepsilon$$

for every finite number $P_1, \ldots, P_l$ *of disjoint parts for which*

$$\sum_{i=1}^{l} V(P_i) < \delta.$$

We say that the Cauchy flux Q is **weakly balanced** if it satisfies the three equivalent conditions of the preceding theorem. The function b as in Condition (1) is called the **volume density of the flux** Q. Of the three equivalent conditions, Condition (1) is most easily handled. It will be precisely this condition that, together with Proposition 2, below, will be used in the following section to establish the existence of the flux vector. Note, however, that the important feature of Conditions (2), (3) is that they do not postulate the additivity of $Q(\partial P)$ on disjoint parts.

The present definition of weak balancing is less restrictive than the definitions in [1], [2], [3], wherein the following stronger version of Condition (2) is adopted as the definition: there is a constant C such that

$$|Q(\partial P)| \leqq C V(P) \tag{4.2}$$

for every part of the body.

Proposition 2. *If* Q *is a weakly balanced Cauchy flux, then*

$$Q(-S) = -Q(S) \tag{4.3}$$

for every $S \in \mathcal{S}$ *with* $-S \in \mathcal{S}$. *Hence the surface density satisfies*

$$q^{-S}(x) = -q^{S}(x) \tag{4.4}$$

for A-a.e. $x \in S$.

Proof of Theorem 1 & Proposition 2. The implication (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) is routine in measure theory. Concerning the implication (3) $\Rightarrow$ (1), we note that (3) is a general necessary and sufficient condition under which an additive set function defined on a ring generating the measurable sets can be represented as in (1); see [15]. In the present situation we apply this condition to the set function M, given by

$$M(P) = Q(\partial P), \qquad P \in \mathcal{P}.$$

However, we must verify that M is additive on $\mathcal{P}$. In order to do that, we first prove that if (3) holds, then the conclusion of Proposition 2 holds. This proof is easy but long at the present level of generality, for non-smooth material surfaces may occur. I shall only sketch the basic idea, or, rather, I shall indicate how the proof can be reduced to the verification of (4.3), (4.4) for smooth material surfaces only. Namely, if S is smooth and its relative $(N-2)$-dimensional boundary is sufficiently regular, then we can use almost the same argument as in [8], [1] to prove (4.3). I note that the proofs in [8], [1] are given under assumptions stronger than the present concerning Q, but in this case the generalization is easy. Now the validity of (4.3) for every smooth material surface with sufficiently regular relative boundary suffices to prove, by using the derivatives with respect to regular families shrinking to a point $x \in S$, the validity of (4.4) for A-a.e. point of a smooth surface S. By integrating over a Borel subset S' of S, we prove the validity of (4.3) for a general S' contained in a smooth material surface. Finally, the proof of (4.3) for a general non-smooth material surface S is completed by using the nontrivial fact that A-almost all of any $S \in \mathcal{S}$ can be covered by a countable family of smooth material surfaces S [11]. This implies that, to within a possible set of area zero, S can be written as a union of a compatible, disjoint, countable family of material surfaces that are contained in appropriate smooth material surfaces. Then, by using the countable additivity of Q, absolute continuity, and the validity of (4.3) for surfaces contained in smooth surfaces, we establish (4.3) generally.

Note. Another proof of Proposition 2 is possible (under the assumption that Condition (3) is valid) which does not use the non-trivial result about covering a general S by smooth material surfaces. It is based on the observation that if P is any part of the body, then for A-a.e. $x \in \partial P$ the following quantities tend to 0 as $r \to 0+$:

$$r^{-N+1}Q(\partial B(x, r)), \qquad r^{-N+1}Q(\partial(P \cap B(x, r))), \qquad r^{-N+1}Q(\partial(P^c \cap B(x, r))),$$

where $B(x, r)$ is the open ball of radius r centered at x, and P^c is the complement of P in B. This is proved from Condition (3) by using directly the definition of Hausdorff measure, the equality $V(\partial P) = 0$, and an argument similar to the one given in FEDERER [11], pp. 179–181. The details are omitted. Next one splits the boundary of $P \cap B(x, r)$ into two parts, one being the portion of the boundary contained in $B(x, r)$ and the other being the portion contained in $\partial B(x, r)$. The same is done for $P^c \cap B(x, r)$. The portions of the boundaries contained in $B(x, r)$ are opposite each to other in the sense defined in section 2. Then, using the additivity of Q, and the fact that the limits of the quantities indicated above are 0, we may evoke the basic result on the differentiation of set functions to prove (4.4) for a general boundary P. The result then follows.

Now the conclusion of Proposition 2 is used to find that the set function M is indeed additive: if P_1 and P_2 are two disjoint parts, then the contributions to the sum $Q(\partial P_1) + Q(\partial P_2)$ from the overlapping parts of the boundaries cancel in view of (4.3) and their opposite orientations.The proof is complete.

5. The flux vector and the divergence theorem

To prove the existence of a flux vector that satisfies the divergence theorem, we have to impose a further condition on the Cauchy flux. This condition is embodied by the definition of the Cauchy flux of class L^1, below. We first introduce the following terminology. We say that a material surface $S = (\hat{S}, \boldsymbol{n})$ is planar if there is a hyperplane H_0 in $\mathbb{R}^N$ for which $\hat{S} \subset H_0$. Given a unit vector $\boldsymbol{n}^* \in \mathbb{S}^{N-1}$, we say that a planar material surface $S = (\hat{S}, \boldsymbol{n})$ is perpendicular to $\boldsymbol{n}^*$ if $\boldsymbol{n}(x) = \boldsymbol{n}^*$ for every $x \in \hat{S}$. A Cauchy flux Q is said to be **of class L^1** if there are N linearly independent vectors $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_N \in \mathbb{S}^{N-1}$ and a non-negative Borel function $h \in L^1(B, V)$ such that

$$|Q(S)| \leqq \int_S h \, dA \tag{5.1}$$

for every planar material surface S perpendicular to one of the vectors $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_N$. Since h is non-negative and Borel measurable, the surface integral in (5.1) is meaningful, but it can be infinite for certain surfaces S. Such a situation is not excluded in the above definition. However, using the condition $h \in L^1(B, V)$ and Fubini's theorem, one can prove that $h \in L^1(H_0 \cap B, A)$ for "almost every" hyperplane H_0 perpendicular to one of the vectors $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_N$. For material surfaces contained in such hyperplanes the finiteness of the integral in (5.1) is then guaranteed. The condition (5.1) thus introduces some uniformity on the variations of the Cauchy flux on hyperplanes perpendicular to the vectors $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_N$.

If the Cauchy flux satisfies the assumption of area Lipschitz continuity (3.2) employed in [1], [2], [3], then Q is of class L^1 and satisfies (5.1) for every material surface, not only for the planar material surfaces perpendicular to the vectors $\boldsymbol{n}_i$. It is precisely the passage from (3.2) to (5.1) that enables us to obtain a larger class of flux vector fields including the unbounded ones. Actually the results to be given below can be established under even less restrictive assumption of local summability, defined in an appropriate way.

The main conclusion in the present section deals with vector fields of class L^1 over B. A vector field $\boldsymbol{q}$ of class L^1 on B is an N-tuple $\boldsymbol{q} = (q_1, \ldots, q_N)$ of measurable functions each of which belongs to $L^1(B, V)$; we write $\boldsymbol{q} \in L^1(B, V)$

in this case. Let $C_0^\infty(B)$ be the set of all infinitely differentiable functions φ: $\mathbb{R}^N \to \mathbb{R}$ with compact support in B. The divergence in the sense of distributions of the vector field $\boldsymbol{q} \in L^1(B, V)$ is a linear functional $\operatorname{div} \boldsymbol{q}$ on $C_0^\infty(B)$ whose value $(\operatorname{div} \boldsymbol{q}, \varphi)$ on a general $\varphi \in C_0^\infty(B)$ is given by

$$(\operatorname{div} \boldsymbol{q}, \varphi) = -\int_B \boldsymbol{q} \cdot \nabla\varphi \, dV,$$

where $\nabla\varphi$ is the gradient of φ. We say that the vector field $\boldsymbol{q}$ has divergence of class L^1 if there is a function $b \in L^1(B, V)$ such that

$$(\operatorname{div} \boldsymbol{q}, \varphi) = \int_B b\varphi \, dV$$

for every $\varphi \in C_0^\infty(B)$. The function b is defined uniquely to within to values on a set of Lebesgue measure zero, and we identify it with the divergence,

$$\operatorname{div} \boldsymbol{q} = b.$$

The fact that $\boldsymbol{q}$ has divergence of class L^1 can be expressed by writing $\operatorname{div} \boldsymbol{q} \in L^1(B, V)$. We easily verify the following: if $\boldsymbol{q}$ has divergence of class L^1 and if v is a Lipschitz continuous function on B with compact support, then also $v\boldsymbol{q}$ has divergence of class L^1 and

$$\operatorname{div}(v\boldsymbol{q}) = v \operatorname{div} \boldsymbol{q} + \nabla v \cdot \boldsymbol{q} \tag{5.2}$$

where ∇v is the gradient of the function v, which, according to the theorem of Rademacher, exists V-a.e. on B.

The definition of Cauchy flux implies that Q induces a Borel measure on each material surface S; if v is a bounded Borel function defined on S, then the symbol $\int_S v \, dQ$ will denote the integral of v with respect to the measure induced by Q on S:

$$\int_S v \, dQ = \int_S v \, q^S \, dA,$$

where q^S is the surface density of the flux.

Theorem 2. *Let Q be a weakly balanced Cauchy flux of class L^1. Then there is a Borel-measurable vector field $\boldsymbol{q}$ with $\boldsymbol{q} \in L^1(B, V)$ and $\operatorname{div} \boldsymbol{q} \in L^1(B, V)$ such that*

$$\int_{\partial P} v \, dQ = \int_P \operatorname{div}(v\boldsymbol{q}) \, dV \tag{5.3}$$

for every part P and every Lipschitz continuous function v on B. The field $\boldsymbol{q}$ also satisfies the local form of the balance law:

$$\operatorname{div} \boldsymbol{q} = b \tag{5.4}$$

where b is the volume density of Q.

Any function $\boldsymbol{q}$ satisfying the assertions of the above theorem will be called a **flux vector** for Q. Any two flux vectors differ at most on a set of Lebesgue measure zero.

Under stronger assumptions about the flux ZIEMER [3] proves that div $q \in L^1(B, V)$ (in fact, under his assumptions, div $q \in L^\infty(B, V)$), but his method of proof, following essentially the traditional line, does not lead to the divergence theorem (5.3).

Proof. By using a suitable affine transformation, one can assume that the vectors occurring in the definition of the Cauchy flux of class L^1 are the natural basis vectors $e_1, \ldots, e_N$ in $\mathbb{R}^N$.

We shall first prove that for each i ($1 \leqq i \leqq N$) there is a Borel function $q_i \in L^1(B, V)$ such that

$$\int_{\partial P} x_i^* \, dQ = \int_P (x_i^* b + q_i) \, dV \tag{5.5}$$

for every $P \in \mathscr{P}$, where $x_i^*: \mathbb{R}^N \to \mathbb{R}$ is a natural coordinate function, given by

$$x_i^*(x) = x_i, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$$

Then we shall define a vector field $q \in L^1(B, V)$ by

$$q(x) = (q_1(x), \ldots, q_N(x)), \quad x \in B, \tag{5.6}$$

and prove that

$$\int_{\partial P} v \, dQ = \int_P (vb + \nabla v \cdot q) \, dV \tag{5.7}$$

for every $P \in \mathscr{P}$ and every Lipschitz continuous function v on B.

To prove (5.5), let i ($1 \leqq i \leqq N$) be fixed. Consider, for each $t \in \mathbb{R}$, the closed half-space $R(t)$ in $\mathbb{R}^N$ given by $R(t) = \{x \in \mathbb{R}^N : x_i^*(x) \leqq t\}$, the corresponding open half-space $R^0(t) = \{x \in \mathbb{R}^N : x_i^*(x) < t\}$, and the boundary hyperplane $N(t) = \{x \in \mathbb{R}^N : x_i^*(x) = t\}$.

Let P be any part of the body. Denote by $P(t)$ the intersection of P with the closed half-space $R(t)$,

$$P(t) = P \cap R(t).$$

It can be shown that $P(t)$ is a set of finite perimeter; hence $P(t) \in \mathscr{P}$ for every $t \in \mathbb{R}$. The boundary of $P(t)$ in the sense of measure theory is given by

$$\partial P(t) = S(t) \cup T(t),$$

where $S(t)$ is the portion of the boundary ∂P of the original set P that is contained in the open half-space $R^0(t)$,

$$S(t) = \partial P \cap R^0(t),$$

and $T(t)$ is the remaining part of $\partial P(t)$; one has, for almost every $t \in \mathbb{R}$,

$$T(t) = P \cap N(t) \tag{5.8}$$

to within a set of zero area. All these essentially geometric facts are intuitively clear and can be verified formally by using the definition of a set of finite perimeter. Note also that most of the above facts are special cases of a general method of "slicing" described in FEDERER [11].

We put

$$F(t) = Q(S(t)),$$
$$G(t) = Q(T(t)),$$
$$H(t) = \int_{P(t)} b\, dV.$$

The boundedness of P and (5.8) imply that G vanishes outside some compact interval. Applying Condition (1) of Theorem 1 to $P(t)$ yields

$$F(t) + G(t) = H(t) \tag{5.9}$$

for every $t \in \mathbb{R}$. Applying (5.2) to $T(t)$ yields

$$|G(t)| \leqq \int_{T(t)} h\, dA. \tag{5.10}$$

Now F is the distribution function of the function x_i^* with respect to the measure induced by Q on ∂P and H is the distribution function of x_i^* with respect to the measure $L \mapsto \int_{L \cap P} b\, dV$, $L \subset \mathbb{R}^N$, a Borel set. Hence F and H have bounded variation. (5.9) implies that also G has bounded variation. The general properties of distribution functions enable us to express the integrals of x_i^* with respect to the indicated measures through the corresponding distribution functions as the "first moments":

$$\int_{\partial P} x_i^*\, dQ = \int_R t\, dF(t),$$
$$\int_P x_i^* b\, dV = \int_R t\, dH(t).$$

The last two formulas, (5.9), integration by parts, and the fact that G vanishes outside some compact interval justify the following computation:

$$\begin{aligned}\int_{\partial P} x_i^*\, dQ - \int_P x_i^* b\, dV &= \int_{\mathbb{R}} t\, d(F - H)(t)\\ &= -\int_{\mathbb{R}} t\, dG(t)\\ &= \int_{\mathbb{R}} G(t)\, dt\end{aligned}$$

Inequality (5.10), relation (5.8) and Fubini's theorem establish the inequality

$$\begin{aligned}\left|\int_{\partial P} x_i^*\, dQ - \int_P x_i^* b\, dV\right| &\leqq \int_{\mathbb{R}} |G(t)|\, dt\\ &\leqq \int_{\mathbb{R}} \left(\int_{T(t)} h\, dA\right) dt\\ &= \int_{\mathbb{R}} \left(\int_{P \cap N(t)} h\, dA\right) dt\\ &= \int_P h\, dV,\end{aligned}$$

i.e.,

$$\left| \int_{\partial P} x_i^* \, dQ - \int_P x_i^* b \, dV \right| \leqq \int_P h \, dV \tag{5.11}$$

for every part $P \in \mathscr{P}$.

It is now observed that the set function

$$P \mapsto \int_{\partial P} x_i^* \, dQ - \int_P x_i^* b \, dV, \qquad P \in \mathscr{P}, \tag{5.12}$$

is additive. Indeed, it is obvious that the volume integral is additive. The additivity of the surface integral in (5.12) is verified by using (4.4) to cancel the contributions to the sum from the overlapping parts of the boundaries.

To summarize, the set function (5.12) is additive and satisfies (5.11). The result of FUGLEDE [15] then implies the existence of a Borel function $q_i \in L^1(B, V)$ such that (5.5) holds.

We now define $\boldsymbol{q}$ by (5.6) and prove (5.7). Noting that $\nabla x_i^* = \boldsymbol{e}_i$, one sees that (5.5) is precisely (5.7) for $v = x_i^*$. Further, (5.7) is also satisfied by constant functions, for in this case the gradient vanishes and (5.7) reduces to the already established equality (4.1) of Theorem 1. But this proves that (5.7) holds for all affine functions since an affine function is a linear combination of $x_1^*, \ldots, x_N^*$ and of constants.

Next, if v is piecewise affine, we establish the validity of (5.7) for a general part P by dividing it into smaller parts R_k on which v is affine, and using the already established validity of (5.7) for affine functions. Then adding the equalities (5.7) for each R_k and using the additivity of the expressions on both sides of the equality, we find that (5.7) holds. (To prove the additivity of the left-hand side, we must evoke (4.4) to cancel the contributions of the overlapping parts of the boundaries of R_k's.)

Finally, if v is a general Lipschitz continuous function on B, then there is a sequence of piecewise affine functions v_n such that

$$v_n \to v \qquad \text{uniformly on } B$$

$$\nabla v_n \to \nabla v \qquad V\text{-a.e. on } B$$

and

$$|\nabla v_n| \leqq C \qquad V\text{-a.e. on } B,$$

with C a constant independent of n. (See *e.g.* EKELAND & TEMAM [16].) Applying (5.7) to v_n and letting n tend to ∞ then yields (5.7) in the general case.

We now prove (5.4). For every $\varphi \in C_0^\infty(B)$ there is a part P of the body such that the support of φ is contained in the interior of P. Hence the function φ vanishes on the boundary of P, and applying (5.7) to P and the function φ yields

$$0 = \int_{\partial P} \varphi \, dQ = \int_P \varphi b + \nabla \varphi \cdot \boldsymbol{q} \, dV = \int_B \varphi b + \nabla \varphi \cdot \boldsymbol{q} \, dV.$$

But this equality means precisely that the divergence of $\boldsymbol{q}$ in the sense of distributions is of class L^1 and equals b.

Finally, (5.4) and (5.2) enable us to reduce the right-hand side of (5.7) to the right-hand side of (5.3), and the proof is complete.

6. The surface densities

In this section we consider a group of results associated with the expression for the surface density in terms of the flux vector.

Theorem 3. *Let Q be a weakly balanced Cauchy flux of class L^1 with the corresponding flux vector $\boldsymbol{q}$. Then there is a Borel subset $N_0 \subset B$ of Lebesgue measure zero such that*

$$Q(S) = \int_{\hat{S}} \boldsymbol{q} \cdot \boldsymbol{n} \, dA \tag{6.1}$$

for every material surface $S = (\hat{S}, \boldsymbol{n})$ satisfying

$$A(\hat{S} \cap N_0) = 0. \tag{6.2}$$

In other words, the surface density corresponding to any such a material surface is given by

$$q^S(x) = \boldsymbol{q}(x) \cdot \boldsymbol{n}(x) \qquad A\text{-a.e. on } S. \tag{6.3}$$

Proof. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a non-negative, infinitely differentiable, spherically symmetric function with compact support satisfying

$$\int_{\mathbb{R}^{N-1}} \varphi(y_1, \dots, y_{N-1}, 0) \, dy_1 \dots dy_{N-1} = 1, \tag{6.4}$$

and let P be any part of the body. In this situation, the following lemma holds.

Lemma 1. *For A-a.e. $x \in \partial P$,*

$$\lim_{r \to 0+} r^{-N} \int_P \nabla \varphi(r^{-1}(x - y)) \, dV(y) = \boldsymbol{n}^P(x) \tag{6.5}$$

and

$$\lim_{r \to 0+} r^{-N+1} \int_{\partial P} \varphi(r^{-1}(x - y)) \, dA(y) = 1. \tag{6.6}$$

The first assertion of the lemma is verified by a direct computation if P is a half-space. The general case is then verified by using this special case and the definition of the normal in the sense of measure theory. The details are omitted. By applying the Gauss-Green theorem (FEDERER [11]), we can transform the volume integral in (6.5) into a surface integral to restate (6.5) in the form

$$\lim r^{-N+1} \int_{\partial P} \varphi(r^{-1}(x - y)) \, \boldsymbol{n}^P(y) \, dA(y) = \boldsymbol{n}^P(x)$$

(from now on I omit the symbol $r \to 0+$ in the limits.) This statement implies (6.6) at every Lebesgue point of $\boldsymbol{n}^P$ relative to A. The lemma is proved.

The proof of Theorem 3 is now easily completed. Let N_0 be the complement (in B) of the set of all Lebesgue points x for $\boldsymbol{q}$ that satisfy

$$\lim r^{-N+1} \int_B \varphi(r^{-1}(x-y))\, b(y)\, dV(y) = 0 \tag{6.7}$$

We easily find that (6.7) is satisfied at every Lebesgue point x for b and since V-a.e. point of B is simultaneously a Lebesgue point for both $\boldsymbol{q}$ and b, we have $V(N_0) = 0$.

Now let S satisfy (6.2). Then A-a.e. point $x \in S$ is a Lebesgue point for $\boldsymbol{q}$ and satisfies (6.7). In virtue of this and in view of (6.5) then, if P is any part with $S \subset \partial P$,

$$\lim r^{-N} \int_P \nabla\varphi(r^{-1}(x-y)) \cdot \boldsymbol{q}(y)\, dV(y) = \boldsymbol{q}(x) \cdot \boldsymbol{n}^P(x) \tag{6.8}$$

and similarly, by (6.6),

$$\lim r^{-N+1} \int_{\partial P} \varphi(r^{-1}(x-y))\, q^{\partial P}(y)\, dA(y) = q^{\partial P}(x) \tag{6.9}$$

for A-a.e. $x \in S$. Application of Theorem 2 (in the form of the equality (5.7)) to the function $v(y) = r^{-N+1}\varphi(r^{-1}(x-y))$ and then use of (6.7), (6.8) and (6.9) will reduce (5.7) with our special choice of v to

$$q^{\partial P}(x) = \boldsymbol{q}(x) \cdot \boldsymbol{n}^P(x) \qquad \text{for } A\text{-a.e. } x \in S,$$

and the results follow.

Unfortunately it is not known whether one can change the flux vector on a set of zero Lebesgue measure so as to make (6.1) hold for every material surface. The set N_0 was defined as the complement of the set of all Lebesgue points for $\boldsymbol{q}$ that satisfy (6.7). The following proposition gives some information about the size of the set of points x that satisfy (6.7).

Lemma 2. *If* $b \in L^1(B, V)$, *then*

$$\lim r^{-N+1} \int_{B(x,r)} |b(y)|\, dV(y) = 0 \tag{6.10}$$

for A-a.e. *point* $x \in B$.

According to this lemma, not only the "volume", but also the "area" of the set of points that do not satisfy (6.10) is zero. Note that if b is the volume density of a weakly balanced Cauchy flux, then the lemma implies that

$$\lim r^{-N+1} Q(\partial B(x,r)) = 0$$

for A-a.e. $x \in B$ and this information is a strengthened version of the information obtained in the "Note" at the end of Section 4 concerning the vanishing of the limits of the quantities indicated there. In that note we derived this information from Condition (3) of Theorem 1 without having b at our disposal.

Proof. Let M be the set of all points of B where (6.10) holds. As has been pointed out in the proof of Theorem 3, the complement of M satisfies

$$V(B - M) = 0. \tag{6.11}$$

We apply FEDERER [11, Thm. 2.10.18, item (2)] to the present situation. Namely, we identify his space X with our B, his measure μ with the indefinite integral

$$\mu(Z) = \int_{Z \cap B} |b| \, dV$$

for any Borel subset $Z \subset B$, choose his function ζ to be given by $\zeta(S) = (\operatorname{diam} S)^{N-1}$, the family C to be the family of all balls, and the set A to be our M. The measure ψ occurring in the indicated theorem then must be our measure A. With this choice of the objects the theorem asserts that

$$\lim r^{-N+1} \int_{M \cap B(x,r)} |b| \, dV = 0$$

for A-a.e. $x \in B - M$. In virtue of (6.11) the limit in the last equality is the same as the limit in (6.10), and hence (6.10) holds for A-a.e. $x \in B - M$. However, since according to our definition of M (6.10) holds at *no* point of $B - M$, we conclude that $A(B - M) = 0$, and the proof is complete.

According to this lemma, we may concentrate our attention entirely on the complement of the set of Lebesgue points of $\boldsymbol{q}$ since adding the points that do not satisfy (6.7) increases negligeably the size of the set N_0 from the point of view of surface integration. The set of all Lebesgue points of $\boldsymbol{q}$ will be the whole of B if the flux vector is continuous, and that in turn is true if the flux Q is continuous in the following sense. A Cauchy flux Q of class L^1 is said to be **continuous** if there are N linearly independent vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_N \in \mathbb{S}^{N-1}$ and continuous functions $q_1^*, \dots, q_N^*$ on B such that

$$Q(S) = \int_S q_i^* \, dA$$

for every planar material surface element S perpendicular to $\boldsymbol{n}_i$.

Theorem 4. *Let Q be a weakly balanced continuous Cauchy flux. Then its flux vector is continuous, and* (6.1) *holds for every material surface.*

This is essentially a result of NOLL [5] under slightly weaker assumptions. (Note that Lemma 2 can be avoided completely if Q satisfies the conditions of Lipschitz continuity with respect to volume.)

We conclude with another theorem generalizing a known result (NOLL [5], ZIEMER [3]). To state the result, we say that a Cauchy flux is **of class L^∞** if there are N linearly independent vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_N \in \mathbb{S}^{N-1}$ and a constant C such that

$$|Q(S)| \leq CA(S)$$

for every planar material surface S perpendicular to one of the vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_N$.

Theorem 5. *Let Q be a weakly balanced Cauchy flux of class L^∞. Then there is a bounded function* $q^*: B \times \mathbb{S}^{N-1} \to \mathbb{R}$ *such that*

$$Q(S) = \int_S q^*(x, \boldsymbol{n}(x))\, dA$$

for every material surface S.

The proof is based on essentially the same idea as the proof of Theorem 3 and will be omitted. Note again that the present version of the result generalizes the existing ones since the condition of Lipschitz continuity with respect to volume is not imposed and Lipschitz continuity with respect to area is postulated only for planar material surfaces perpendicular to the vectors $\boldsymbol{n}$. Also it is worth stressing that the previous works use this result as an important intermediate step in proving the more concrete results of the type of Theorems 3, 4, while here we have found a way to avoid use of the function $q^*(x, \boldsymbol{n})$. Theorem 5 is included for completeness and to indicate that no information was lost by following the path indicated in this paper.

Acknowledgements. I thank Professors WALTER NOLL and MARIO PITTERI for discussions on a previous draft of the paper during my stays at CNUCE, the institute of CNR in Pisa, and at the University of Padova. These discussions helped me in particular to make a final choice between the two possible constructions of the flux vector I had. I also thank Professor CLIFFORD TRUESDELL for improving the English of the paper and for a discussion on the historical position of Cauchy's postulate.

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Mathematical Institute
Czechoslovak Academy of Sciences
Prague

(Received December 28, 1984)